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ORIGINAL ARTICLE

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# Iterative methods for nonlocal elasticity problems

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**Abstract** Nonlocal elasticity is addressed in the context of geometrically linearised structural models with linear and symmetric constitutive relations between dual fields, with physical interpretation of stress and elastic strain. The theory applies equally well to formulations as integral convolutions with an averaging symmetric kernel and to those involving an elastic potential depending on local gradients. According to the original strain-driven nonlocal elastic model, input and output of the stiffness operator are, respectively, the elastic strain field and the stress field. The swapped correspondence is assumed in the recently proposed stress-driven nonlocal elastic model, with input and output of the compliance operator, respectively, given by the stress field and the elastic strain field. Two distinct nonlocal elasticity models, not the inverse of one another, may thus be considered. The strain-driven model leads to nonlocal elastostatic problems which may not admit solution due to conflicting requirements imposed on the stress field by constitutive law and equilibrium. To overcome this obstruction, several modifications of the original scheme have been proposed, including mixtures of local/nonlocal elastic laws and compensation of boundary effects. On the other hand, the stress-driven model leads to nonlocal elastostatic problems which are consistent and well-posed. Two iterative methods of solution, respectively, for stress-driven and mixture strain-driven models, are here contributed and analysed. The relevant algorithms require only solutions of standard local elastostatic problems. Fixed points of the algorithms are shown to be coincident with solutions of the pertinent nonlocal elastostatic problem. It is also proven that, for statically determinate structural models, the iterative method pertaining to the nonlocal stress-driven model converges to the solution just at the first step. For statically indeterminate ones, the computations reveal that convergence is asymptotic but very fast. Clamped-free and clamped-supported nano-beams are investigated as standard examples in order to provide evidence of the theoretical results and to test and show performance of the iterative procedures.

**Keywords** Nonlocal elasticity · Strain-driven model · Stress-driven model · Fixed points algorithms · Iterative methods

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## 1 Introduction

Nonlocal elastic models were originally proposed in the literature as a viable alternative for investigating problems involving dynamical properties of atomic lattices. A basic reference is [1] where an integral convolution between the elastic strain and an averaging kernel was proposed to simulate long-range effects on the evaluation of the stress field. This scheme was referred to as the *strain-driven* constitutive model in [2,3].

Equivalent differential formulations discussed in [1] are based on linearity of the convolution operator and on the property that the kernel appearing in the convolution is the fundamental solution of a differential operator.

This equivalence is a well-known result of potential theory in linear spaces, for problems in unbounded domains where the involved fields are rapidly decreasing at infinity. For problems on $\Re^n$ fundamental solutions of several differential operators are available [4]. On the other way, for bounded domains fundamental solutions are available only in special cases.

In more recent times, the differential equation associated with a strain-driven model with Helmholtz kernel was diffusely applied in static and dynamic investigations of nanostructures [5–7].

Most treatments were, however, not aware of the fact that, for bounded domains, constitutive boundary conditions must also be imposed to obtain the closure of the constitutive law, thus ensuring equivalence to the integral convolution constitutive law [2,8–10].

Paradoxical results detected in the literature are explained by the observation that stress fields, output by the *strain-driven* model, are not able to fulfil equilibrium conditions, so that no solution to the elastostatic problem exists [11]. When constitutive boundary conditions are compared with requirements dictated by equilibrium, the conflict is evident [2,9,10].

The pertinent literature amounts nowadays to a huge collection and is ever increasing, notwithstanding the serious obstructions illustrated above and put into evidence by recent contributions [2,3,11–14].

Several skilful modifications have been proposed to overcome the inherent ill-posedness of the strain-driven model.

A first expedient was the formulation of mixtures of local–nonlocal elastic behaviours [9,10,15–19]. This modification is able to ensure existence of a solution, but leads to singularities for small percentage of the local elastic model.

More recently, compensations of boundary effects were conceived in [20–22]. The modified kernel suggested in [20] generates a nonsymmetric kernel and hence a nonsymmetric response operator.

An improvement was brought about by the modified response contributed in [21,22] which is effective in preserving uniformity of the output field generated by a uniform input field, and generates a symmetric response operator, so that existence of an elastic potential is assured. It may, however, lead to drastic reductions of nonlocal size effects [2,14]. Mathematical aspects of thin structures have been investigated in [23,24].

Discrete formulations, for instance, by finite element method (FEM), transform continuum problems into solvable algebraic ones [25,26]. Indeed, difficulties inherent to nonlocal strain-driven continuum formulations are hidden in a decisive way since equilibrium requirements are drastically relaxed by discretisation. Moreover, equilibrium features of stress distributions emerging from numerical computations are not discussed, with only displacement solutions explicitly displayed and commented upon.

A decisive contribution for setting up a well-posed theory of nonlocal elasticity was the innovative proposal of a *stress-driven* model of nonlocal elasticity, made in [13].

The idea consists in swapping the roles of the involved fields, with the stress as input and the elastic strain as output. This new model is in line with the theory of incremental elasticity developed in [27,28].

It is to be underlined that strain-driven and stress-driven models may not be the inverse of one another and are not such, in particular, in the case of integral convolutions [3].

An iterative scheme of solution for strain-driven nonlocal elastostatic problems was early proposed in [29] with a misplaced sign of the correction strain. A change of sign is in fact needed to recover kinematic compatibility, as illustrated in Sect. 6, Eq. (72). Basic difficulties due to nonexistence of a solution, for pure strain-driven models, were also not evidenced since the iterative scheme was not put into operation.

Recently, the iterative scheme in [29] has been resorted to in [30–32] with the intent of proposing a remedy to overcome obstructions against the adoption of a pure strain-driven model. The strategy there adopted is declaredly founded on the hope that an iterative scheme could help in overcoming these basic obstructions. We cannot give any credit to this unsupported hope since the problem at hand is not modified and only an alternative method of solution (if any) is conceived.

As a matter of fact, the proposals in [30–32] were somewhat unclearly and incompletely expressed and somewhere incorrectly formulated or codified. The improper sign in the definition of the correction strain, inherited from [29], is still present in these treatments. It will unavoidably generate incorrect results when put into operation in the iterative procedure.

In the recent treatment [32] diagrams of bending fields, generated by the proposed iterative procedure for nonlocal cantilevers and simply supported beams, were displayed revealing a dependence on the nonlocality parameter. This outcome crashes against the fact that in statically determinate structures there are no nontrivial self-bending fields, so that the bending field is univocally determined by equilibrium conditions.

This is what elementary structural analysis *docet*.

We revisit here the whole matter afresh, by providing algorithmic procedures and iterative schemes for nonlocal elasticity problems formulated according to both stress-driven models and modified strain-driven models.

Theoretical results, concerning coincidence of the fixed point for the algorithms with the solutions of the nonlocal problems, are contributed. Notions and definitions introduced in [2,3,13] are recalled in Sect. 2.

Nonexistence of solution for pure strain-driven nonlocal elastic problems on bounded domains, put into evidence in [12] and acknowledged in [33], is confirmed by lack of convergence of the relevant iterative procedure.

On the other way, for stress-driven and modified strain-driven nonlocal elastic problems, the conceived iterative procedures are effective in providing the solution as fixed point of the relevant algorithm.

The performed computational tests show that convergence to the solution is quite fast so that, from the applicative standpoint, iterations can be effectively stopped after the very first steps.

In particular, for statically determinate structural problems, such as cantilevers (CF) or simply supported beams (SS), where self-equilibrated bending fields vanish identically, the iterative procedure for stress-driven models yields the exact solution just at the first step of iteration.

Iterative procedures relevant to mixture strain-driven model do not enjoy such optimality property and convergence is always asymptotic, but still fast.

Computations and graphic output were carried out by WOLFRAM's code MATHEMATICA® [34].

## 2 Prolegomena

Let us consider a geometrically linearised structural model occupying a configuration $\boldsymbol{\Omega}$ in the EUCLID space $\Sigma$ and subject to linear constraints. Regularity of the boundary $\partial\boldsymbol{\Omega}$ may be expressed by the cone property [35].

*Conforming* kinematic fields belong then to a closed linear subspace $\mathcal{L} \subset \mathcal{V}$ of a linear kinematic space HILBERT $\mathcal{V}$ with dual force space $\Lambda = \mathcal{V}'$.

The kinematic operator

$$\mathbf{B} : \mathcal{V} \mapsto \mathcal{D}, \tag{1}$$

is a linear map with a closed range in the HILBERT space $\mathcal{D}$.

It evaluates the (small) strain $\mathbf{B}(\mathbf{u}) \in \mathcal{D}$ corresponding to a given (small) displacement $\mathbf{u} \in \mathcal{V}$.

Stress and strain spaces $\Sigma$ and $\mathcal{D}$ are HILBERT spaces in duality. Duality pairings will all be denoted by the symbol:

$$\langle \cdot, \cdot \rangle. \tag{2}$$

The dual operator $\mathbf{B}' : \Sigma \mapsto \Lambda$ evaluates the force $\mathbf{f} \in \Lambda$ in equilibrium with a given stress field $\boldsymbol{\sigma} \in \Sigma$ and is uniquely defined by the virtual power identity:

$$\langle \boldsymbol{\sigma}, \mathbf{B}(\mathbf{u}) \rangle = \langle \mathbf{B}'\boldsymbol{\sigma}, \mathbf{u} \rangle, \quad \forall \boldsymbol{\sigma} \in \Sigma, \quad \forall \mathbf{u} \in \mathcal{V}. \tag{3}$$

From this basic duality relation, two polarity properties are directly deduced:

$$\begin{cases} \mathrm{Ker}(\mathbf{B}) = \big(\mathrm{Im}(\mathbf{B}')\big)^{\circ}, \\ \mathrm{Ker}(\mathbf{B}') = \big(\mathrm{Im}(\mathbf{B})\big)^{\circ}. \end{cases} \tag{4}$$

In a HILBERT space $\mathcal{X}$, the polar $\mathcal{A}^{\circ} \subset \mathcal{X}'$ of a given set $\mathcal{A} \subset \mathcal{X}$ is the linear subset of the elements in the dual space $\mathcal{X}'$ which have a null duality interaction with all elements of $\mathcal{A} \subset \mathcal{X}$:

$$\mathcal{A}^{\circ} := \{\mathbf{x}' \in \mathcal{X}' \,:\, \langle \mathbf{x}', \mathbf{x} \rangle = 0, \quad \forall \mathbf{x} \in \mathcal{A}\}, \tag{5}$$

and similarly for the polar $\mathcal{B}^\circ \subset \mathcal{X}$ of a set $\mathcal{B} \subset \mathcal{X}'$.

By BANACH closed range theorem [36,37], the equilibrium operator $\mathbf{B}' : \varSigma \mapsto \varLambda$ has closed range too and the following complementary polarity properties hold true:

$$\begin{cases} \mathrm{Im}(\mathbf{B}') = \big(\mathrm{Ker}(\mathbf{B})\big)^\circ, \\ \mathrm{Im}(\mathbf{B}) = \big(\mathrm{Ker}(\mathbf{B}')\big)^\circ. \end{cases} \tag{6}$$

The relations in Eq. (6) provide the basic existence results in continuum mechanics. In the geometrically linearised theory the assigned data are:

- a prescribed (small) displacement $\mathbf{w} \in \mathcal{V}$,
- a prescribed (small) strain $\boldsymbol{\eta} \in \mathcal{D}$,
- a loading functional:

$$\ell \in \mathcal{L}' \equiv \varLambda / \mathcal{L}^\circ, \tag{7}$$

where $\mathcal{L}^\circ$ is the linear subspace of reactive forces, those performing no virtual power for any *conforming* virtual velocity field:

$$\mathbf{r} \in \mathcal{L}^\circ \subset \varLambda \iff \langle \mathbf{r}, \delta\mathbf{v} \rangle = 0, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}. \tag{8}$$

Displacement fields belonging to the variety $\mathbf{w} + \mathcal{L}$ are said to be *admissible*.

