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ORIGINAL ARTICLE

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A microstructure continuum approach to electromagneto-elastic conductors

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Abstract A micromorphic continuum model of a deformable electromagnetic conductor is established introducing microdensities of bound and free charges. The conductive part of electric current consists of contributions due to free charges and microdeformation. Beside the conservation of charge, we derive suitable evolution equations for electric multipoles which are exploited to obtain the macroscopic form of Maxwell's equations. A constitutive model for electromagneto-elastic conductors is considered which allows for a natural characterization of perfect conductors independently on the form of the constitutive equation for the conduction current. A generalized Ohm's law is also derived for not ideal conductors which accounts for relaxation effects. The consequences of the linearized Ohm's law on the classic magnetic transport equation are shown.

Keywords Electromagneto-elastic continua · Continua with microstructure · Generalized Ohm's law

1 Introduction

The coupling between elastic and electromagnetic fields in continuum mechanics has a relevant role in modeling a wide class of solid and fluid materials. Physical interactions due to the presence of electric and magnetic fields in matter are introduced in current macroscopic theories on the basis of Maxwell's equations, thermodynamic laws, and suitable constitutive assumptions accounting for the intrinsic material structure. Along this line, many contributions to model various thermo-electromagneto-elastic effects have been given and comprehensive phenomenological continuum theories are established (see, e.g., [1,2]).

A further refinement of mechanical continuum theories has been proposed which accounts for additional degrees of freedom arising from the assumption that the continuum element has a microstructure [3,4]. The extension of this approach to electromagnetic microcontinua has been performed by the inclusion of the pertinent couplings into the balance equations and the constitutive equations [5,6]. In spite of aiming at improving the accuracy of the physical model, this last generalization implies an increasing number of coupling constitutive parameters which may prevent the model to be effective.

A different electromagnetic micromorphic model has been recently introduced where the electromagnetic properties of the continuum are directly connected to its microstructure through the microcharge density. This allows to account for polarization and magnetization via electric multipoles, avoiding the introduction of additional constitutive equations [7]. This approach has been exploited to formulate alternative phenomenological models for piezoelectric and ferroelectric materials [8,9]. In particular, the micropolar reduction in such model has shown to represent an effective alternative to polarization gradient models of dielectrics (see [1,10,11]).

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In the present paper, we propose an extension of the previous model to electromagnetic conductors. On the basis of a general micromorphic continuum theory, we introduce the electric current as the flow of a continuum of charge carriers which is superimposed to the microstructured electromagnetic medium. The basic quantities and a suitable definition of electric current via charge microdensities are given in Sect. 2. The carriers are supposed to be massless and interact with bound charges via the electromotive field. As a first step, in Sect. 3, we derive the conservation of total charge in the form of the usual balance law and a couple of evolution equations for electric dipole and quadrupole. Besides, we arrive at an explicit expression of the conductive part of the current. Section 4 is devoted to the derivation of macroscopic Maxwell's equations starting from their elementary microscopic form. This result is obtained by means of the previous evolution equations for electric dipole and quadrupole densities, and represents the micromorphic analogue to the space or statistical averaging techniques proposed in the past [1,12]. According to [7], in Sect. 5, we give the complete set of mechanical balance laws for a micromorphic conductor. In Sect. 6, we formulate the constitutive model which specifically accounts for the interaction between elastic continuum and the continuum of free charges. This allows for a rigorous characterization of perfect (ideal) conductors. The constitutive dependence and the thermodynamic restrictions on the conductive current are then derived for not ideal conductors in Sect. 7, in order to state a generalized Ohm's law. In the final part of that section, we obtain a linearized Ohm's law arriving at a generalization of the usual magnetic transport equation for deformable conductors.

