Continuum Mech. Thermodyn. (2019) 31:1087–1097
https://doi.org/10.1007/s00161-018-0679-x

ORIGINAL ARTICLE

Milad Shirani · Cheng Luo · David J. Steigmann

Cosserat elasticity of lattice shells with kinematically independent flexure and twist

Received: 10 April 2018 / Accepted: 3 May 2018 / Published online: 12 May 2018

Abstract A model of nonlinearly elastic surfaces composed of continuously distributed embedded fibers is formulated. This takes account of the elastic resistance of the fibers to extension, bending and twist. Twist is regarded as being kinematically independent of surface deformation, just as the twist in a spatial Kirchhoff rod is independent of the deformation of the curve of the rod.

Keywords Cosserat elasticity · Elastic shells · Flexure · Twist

1 Introduction

The mechanics of fiber-reinforced plates and shells has recently received renewed attention [1–5] from the perspective of the theory of elastic surfaces. In these works the fibers constituting the shell are regarded as belonging to two families of fibers forming a continuously distributed network over the surface; these are regarded as spatial rods that are orthogonal, straight and untwisted in the reference configuration. Upon the introduction of certain constraints, the model is reduced to a special form in which the constitutive functions depend only on the first and second gradients of the surface deformation, culminating in a strain-gradient theory of elastic shells. Such constraints are appropriate in the case of a lattice in which the constituent fibers are pinned at their points of intersection so as to pivot about the evolving surface normal. Detailed discussions may be found in [1–5], and extensions to accommodate dynamics and vibrations are given in [4]. The constrained theory also models fabrics in which the fibers of the yarn have elliptical or rectangular cross sections; in this case the geometry of the fibers of the woven fabric precludes independent fiber rotations at their points of mutual contact. However, in a fabric consisting of fibers having circular cross sections, it is conceivable that the fibers may pivot freely about their respective tangent directions while remaining congruent to the deforming surface. This implies that fiber twist cannot be determined by surface deformation, but rather by a rotation field, one for each fiber family. To cover this case, a more general model of the Cosserat type is required [6,7], the development of which is the objective of the present work.

To this end the fibers are regarded as continuously distributed spatial rods of the Kirchhoff type in which the kinematics are based on a position field and an orthonormal triad field [8,9]. Variation of the triad along the length of a fiber accounts for flexure and twist, while the deformation of the underlying surface generates fiber stretches and fiber shear. We generalize the considerations of [1–5] by allowing the fibers to form a

Communicated by Francesco dell'Isola.

M. Shirani · D. J. Steigmann (✉)
Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA
E-mail: dsteigmann@berkeley.edu

C. Luo
Department of Mechanical Engineering, University of Texas, Arlington, TX 76019, USA

nonorthogonal grid in the reference placement and to be bent and twisted in their natural configurations. We also relax the constraints on the fiber rotations to require only that the fibers remain embedded in the surface as it deforms, while allowing them to twist independently of the surface deformation.

As a prelude to our main development, in Sect. 2 we summarize the equilibrium theory of Kirchhoff rods, using an appropriate version of the virtual-work principle [10–13]. This serves as a guide for the construction, in Sect. 3, of the proposed shell model. Section 4 is devoted to a discussion of the global balances of force and moment emerging from the virtual-work statement. In contrast to the situation in conventional elasticity, these global balances are found to be necessary, but not sufficient, for equilibrium. Finally, in Sect. 5 we apply the shell model to the solution of an example in which an initially plane, rectangular net of fibers is mapped to a helicoidal surface.

Concerning notation, we treat Latin indices, taking values in {1, 2, 3}, as being Cartesian in nature and thus make no distinction between co- and contravariance. However, this distinction is maintained for variables bearing Greek indices ranging over {1, 2}. Repeated indices are summed over their range.

Some elements of the theory described here have been discussed in [14,15], and attention is confined in this work to the equilibrium theory.

2 Kirchhoff's theory for spatial rods

In Kirchhoff theory the rod is regarded as a spatial curve endowed with an elastic energy density, per unit initial length, that depends on the curvature and twist of the rod. According to the derivation from conventional three-dimensional nonlinear elasticity given in [8], this theory also accommodates small axial strain along the rod, although this effect is suppressed in derivations based on asymptotic analysis or the method of gamma convergence. We include it here, however. Further, we forego any discussion of the connection between Kirchhoff theory and three-dimensional elasticity and simply regard the rod as a directed curve [9] in which certain a priori constraints are imposed. It is possible, of course, to base a rod theory on the general Cosserat theory of directed curves, but this requires additional constitutive data and so, for the sake of simplicity, we do not pursue such a development here.

2.1 Kinematics

The basic kinematic variables in the theory are a (deformed) position field $\mathbf{r}(s)$, where $s \in [0, l]$ and l is the length of the rod in a reference configuration, and a right-handed, orthonormal triad $\{\mathbf{d}_i(s)\}$ in which $\mathbf{d}_3 = \mathbf{d}$, where $\mathbf{d}$ is the unit vector defined by

$$\mathbf{r}'(s) = \lambda\mathbf{d}, \quad \text{and} \quad \lambda = \left|\mathbf{r}'(s)\right| \tag{1}$$

is the *stretch* of the rod. Thus $\mathbf{d}_3$ is the unit tangent to the rod in a deformed configuration, and $\mathbf{d}_\alpha$ $(\alpha = 1, 2)$ span its cross-sectional plane at arclength station s.

A central aspect of the Kirchhoff theory is the stipulation that each cross section deforms as a rigid disk, the rigid deformation varying from one section to another. Accordingly, there is a rotation field $\mathbf{R}(s)$ such that $\mathbf{d}_i = \mathbf{R}\mathbf{D}_i$, where $\mathbf{D}_i(s)$ are the values of $\mathbf{d}_i(s)$ in a reference configuration. Using the representation $\mathbf{D}_i \otimes \mathbf{D}_i$ for the three-dimensional identity, we infer that

$$\mathbf{R} = \mathbf{d}_i \otimes \mathbf{D}_i. \tag{2}$$

The curvature and twist of the rod are embodied in the derivatives $\mathbf{d}_i'(s)$, where

$$\mathbf{d}_i' = \mathbf{R}'\mathbf{D}_i + \mathbf{R}\mathbf{D}_i'. \tag{3}$$

Let $\{\mathbf{E}_i\}$ be a fixed right-handed background frame. Then $\mathbf{D}_i(s) = \mathbf{A}(s)\mathbf{E}_i$ for some rotation field $\mathbf{A}$, yielding

$$\mathbf{d}_i' = \mathbf{W}\mathbf{d}_i = \mathbf{w} \times \mathbf{d}_i, \tag{4}$$

where

$$\mathbf{W} = \mathbf{R}'\mathbf{R}^t + \mathbf{R}\mathbf{A}'\mathbf{A}^t\mathbf{R}^t \tag{5}$$

is a skew tensor and $\mathbf{w}$ is its axial vector. If the rod is straight and untwisted in the reference configuration; i.e., if $\mathbf{D}_i' = \mathbf{0}$, then we have simply $\mathbf{W} = \mathbf{R}'\mathbf{R}^t$.