The kinematic operator $\mathbf{B}_{\mathcal{L}} : \mathcal{L} \mapsto \mathcal{D}$, restriction of the operator $\mathbf{B} : \mathcal{V} \mapsto \mathcal{D}$ to the conformity subspace $\mathcal{L} \subset \mathcal{V}$, is the dual of the static operator $\mathbf{B}'_{\mathcal{L}} : \varSigma \mapsto \mathcal{L}'$ so that by Eq. (6):

$$\begin{cases} \mathrm{Im}(\mathbf{B}'_{\mathcal{L}}) = \big(\mathrm{Ker}(\mathbf{B}_{\mathcal{L}})\big)^\circ, \\ \mathrm{Im}(\mathbf{B}_{\mathcal{L}}) = \big(\mathrm{Ker}(\mathbf{B}'_{\mathcal{L}})\big)^\circ. \end{cases} \tag{9}$$

Without loss of generality, it can be assumed that rigid conforming displacements vanish identically:

$$\mathrm{Ker}(\mathbf{B}_{\mathcal{L}}) = \mathcal{L} \cap \mathrm{Ker}(\mathbf{B}) = \{\mathbf{0}\}_{\mathcal{V}}, \tag{10}$$

so that

$$\mathrm{Im}(\mathbf{B}'_{\mathcal{L}}) = \varLambda. \tag{11}$$

Let $\varSigma_\ell$ be the affine variety of stress fields in equilibrium with a current loading $\ell \in \varLambda$. Under condition in Eq. (10), the variety $\varSigma_\ell$ is nonempty for all $\ell \in \varLambda$, since Eq. (11) holds.

This means that, for any conforming virtual velocity, the following virtual power principle holds true:

$$\ell \in \mathrm{Im}(\mathbf{B}'_{\mathcal{L}}), \tag{12}$$

or explicitly:

$$\langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v}) \rangle = \langle \ell, \delta\mathbf{v} \rangle, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}. \tag{13}$$

Stress fields in $\varSigma_0 := \mathrm{Ker}(\mathbf{B}'_{\mathcal{L}})$ are self-equilibrated. If $\varSigma_0 = \{\mathbf{0}\}$, the structural problem is qualified as *statically determinate*.

## 3 Local and nonlocal elasticity

### 3.1 Local elasticity

The standard local elastic relation is governed by an invertible linear, symmetric and positive definite stiffness operator $E : \mathcal{D} \mapsto \varSigma$ with inverse compliance $C : \varSigma \mapsto \mathcal{D}$:

$$\boldsymbol{\sigma} = E\mathbf{e} \iff \mathbf{e} = C\boldsymbol{\sigma}. \tag{14}$$

The local elastostatic operator $\mathbf{P}_{\text{LOC}} : \mathcal{Y} \mapsto \mathcal{X}$, defined by

$$\mathbf{P}_{\text{LOC}}(\mathbf{d}) = \mathbf{t}, \quad \mathbf{t} \in \mathcal{X}, \quad \mathbf{d} \in \mathcal{Y}, \tag{15}$$

when acting on a data triplet in the source linear space:

$$\mathbf{d} \in \mathcal{Y} \iff (\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathcal{V}, \mathcal{D}, \Lambda), \tag{16}$$

made of imposed displacement field $\mathbf{w} \in \mathcal{V}$, impressed strain field $\boldsymbol{\eta} \in \mathcal{D}$ and prescribed loading $\ell \in \Lambda$ yields the target displacement–stress pair:

$$\mathbf{t} \in \mathcal{X} \iff \{\mathbf{u}, \boldsymbol{\sigma}\} \in (\mathcal{V}, \mathcal{S}), \tag{17}$$

made of the admissible displacement field $\mathbf{u} \in \mathbf{w} + \mathcal{L}$, and the stress field $\boldsymbol{\sigma} \in \Sigma$:

$$\{\mathbf{u}, \boldsymbol{\sigma}\} = \mathbf{P}_{\text{LOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathbf{w} + \mathcal{L}) \times \Sigma, \tag{18}$$

solution of the local elastic problem in Eqs. (13)–(14)–(20):

$$\boxed{\begin{cases} \langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v}) \rangle = \langle \ell, \delta\mathbf{v} \rangle, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}, \\ \boldsymbol{\sigma} = E\mathbf{e}, \\ \mathbf{e} = \mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}, \quad \mathbf{u} \in \mathbf{w} + \mathcal{L}. \end{cases}} \tag{19}$$

Existence and uniqueness of the solution $\{\mathbf{u}, \boldsymbol{\sigma}\} \in (\mathbf{w} + \mathcal{L}) \times \Sigma$ for any data $(\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathcal{V}, \mathcal{D}, \Lambda)$ are a standard result of the linear theory of local elasticity [35].

The elastic strain field $\mathbf{e} \in \mathcal{D}$ corresponding to a given displacement field $\mathbf{u} \in \mathcal{V}$ is defined as difference between the geometric strain $\mathbf{B}(\mathbf{u}) \in \mathcal{D}$ and a prescribed strain $\boldsymbol{\eta} \in \mathcal{D}$:

$$\mathbf{e} := \mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}. \tag{20}$$

The condition of *kinematic compatibility* on the total strain $\mathbf{e} + \boldsymbol{\eta} \in \mathcal{D}$, sum of elastic and anelastic contributions, expressed by Eq. $(19)_3$, can be formulated in variational terms by requiring that

$$\mathbf{e} + \boldsymbol{\eta} - \mathbf{B}(\mathbf{w}) \in \operatorname{Im}(\mathbf{B}_{\mathcal{L}}) = \operatorname{Ker}(\mathbf{B}'_{\mathcal{L}})^{\circ}. \tag{21}$$

The polarity condition in Eq. (21) means that the duality interaction with any self-equilibrated stress field is vanishing:

$$\langle \delta\boldsymbol{\sigma}, \mathbf{e} + \boldsymbol{\eta} - \mathbf{B}(\mathbf{w}) \rangle = 0, \quad \forall\, \delta\boldsymbol{\sigma} \in \Sigma_0 = \operatorname{Ker}(\mathbf{B}'_{\mathcal{L}}). \tag{22}$$

## 3.2 Nonlocal elasticity

A nonlocal constitutive relation may be formally written by stating that an output field $f \in \mathcal{F}$ over the domain $\boldsymbol{\Omega}$ is obtained by acting with a linear operator $\mathcal{R}$ on a source field $s \in \mathcal{S}$ over the same domain:

$$\langle f, \delta s \rangle = \langle \mathcal{R}(s), \delta s \rangle, \quad \forall\, \delta s \in \delta\mathcal{S}. \tag{23}$$

The linear operator $\mathcal{R} : \mathcal{S} \mapsto \mathcal{F}$ is assumed to be an injective mapping between Hilbert spaces in duality, made of fields from $\boldsymbol{\Omega}$ to a target finite-dimensional tensor space $E$. The inverse mapping may not be available, in general.

The test fields $\delta s \in \delta\mathcal{S}$ belong to a suitable linear test space $\delta\mathcal{S} \subseteq \mathcal{S}$. If the injection $\delta\mathcal{S} \hookrightarrow \mathcal{S}$ is continuous and dense, the variational condition in Eq. (23) can be written as an equality:

$$f = \mathcal{R}(s). \tag{24}$$

For application to nonlocal elasticity, the response operator $\mathcal{R}$ is assumed to be symmetric:

$$\langle \mathcal{R}(s_1), s_2 \rangle = \langle \mathcal{R}(s_2), s_1 \rangle. \tag{25}$$

It is then derivable from a quadratic potential $\Phi : \mathcal{S} \mapsto \Re$:[1]

$$\langle f, \delta s \rangle = \langle \nabla\Phi(s), \delta s \rangle, \quad \forall\, \delta s \in \delta\mathcal{S}, \tag{26}$$

with

$$\Phi(s) = \tfrac{1}{2}\langle \mathcal{R}(s), s \rangle. \tag{27}$$

The law in Eq. (26) with the potential in Eq. (27) is equivalent to Eq. (23). In this general framework, the results exposed in the sequel may be equally well applied to the following models of nonlocal elasticity.

[1] The symbol $\delta$ is just a prefix emphasising that arbitrary test fields belonging to a linear test space are involved in the analysis.

1. A first model of nonlocal elasticity involves an integral convolution with an averaging kernel. Accordingly, the point values of the field $f$ output by the law in Eq. (24) are obtained by the relation:[2]

$$f_{\mathbf{x}} = \mathcal{R}(s)_{\mathbf{x}} = (\varphi * Ks)(\mathbf{x}) := \int_{\boldsymbol{\Omega}} \varphi(\|\mathbf{x}-\mathbf{y}\|) \cdot Ks(\mathbf{y})\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{y}}. \tag{28}$$

The integral convolution involves a smooth positive kernel function $\varphi : \Re_{\geq 0} \mapsto \Re_{>0}$ which is decreasing for increasing distance of $\mathbf{y} \in \boldsymbol{\Omega}$ from $\mathbf{x} \in \boldsymbol{\Omega}$ and a uniform constitutive operator $K : \mathcal{S}_{\mathbf{x}} \mapsto \mathcal{F}_{\mathbf{x}}$, which is independent of $\mathbf{x} \in \boldsymbol{\Omega}$, symmetric and positive definite:

$$\begin{cases} K = K^A \iff \langle K s_{1\mathbf{x}}, s_{2\mathbf{x}} \rangle = \langle K s_{2\mathbf{x}}, s_{1\mathbf{x}} \rangle, \\ s_{\mathbf{x}} \neq \mathbf{0} \implies \langle K s_{\mathbf{x}}, s_{\mathbf{x}} \rangle > 0. \end{cases} \tag{29}$$

The standard local elasticity is recovered, to within boundary effects, by imposing that the kernel depends on a scale parameter $\lambda > 0$ and fulfils the following *impulsivity condition* (IC):

$$\begin{aligned} \lim_{\lambda \to 0^+} (\varphi_\lambda * Ks)_{\mathbf{x}} &= \lim_{\lambda \to 0^+} \int_{\boldsymbol{\Omega}} \varphi_\lambda(\|\mathbf{x}-\mathbf{y}\|) \cdot Ks(\mathbf{y})\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{y}} \\ &= \Theta\, Ks(\mathbf{x}), \quad \forall \mathbf{x} \in \boldsymbol{\Omega}, \end{aligned} \tag{30}$$

with $\Theta = 1$ at inner points in $\boldsymbol{\Omega}$. It follows that on the boundary $\Theta = 1/2$ at regular points [2], while $\Theta < 1/2$ is equal to the fraction of solid inward angle at singular points [14].
This means that for $\lambda \to 0^+$ the response operator tends to a DIRAC impulse at interior points and to a fraction of it at boundary points, with a reduction factor not less than one half.
A convex combination of local/nonlocal elastic models, with $0 \leq m \leq 1$ mixture parameter, can be considered by setting:

$$f = m \cdot s + (1-m) \cdot (\varphi * Ks). \tag{31}$$

For $m = 0$ the standard nonlocal integral convolution is recovered:

$$f = \varphi * Ks. \tag{32}$$

2. A second model is gradient elasticity in which point values of a differentiable scalar elastic potential $\Phi_{\mathbf{x}}$ depend on point values of the source field and of its gradient:

$$\Phi(s)_{\mathbf{x}} = \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s), \tag{33}$$

with the global potential given by [38]:

$$\Phi(s) := \int_{\boldsymbol{\Omega}} \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s)\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{x}}. \tag{34}$$