2 Micromorphic electromagneto-elasticity

We consider here a continuum body $\mathcal{B}$ equipped with an internal structure according to the micromorphic theory of continua [5]. It is assumed that each "point" in $\mathcal{B}$ has the structure of a particle $\mathcal{P}$ whose center of mass is given by the position vector $\mathbf{x}$ of the continuum. A vector $\boldsymbol{\xi}$ denotes the relative position of a point within $\mathcal{P}$, with respect to $\mathbf{x}$. Denoting by $\mathbf{X}$ and Ξ the vectors of the reference (material) configuration corresponding to $\mathbf{x}$ and $\boldsymbol{\xi}$, we suppose that macro- and microdeformations are allowed in $\mathcal{B}$, defined by smooth functions $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$, $\boldsymbol{\xi} = \bar{\boldsymbol{\xi}}(\mathbf{X}, \Xi, t)$, respectively. As usual, we introduce the macrodeformation gradient $\mathbf{F} = (\nabla_{\mathbf{X}}\bar{\mathbf{x}})^T$ and pose $J = \det\mathbf{F}$. Besides, a microdeformation tensor $\boldsymbol{\chi}$ is defined assuming $\bar{\boldsymbol{\xi}}$ to be linear in Ξ, that is, $\bar{\boldsymbol{\xi}}(\mathbf{X}, \Xi, t) = \boldsymbol{\chi}(\mathbf{X}, t)\Xi$ and pose $j = \det\boldsymbol{\chi}$. In components, we have

$$F_{iK} = \frac{\partial \bar{x}_i}{\partial X_K}, \quad \bar{\xi}_i = \chi_{iK}\Xi_K.$$

We denote by $\boldsymbol{\mathfrak{X}}$ the inverse of $\boldsymbol{\chi}$, such that $\chi_{iJ}\mathfrak{X}_{Jk} = \delta_{ik}$. A set of strain measures can be introduced by means of the following material tensors

$$\mathcal{C} = \boldsymbol{\chi}^T\boldsymbol{\chi}, \quad \mathfrak{C} = \mathbf{F}^T\boldsymbol{\mathfrak{X}}^T, \quad \boldsymbol{\Gamma} = \boldsymbol{\mathfrak{X}}(\nabla_{\mathbf{X}}\boldsymbol{\chi})^T \tag{2.1}$$

known, respectively, as microdeformation strain, deformation strain, and wryness tensor. From the previous definitions, spatial derivatives of $\boldsymbol{\xi}$ are obtained in terms of the wryness tensor as

$$\xi_{i,j} = \chi_{iH}\Gamma_{HKL}F^{-1}_{Lj}\Xi_K,$$

and, posing $\gamma_{ijh} = \chi_{iH}\Gamma_{HKL}F^{-1}_{Lj}\mathfrak{X}_{Kh}$, we can write

$$(\nabla\boldsymbol{\xi})^T = \boldsymbol{\gamma}\,\boldsymbol{\xi}. \tag{2.2}$$

According to the definition of microgyration tensor $\mathbf{N}(\mathbf{x}, t)$, (see [5]), we have

$$\dot{\boldsymbol{\chi}} \equiv \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\chi} = \mathbf{N}\boldsymbol{\chi},$$

so that

$$\dot{\boldsymbol{\xi}} = \mathbf{N}\boldsymbol{\xi}. \tag{2.3}$$

Owing to this result, we find the following time rates of the strain tensors (2.1)

$$\dot{\mathcal{C}} = 2\boldsymbol{\chi}^T(\mathrm{Sym}\mathbf{N})\boldsymbol{\chi}, \quad \dot{\mathfrak{C}} = [\boldsymbol{\mathfrak{X}}(\mathbf{L} - \mathbf{N})\mathbf{F}]^T, \quad \dot{\boldsymbol{\Gamma}} = [\boldsymbol{\mathfrak{X}}(\nabla\mathbf{N})^T\boldsymbol{\chi}]\mathbf{F}, \tag{2.4}$$

where $\mathbf{L} = (\nabla \mathbf{v})^T$ is the velocity gradient and $\mathbf{v} = \dot{\mathbf{x}}$ is the usual macroscopic velocity. On the basis of this physical picture of a micromorphic continuum, we are assuming different orders of magnitude for macro- and microdeformation scales, so that $\|\mathrm{d}\boldsymbol{\xi}\| \ll \|\mathrm{d}\mathbf{x}\|$. Equivalently, in view of (2.2), we require the following condition on the tensor $\boldsymbol{\gamma}\,\boldsymbol{\xi}$,