2.2 Strain energy function

We assume the strain energy S stored in a segment $[l_1, l_2] \subset [0, l]$ of a rod of length l to be expressible in the form

$$S = \int_{l_1}^{l_2} U \mathrm{d}s, \tag{6}$$

where U, the strain energy per unit length, is a function of the list $\{\mathbf{R}, \mathbf{R}', \mathbf{r}'\}$, possibly depending explicitly on s. Explicit s-dependence may arise from the initial curvature or twist of the rod, or from possibly nonuniform material properties.

We require U to be Galilean invariant and thus stipulate that its values be unaffected by the substitution $\{\mathbf{R}, \mathbf{R}', \mathbf{r}'\} \to \{\mathbf{QR}, \mathbf{QR}', \mathbf{Qr}'\}$, where $\mathbf{Q}$ is an arbitrary *uniform* rotation. Because U is defined pointwise, to derive a necessary condition we may choose the particular rotation $\mathbf{Q} = \mathbf{R}^t_{|s}$ and conclude that U is determined by the list $\{\mathbf{R}^t\mathbf{R}', \mathbf{R}^t\mathbf{r}'\}$. This list is easily shown to be Galilean invariant, and so our necessary condition is also sufficient. It is equivalent to $\{\mathbf{R}^t\mathbf{WR} - \mathbf{A}'\mathbf{A}^t, \lambda\mathbf{D}\}$, where $\mathbf{D} = \mathbf{D}_3$ and $\mathbf{R}^t\mathbf{WR} - \mathbf{A}'\mathbf{A}^t$ is a Galilean-invariant measure of the *relative* flexure and twist of the rod due to deformation. Here $\mathbf{D}$ and $\mathbf{A}'\mathbf{A}^t$ are independent of the deformation and serve to confer an explicit s-dependence on the strain energy function. Accordingly, we write $U = U(\mathbf{R}^t\mathbf{WR}, \lambda; s)$.

We observe that

$$\mathbf{R}^t\mathbf{WR} = W_{ij}\mathbf{D}_i \otimes \mathbf{D}_j, \quad \text{with} \quad W_{ij} = \mathbf{d}_i \cdot \mathbf{W}\mathbf{d}_j = \mathbf{d}_i \cdot \mathbf{d}'_j. \tag{7}$$

Assuming the explicit s-dependence to be due solely to the initial geometry of the rod in the manner described above, we then have

$$U = W(\lambda, \boldsymbol{\kappa} - \bar{\boldsymbol{\kappa}}), \tag{8}$$

where $\boldsymbol{\kappa}(= \kappa_i\mathbf{D}_i)$ is the axial vector of $\mathbf{R}^t\mathbf{WR}$; i.e.,

$$\kappa_i = \tfrac{1}{2}e_{ijk}W_{kj}, \tag{9}$$

and where $\bar{\boldsymbol{\kappa}}(= \bar{\kappa}_i\mathbf{D}_i)$ is the axial vector of $\mathbf{A}'\mathbf{A}^t$; i.e.,

$$\bar{\kappa}_i = \tfrac{1}{2}e_{ijk}\mathbf{D}_k \cdot \mathbf{D}'_j. \tag{10}$$

Here e_{ijk} is the Levi-Civita permutation symbol ($e_{123} = +1$, etc.), κ_3 is the twist of the rod, and κ_α are the curvatures.

2.3 Variational theory

Although the equilibrium equations of the Kirchhoff theory are well known and easily derived from elementary considerations, for our present purposes it is instructive to derive them from a variational argument. Thus let $\epsilon \in (-\epsilon_0, \epsilon_0)$ be a parameter and consider the configuration defined by $\mathbf{r}^*(s;\epsilon)$ and $\mathbf{d}^*_i(s;\epsilon)$, with [cf. (1)]

$$(\mathbf{r}^*)' = \lambda^*\mathbf{d}^*, \quad \lambda^* = \left|(\mathbf{r}^*)'\right|, \tag{11}$$

where $\mathbf{r}^*(s;0) = \mathbf{r}(s)$ and $\mathbf{d}^*_i(s;0) = \mathbf{d}_i(s)$. We take the state defined by $\epsilon = 0$ to be an equilibrium configuration.

We assume, for ϵ close to zero, that

$$\begin{aligned} \mathbf{r}^*(s;\epsilon) &= \mathbf{r}(s) + \epsilon\mathbf{u}(s) + o(\epsilon), \\ \mathbf{d}^*_i(s;\epsilon) &= \mathbf{d}_i(s) + \epsilon\dot{\mathbf{d}}_i(s) + o(\epsilon), \end{aligned} \tag{12}$$

where superposed dots identify ϵ-derivatives, evaluated at $\epsilon = 0$; thus, $\mathbf{u} = \dot{\mathbf{r}}$. The associated strain energy is

$$S^* = \int_{l_1}^{l_2} W(\boldsymbol{\kappa}^* - \bar{\boldsymbol{\kappa}}, \lambda^*)\mathrm{d}s = S + \epsilon\dot{S} + o(\epsilon), \tag{13}$$

where

$$\dot{S} = \int_{l_1}^{l_2} (W_\lambda\dot{\lambda} + \mu_i\dot{\kappa}_i)\mathrm{d}s, \tag{14}$$

with

$$\mu_i = \partial W/\partial(\kappa_i - \bar{\kappa}_i) \quad \text{and} \quad W_\lambda = \partial W/\partial\lambda, \tag{15}$$

is the first variation of the energy.