Then, from LEIBNIZ rule and integration by parts, denoting by $\nabla^A$ the formal adjoint of $\nabla$, we have:

$$\begin{aligned} &\langle \nabla \Phi(s), \delta s \rangle_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \langle \nabla_1 \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s), \delta s_{\mathbf{x}} \rangle + \langle \nabla_2 \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s), \nabla_{\mathbf{x}} \delta s \rangle\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{x}} \\ &= \int_{\boldsymbol{\Omega}} \langle \nabla_1 \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s), \delta s_{\mathbf{x}} \rangle - \langle \nabla_{\mathbf{x}}^A \Big( \nabla_2 \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s) \Big), \delta s_{\mathbf{x}} \rangle\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{x}} \\ &\quad + \oint_{\partial \boldsymbol{\Omega}} \langle \nabla_2 \Phi_{\mathbf{x}}(s_{\mathbf{x}}, \nabla_{\mathbf{x}} s) \cdot \mathbf{n}, \delta s_{\mathbf{x}} \rangle\, \mathrm{d}\partial \boldsymbol{\mu}_{\mathbf{x}}. \end{aligned} \tag{35}$$

Considering suitable linear test spaces $\delta\mathcal{S} \subseteq \mathcal{S}$, the localisation of Eq. (35) provides the corresponding differential and boundary conditions.

[2] Here $*$ denotes the integral convolution and $\boldsymbol{\mu}$ is the standard volume form. Accordingly the source field is a density *per unit volume*.

Two distinct constitutive models are associated with the law in Eq. (23), depending on whether the first or the second of the following choices is made:

$$\begin{aligned} &s = \mathbf{e}, \quad f = \boldsymbol{\sigma}, \quad K = E \quad \text{(strain-driven)}, \\ &s = \boldsymbol{\sigma}, \quad f = \mathbf{e}, \quad K = C \quad \text{(stress-driven)}. \end{aligned} \tag{36}$$

In this paper attention and computations will be especially focused on these two models of nonlocal elasticity involving an integral convolution with an averaging kernel, as in Eqs. (31) or (32).

### 3.3 Strain-driven model

In a *mixture strain-driven* nonlocal elastic model, the stress field is provided by the constitutive relation in Eq. (24) where, $f = \boldsymbol{\sigma}$ and $s = \mathbf{e}$:

$$\boldsymbol{\sigma} = m \cdot E\mathbf{e} + (1 - m) \cdot \mathcal{R}(\mathbf{e}). \tag{37}$$

The integral convolution model adopted in [1] is obtained by setting:

$$\mathcal{R}(\mathbf{e})_{\mathbf{x}} = \big(\varphi * E\mathbf{e}\big)_{\mathbf{x}} := \int_{\boldsymbol{\Omega}} \varphi(\|\mathbf{x} - \mathbf{y}\|) \cdot E\mathbf{e}(\mathbf{y})\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{y}}. \tag{38}$$

A strain-driven nonlocal elastic problem consists of the following set of equations:

$$\boxed{\begin{cases} \langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v})\rangle = \langle \ell, \delta\mathbf{v}\rangle, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}, \\ \boldsymbol{\sigma} = E_{\mathrm{NLOC}}(\mathbf{e}), \\ \mathbf{e} = \mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}, \quad \mathbf{u} \in \mathbf{w} + \mathcal{L}. \end{cases}} \tag{39}$$

with the nonlocal stiffness defined by the local/nonlocal mixture:

$$E_{\mathrm{NLOC}}(\mathbf{e}) = m \cdot E\mathbf{e} + \big(1 - m\big) \cdot (\varphi * E\mathbf{e}). \tag{40}$$

For $m = 1$ the local elastic problem in Eq. (19) is recovered.

A main drawback of models of this kind is the following.

For a pure nonlocal model ($m = 0$), the stress fields

$$\boldsymbol{\sigma} = \mathcal{R}(\mathbf{e}), \tag{41}$$

with $\mathbf{e} = \mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}$ for any admissible displacement $\mathbf{u} \in \mathbf{w} + \mathcal{L}$, may not (and generally will not) be able to fulfil the equilibrium condition in Eq. $(39)_1$.

Accordingly, pure strain-driven nonlocal elastic problems will admit no solution and mixed models will manifest a singular behaviour when $m \to 0$.

### 3.4 Stress-driven model

To overcome this obstruction, a new stress-driven constitutive model was introduced in [13] by interchanging the roles of stress and elastic strains as input and output fields to define the nonlocal compliance:

$$\mathbf{e} = C_{\mathrm{NLOC}}(\boldsymbol{\sigma}), \tag{42}$$

In case of an integral convolution we have:

$$C_{\mathrm{NLOC}}(\boldsymbol{\sigma})_{\mathbf{x}} = \big(\varphi * C\boldsymbol{\sigma}\big)_{\mathbf{x}} := \int_{\boldsymbol{\Omega}} \varphi(\|\mathbf{x} - \mathbf{y}\|) \cdot C\boldsymbol{\sigma}(\mathbf{y})\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{y}}. \tag{43}$$

All difficulties met by the strain-driven model are overcome by adopting the new model.

We explicitly remark that the nonlocal elastic laws, described by the stress-driven model of Eq. (43) and by the strain-driven model of Eq. (38), are not one the inverse of the other.

A *stress-driven nonlocal elastic problem* $\mathbf{P}_{\text{NLOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell)$ consists of the following set of equations (Eqs. (13)–(20)–(43)):

$$\boxed{\begin{cases} \langle \boldsymbol{\sigma}, \mathbf{B}(\delta\mathbf{v}) \rangle = \langle \ell, \delta\mathbf{v} \rangle, \quad \forall\, \delta\mathbf{v} \in \mathcal{L}, \\ \mathbf{e} = C_{\text{NLOC}}(\boldsymbol{\sigma}), \\ \mathbf{e} = \mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}, \quad \mathbf{u} \in \mathbf{w} + \mathcal{L}, \end{cases}} \tag{44}$$

with the nonlocal compliance of the integral convolution stress-driven model defined by

$$C_{\text{NLOC}}(\boldsymbol{\sigma}) = \varphi * C\boldsymbol{\sigma}. \tag{45}$$

## 4 Iterative methods

### 4.1 Algorithms, fixed points and iterative schemes

Let us consider a problem whose solution $\mathbf{t} \in \mathcal{X}$ is provided by an operator $\mathbf{P} : \mathcal{Y} \mapsto \mathcal{X}$ with $\mathcal{X}, \mathcal{Y}$ HILBERT spaces:

$$\mathbf{P}(\mathbf{d}) = \mathbf{t}, \quad \mathbf{t} \in \mathcal{X}, \quad \mathbf{d} \in \mathcal{Y}. \tag{46}$$

Iterative methods are based on the definition of an algorithm $\mathcal{A} : \mathcal{X} \mapsto \mathcal{X}$ such that the solution $\mathbf{t} \in \mathcal{X}$ given by Eq. (46) is a fixed point of the algorithm and vice versa:

$$\mathbf{P}(\mathbf{d}) = \mathbf{t} \iff \mathcal{A}(\mathbf{t}) = \mathbf{t}. \tag{47}$$

The relevant iterative procedure consists of an initial guess $\mathbf{t}_0 \in \mathcal{X}$ and of the recursive relation:

$$\mathbf{t}_k = \mathcal{A}(\mathbf{t}_{k-1}), \quad k = 1, 2, \ldots. \tag{48}$$

Existence of fixed points, convergence of the iterative procedure and estimates of convergence speed are well investigated but most often challenging mathematical problems.

A classic powerful result is BANACH fixed point theorem for a contraction mapping (the algorithm $\mathcal{A}$) in a complete metric space $\mathcal{X}$ [36,37].

In problems of structural mechanics, as already specified in Sect. 3.1, the linear space $\mathcal{X}$ can be identified with the displacement–stress space:

$$\mathbf{t} \in \mathcal{X} \to \{\mathbf{u}, \boldsymbol{\sigma}\} \in \{\mathbf{w} + \mathcal{L}, \Sigma\}, \tag{49}$$

and the linear space $\mathcal{Y}$ is made of the data triplets

$$\mathbf{d} \in \mathcal{Y} \to (\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathcal{V}, \mathcal{D}, \Lambda). \tag{50}$$

In structural mechanics two iterative schemes for nonlocal elasticity can be envisaged, depending on whether a mixed strain-driven or a stress-driven model is assumed.

## 5 Iterative scheme: strain-driven

In the strain-driven model, the nonlocal elastic structural problem is defined by the set of equations (Eq. (39)). The corresponding algorithm

$$\mathcal{A} : \mathcal{X} \mapsto \mathcal{X}, \quad \text{with} \quad \mathcal{X} = \{\mathbf{w} + \mathcal{L}, \Sigma\}, \tag{51}$$

is defined by the chain of operational steps:

$$
\boxed{\begin{aligned}
&\{\mathbf{u}\,,\boldsymbol{\sigma}\}\in\{\mathbf{w}+\mathcal{L}\,,\Sigma_\ell\}\\
&\to\mathbf{e}=\mathbf{B}(\mathbf{u})-\boldsymbol{\eta}\\
&\to\boldsymbol{\sigma}_{\text{NLOC}}=E_{\text{NLOC}}(\mathbf{e})\\
&\to\langle\Delta\boldsymbol{\rho}_{\text{RES}},\delta\mathbf{v}\rangle=\langle\ell,\delta\mathbf{v}\rangle-\langle\boldsymbol{\sigma}_{\text{NLOC}},\mathbf{B}(\delta\mathbf{v})\rangle,\quad\forall\,\delta\mathbf{v}\in\mathcal{L}\\
&\to\{\Delta\mathbf{u}\,,\Delta\boldsymbol{\sigma}\}=\mathbf{P}_{\text{LOC}}(\mathbf{0},\mathbf{0},\Delta\boldsymbol{\rho}_{\text{RES}})\in\{\mathcal{L}\,,\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}\}\\
&\to\mathcal{A}\{\mathbf{u}\,,\boldsymbol{\sigma}\}=\{\mathbf{u}+\Delta\mathbf{u}\,,\boldsymbol{\sigma}_{\text{NLOC}}+\Delta\boldsymbol{\sigma}\}\in\{\mathbf{w}+\mathcal{L}\,,\Sigma_\ell\}
\end{aligned}}
\tag{52}
$$

In mixed strain-driven models, the nonlocal stiffness is:

$$E_{\text{NLOC}}(\mathbf{e}):=m\,(E\mathbf{e})+(1-m)\cdot(E\varphi*\mathbf{e}). \tag{53}$$

The loading system $\Delta\boldsymbol{\rho}_{\text{RES}}\in\mathcal{L}'$ defined by the gap of equilibrium in Eq. $(52)_3$ is named *static residual*. The third and fourth algorithmic steps can be conveniently performed as follows. First, we set:

$$\Delta\boldsymbol{\sigma}_{\text{RES}}:=\boldsymbol{\sigma}_\ell-\boldsymbol{\sigma}_{\text{NLOC}}\in\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}, \tag{54}$$

with $\boldsymbol{\sigma}_\ell\in\Sigma_\ell$ and $\boldsymbol{\sigma}_{\text{NLOC}}\in\Sigma_{\ell-\Delta\boldsymbol{\rho}_{\text{RES}}}$ by Eq. $(52)_3$.