$$\|\boldsymbol{\gamma}\,\boldsymbol{\xi}\| \ll 1. \tag{2.5}$$

Microdensities of mass and electric bound charge, respectively, ρ' and σ', are here introduced as smooth functions of $\mathbf{x}+\boldsymbol{\xi}$ and t in the current configuration of $\mathcal{B}$. Here, we distinguish bound charges from free charges responsible for conduction current, which will be introduced later. Denoting by $\mathcal{P}_t$ the current configuration of the particle $\mathcal{P}$, we define the (macroscopic) density of mass and density of electric bound charge as

$$\rho(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \rho'(\mathbf{x} + \boldsymbol{\xi}, t)\, \mathrm{d}v', \quad q'(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\, \mathrm{d}v', \tag{2.6}$$

where the integration vector variable is $\boldsymbol{\xi}$ and $\Delta v' = \mathrm{vol}(\mathcal{P}_t)$. These quantities represent mean values of mass and charge on the particle's volume. Similarly, we can represent densities of inertial and electric quantities in terms of ρ' and σ', respectively. In this respect, we observe that, since $\mathbf{x}$ is the center of mass of $\mathcal{P}_t$,

$$\frac{1}{\Delta v'} \int_{\mathcal{P}_t} \rho'(\mathbf{x} + \boldsymbol{\xi}, t)\boldsymbol{\xi}\, \mathrm{d}v' = \mathbf{0}. \tag{2.7}$$

Then, we define the microinertia tensor $\boldsymbol{\mathcal{I}}$ per unit mass and the electric dipole and quadrupole densities $\mathbf{p}$ and $\mathbf{Q}$ as follows:

$$\boldsymbol{\mathcal{I}}(\mathbf{x}, t) = \frac{1}{\rho \Delta v'} \int_{\mathcal{P}_t} \rho'(\mathbf{x} + \boldsymbol{\xi}, t)\boldsymbol{\xi} \otimes \boldsymbol{\xi}\, \mathrm{d}v', \tag{2.8}$$

$$\mathbf{p}(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\boldsymbol{\xi}\, \mathrm{d}v', \qquad \mathbf{Q}(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma'(\mathbf{x} + \boldsymbol{\xi}, t)\boldsymbol{\xi} \otimes \boldsymbol{\xi}\, \mathrm{d}v'. \tag{2.9}$$

Quantities of order greater than 2 with respect to $\boldsymbol{\xi}$ can be defined in a similar way (see [9]).

According to (2.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x} + \boldsymbol{\xi}) = \mathbf{v} + \mathbf{N}\boldsymbol{\xi}. \tag{2.10}$$

It is natural to define the density of electric current as the sum of a mean value, over the particle's volume, of a microcurrent due to bound charge microdensity σ' and a conduction current due to a microdensity $\hat{\sigma}$ of a continuum of free charges with negligible mass density and with relative velocity $\hat{\mathbf{v}}$ with respect to $\mathbf{x}$,

$$\mathbf{J}(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma'(\mathbf{x} + \boldsymbol{\xi}, t)(\mathbf{v} + \mathbf{N}\boldsymbol{\xi})\, \mathrm{d}v' + \frac{1}{\Delta v'} \int_{\hat{\mathcal{P}}_t} \hat{\sigma}(\mathbf{x} + \hat{\boldsymbol{\xi}}, t)[\mathbf{v} + \hat{\mathbf{v}}(\mathbf{x} + \hat{\boldsymbol{\xi}})]\, \mathrm{d}v', \tag{2.11}$$

where $\hat{\mathcal{P}}_t$ is the portion of continuum of free charges coinciding with $\mathcal{P}_t$ at time t and $\hat{\boldsymbol{\xi}}$ the corresponding relative vector position. Denoting by $q = q' + \hat{q}$ the total charge density, where

$$\hat{q}(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\hat{\mathcal{P}}_t} \hat{\sigma}(\mathbf{x} + \hat{\boldsymbol{\xi}}, t)\, \mathrm{d}v',$$

using $(2.6)_2$ and $(2.9)_1$, we can write

$$\mathbf{J} = q\mathbf{v} + \mathbf{N}\mathbf{p} + \hat{\mathbf{J}}, \tag{2.12}$$

where

$$\hat{\mathbf{J}}(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\hat{\mathcal{P}}_t} \hat{\sigma}(\mathbf{x} + \hat{\boldsymbol{\xi}}, t)\, \hat{\mathbf{v}}(\mathbf{x} + \hat{\boldsymbol{\xi}})\, \mathrm{d}v'. \tag{2.13}$$

The first term in the right-hand side of (2.12) represents the common convective electric current, while the second term accounts for internal relative motion of charges and represents a specific contribution of the microstructure.