Let

$$\mathbf{R}^*(s;\epsilon) = \mathbf{d}_i^*(s;\epsilon) \otimes \mathbf{D}_i(s). \tag{16}$$

Then,

$$\dot{\mathbf{d}}_i = \mathbf{a} \times \mathbf{d}_i, \tag{17}$$

where $\mathbf{a}(s)$ is the axial vector of the skew tensor $\dot{\mathbf{R}}\mathbf{R}^t$. Let $a(S) = \dot{\lambda}$; i.e.,

$$\lambda^*(s;\epsilon) = \lambda(s) + \epsilon a(s) + o(\epsilon). \tag{18}$$

On combining these formulae we have

$$\dot{S} = I[a, \mathbf{a}], \tag{19}$$

where

$$I[a, \mathbf{a}] = \int_{l_1}^{l_2} (W_\lambda a + \boldsymbol{\mu} \cdot \mathbf{a}')\mathrm{d}s, \tag{20}$$

with

$$\boldsymbol{\mu} = \mu_i \mathbf{d}_i. \tag{21}$$

It follows from (7) and (9) that

$$\begin{aligned} \dot{\kappa}_i &= \tfrac{1}{2} e_{ijk}(\dot{\mathbf{d}}_k \cdot \mathbf{d}_j' + \mathbf{d}_k \cdot \dot{\mathbf{d}}_j') \\ &= \tfrac{1}{2} e_{ijk}[\mathbf{a} \times \mathbf{d}_k \cdot \mathbf{d}_j' + \mathbf{d}_k \cdot (\mathbf{a}' \times \mathbf{d}_j + \mathbf{a} \times \mathbf{d}_j')], \end{aligned} \tag{22}$$

in which the terms involving $\mathbf{a}$ cancel; one of the $e - \delta$ identities then yields

$$\dot{\kappa}_i = \mathbf{d}_i \cdot \mathbf{a}'. \tag{23}$$

Further, from (1) we have that

$$\mathbf{r}' \cdot \mathbf{d}_\alpha = 0 \tag{24}$$

for $\alpha = 1, 2$. To accommodate these constraints in the variational formulation we introduce the augmented energy

$$\bar{S} = S + \int_{l_1}^{l_2} f^\alpha \mathbf{r}' \cdot \mathbf{d}_\alpha \mathrm{d}s, \tag{25}$$

where $f^\alpha(s)$ are Lagrange multipliers. This is an extension of the actual energy from the class of deformations defined by the constraints to arbitrary deformations.

We propose that equilibrium of the configuration corresponding to $\epsilon = 0$ is equivalent to the virtual-work equality

$$(\bar{S})^{\cdot} = P, \tag{26}$$

where P is the virtual power of the loads, the form of which is made explicit below.

First, we observe that variations with respect to the Lagrange multipliers simply return the constraints as the relevant Euler–Lagrange equations; thus, we do not make these explicit. Further, the stationarity of $\bar{S}$ with respect to arbitrary variations implies stationarity with respect to variations satisfying the constraints; this in turn ensures that $\dot{S} = P$—and hence equilibrium—for the actual constrained problem. Conversely, if the Euler–Lagrange equations and natural boundary conditions for $\bar{S}$ are satisfied, which are necessary and sufficient for (26) against arbitrary variations, then (26) is satisfied by constrained variations in particular, and this ensures equilibrium.

Proceeding to the examination of (26), we have

$$I[a, \mathbf{a}] + \int_{l_1}^{l_2} f^\alpha(\mathbf{u}' \cdot \mathbf{d}_\alpha + \mathbf{r}' \cdot \mathbf{a} \times \mathbf{d}_\alpha)\mathrm{d}s = P, \tag{27}$$

which is equivalent to

$$\int_{l_1}^{l_2} (\mathbf{f} \cdot \mathbf{u}' + \boldsymbol{\mu} \cdot \mathbf{a}' + \mathbf{a} \cdot \mathbf{f} \times \mathbf{r}')\mathrm{d}s = P, \tag{28}$$

where

$$\mathbf{f} = W_\lambda \mathbf{d} + f^\alpha \mathbf{d}_\alpha. \tag{29}$$

This in turn is equivalent to

$$(\mathbf{f} \cdot \mathbf{u} + \boldsymbol{\mu} \cdot \mathbf{a}) \,|_{l_1}^{l_2} - \int_{l_1}^{l_2} \{\mathbf{u} \cdot \mathbf{f}' + \mathbf{a} \cdot (\boldsymbol{\mu}' - \mathbf{f} \times \mathbf{r}')\} \mathrm{d}s = P, \tag{30}$$

which implies that the virtual power is expressible in the form

$$P = (\mathbf{t} \cdot \mathbf{u} + \mathbf{c} \cdot \mathbf{a}) \,|_{l_1}^{l_2} + \int_{l_1}^{l_2} (\mathbf{u} \cdot \mathbf{g} + \mathbf{a} \cdot \mathbf{m}) \mathrm{d}s, \tag{31}$$

in which $\mathbf{t}$ and $\mathbf{c}$ represent forces and couples acting at the ends of the segment and $\mathbf{g}$ and $\mathbf{m}$ are force and couple distributions over the interior.

By the fundamental lemma, the Euler equations holding at points in the interior of the rod are

$$\mathbf{f}' + \mathbf{g} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}' + \mathbf{m} + \mathbf{r}' \times \mathbf{f} = \mathbf{0}, \tag{32}$$

and the endpoint conditions are

$$\mathbf{f} = \mathbf{t} \quad \text{and} \quad \boldsymbol{\mu} = \mathbf{c}, \tag{33}$$

provided that neither position nor section orientation is assigned at the endpoints. These are the equilibrium conditions of classical rod theory [8,9] in which $\mathbf{f}$ and $\boldsymbol{\mu}$, respectively, are the cross-sectional force and moment transmitted by the segment $(s, l]$ on the part $[0, s]$. From (29) we have the interpretation of the Lagrange multipliers f^α as transverse shear forces acting on a fiber cross section.

Other boundary conditions are, of course, possible. For example, if the tangent direction $\mathbf{d}$ is assigned at a boundary point, then its variation $\mathbf{a} \times \mathbf{d}$ vanishes there, leaving $\mathbf{a} = \alpha \mathbf{d}$ in which α is arbitrary. In this case (30) and (31) furnish the boundary condition

$$\boldsymbol{\mu} \cdot \mathbf{d} = c, \tag{34}$$

in which $c = \mathbf{c} \cdot \mathbf{d}$ is the twisting moment applied at the boundary.