The equilibrium condition in Eq. (54) may be written as

$$\langle\Delta\boldsymbol{\sigma}_{\text{RES}},\mathbf{B}(\delta\mathbf{v})\rangle=\langle\Delta\boldsymbol{\rho}_{\text{RES}},\delta\mathbf{v}\rangle,\quad\forall\,\delta\mathbf{v}\in\mathcal{L}. \tag{55}$$

The solution $\{\Delta\mathbf{u}\,,\Delta\boldsymbol{\sigma}\}$ provided by $\mathbf{P}_{\text{LOC}}(\mathbf{0},\mathbf{0},\Delta\boldsymbol{\rho}_{\text{RES}})$ in Eq. $(52)_4$ can be found by imposing on the strain increment:

$$\Delta\boldsymbol{\varepsilon}\in C\cdot(\Delta\boldsymbol{\sigma}_{\text{RES}}+\Sigma_0), \tag{56}$$

the kinematic compatibility condition:

$$\langle\Delta\boldsymbol{\varepsilon},\delta\boldsymbol{\sigma}\rangle=0,\quad\forall\,\delta\boldsymbol{\sigma}\in\Sigma_0. \tag{57}$$

This condition is equivalent to performing an orthogonal projection, in local elastic energy, of the stress increment $\Delta\boldsymbol{\sigma}_{\text{RES}}$ on the linear subspace $\Sigma_0$ of self-stresses. The residual to the orthogonal projection $\Delta\boldsymbol{\sigma}_0\in\Sigma_0$ yields the stress increment by the formula:

$$\Delta\boldsymbol{\sigma}=\Delta\boldsymbol{\sigma}_{\text{RES}}-\Delta\boldsymbol{\sigma}_0\in\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}, \tag{58}$$

and the compatible strain increment is given by:

$$\Delta\boldsymbol{\varepsilon}=\Delta\mathbf{e}=C\cdot\Delta\boldsymbol{\sigma}\in(\Sigma_0)^\circ. \tag{59}$$

The updated stress, given by:

$$\boldsymbol{\sigma}_{\text{NLOC}}+\Delta\boldsymbol{\sigma}\in\Sigma_\ell, \tag{60}$$

is equilibrated, being sum of $\boldsymbol{\sigma}_{\text{NLOC}}\in\Sigma_{\ell-\Delta\boldsymbol{\rho}_{\text{RES}}}$ and $\Delta\boldsymbol{\sigma}\in\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}$.

To the stress $\boldsymbol{\sigma}_\ell\in\Sigma_\ell$ an arbitrary self-stress $\boldsymbol{\sigma}_0\in\Sigma_0$ could be added but the solution $\Delta\boldsymbol{\sigma}\in\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}$ would still be the same:

$$\Delta\boldsymbol{\sigma}=\Delta\boldsymbol{\sigma}_{\text{RES}}+\boldsymbol{\sigma}_0-(\Delta\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_0)\in\Sigma_{\Delta\boldsymbol{\rho}_{\text{RES}}}. \tag{61}$$

The procedure described above is the one adopted to evaluate the nonlocal iterations in the examples of Sect. 8. The iterative scheme for approaching the fixed point of the algorithm is made of the following steps.

1. Start by computing the unique solution $\{\mathbf{u}_0\,,\boldsymbol{\sigma}_0\}=\mathbf{P}_{\text{LOC}}(\mathbf{w},\boldsymbol{\eta},\ell)$ of the local elastic problem in Eq. (19).
2. At the $k$-th step, with $k=1,2,3,\ldots$, apply the algorithm to obtain:

$$\{\mathbf{u}_k\,,\boldsymbol{\sigma}_k\}=\mathcal{A}\{\mathbf{u}_{k-1}\,,\boldsymbol{\sigma}_{k-1}\}. \tag{62}$$

3. Check whether the chosen suitable norm of the static residual is below the prescribed tolerance threshold value ($\Delta\boldsymbol{\rho}_{\mathrm{RES}k} \approx \mathbf{0}$). If this is the case the static residual may be taken to be vanishing ($\Delta\boldsymbol{\rho}_{\mathrm{RES}} = \mathbf{0}$), so that $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\} = \{\mathbf{0}\,,\,\mathbf{0}\}$ and hence $\mathcal{A}\{\mathbf{u}\,,\,\boldsymbol{\sigma}\} = \{\mathbf{u}\,,\,\boldsymbol{\sigma}\}$.
4. Otherwise increment the index $k \rightarrow k+1$ and loop to 1.

By the recursive definition

$$\begin{cases} (\boldsymbol{\rho}_{\mathrm{RES}})_0 &= \mathbf{0}, \\ (\boldsymbol{\rho}_{\mathrm{RES}})_k &= (\boldsymbol{\rho}_{\mathrm{RES}})_{k-1} + (\Delta\boldsymbol{\rho}_{\mathrm{RES}})_k, \end{cases} \tag{63}$$

the incremental static residual is accumulated on the null distribution towards the asymptotic value of the static residual:

$$\lim_{k\rightarrow\infty} (\boldsymbol{\rho}_{\mathrm{RES}})_k = (\boldsymbol{\rho}_{\mathrm{RES}})_\infty, \tag{64}$$

which is the additional loading to be imposed to make the local elastic problem equivalent to the nonlocal one:

$$\mathbf{P}_{\mathrm{NLOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell) = \mathbf{P}_{\mathrm{LOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell + (\boldsymbol{\rho}_{\mathrm{RES}})_\infty). \tag{65}$$

Validity of the algorithm in Eq. (52) for strain-driven problems is based on the equivalence property in Eq. (47), which is proven in the next proposition.

**Proposition 1** *A fixed point (if any) of the algorithm $\mathcal{A}$ for the strain-driven model defined by Eq. (52) is solution of the nonlocal elastic problem in Eq. (39) and vice versa.*

*Proof* If the pair $\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}$ is a fixed point for $\mathcal{A}$, then $\Delta\mathbf{u} = \mathbf{0}$ and hence, by Eq. (19) also $\Delta\boldsymbol{\sigma} = E\mathbf{B}(\Delta\mathbf{u}) = \mathbf{0}$ and $\Delta\boldsymbol{\rho}_{\mathrm{RES}} = \mathbf{0}$. Then Eq. $(52)_3$ gives $\boldsymbol{\sigma}_{\mathrm{NLOC}} \in \Sigma_\ell$. Since also by Eq. $(52)_{12}$

$$\boldsymbol{\sigma}_{\mathrm{NLOC}} = E_{\mathrm{NLOC}}(\mathbf{B}(\mathbf{u}) - \boldsymbol{\eta}), \tag{66}$$

the pair $\{\mathbf{u}\,,\,\boldsymbol{\sigma}_{\mathrm{NLOC}}\}$ is the unique solution of the strain-driven nonlocal elastic problem in Eq. (39) and hence $\{\mathbf{u}\,,\,\boldsymbol{\sigma}_{\mathrm{NLOC}}\} = \{\mathbf{u}\,,\,\boldsymbol{\sigma}\} \in \{\mathbf{w} + \mathcal{L}\,,\,\Sigma_\ell\}$. Vice versa, if the pair $\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}$ is solution of the nonlocal elastic problem in Eq. (39), then we would have $\Delta\boldsymbol{\rho}_{\mathrm{RES}} = \mathbf{0}$ and $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\} = \{\mathbf{0}\,,\,\mathbf{0}\}$ so that $\mathcal{A}\{\mathbf{u}\,,\,\boldsymbol{\sigma}\} = \{\mathbf{u}\,,\,\boldsymbol{\sigma}\}$. □

## 6 Iterative scheme: stress-driven

In the stress-driven model [2,3,13], the nonlocal elastic structural problem is defined by the set of equations Eq. (44).

The corresponding algorithm

$$\mathcal{A} : \mathcal{X} \mapsto \mathcal{X}, \quad \text{with} \quad \mathcal{X} = \{\mathbf{w} + \mathcal{L}\,,\,\Sigma\}, \tag{67}$$

is defined by the chain of operations:

$$\boxed{\begin{aligned} &\{\mathbf{u}\,,\,\boldsymbol{\sigma}\} \in \{\mathbf{w} + \mathcal{L}\,,\,\Sigma_\ell\} \\ &\rightarrow \mathbf{e} = C_{\mathrm{NLOC}}(\boldsymbol{\sigma}) \\ &\rightarrow \Delta\mathbf{e}_{\mathrm{RES}} = \mathbf{e} - \Big(\mathbf{B}(\mathbf{u}) - \eta\Big) \\ &\rightarrow \{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\} = \mathbf{P}_{\mathrm{LOC}}(\mathbf{0}, \Delta\mathbf{e}_{\mathrm{RES}}, \mathbf{0}) \in \{\mathcal{L}\,,\,\Sigma_0\} \\ &\rightarrow \mathcal{A}\{\mathbf{u}\,,\,\boldsymbol{\sigma}\} = \{\mathbf{u} + \Delta\mathbf{u}\,,\,\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}\} \in \{\mathbf{w} + \mathcal{L}\,,\,\Sigma_\ell\} \end{aligned}} \tag{68}$$

The operator $\mathbf{P}_{\mathrm{LOC}}(\mathbf{0}, \Delta\mathbf{e}_{\mathrm{RES}}, \mathbf{0})$ is evaluated as a special case of Eq. (19) by setting there $\mathbf{w} = \mathbf{0}$, $\boldsymbol{\eta} = \Delta\mathbf{e}_{\mathrm{RES}}$ and $\ell = \mathbf{0}$.

In the local problem in Eq. $(68)_3$, $\Delta\mathbf{e}_{\mathrm{RES}}$ is an imposed strain equal to the strain residual.

According to Eq. (68)$_3$, the increment $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\}\in\{\mathcal{L}\,,\,\Sigma_0\}$ is solution of the local elastostatic problem with a prescribed strain $\Delta\mathbf{e}_{\mathrm{RES}}\in\mathcal{D}$:

$$\boxed{\begin{cases}\langle\,\Delta\boldsymbol{\sigma}\,,\mathbf{B}(\delta\mathbf{v})\,\rangle=0, \quad \forall\,\delta\mathbf{v}\in\mathcal{L},\\ \Delta\boldsymbol{\sigma}=E\,\Delta\mathbf{e}\in\Sigma_0,\\ \Delta\mathbf{e}=\mathbf{B}(\Delta\mathbf{u})-\Delta\mathbf{e}_{\mathrm{RES}}, \quad \Delta\mathbf{u}\in\mathcal{L}.\end{cases}} \tag{69}$$

The strain field $\Delta\mathbf{e}_{\mathrm{RES}}\in\mathcal{D}$ defined by the gap of kinematic compatibility in Eq. (68)$_3$ is named *kinematic residual*.