It is worth remarking that the continuum of free charges does not possess a microstructure. The previous expressions for $\hat{q}$ and $\hat{\mathbf{J}}$, which formally reflects the superposition of $\hat{\mathcal{P}}_t$ on the microcontinuum particle of $\mathcal{B}$, will be developed in the next sections to deal with a pertinent model of continuum electric conductors.

3 Conservation of mass and charge, evolution equations

Let us consider a scalar or vector or tensor microfield density Φ' defined on $\mathcal{B}$, for instance one of the microdensities ρ' or σ' of the previous section, and represent it as a smooth function $\Phi'(\mathbf{x}+\boldsymbol{\xi}, t)$. The mean value of Φ' on $\mathcal{P}_t$ is the macroscopic field

$$\Phi(\mathbf{x}, t) = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \Phi'(\mathbf{x}+\boldsymbol{\xi}, t)\, \mathrm{d}v'. \tag{3.1}$$

We pose $\tilde{\mathbf{x}} = \mathbf{x} + \xi$, or, more explicitly

$$\tilde{\mathbf{x}}(\mathbf{X}, \Xi, t) = \hat{\mathbf{x}}(\mathbf{X}, t) + \boldsymbol{\chi}(\mathbf{X}, t)\Xi,$$

and consider the deformation gradient

$$\widetilde{\mathbf{F}} = (\nabla_{\mathbf{X}} \tilde{\mathbf{x}})^T, \quad \widetilde{J} = \det\widetilde{\mathbf{F}}.$$

By a standard derivation, accounting for Eq. (2.10), we obtain

$$(\det\widetilde{\mathbf{F}})^{\cdot} = (\det\widetilde{\mathbf{F}})\dot{\widetilde{\mathbf{F}}}\widetilde{\mathbf{F}}^{-1} = \widetilde{J}[\nabla\cdot\mathbf{v} + \widetilde{\nabla}\cdot(\mathbf{N}\boldsymbol{\xi})], \tag{3.2}$$

where $\widetilde{\nabla} \equiv \nabla_{\tilde{\mathbf{x}}}$ and where we accounted for the independence of $\mathbf{v}$ on $\boldsymbol{\xi}$. Let $\mathcal{A}$ be an arbitrary part of $\mathcal{B}$ and consider the integral of $\Phi'(\tilde{\mathbf{x}}, t)$ over $\mathcal{A}_t$, denoting the element of volume by $\mathrm{d}\tilde{v}$. Evaluating the material time derivative by means of the transport theorem and Eq. (3.2), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{A}_t} \Phi'(\tilde{\mathbf{x}}, t)\,\mathrm{d}\tilde{v} &= \int_{\mathcal{A}_t}\left[\frac{\partial\Phi'}{\partial t} + (\mathbf{v}+\mathbf{N}\boldsymbol{\xi})\cdot\widetilde{\nabla}\Phi' + \Phi'\nabla\cdot\mathbf{v} + \Phi'\widetilde{\nabla}\cdot(\mathbf{N}\boldsymbol{\xi})\right]\mathrm{d}\tilde{v}\\ &= \int_{\mathcal{A}_t}\left[\frac{\partial\Phi'}{\partial t} + \mathbf{v}\cdot\widetilde{\nabla}\Phi' + \Phi'\nabla\cdot\mathbf{v} + \widetilde{\nabla}\cdot(\Phi'\mathbf{N}\boldsymbol{\xi})\right]\mathrm{d}\tilde{v},\end{aligned} \tag{3.3}$$