3 An elastic surface formed by two families of Kirchhoff rods

3.1 Kinematics

We construct a model of elastic surfaces formed by two families of Kirchhoff rods of the kind considered in the foregoing. The rods, or fibers, are regarded as being continuously distributed, so that every material point of the surface, ω say, lies at the intersection of two rods, one from each family. Thus, let $\{\mathbf{l}_i\}$ and $\{\mathbf{m}_i\}$ be triads of orthonormal vectors associated with the two fiber families; these are the surface analogues of the triad $\{\mathbf{d}_i\}$ discussed above. Their pre-images on a reference surface Ω are $\{\mathbf{L}_i\}$ and $\{\mathbf{M}_i\}$. The various triads are regarded as functions of (convected) surface coordinates θ^α; $\alpha = 1, 2$ and $\mathbf{r}(\theta^\alpha)$ is the position field on ω. We take $\mathbf{l}(= \mathbf{l}_3)$ and $\mathbf{m}(= \mathbf{m}_3)$ to be the unit tangents to the fiber trajectories on ω; the unit tangents to the fibers on Ω are $\mathbf{L}(= \mathbf{L}_3)$ and $\mathbf{M}(= \mathbf{M}_3)$. Accordingly, by analogy with (1) we have [16]

$$\lambda \mathbf{l} = L^\alpha \mathbf{r}_{,\alpha} \quad \text{and} \quad \mu \mathbf{m} = M^\alpha \mathbf{r}_{,\alpha}, \tag{35}$$

where λ and μ are the fiber stretches. Here L^α and M^α are the contravariant components of $\mathbf{L}$ and $\mathbf{M}$ on Ω, and the derivatives on the right-hand side of these equations are directional derivatives along the fiber trajectories on Ω. Hence the constraints (cf. (24))

$$L^\alpha \mathbf{r}_{,\alpha} \cdot \mathbf{l}_\beta = 0 \quad \text{and} \quad M^\alpha \mathbf{r}_{,\alpha} \cdot \mathbf{m}_\beta = 0, \tag{36}$$

associated with the fact that the fiber 'cross sections' remain orthogonal to their tangents as the surface deforms.

3.2 Strain energy

To accommodate these constraints we introduce augmented energy (cf. (25))

$$\bar{S} = S + \int_{\Pi} \left[f_{(l)}^{\beta} \mathbf{l}_{\beta} \cdot L^{\alpha} \mathbf{r}_{,\alpha} + f_{(m)}^{\beta} \mathbf{m}_{\beta} \cdot M^{\alpha} \mathbf{r}_{,\alpha} \right] \mathrm{d}a, \tag{37}$$

where $\Pi \subset \Omega$ is an arbitrary simply connected subregion of Ω; $f_{(l)}^{\beta}$ and $f_{(m)}^{\beta}$ are Lagrange multipliers defined on Ω; and

$$S = \int_{\Pi} W \mathrm{d}a \tag{38}$$

is the strain energy, in which W, the strain energy per unit area of Ω, is presumed to depend on the surface metric $a_{\alpha\beta} = \mathbf{r}_{,\alpha} \cdot \mathbf{r}_{,\beta}$ and the fiber curvature-twist vectors $\boldsymbol{\kappa}_l = \kappa_{(l)i} \mathbf{L}_i$ and $\boldsymbol{\kappa}_m = \kappa_{(m)i} \mathbf{M}_i$, defined (cf. (7) and (9)) by

$$\kappa_{(l)i} = \tfrac{1}{2} e_{ijk} W_{(l)kj} \quad \text{and} \quad \kappa_{(m)i} = \tfrac{1}{2} e_{ijk} W_{(m)kj}, \tag{39}$$

with

$$W_{(l)kj} = \mathbf{l}_k \cdot L^{\alpha} \mathbf{l}_{j,\alpha} \quad \text{and} \quad W_{(m)kj} = \mathbf{m}_k \cdot M^{\alpha} \mathbf{m}_{j,\alpha}. \tag{40}$$

We assume that W depends on the surface metric via the fiber stretches λ and μ and the fiber shear angle γ, defined by

$$\sin \gamma = \mathbf{l} \cdot \mathbf{m}. \tag{41}$$

In particular, we note the connection [16]

$$\lambda \mu \left| \cos \gamma \right| = J, \tag{42}$$

where

$$J = \sqrt{a/e}, \tag{43}$$

in which $a = \det(a_{\alpha\beta})$ and e is the value of a on Ω, and we take the strain energy to be given by the function

$$W(\lambda, \mu, J, \boldsymbol{\kappa}_l - \bar{\boldsymbol{\kappa}}_l, \boldsymbol{\kappa}_m - \bar{\boldsymbol{\kappa}}_m), \tag{44}$$

where

$$\bar{\boldsymbol{\kappa}}_l = \bar{\kappa}_{(l)i} \mathbf{L}_i \quad \text{and} \quad \bar{\boldsymbol{\kappa}}_m = \bar{\kappa}_{(m)i} \mathbf{M}_i, \tag{45}$$

with

$$\bar{\kappa}_{(l)i} = \tfrac{1}{2} e_{ijk} \mathbf{L}_k \cdot L^{\alpha} \mathbf{L}_{j,\alpha} \quad \text{and} \quad \bar{\kappa}_{(m)i} = \tfrac{1}{2} e_{ijk} \mathbf{M}_k \cdot L^{\alpha} \mathbf{M}_{j,\alpha}. \tag{46}$$

3.3 Variational theory

As in the case of a single rod, we suppose that equilibrium of a configuration defined by $\{\mathbf{r}, \mathbf{l}_i, \mathbf{m}_i\}$ is expressed by the virtual-work statement

$$(\bar{S})^{\cdot} = P \tag{47}$$

for arbitrary $\Pi \subset \Omega$ in which P is the virtual power of the loads, the form of which will be deduced in what follows.