The solution of local elastostatic problem in Eq. (69) is equivalent to the orthogonal projection of the kinematic residual $\Delta\mathbf{e}_{\mathrm{RES}}\in\mathcal{D}$ onto the self-stress subspace $\Sigma_0$ in local elastic energy:

$$\mathbf{B}(\Delta\mathbf{u})=\Delta\mathbf{e}_{\mathrm{RES}}+C\,\Delta\boldsymbol{\sigma}\in(\Sigma_0)^{\circ}. \tag{70}$$

Addition of increments $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\}\in\{\mathcal{L}\,,\,\Sigma_0\}$ in Eq. (68)$_4$ is motivated by the next relations which collect the kinematic residual with the proper sign of Eq. (68)$_2$ and the expression in Eq. (69)$_3$:

$$\begin{cases}\mathbf{e}\;=\mathbf{B}(\mathbf{u})-\boldsymbol{\eta}+\Delta\mathbf{e}_{\mathrm{RES}},\\ \Delta\mathbf{e}\;=\mathbf{B}(\Delta\mathbf{u})-\Delta\mathbf{e}_{\mathrm{RES}}.\end{cases} \tag{71}$$

The recover of kinematic compatibility is thus ensured:

$$\begin{aligned}\mathbf{e}+\Delta\mathbf{e}\;&=\mathbf{B}(\mathbf{u})-\boldsymbol{\eta}+\Delta\mathbf{e}_{\mathrm{RES}}+\mathbf{B}(\Delta\mathbf{u})-\Delta\mathbf{e}_{\mathrm{RES}}\\ &=\mathbf{B}(\mathbf{u}+\Delta\mathbf{u})-\boldsymbol{\eta}.\end{aligned} \tag{72}$$

The above evaluation clarifies why the sign of the correction strain exposed in [29] should be amended.

Validity of the algorithm in Eq. (68) for stress-driven problem is based on the equivalence property in Eq. (47), as ensured by the next proposition.

**Proposition 2** *A fixed point of the algorithm $\mathcal{A}$ for the stress-driven model defined by Eq.* (68)*, is solution of the nonlocal elastic problem in Eq.* (44) *and vice versa.*

*Proof* If the pair $\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}\in\{\mathbf{w}+\mathcal{L}\,,\,\Sigma_\ell\}$ is a fixed point for the algorithm $\mathcal{A}$, then $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\}=\{\mathbf{0}\,,\,\mathbf{0}\}$ so that by Eq. (69) the kinematic residual vanishes ($\Delta\mathbf{e}_{\mathrm{RES}}=\mathbf{0}$). From Eq. (68) we infer that

$$C_{\mathrm{NLOC}}(\boldsymbol{\sigma})=\mathbf{B}(\mathbf{u})-\boldsymbol{\eta}. \tag{73}$$

Thus, Eq. (44), defining the stress-driven nonlocal elastic problem, is fulfilled by the pair $\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}\in\{\mathbf{w}+\mathcal{L}\,,\,\Sigma_\ell\}$. The converse implication is trivially verified. □

The iterative procedure for approaching the fixed point of the algorithm is made of the following steps.

1. Start by computing the unique solution $\{\mathbf{u}_0\,,\,\boldsymbol{\sigma}_0\}=\mathbf{P}_{\mathrm{LOC}}(\mathbf{w},\boldsymbol{\eta},\ell)$ of the initial local elastic problem in Eq. (19).
2. At the $k$-th step, with $k=1,2,3,\ldots$, apply the algorithm to obtain:

$$\{\mathbf{u}_k\,,\,\boldsymbol{\sigma}_k\}=\mathcal{A}\{\mathbf{u}_{k-1}\,,\,\boldsymbol{\sigma}_{k-1}\}. \tag{74}$$

3. Check whether a chosen suitable norm of the kinematic residual is below the prescribed tolerance threshold value ($(\Delta\mathbf{e}_{\mathrm{RES}})_k\approx\mathbf{0}$). If this is the case, the kinematic residual may be taken to be vanishing ($\Delta\mathbf{e}_{\mathrm{RES}}=\mathbf{0}$), so that $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\}=\{\mathbf{0}\,,\,\mathbf{0}\}$ and hence $\mathcal{A}\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}=\{\mathbf{u}\,,\,\boldsymbol{\sigma}\}$.
4. Otherwise increment the index $k\to k+1$ and loop to 1.

By the recursive definition

$$\begin{cases} (\mathbf{e}_{\text{RES}})_0 &= \mathbf{0}, \\ (\mathbf{e}_{\text{RES}})_k &= (\mathbf{e}_{\text{RES}})_{k-1} + (\varDelta\mathbf{e}_{\text{RES}})_k, \end{cases} \tag{75}$$

the incremental kinematic residual is accumulated on the null field towards the final or asymptotic value of the kinematic residual:

$$(\mathbf{e}_{\text{RES}})_\infty = C_{\text{NLOC}}(\boldsymbol{\sigma}) - C(\boldsymbol{\sigma}), \tag{76}$$

which is the prescribed strain field to be imposed to make the local elastic problem equivalent to the nonlocal one:

$$\mathbf{P}_{\text{NLOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell) = \mathbf{P}_{\text{LOC}}(\mathbf{w}, \boldsymbol{\eta} + (\mathbf{e}_{\text{RES}})_\infty, \ell) \tag{77}$$

The next proposition provides the proof of a noteworthy peculiar property of the iterative procedure for stress-driven nonlocal elasticity problems.

**Proposition 3** *For statically determinate structures, the iterative procedure in Eq.* (74) *based on the algorithm of Eq.* (68)*, yields the displacement solution of the nonlocal problem in Eq.* (44)*, just at the first step* ($k = 1$)*.*

*Proof* In statically determinate structures, self-equilibrated incremental stress fields vanish identically and the stress $\boldsymbol{\sigma}_\ell \in \varSigma$, in equilibrium with the loading $\ell \in \varLambda$, is determined in a unique way. Any strain field is kinematically compatible with a unique admissible displacement field, since rigid and conforming ones are vanishing by assumption in Eq. (10). Given $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_\ell$, we have $\mathbf{e}_1 = \varphi * (C\boldsymbol{\sigma}_\ell)$. The first evaluation of the kinematic residual is

$$(\varDelta\mathbf{e}_{\text{RES}})_1 = \mathbf{e}_1 - \Big(\mathbf{B}(\mathbf{u}_0) - \boldsymbol{\eta}\Big). \tag{78}$$

The solution of the local elastic problem in Eq. (69) gives a null self-equilibrated incremental stress field $\varDelta\boldsymbol{\sigma}_1 = \mathbf{0}$ and an incremental displacement such that

$$\mathbf{B}(\varDelta\mathbf{u}_1) = (\varDelta\mathbf{e}_{\text{RES}})_1. \tag{79}$$

At the end of the first iteration, setting:

$$\begin{cases} \mathbf{u}_1 = \mathbf{u}_0 + \varDelta\mathbf{u}_1, \\ \boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_0 + \varDelta\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_\ell, \end{cases} \tag{80}$$

from Eqs. (78) and (79) we infer that

$$\mathbf{e}_1 = \mathbf{B}(\mathbf{u}_1) - \boldsymbol{\eta} = C_{\text{NLOC}}(\boldsymbol{\sigma}_\ell), \tag{81}$$

so that the pair $\{\mathbf{u}_1, \boldsymbol{\sigma}_1\} = \mathbf{P}_{\text{NLOC}}(\mathbf{w}, \boldsymbol{\eta}, \ell) \in (\mathbf{w} + \mathcal{L}) \times \varSigma_\ell$ is solution of the nonlocal problem in Eq. (44). □

## 7 Exchanging iterative schemes

When the nonlocal law is invertible, and the inverse law is available through a feasible procedure, the algorithms defined by Eq. (52) and by Eq. (68) can be applied to both stress-driven and mixture strain-driven models.

This is the case when a mixture of local/nonlocal elastic laws is considered:

$$f = m \cdot s + \big(1 - m\big) \cdot \big(\varphi_\lambda * Ks\big), \tag{82}$$

with the integral convolution involving the exponential kernel:

$$\varphi_\lambda(x) := \frac{1}{2\lambda} \exp\Big(-\frac{|x|}{\lambda}\Big). \tag{83}$$

For $1 \geq m > 0$, the nonlocal elastic law in Eqs. (82)–(83) can be inverted by solving the differential equation in the source field $s \in \mathcal{S}$ [2]:

$$\frac{f}{\lambda^2} - f'' = \frac{Ks}{\lambda^2} - m\,Ks'', \tag{84}$$

with the boundary condition:

$$\begin{cases} f'(a) - \dfrac{f(a)}{\lambda} = m\left(Ks'(a) - \dfrac{Ks(a)}{\lambda}\right), \\ f'(b) + \dfrac{f(b)}{\lambda} = m\left(Ks'(b) + \dfrac{Ks(b)}{\lambda}\right). \end{cases} \tag{85}$$

The strain-driven model of Eq. (39) is obtained by setting, according to Eq. $(36)_1$:

$$s = \mathbf{e}, \quad f = \boldsymbol{\sigma}, \quad K = E \quad \text{(strain-driven)} \tag{86}$$

For $m = 0$ the boundary condition in Eq. (85) cannot be imposed on the output stress field $f = \boldsymbol{\sigma}$ since a conflict with the boundary condition imposed by equilibrium will in general occur.

The mixture stress-driven model of Eq. (44) is instead obtained by setting, according to Eq. $(36)_2$:

$$s = \boldsymbol{\sigma}, \quad f = \mathbf{e}, \quad K = C \quad \text{(stress-driven).} \tag{87}$$

A stress-driven mixture model can be inverted by solving the problem in Eqs. (84)–(85), for any $1 \geq m \geq 0$.

Indeed, the output field is now the elastic strain $f = \mathbf{e}$ on which the boundary condition in Eq. (85) does not conflict with kinematic compatibility, even when $m = 0$.

## 8 Examples

To provide evidence of convergence properties of iterative methods for mixture strain-driven and stress-driven models of nonlocal elasticity formulated according to an integral convolution with the Helmholtz kernel in Eq. (83), the results of two beam problems (according to Bernoulli-Euler theory) are displayed, and performances of related iterative schemes are compared. Origin of the nondimensional abscissa $0 \leq x \leq 1$ is taken at the left-end side.

A clamper-free (CF) beam (cantilever) with end-point force:

$$\langle \ell, \delta\mathbf{v} \rangle := F(1), \tag{88}$$

and a clamped-supported (CS) beam under uniform loading:

$$\langle \ell, \delta\mathbf{v} \rangle := \int_0^1 p(x) \cdot \delta\mathbf{v}(x)\, \mathrm{d}x, \tag{89}$$

with $p(x)$ constant, are taken as simple benchmarks.

Both examples are carried out by assuming for the nonlocality parameter in the kernel in Eq. (83) of the integral convolution, the value $\lambda = 0.10$.

A local-nonlocal mixture parameter $m = 0.50$ is taken for the strain-driven model.

Six iteration steps have been performed in all cases.

All relevant quantities are assumed dimension free and of unitary value. Bending, displacement, curvature and elastic curvature fields will be denoted by the symbols $\boldsymbol{\sigma}$, $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\mathbf{e}$.

The kinematic operator $\mathbf{B}$ is the second derivative along the beam axis.