If Φ', and in turn Φ, is the density of a field which is conserved during the motion, the last integral on the right-hand side of (3.3) must vanish for any part $\mathcal{A}_t$ and this fact implies

$$\frac{\partial\Phi'}{\partial t} + \mathbf{v}\cdot\widetilde{\nabla}\Phi' + \Phi'\nabla\cdot\mathbf{v} + \widetilde{\nabla}\cdot(\Phi'\mathbf{N}\boldsymbol{\xi}) = \mathbf{0}, \tag{3.4}$$

during the motion. In particular, if we choose $\mathcal{A}_t$ to coincide with $\mathcal{P}_t$, we get

$$\int_{\mathcal{P}_t}\left[\frac{\partial\Phi'}{\partial t} + \mathbf{v}\cdot\widetilde{\nabla}\Phi' + \Phi'\nabla\cdot\mathbf{v} + \widetilde{\nabla}\cdot(\Phi'\mathbf{N}\boldsymbol{\xi})\right]\mathrm{d}v' = \mathbf{0}. \tag{3.5}$$

Now, we observe that in view of the previous positions,

$$\widetilde{\nabla} = [\mathbf{F} + (\nabla_{\mathbf{X}}\boldsymbol{\chi})\Xi]^{-1}\mathbf{F}\nabla = \mathbf{F}^{-1}[\mathbf{I} + \boldsymbol{\gamma}\,\boldsymbol{\xi}]^{-1}\mathbf{F}\nabla.$$

Then, accounting for the condition (2.5), we can replace $\widetilde{\nabla}$ with ∇ in (3.5) to obtain

$$\begin{aligned}\mathbf{0} &= \int_{\mathcal{P}_t}\frac{\partial\Phi'}{\partial t}\mathrm{d}v' + \nabla\cdot\mathbf{v}\int_{\mathcal{P}_t}\Phi'\,\mathrm{d}v' + \mathbf{v}\cdot\nabla\int_{\mathcal{P}_t}\Phi'\,\mathrm{d}v' + \nabla\cdot\int_{\mathcal{P}_t}\Phi'\mathbf{N}\boldsymbol{\xi}\,\mathrm{d}v'\\ &= \Delta v'\left[\frac{\partial\Phi}{\partial t} + \Phi\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla\Phi + \nabla\cdot\left(\mathbf{N}\frac{1}{\Delta v'}\int_{\mathcal{P}_t}\Phi'\boldsymbol{\xi}\,\mathrm{d}v'\right)\right]\end{aligned}$$

or, equivalently

$$\dot{\Phi} + \Phi\nabla\cdot\mathbf{v} + \nabla\cdot\left(\mathbf{N}\frac{1}{\Delta v'}\int_{\mathcal{P}_t}\Phi'\boldsymbol{\xi}\,\mathrm{d}v'\right) = \mathbf{0}. \tag{3.6}$$

The previous analysis can be applied, in particular, to derive the conservation laws of mass and charge.

Letting $\Phi' = \rho'$ and accounting for equality (2.7), from (3.6) and $(2.6)_1$, we obtain the law of mass conservation

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0. \tag{3.7}$$

Concerning the charge density, letting $\Phi' = \sigma'$, accounting for (2.9)$_1$ and (2.6)$_2$, from (3.6), we arrive at the following law of conservation for bound charges

$$\dot{q}' + q'\nabla \cdot \mathbf{v} + \nabla \cdot (\mathbf{N}\mathbf{p}) = 0. \tag{3.8}$$

A similar result holds for free charges. In this case, we can proceed by considering an arbitrary portion $\hat{\mathcal{A}}_t$ and impose the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\mathcal{A}}_t} \hat{\sigma}(\mathbf{x} + \hat{\boldsymbol{\xi}}, t)\,\mathrm{d}v = 0,$$

which yields

$$\int_{\hat{\mathcal{A}}_t}\left[\frac{\partial \hat{\sigma}}{\partial t} + \nabla \cdot (\hat{\sigma}\mathbf{v}) + \nabla \cdot (\hat{\sigma}\hat{\mathbf{v}})\right]\mathrm{d}v = 0. \tag{3.9}$$