Proceeding, we compute the variation

$$\dot{W} = \sigma^{\alpha\beta} \mathbf{a}_{\beta} \cdot \mathbf{u}_{,\alpha} + \mu_{(l)i} \dot{\kappa}_{(l)i} + \mu_{(m)i} \dot{\kappa}_{(m)i}, \tag{48}$$

where $\mathbf{a}_{\beta} = \mathbf{r}_{,\beta}$,

$$\sigma^{\alpha\beta} = \partial W / \partial a_{\alpha\beta} + \partial W / \partial a_{\beta\alpha}, \tag{49}$$

$$\mu_{(l)i} = \partial W / \partial (\kappa_{(l)i} - \bar{\kappa}_{(l)i}) \quad \text{and} \quad \mu_{(m)i} = \partial W / \partial (\kappa_{(m)i} - \bar{\kappa}_{(m)i}), \tag{50}$$

and where

$$\sigma^{\alpha\beta} \mathbf{a}_{\beta} = W_{\lambda} L^{\alpha} \mathbf{l} + W_{\mu} M^{\alpha} \mathbf{m} + J W_J \mathbf{a}^{\alpha} \tag{51}$$

for energies of form (44), in which $\mathbf{a}^\alpha$ are the duals to the $\mathbf{a}_\alpha$ on the tangent planes of ω. This follows from the variational forms of (35); i.e.,

$$L^\alpha \mathbf{u}_{,\alpha} = \dot{\lambda}\mathbf{l} + \lambda\dot{\mathbf{l}} \quad \text{and} \quad M^\alpha \mathbf{u}_{,\alpha} = \dot{\mu}\mathbf{m} + \mu\dot{\mathbf{m}}, \tag{52}$$

together with

$$\dot{J}/J = \mathbf{a}^\alpha \cdot \mathbf{u}_{,\alpha}. \tag{53}$$

Using (23) in the forms

$$\dot{\kappa}_{(l)i} = \mathbf{l}_i \cdot L^\alpha \mathbf{a}_{l,\alpha} \quad \text{and} \quad \dot{\kappa}_{(m)i} = \mathbf{m}_i \cdot M^\alpha \mathbf{a}_{m,\alpha}, \tag{54}$$

where $\mathbf{a}_l$ and $\mathbf{a}_m$ are the virtual fiber rotations defined by

$$\dot{\mathbf{l}}_i = \mathbf{a}_l \times \mathbf{l}_i \quad \text{and} \quad \dot{\mathbf{m}}_i = \mathbf{a}_m \times \mathbf{m}_i, \tag{55}$$

we find that

$$(\bar{S})^\cdot = \int_\Pi (\mathbf{N}^\alpha \cdot \mathbf{u}_{,\alpha} + L^\alpha \boldsymbol{\mu}_l \cdot \mathbf{a}_{l,\alpha} + M^\alpha \boldsymbol{\mu}_m \cdot \mathbf{a}_{m,\alpha})\mathrm{d}a + \int_\Pi (\mathbf{a}_l \cdot \mathbf{f}_l \times L^\alpha \mathbf{r}_{,\alpha} + \mathbf{a}_m \cdot \mathbf{f}_m \times M^\alpha \mathbf{r}_{,\alpha})\mathrm{d}a, \tag{56}$$

where

$$\mathbf{N}^\alpha = J W_J \mathbf{a}^\alpha + L^\alpha \mathbf{f}_l + M^\alpha \mathbf{f}_m, \quad \boldsymbol{\mu}_l = \mu_{(l)i}\mathbf{l}_i \quad \text{and} \quad \boldsymbol{\mu}_m = \mu_{(m)i}\mathbf{m}_i \tag{57}$$

with

$$\mathbf{f}_l = W_\lambda \mathbf{l} + f^\beta_{(l)}\mathbf{l}_\beta \quad \text{and} \quad \mathbf{f}_m = W_\mu \mathbf{m} + f^\beta_{(m)}\mathbf{m}_\beta. \tag{58}$$

Finally, an application of Stokes' theorem leads to

$$(\bar{S})^\cdot = \int_{\partial\Pi} \mathbf{N}^\alpha \nu_\alpha \cdot \mathbf{u}\mathrm{d}s + \int_{\partial\Pi} (L^\alpha \boldsymbol{\mu}_l \cdot \mathbf{a}_l + M^\alpha \boldsymbol{\mu}_m \cdot \mathbf{a}_m)\nu_\alpha \mathrm{d}s - \int_\Pi \Big\{\mathbf{u} \cdot \mathbf{N}^\alpha_{|\alpha} + \mathbf{a}_l \cdot [(L^\alpha \boldsymbol{\mu}_l)_{|\alpha} - \mathbf{f}_l \times L^\alpha \mathbf{r}_{,\alpha}] + \mathbf{a}_m \cdot [(M^\alpha \boldsymbol{\mu}_m)_{|\alpha} - \mathbf{f}_m \times M^\alpha \mathbf{r}_{,\alpha}]\Big\}\mathrm{d}a, \tag{59}$$

where ν_α are the covariant components of the exterior unit normal to $\partial\Pi$, and $\mathbf{N}^\alpha_{|\alpha}$, etc., is the covariant divergence on Ω defined by

$$\mathbf{N}^\alpha_{|\alpha} = e^{-1/2}(e^{1/2}\mathbf{N}^\alpha)_{,\alpha} \tag{60}$$

with e the determinant of the surface metric on Ω.

In view of (47) and (56) we conclude that the virtual power of the loads is expressible in the form

$$P = \int_\Pi \mathbf{g} \cdot \mathbf{u}\mathrm{d}a + \int_\Pi (\mathbf{m}_l \cdot \mathbf{a}_l + \mathbf{m}_m \cdot \mathbf{a}_m)\mathrm{d}a + \int_{\partial\Pi} \mathbf{t} \cdot \mathbf{u}\mathrm{d}s + \int_{\partial\Pi} (\mathbf{c}_l \cdot \mathbf{a}_l + \mathbf{c}_m \cdot \mathbf{a}_m)\mathrm{d}s, \tag{61}$$

where $\mathbf{g}$ is a distributed force acting on the surface and $\mathbf{m}_l$ and $\mathbf{m}_m$ are distributed moments acting on the fibers, all per unit area of Ω; and $\mathbf{t}$ is the traction and $\mathbf{c}_l$ and $\mathbf{c}_m$ are couples applied to the fibers, per unit arclength of the boundary. The fundamental lemma then yields

$$\mathbf{N}^\alpha_{|\alpha} + \mathbf{g} = \mathbf{0}, \tag{62}$$

$$(L^\alpha \boldsymbol{\mu}_l)_{|\alpha} + \mathbf{m}_l + L^\alpha \mathbf{r}_{,\alpha} \times \mathbf{f}_l = \mathbf{0} \quad \text{and} \quad (M^\alpha \boldsymbol{\mu}_m)_{|\alpha} + \mathbf{m}_m + M^\alpha \mathbf{r}_{,\alpha} \times \mathbf{f}_m = \mathbf{0} \tag{63}$$

on Ω, whereas

$$\mathbf{t} = \mathbf{N}^\alpha \nu_\alpha, \quad \mathbf{c}_l = (\mathbf{L} \cdot \boldsymbol{\nu})\boldsymbol{\mu}_l \quad \text{and} \quad \mathbf{c}_m = (\mathbf{M} \cdot \boldsymbol{\nu})\boldsymbol{\mu}_m \tag{64}$$

in which the left-hand sides are assigned at points of the boundary where $\mathbf{r}$, $\{\mathbf{l}_i\}$ or $\{\mathbf{m}_i\}$, respectively, are *not* assigned. These conditions may be compared to (32) and (33), and alternative boundary conditions, such as the analogues of (34), may be easily derived.