Convergence properties are evidenced by discrete plots (right extended) of the standard deviation of stress or strain residuals from the null field, respectively, for strain-driven and stress-driven iterative schemes:

$$\sqrt{\int_0^1 \left(\Delta\boldsymbol{\sigma}_{\mathrm{RES}}(x)\right)^2 \mathrm{d}x}, \qquad \sqrt{\int_0^1 \left(\Delta\boldsymbol{\varepsilon}_{\mathrm{RES}}(x)\right)^2 \mathrm{d}x}. \tag{90}$$

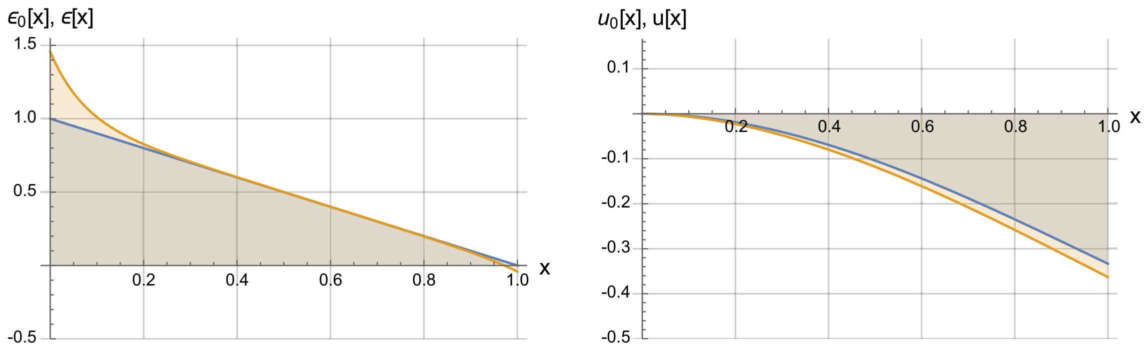


**Fig. 1** Strain-driven. CF: local/nonlocal fields. Left: curvature. Right: displacement

## 8.1 Examples: strain-driven model

The start point is the pair $\{\mathbf{u}\,,\boldsymbol{\sigma}\} \in \{\mathbf{w}+\mathcal{L}\,,\,\Sigma_\ell\}$ solution of the local problem in Eq. (19).

Being $\boldsymbol{\eta} = \mathbf{0}$, the stress field is evaluated by $\boldsymbol{\sigma}_{\mathrm{NLOC}} = E_{\mathrm{NLOC}}(\mathbf{B}(\mathbf{u}))$ and the static residual is given by $\Delta\boldsymbol{\sigma}_{\mathrm{RES}} := \boldsymbol{\sigma}_\ell - \boldsymbol{\sigma}_{\mathrm{NLOC}}$ from Eq. (54).

When the beam is statically indeterminate, with redundancy degree equal to one, the solution pair $\{\Delta\mathbf{u}\,,\Delta\boldsymbol{\sigma}\} \in \mathcal{L} \times \Sigma_{\Delta\boldsymbol{\rho}_{\mathrm{RES}}}$ of the incremental local elastic problem $\mathbf{P}_{\mathrm{LOC}}(\mathbf{0}, \mathbf{0}, \Delta\boldsymbol{\rho}_{\mathrm{RES}})$ is searched for by performing the projection procedure outlined in Sect. 5.

To this end, the stress residual is decomposed as follows

$$\Delta\boldsymbol{\sigma}_{\mathrm{RES}} = E \cdot \Delta\boldsymbol{\varepsilon} + \Delta\boldsymbol{\sigma}_0. \tag{91}$$

Here $\Delta\boldsymbol{\sigma}_0 = \xi \cdot \delta\boldsymbol{\sigma} \in \Sigma_0$ with $\delta\boldsymbol{\sigma} \in \Sigma_0\backslash\{\mathbf{0}\}$. The redundancy parameter $\xi \in \Re$ is evaluated by imposing the kinematic compatibility condition

$$\langle\, \Delta\boldsymbol{\varepsilon}, \delta\boldsymbol{\sigma}\,\rangle = 0, \tag{92}$$

which gives

$$\langle C \cdot \Delta\boldsymbol{\sigma}_{\mathrm{RES}}, \delta\boldsymbol{\sigma}\,\rangle = \xi \cdot \langle C \cdot \delta\boldsymbol{\sigma}\,, \delta\boldsymbol{\sigma}\,\rangle. \tag{93}$$

The incremental stress $\Delta\boldsymbol{\sigma} = E \cdot \Delta\mathbf{e} = E \cdot \Delta\boldsymbol{\varepsilon}$ is expressed by Eq. (91) as

$$\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}_{\mathrm{RES}} - \Delta\boldsymbol{\sigma}_0, \tag{94}$$

with

$$\Delta\boldsymbol{\sigma}_0 = \frac{\langle C \cdot \Delta\boldsymbol{\sigma}_{\mathrm{RES}}, \delta\boldsymbol{\sigma}\,\rangle}{\langle C \cdot \delta\boldsymbol{\sigma}\,, \delta\boldsymbol{\sigma}\,\rangle} \cdot \delta\boldsymbol{\sigma} \in \Sigma_0. \tag{95}$$

The incremental displacement field $\Delta\mathbf{u} \in \mathcal{L}$ may be evaluated by integrating the second-order differential equation

$$\mathbf{B}(\Delta\mathbf{u}) = \Delta\boldsymbol{\varepsilon}, \tag{96}$$

with the kinematic boundary conditions imposed to eliminate rigid displacement fields.

For statically determinate beams $\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}_{\mathrm{RES}}$ and

$$\Delta\mathbf{e} = \Delta\boldsymbol{\varepsilon} = C \cdot \Delta\boldsymbol{\sigma}_{\mathrm{RES}}. \tag{97}$$

### *8.1.1 CF strain-driven*

The displays in Figs. 1, 2 and 3 refer to a cantilever (CF) of under downward end-point loading. The displays of iterations show that the algorithm enjoys a good convergence rate to a fixed point, as quantified by the (right extended) discrete plot in Fig. 3 (Right).

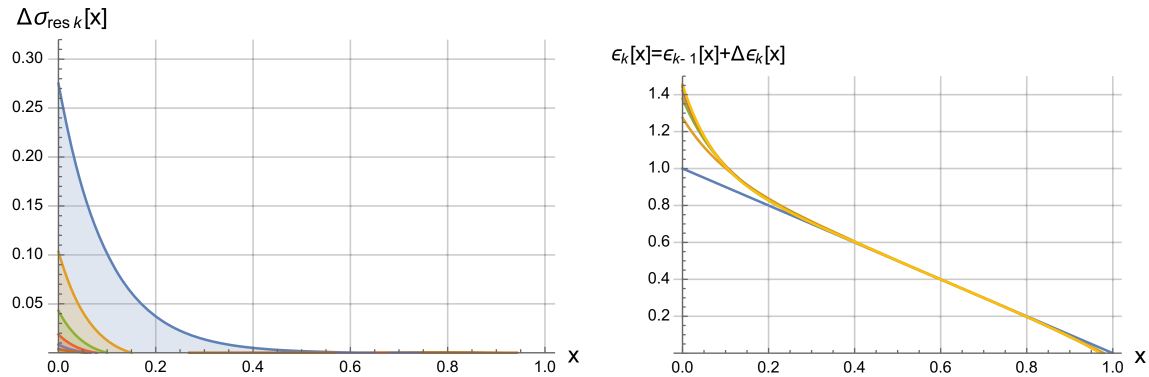


**Fig. 2** Strain-driven. CF: left: stress increments. Right: strain trials

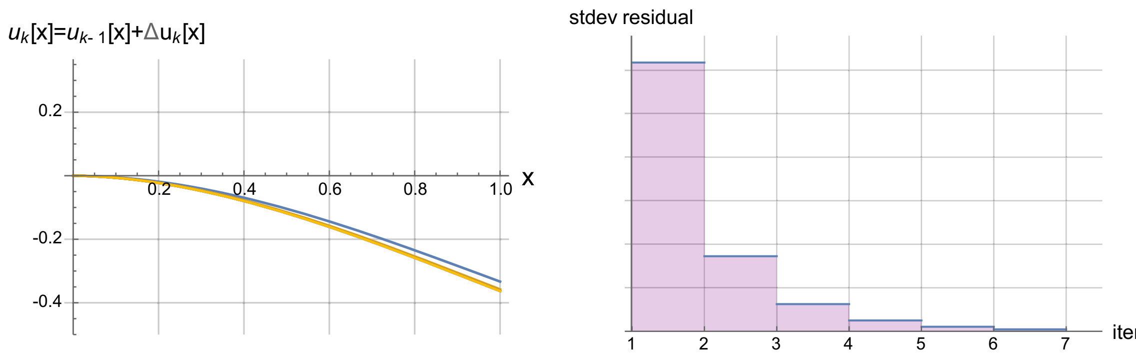


**Fig. 3** Strain-driven. CF: left: displacement trials. Right: standard deviation

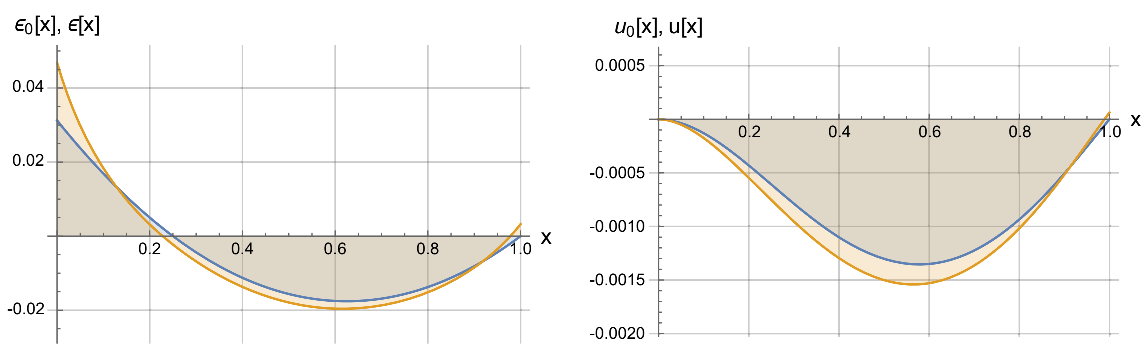


**Fig. 4** Strain-driven. CS: local/nonlocal fields. Left: curvature. Right: displacement

### *8.1.2 CS strain-driven*

A similar computation can be carried out for a clamped-supported (CS) beam under uniform loading. Iterations are described in Figs. 4, 5, 6 and 7, and convergence rate is outlined by a discrete plot of the standard deviation in Fig. 8.

## 8.2 Examples: stress-driven model

The iterative procedure outlined in Sect. 6 is carried out as explicated below.

The start point is the pair $\{\mathbf{u}\,,\boldsymbol{\sigma}\} \in \{\mathbf{w}+\mathcal{L}\,,\,\Sigma_\ell\}$ solution of the local problem. The nonlocal elastic strain is evaluated by $\mathbf{e} = C_{\mathrm{NLOC}}(\boldsymbol{\sigma})$. Being $\boldsymbol{\eta} = \mathbf{0}$, the kinematic residual is given by $\Delta\mathbf{e}_{\mathrm{RES}} = \mathbf{e} - \mathbf{B}(\mathbf{u})$ in Eq. (68).