In particular, if $\hat{\mathcal{A}}_t = \hat{\mathcal{P}}_t$, owing to (2.13), we obtain

$$\dot{\hat{q}} + \hat{q}\nabla \cdot \mathbf{v} + \nabla \cdot \hat{\mathbf{J}} = 0. \tag{3.10}$$

Then, adding term by term Eqs. (3.8), (3.10), and using (2.12), we obtain the law of conservation for the total charge, in the form of the well-known balance law

$$\frac{\partial q}{\partial t} = -\nabla \cdot \mathbf{J}. \tag{3.11}$$

Dipole and quadrupole can be viewed as internal variables of the microcontinuum model. Here, we derive suitable evolution equations for these quantities. From (2.9)$_1$, by taking the material time derivative, we get

$$\Delta v'(\dot{\mathbf{p}} + (\mathrm{tr}\mathbf{N})\mathbf{p}) = \int_{\mathcal{P}_t}\left[\frac{\partial \sigma'}{\partial t}\boldsymbol{\xi} + (\mathbf{v} + \mathbf{N}\boldsymbol{\xi}) \cdot (\widetilde{\nabla}\sigma')\boldsymbol{\xi} + \sigma'\mathbf{N}\boldsymbol{\xi} + \sigma'\boldsymbol{\xi}\,\mathrm{tr}\mathbf{N}\right]\mathrm{d}v' \tag{3.12}$$

where, in view of (2.10), we have exploited the result $(j)^{\cdot} = j(\mathrm{tr}\mathbf{N})$ and applied the transport theorem. From (3.4) applied to the charge density, we have

$$\frac{\partial \sigma'}{\partial t} + (\mathbf{v} + \mathbf{N}\boldsymbol{\xi}) \cdot \widetilde{\nabla}\sigma' = -\sigma'\nabla \cdot \mathbf{v} - \sigma'\widetilde{\nabla} \cdot (\mathbf{N}\boldsymbol{\xi}).$$

Substituting into (3.12), we obtain

$$\dot{\mathbf{p}} + \mathbf{p}\nabla \cdot \mathbf{v} = \mathbf{N}\mathbf{p} - \frac{1}{\Delta v'}\int_{\mathcal{P}_t} \sigma'[\widetilde{\nabla} \cdot (\mathbf{N}\boldsymbol{\xi})]\boldsymbol{\xi}\,\mathrm{d}v' \tag{3.13}$$

By applying the same arguments on $\widetilde{\nabla}$, exploited to obtain (3.6), we can recast Eq. (3.13) in the form

$$\dot{\mathbf{p}} + \mathbf{p}\nabla \cdot \mathbf{v} = \mathbf{N}\mathbf{p} - (\nabla \cdot \mathbf{N})\mathbf{Q} - \mathbf{N} : \boldsymbol{\gamma}\,\mathbf{Q} \tag{3.14}$$

By the same token, we arrive at the following evolution equation for the quadrupole density

$$\dot{\mathbf{Q}} + \mathbf{Q}\nabla \cdot \mathbf{v} = 2\mathrm{Sym}(\mathbf{N}\mathbf{Q}) - (\nabla \cdot \mathbf{N})\,\boldsymbol{\mathcal{Q}} - \mathbf{N} : \boldsymbol{\gamma}\,\boldsymbol{\mathcal{Q}} \tag{3.15}$$

where

$$\boldsymbol{\mathcal{Q}} = \frac{1}{\Delta v'}\int_{\mathcal{P}_t} \sigma'\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \boldsymbol{\xi}\,\mathrm{d}v'$$

is the electric octupole density. We finally observe that a similar evolution equation holds for the microinertia tensor $\boldsymbol{\mathcal{I}}$. In this case, starting from (2.8), we obtain

$$\dot{\boldsymbol{\mathcal{I}}} = 2\mathrm{Sym}(\mathbf{N}\boldsymbol{\mathcal{I}}) - (\nabla \cdot \mathbf{N})\mathfrak{I} - \mathbf{N} : \boldsymbol{\gamma}\,\mathfrak{I} \tag{3.16}$$

where

$$\mathfrak{I} = \frac{1}{\rho\Delta v'}\int_{\mathcal{P}_t} \rho'\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \boldsymbol{\xi}\,\mathrm{d}v'.$$