Distributed twisting moments $\mathbf{l} \cdot \mathbf{m}_l$ and $\mathbf{m} \cdot \mathbf{m}_m$ may arise, for example, from friction generated as the fibers pivot about their respective tangents independently of the underlying surface deformation. However, by confining attention to a single surface deformation field, we have effectively precluded the possibility of relative slipping of the fibers. This reflects the tacit assumption that sufficient friction is available to suppress slipping. The present model may be generalized to accommodate this effect, however; see [17] for relevant developments.

4 Global balances of force and moment

In conventional elasticity the local equilibrium equations are equivalent, granted suitable regularity, to global balances of force and moment for arbitrary subdomains of the body. The latter also follow from the virtual-work principle when restricted to rigid-body virtual displacements. To explore the consequences of the virtual-work statement with respect to such displacements in the present setting, we consider rigid-body deformations

$$\mathbf{r}^*(\theta^\alpha;\epsilon) = \mathbf{Q}(\epsilon)\mathbf{r}(\theta^\alpha) + \mathbf{b}^*(\epsilon), \quad \mathbf{l}_i^*(\theta^\alpha;\epsilon) = \mathbf{Q}(\epsilon)\mathbf{l}_i(\theta^\alpha), \quad \mathbf{m}_i^*(\theta^\alpha;\epsilon) = \mathbf{Q}(\epsilon)\mathbf{m}_i(\theta^\alpha), \tag{65}$$

where $\mathbf{Q}(\epsilon)$ and $\mathbf{b}^*(\epsilon)$, respectively, are one-parameter families of rotations and displacements, with $\mathbf{Q}(0) = \mathbf{I}$—the three-dimensional identity—and $\mathbf{b}^*(0) = \mathbf{0}$. The associated variations, evaluated at the equilibrium state $\epsilon = 0$, are

$$\mathbf{u}(\theta^\alpha) = \boldsymbol{\omega} \times \mathbf{r}(\theta^\alpha) + \mathbf{b}, \quad \dot{\mathbf{l}}_i(\theta^\alpha) = \boldsymbol{\omega} \times \mathbf{l}_i(\theta^\alpha) \quad \text{and} \quad \dot{\mathbf{m}}_i(\theta^\alpha) = \boldsymbol{\omega} \times \mathbf{m}_i(\theta^\alpha), \tag{66}$$

where $\boldsymbol{\omega}$ is the axial vector of the fixed skew tensor $\boldsymbol{\Omega} = \dot{\mathbf{Q}}(0)$ and $\mathbf{b}$ is a fixed vector. Accordingly, the virtual fiber rotations are equal and uniform: $\mathbf{a}_l = \mathbf{a}_m = \boldsymbol{\omega}$.

Because the strain energy is invariant under such deformations, and assuming the constraints to be in force, we have $(\bar{S})^{\cdot} = 0$ and virtual-work statement (47) reduces to $P = 0$; i.e.,

$$\mathbf{b} \cdot \left(\int_\Pi \mathbf{g} da + \int_{\partial\Pi} \mathbf{t} ds \right) + \boldsymbol{\omega} \cdot \left\{ \int_\Pi (\mathbf{r} \times \mathbf{f} + \mathbf{m}) da + \int_{\partial\Pi} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds \right\} = \mathbf{0}, \tag{67}$$

in which $\mathbf{b}$ and $\boldsymbol{\omega}$ are arbitrary, and

$$\mathbf{m} = \mathbf{m}_l + \mathbf{m}_m, \quad \mathbf{c} = \mathbf{c}_l + \mathbf{c}_m, \tag{68}$$

are the net distributed moment and edge couple, respectively. Hence the global force and moment balances

$$\int_\Pi \mathbf{g} da + \int_{\partial\Pi} \mathbf{t} ds = \mathbf{0} \quad \text{and} \quad \int_\Pi (\mathbf{r} \times \mathbf{g} + \mathbf{m}) da + \int_{\partial\Pi} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds = \mathbf{0}. \tag{69}$$

Using (62) we find that

$$\int_\Pi \mathbf{g} da + \int_{\partial\Pi} \mathbf{t} ds = -\int_\Pi \mathbf{N}^\alpha_{|\alpha} da + \int_{\partial\Pi} \mathbf{N}^\alpha \nu_\alpha ds, \tag{70}$$

which vanishes by virtue of Stokes' theorem; accordingly, $(69)_1$ is identically satisfied. Further, from (63) and $(68)_1$,

$$\mathbf{m} = -\boldsymbol{\mu}^\alpha_{|\alpha} - \mathbf{a}_\alpha \times \mathbf{f}^\alpha, \tag{71}$$

where

$$\boldsymbol{\mu}^\alpha = L^\alpha \boldsymbol{\mu}_l + M^\alpha \boldsymbol{\mu}_m \quad \text{and} \quad \mathbf{f}^\alpha = L^\alpha \mathbf{f}_l + M^\alpha \mathbf{f}_m \tag{72}$$

are couple-stress and fiber-stress vectors. Accordingly, with (62),

$$\mathbf{r} \times \mathbf{g} + \mathbf{m} = -\boldsymbol{\mu}^\alpha_{|\alpha} - (\mathbf{r} \times \mathbf{N}^\alpha)_{|\alpha} + \mathbf{a}_\alpha \times (\mathbf{N}^\alpha - \mathbf{f}^\alpha). \tag{73}$$

Similarly, (64) and $(68)_2$ combine to give

$$\mathbf{r} \times \mathbf{t} + \mathbf{c} = (\mathbf{r} \times \mathbf{N}^\alpha + \boldsymbol{\mu}^\alpha)\nu_\alpha, \tag{74}$$

and we conclude, with (73), that

$$\int_\Pi (\mathbf{r} \times \mathbf{g} + \mathbf{m}) da + \int_{\partial\Pi} (\mathbf{r} \times \mathbf{t} + \mathbf{c}) ds = \int_\Pi J W_J \mathbf{a}_\alpha \times \mathbf{a}^\alpha da. \tag{75}$$

Here $\mathbf{a}_\alpha \times \mathbf{a}^\alpha = \mathbf{a}_\alpha \times \mathbf{a}_\beta a^{\alpha\beta}$, where $a^{\alpha\beta}$ is the symmetric dual metric on ω. Thus, the right-hand side vanishes and $(69)_2$ reduces to an identity.