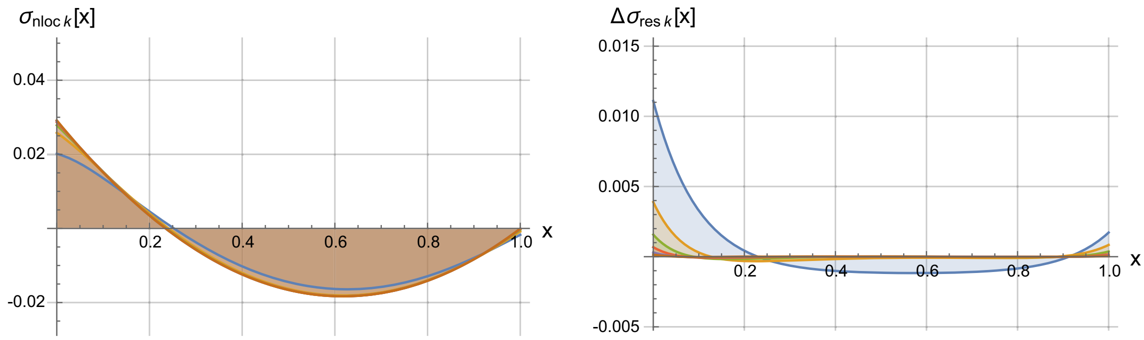


**Fig. 5** Strain-driven. CS: trials. Left: nonlocal stress. Right: stress residuals

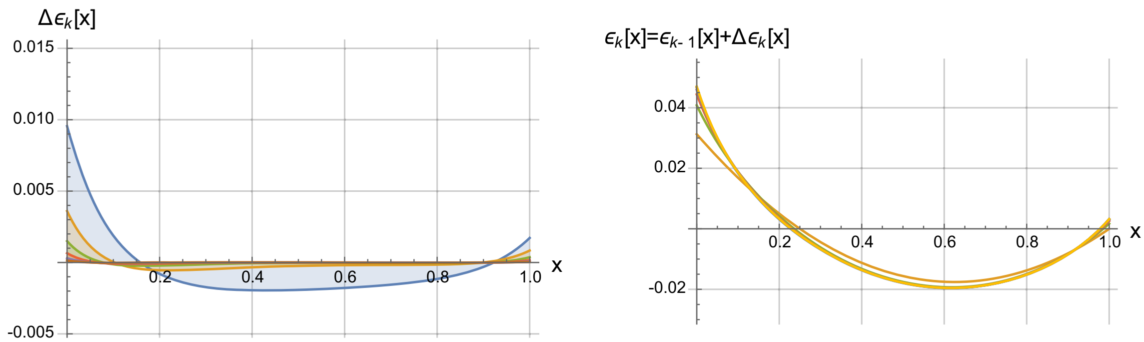


**Fig. 6** Strain-driven. CS: trials. Left: strain increment. Right: strain

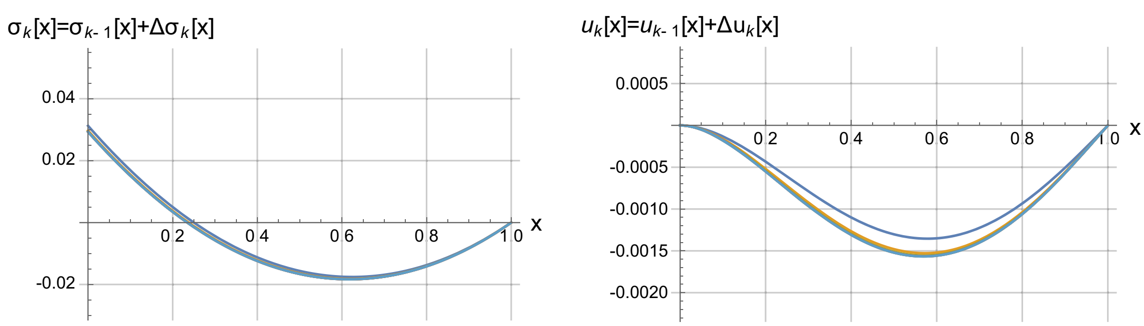


**Fig. 7** Strain-driven. CS: trials. Left: stress. Right: displacement

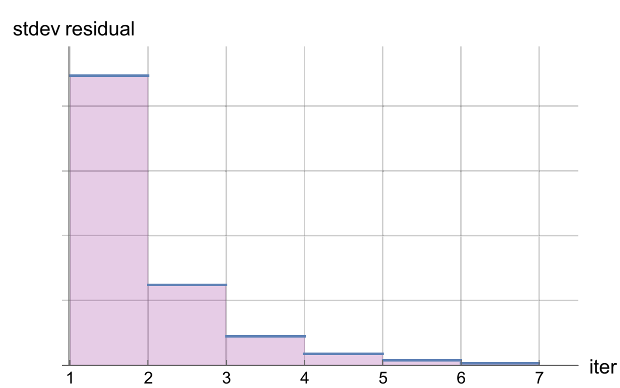


**Fig. 8** Strain-driven. CS: standard deviation of the static residual

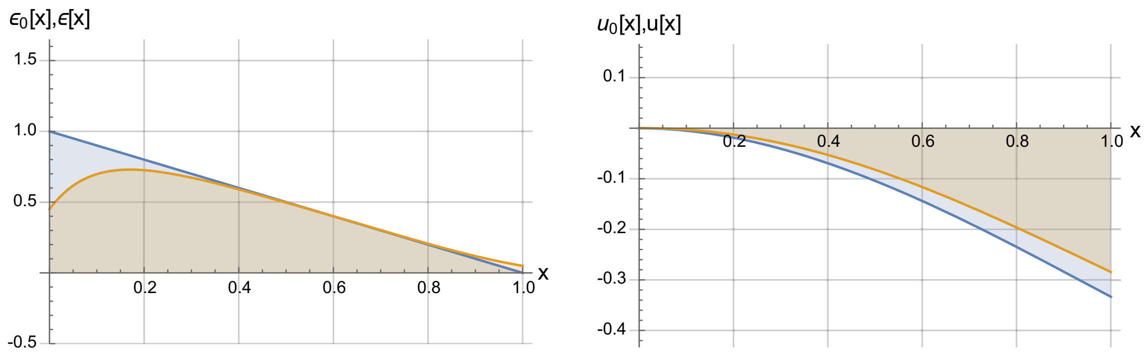


**Fig. 9** Stress-driven. CF local/nonlocal fields: left: curvature. Right: displacement

When the beam is statically indeterminate, with redundancy degree equal to one, the solution of the incremental local elastic is carried out by performing the projection procedure outlined in Sect. 6, as explicated below.

The pair $\{\Delta\mathbf{u}\,,\,\Delta\boldsymbol{\sigma}\}\in\mathcal{L}\times\Sigma_0$, solution of problem $\mathbf{P}_{\text{LOC}}(\mathbf{0},\,\Delta\mathbf{e}_{\text{RES}},\,\mathbf{0})$ of Eq. (69), is detected by considering a nonnull self-stress field $\delta\boldsymbol{\sigma}\in\Sigma_0\backslash\{\mathbf{0}\}$.

Then, setting

$$\Delta\boldsymbol{\sigma}=\xi\cdot\delta\boldsymbol{\sigma}\,, \tag{98}$$

the redundancy parameter $\xi\in\Re$ is evaluated by imposing the kinematic compatibility condition

$$\langle\,\Delta\boldsymbol{\varepsilon},\,\delta\boldsymbol{\sigma}\,\rangle=0, \tag{99}$$

on the total incremental strain:

$$\Delta\boldsymbol{\varepsilon}=\Delta\mathbf{e}_{\text{RES}}-\Delta\mathbf{e}, \tag{100}$$

with the incremental elastic strain given by $\Delta\mathbf{e}=C\cdot\Delta\boldsymbol{\sigma}=\xi\cdot C\cdot\delta\boldsymbol{\sigma}$. Then

$$\xi\cdot\langle\,C\cdot\delta\boldsymbol{\sigma}\,,\,\delta\boldsymbol{\sigma}\,\rangle=\langle\,\Delta\mathbf{e}_{\text{RES}},\,\delta\boldsymbol{\sigma}\,\rangle. \tag{101}$$

The incremental elastic strain is therefore expressed by

$$\Delta\mathbf{e}=\xi\cdot C\cdot\delta\boldsymbol{\sigma}=\frac{\langle\,\Delta\mathbf{e}_{\text{RES}},\,\delta\boldsymbol{\sigma}\,\rangle}{\langle\,C\cdot\delta\boldsymbol{\sigma}\,,\,\delta\boldsymbol{\sigma}\,\rangle}\cdot C\cdot\delta\boldsymbol{\sigma}\,, \tag{102}$$

and the incremental displacement field $\Delta\mathbf{u}\in\mathcal{L}$ may be evaluated by integrating the second-order differential equation

$$\mathbf{B}(\Delta\mathbf{u})=\Delta\boldsymbol{\varepsilon}, \tag{103}$$

with the kinematic boundary conditions imposed to eliminate rigid displacement fields. For statically determinate beams $\Delta\boldsymbol{\sigma}\in\Sigma_0=\{\mathbf{0}\}$ and hence $\Delta\mathbf{e}=\mathbf{0}$ and $\Delta\boldsymbol{\varepsilon}=\Delta\mathbf{e}_{\text{RES}}$.

### *8.2.1 CF stress-driven*

The displays in Figs. 9, 10, 11 and 12 refer to a cantilever (CF) of under downward end-point loading. The displays of iterations show that the algorithm enjoys a full convergence just after the first step, as predicted by Proposition 3.

### *8.2.2 CS stress-driven*

The displays in Figs. 13, 14, 15, 16 and 17 refer to a clamped-supported (CS) beam of under downward distributed loading. The displays of iterations shows that the algorithm enjoys a quite fast convergence rate.

A comparison between Figs. 8 and 17 (Right) reveals that convergence of the stress-driven algorithm is faster than the one of the strain-driven algorithm.

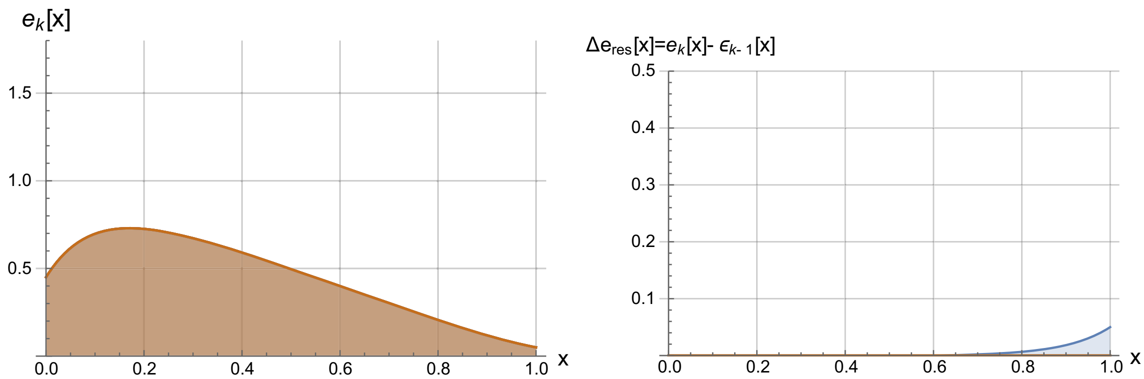


**Fig. 10** Stress-driven. CF: trials $k = 0, 1$. Left: elastic curvature $\mathbf{e}_k$. Right: kinematic residual $(\Delta\mathbf{e}_{\text{RES}})_k$

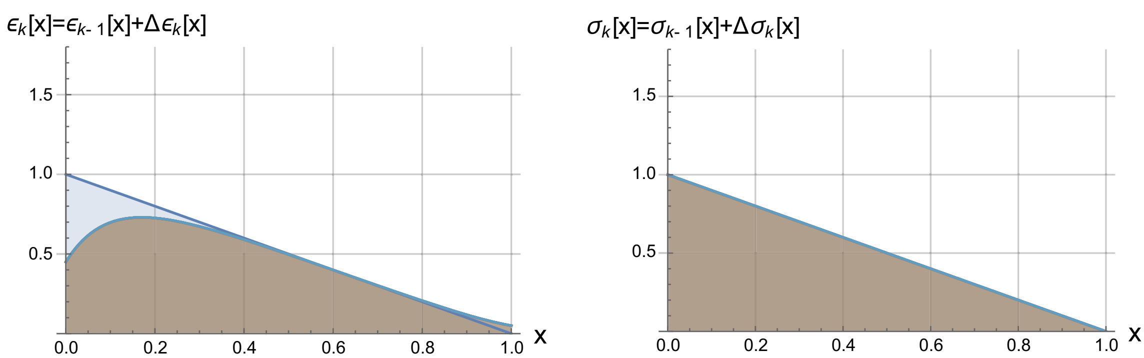


**Fig. 11** Stress-driven. CF: trials $k = 0, 1$. Left: curvature $\Delta\boldsymbol{\epsilon}_k$. Right: bending $\Delta\boldsymbol{\sigma}_k$

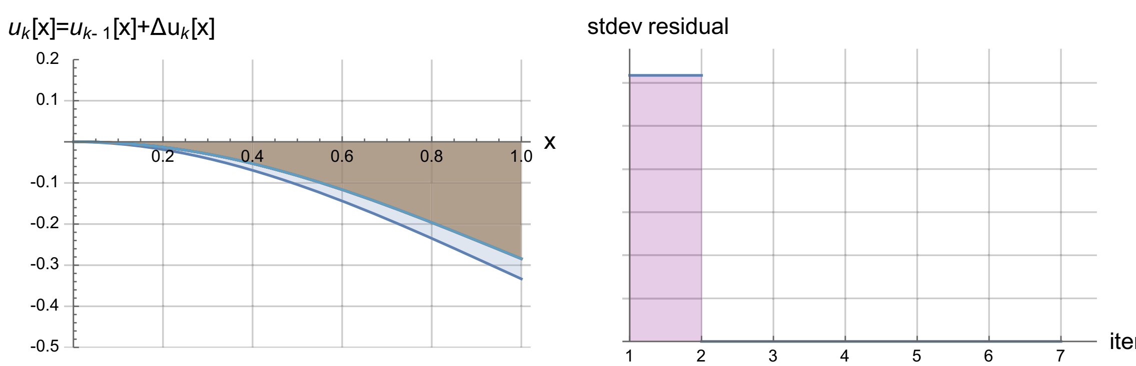


**Fig. 12** Stress-driven. CF: trials. Left: displacement. Right: standard deviation

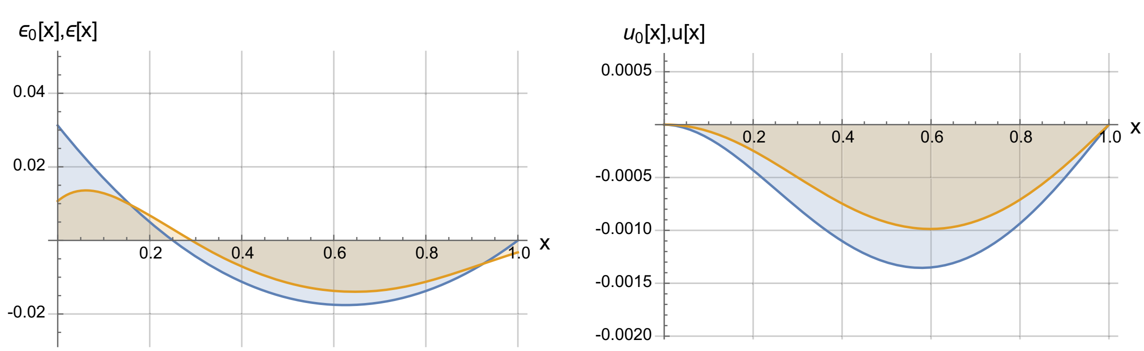


**Fig. 13** Stress-driven. CS: local/nonlocal fields. Left: curvature. Right: displacement

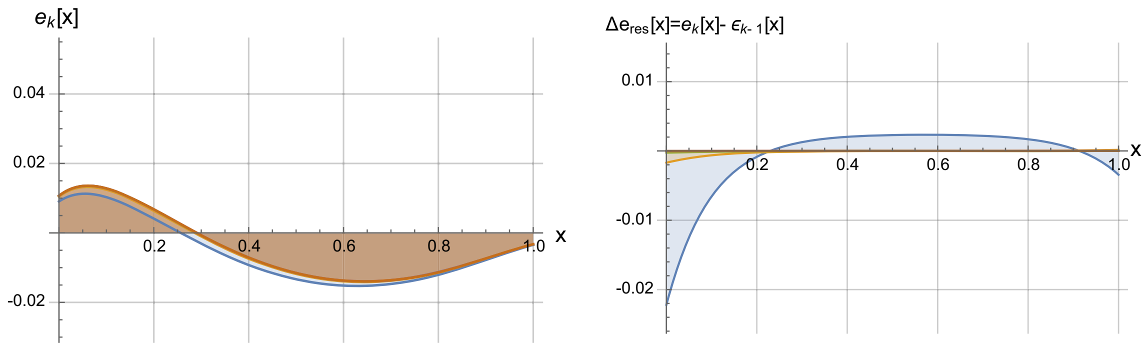


**Fig. 14** Stress-driven. CS: trials. Left: elastic strain. Right: increment of strain residual

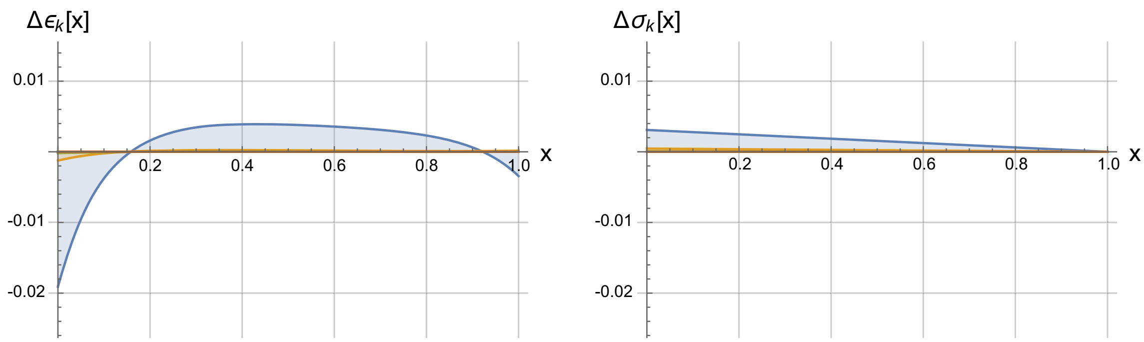


**Fig. 15** Stress-driven. CS: increments. Left: strain. Right: stress

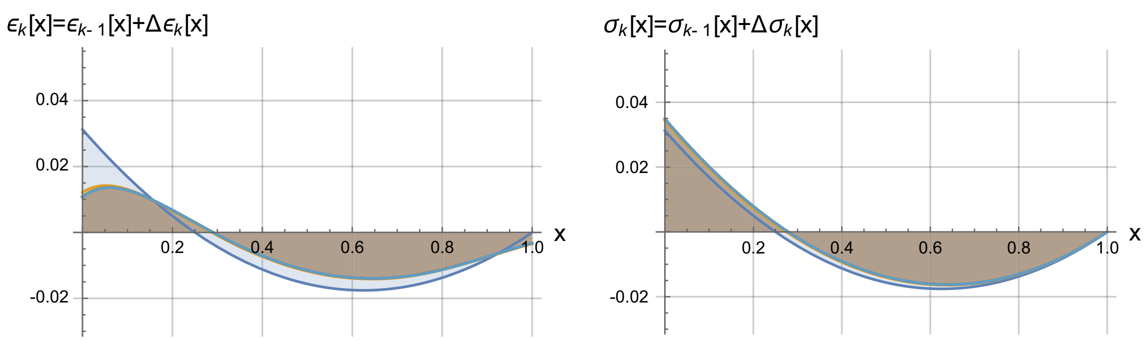


**Fig. 16** Stress-driven. CS: trials. Left: strain. Right: stress

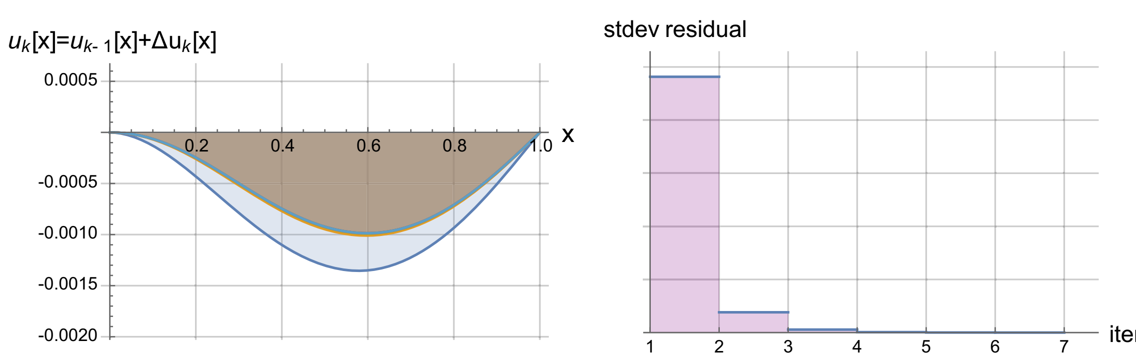


**Fig. 17** Stress-driven. CS: trials. Left: displacement. Right: standard deviation of the kinematic residual

## 9 Conclusions

The algorithmic formulation of strain-driven and stress-driven model of nonlocal elasticity leads to a simple and effective iterative computational scheme, with strikingly well-behaved convergence properties.

The special effectiveness of the iterative scheme for the stress-driven model is exploited by the theoretical analysis developed in Proposition 3, with reference to statically determinate structures, and confirmed by the computations performed in Sect. 8 on a simple cantilever (CF) and a clamped-supported (CS) beam.

This analysis reveals that the solution is reached after quite a few steps of iterative procedure.

For statically determinate structures, the solution is obtained at the very first step of iteration.

For statically indeterminate ones, the convergence is asymptotic but so fast that very few iterations could suffice.

Iterative methods for nonlocal elastic problems require the solution of the initial local elastic problem, and at each iteration, the solution of the local elastic problem devoted to manage the static or kinematic residuals.

In general the algorithm for the stress-driven model shows better convergence properties than the one for the mixed strain-driven model.

Together with general applicability, this feature is a further point in favour of stress-driven elastic models, in comparison with strain-driven ones.

This fact is in agreement with the conclusions of the incremental theory of elasticity exposed in [27,28, 37,39].

According to this formulation, the elastic law is conceived as a stress-dependent linear compliance operator which, acting on the time derivative of the stress state along the motion, generates the time derivative of the elastic state along the motion.

Time independence and integrability of the compliance operator are essential requisites characterising the elastic behaviour.

The stress state plays then the natural role of primary variable, with the elastic state being a derived notion.

This is the constitutive scheme of stress-driven models of elasticity.

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