We conclude that the global force and moment balances are consequences of general variational statement (47). The converse is not true, however, as (71) does not yield (63) or (64). This state of affairs is to be expected, as (62)–(64) follow by imposing (47) for arbitrary variations, whereas special (i.e., rigid-body) variations were used to derive the global force and moment balances. Consequently, in contrast to the situation in conventional elasticity theory, global balances of force and moment are necessary, but not sufficient, for equilibrium in the sense of (47).

5 Example: bending and twisting of a rectangular strip to a helicoid

Consider the rectangular strip defined by $l_1 \leq x \leq l_2$, $-h \leq y \leq h$, where $l_{1,2}$ and h are positive constants and x, y are Cartesian coordinates on a plane. We take this strip to be the reference configuration of the elastic surface with fibers lying parallel to the coordinate axes; i.e., $\mathbf{L} = \mathbf{i}$ and $\mathbf{M} = \mathbf{j}$, the unit vectors aligned with the coordinate directions. The fibers are straight and untwisted in this configuration ($\bar{\boldsymbol{\kappa}}_l = \bar{\boldsymbol{\kappa}}_m = \mathbf{0}$), and we consider strain energy functions of the form

$$W = w(\lambda, \mu, J) + \tfrac{1}{2}k[\kappa_{(l)\alpha}\kappa_{(l)\alpha} + \kappa_{(m)\alpha}\kappa_{(m)\alpha}] + \tfrac{1}{2}\bar{k}\left(\tau_l^2 + \tau_m^2\right), \tag{76}$$

where k and $\bar{k}$ are the fiber flexural and torsional rigidities, assumed to be common to the two fiber families, and $\tau_l = \kappa_{(l)3}$ and $\tau_m = \kappa_{(m)3}$ are the fiber twists. This energy is appropriate for fibers of circular cross section that are straight and untwisted in their natural states [8]. Constitutive equations (50) furnish the fiber couple vectors

$$\boldsymbol{\mu}_l = k\mathbf{l} \times L^\alpha \mathbf{l}_{,\alpha} + \bar{k}\tau_l \mathbf{l} \quad \text{and} \quad \boldsymbol{\mu}_m = k\mathbf{m} \times M^\alpha \mathbf{m}_{,\alpha} + \bar{k}\tau_m \mathbf{m}, \tag{77}$$

with $L^\alpha = \delta_1^\alpha$ and $M^\alpha = \delta_2^\alpha$; δ_β^α being the usual Kronecker delta.

We take the convected coordinates to be $\theta^1 = x$ and $\theta^2 = y$, and consider the deformation

$$\mathbf{r}(x, y) = r(x)\mathbf{e}_r(\theta(y)) + \beta\theta(y)\mathbf{k}; \quad \theta(y) = \alpha y, \quad r'(x) > 0, \tag{78}$$

where $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ is the unit normal to the reference plane, $\mathbf{e}_r(\theta) = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is a unit vector directed along a ray $\theta = const.$, and α and β are positive constants. The deformed surface is a helicoid on which the fibers $x = const.$ are helices of constant pitch and the fibers $y = const.$ are their straight orthogonal trajectories. We readily compute

$$\mathbf{a}_1 = r'\mathbf{e}_r \quad \text{and} \quad \mathbf{a}_2 = \alpha(r\mathbf{e}_\theta + \beta\mathbf{k}), \tag{79}$$

where $\mathbf{e}_\theta = \mathbf{e}_r'(\theta) = \mathbf{k} \times \mathbf{e}_r$. From (35) it then follows that

$$\lambda = r', \quad \mu = \alpha\sqrt{r^2 + \beta^2}, \quad \mathbf{l} = \mathbf{e}_r \quad \text{and} \quad \mathbf{m} = (\alpha/\mu)(r\mathbf{e}_\theta + \beta\mathbf{k}). \tag{80}$$

Accordingly, the fiber stretches and $J = \lambda\mu$ depend only on x, while $\mathbf{l}$ is a function only of y. Equations (77) furnish

$$\boldsymbol{\mu}_l = \bar{k}\tau_l\mathbf{e}_r \quad \text{and} \quad \boldsymbol{\mu}_m = \bar{k}\tau_m\mathbf{m} + k\mathbf{m} \times \mathbf{m}_{,y}, \quad \text{where} \quad \mathbf{m}_{,y} = (-\alpha^2/\mu)r\mathbf{e}_r. \tag{81}$$

We seek a solution in which the transverse shear forces $f_{(l)}^\alpha$ and $f_{(m)}^\alpha$ acting on the fibers vanish together with the distributed couples $\mathbf{m}_l$ and $\mathbf{m}_m$. In these circumstances eqs. (63) reduce to

$$\boldsymbol{\mu}_{l,x} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}_{m,y} = \mathbf{0}, \tag{82}$$

or

$$\tau_{l,x}\mathbf{e}_r = \mathbf{0} \quad \text{and} \quad \bar{k}(\tau_{m,y}\mathbf{m} + \tau_m\mathbf{m}_{,y}) + k\mathbf{m} \times \mathbf{m}_{,yy} = \mathbf{0}, \tag{83}$$

yielding $\tau_{l,x} = 0$ and $\tau_{m,y} = 0$—implying that a given fiber is uniformly twisted along its length—together with

$$\tfrac{\alpha^2 r}{\mu}\left(k\tfrac{\alpha^2\beta}{\mu} - \bar{k}\tau_m\right)\mathbf{e}_r = \mathbf{0}, \tag{84}$$

which furnishes the twist

$$\tau_m(x) = k\alpha^2\beta/\bar{k}\mu(x). \tag{85}$$

The fiber stretch $\mu(x)$ is determined by solving (62). Using (57) and

$$\mathbf{a}^\alpha = l^\alpha\mathbf{l} + m^\alpha\mathbf{m}, \tag{86}$$

which follow from the orthonormality of $\{\mathbf{l}, \mathbf{m}\}$ in the present context, together with $\lambda l^\alpha = L^\alpha$ and $\mu m^\alpha = M^\alpha$, which follow from (35), we find that

$$\mathbf{N}^\alpha = (w_\lambda + \mu w_J)L^\alpha\mathbf{l} + (w_\mu + \lambda w_J)M^\alpha\mathbf{m} \tag{87}$$

and that (62) reduces to

$$(w_\lambda + \mu w_J)_{,x}\mathbf{l} + (w_\mu + \lambda w_J)_{,y}\mathbf{m} + (w_\mu + \lambda w_J)\mathbf{m}_{,y} = \mathbf{0}, \tag{88}$$

assuming the distributed force $\mathbf{g} = \mathbf{0}$. With our previous results this may be further reduced to

$$[(w_\lambda + \mu w_J)_{,x} - \tfrac{\alpha^2 r}{\mu}(w_\mu + \lambda w_J)]\mathbf{l} + (w_\mu + \lambda w_J)_{,y}\mathbf{m} = \mathbf{0}, \tag{89}$$

and we conclude that

$$(w_\lambda + \mu w_J)_{,x} = \tfrac{\alpha^2 r}{\mu}(w_\mu + \lambda w_J) \quad \text{and} \quad (w_\mu + \lambda w_J)_{,y} = 0. \tag{90}$$

Because the stretches depend only on x, the second result reduces to an identity if the material is uniform (w not explicitly dependent on x or y), whereas the first reduces to a nonlinear second-order ordinary differential equation for $r(x)$, which may be solved in principle once the function w is specified explicitly.

Concerning edge conditions, on the edge $x = l_2$ we have the traction

$$\mathbf{t}_2(y) = (w_\lambda + \mu w_J)_{|l_2}\mathbf{e}_r(\alpha y). \tag{91}$$

The traction at the opposite edge $x = l_1$ is

$$\mathbf{t}_1(y) = -(w_\lambda + \mu w_J)_{|l_1}\mathbf{e}_r(\alpha y). \tag{92}$$

Specification of these tractions provides data for the integration of the equation for $r(x)$. In this specification it is necessary that $\left|\mathbf{t}_{1,2}(y)\right|$ be uniform. The solution determines the tractions at the edges $y = \pm h$:

$$\mathbf{t}_\pm(x) = \pm(w_\mu + \lambda w_J)\mathbf{m}(x, \pm h), \tag{93}$$

where

$$\mathbf{m}(x, \pm h) = (\alpha/\mu)\{r\mathbf{e}_\theta(\pm\alpha h) + \beta\mathbf{k}\}. \tag{94}$$

The solution $r(x)$ also determines the twist $\tau_m(x)$ in accordance with $(80)_2$ and (85), which generates the couples

$$\mathbf{c}_m = \pm\boldsymbol{\mu}_m \tag{95}$$

at the edges $y = \pm h$; these contain both bending and twisting components. The couple at the edge $x = l_2$ is the pure twisting moment

$$\mathbf{c}_l(y) = \bar{k}\tau_l(y)\mathbf{e}_r(\alpha y), \tag{96}$$

specification of which yields the twist $\tau_l(y)$ throughout the sheet. Finally, the couple acting at $x = l_1$ is $-\mathbf{c}_l(y)$.

Acknowledgements The support of the US National Science Foundation through Grant Number CMMI-1538228 is gratefully acknowledged. We are especially grateful to Professor Francesco dell'Isola for extensive discussions concerning the topic of this paper during his Visiting Professorship at UC Berkeley.

References

1. Steigmann, D.J., dell'Isola, F.: Mechanical response of fabric sheets to three-dimensional bending, twisting and stretching. Acta. Mech. Sin. **31**, 373–382 (2015)
2. Steigmann, D.J.: Equilibrium of elastic lattice shells. J. Eng. Math. **109**, 47–61 (2018)
3. Steigmann, D.J.: Continuum theory for elastic sheets formed by inextensible crossed elasticae. Int. J. Non Linear Mech. https://doi.org/10.1016/j.ijnonlinmec.2018.03.012
4. Giorgio, I., Rizzi, N.L., Turco, E.: Continuum modelling of pantographic sheets for out-of-plane bifurcation and vibrational analysis. Proc. R. Soc. Lond. A **473**(2207), 21 (2017)
5. Turco, E., dell'Isola, F., Cazzani, A., Rizzi, N.L.: Hencky-type discrete model for pantographic structures: numerical comparison with second gradient continuum models. Z. Angew. Math. Phys. **67**(4), 28 (2016)
6. Toupin, R.A.: Elastic materials with couple-stresses. Arch. Ration. Mech. Anal. **11**, 385–414 (1962)
7. Toupin, R.A.: Theories of elasticity with couple-stress. Arch. Ration. Mech. Anal. **17**, 85–112 (1964)
8. Dill, E.H.: Kirchhoff's theory of rods. Arch. Hist. Exact Sci. **44**, 1–23 (1992)
9. Antman, S.S.: Nonlinear Problems of Elasticity, 2nd edn. Springer, New York (2005)
10. Germain, P.: The method of virtual power in continuum mechanics, part 2: microstructure. SIAM J. Appl. Math. **25**, 556–575 (1973)
11. dell'Isola, F., Seppecher, P.: Edge contact forces and quasi-balanced power. Meccanica **32**, 33–52 (1997)
12. Maugin, G.A.: The principle of virtual power: from eliminating metaphysical forces to providing an efficient modeling tool. Cont. Mech. Thermodyn. **25**, 127–146 (2013)
13. Eugster, S.R., dell'Isola, F.: Exegesis of the introduction and sect. I from "Fundamentals of the Mechanics of Continua", by E. Hellinger. ZAMM **97**, 477–506 (2017)
14. Luo, C.: Nonlinear three-dimensional mechanics of fabrics. Dissertation, University of California at Berkeley (2000)

15. Luo, C., Steigmann, D.J.: Bending and twisting effects in the three-dimensional finite deformations of an inextensible network. In: Durban, D., et al. (eds.) Developments in Shell Theory, Technion Volume to Commemorate the 70th Birthday of A. Libai, Advances in the Mechanics of Plates and Shells, pp. 213–228. Kluwer Academic Publishers, London (2001)
16. Steigmann, D.J., Pipkin, A.C.: Equilibrium of elastic nets. Philos. Trans. R. Soc. Lond. **A335**, 419–454 (1991)
17. Nadler, B.: A theory of the mechanics of two coupled surfaces. Math. Mech. Solids **14**, 456–473 (2009)