If terms of highest order in $\boldsymbol{\xi}$ are neglected in each equation, (3.14)–(3.16) reduce to

$$\begin{aligned}&\dot{\mathbf{p}}+\mathbf{p}\nabla\cdot\mathbf{v}-\mathbf{N}\mathbf{p}=\mathbf{0},\\&\dot{\mathbf{Q}}+\mathbf{Q}\nabla\cdot\mathbf{v}-2\mathrm{Sym}(\mathbf{N}\mathbf{Q})=\mathbf{0},\\&\dot{\mathcal{I}}-2\mathrm{Sym}(\mathbf{N}\mathcal{I})=\mathbf{0}.\end{aligned}\tag{3.17}$$

We finally observe that the previous derivations can be also applied to quantities pertaining to the continuum of free charges, if we replace the time derivative of $\hat{\boldsymbol{\xi}}$ by the velocity $\mathbf{v}+\hat{\mathbf{v}}$. In particular, we exploit the quantities

$$\hat{\mathbf{p}}(\mathbf{x},t)=\frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\hat{\sigma}(\mathbf{x}+\hat{\boldsymbol{\xi}},t)\,\hat{\boldsymbol{\xi}}\,\mathrm{d}v',\qquad \hat{\mathbf{Q}}(\mathbf{x},t)=\frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\hat{\sigma}(\mathbf{x}+\hat{\boldsymbol{\xi}},t)\,\hat{\boldsymbol{\xi}}\otimes\hat{\boldsymbol{\xi}}\,\mathrm{d}v',\tag{3.18}$$

to obtain a suitable expression for the conduction current $\hat{\mathbf{J}}$. Taking the material time derivative of $(3.18)_1$, accounting for (3.9) and (2.13), we get

$$\dot{\hat{\mathbf{p}}}+\hat{\mathbf{p}}(\nabla\cdot\mathbf{v})=\hat{\mathbf{J}}-\frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\hat{\sigma}\hat{\boldsymbol{\xi}}\,\nabla\cdot\hat{\mathbf{v}}\,\mathrm{d}v'.$$

If we expand $\nabla\cdot\hat{\mathbf{v}}$ about $\hat{\boldsymbol{\xi}}=\mathbf{0}$, up to the second order, we obtain

$$\hat{\mathbf{J}}=\dot{\hat{\mathbf{p}}}+\hat{\mathbf{p}}\,\nabla\cdot(\mathbf{v}+\hat{\mathbf{v}})+(\hat{\mathbf{Q}}\cdot\nabla)(\nabla\cdot\hat{\mathbf{v}}).\tag{3.19}$$

Equation (3.19) shows that, beside $\hat{\mathbf{v}}$, the quantities $\hat{\mathbf{p}}$ and $\hat{\mathbf{Q}}$ characterize the conductive properties of the present continuum model, describing the interaction of free charges with the microcontinuum.

4 Maxwell's equations

In this section, we derive the actual form of Maxwell's equation for the continuum in terms of its microstructure. Our main objective was to arrive at the pertinent expressions for polarization and magnetization for a consistent electromagnetic micromorphic theory. Really, we restate, in a somewhat alternative way, the problem of a microscopic derivation of Maxwell's equations in continuous matter which traces back to the Lorentz electron theory. Space averaging and statistical ensemble averaging have been exploited to this end (see [12,13]), and later, a consistent formulation for classical continua has been obtained on the basis of the previous averaging techniques (see [1] and references therein).

We denote by $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ the electric field and the magnetic induction valued at the center of mass of the continuum particle and write the Maxwell's equations pertaining to the material contribution of charge and current densities valued at $\mathbf{x}$, i.e., for $\boldsymbol{\xi}=0$. Omitting the arguments $(\mathbf{x},t)$ in each term in the following equations, using Heaviside–Lorentz electromagnetic units and denoting by c the speed of light in vacuum, we have

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0}\tag{4.1}$$

$$\nabla\cdot\mathbf{E}=\sigma'+\hat{q},\quad \nabla\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{c}\mathbf{J}'+\frac{1}{c}(\hat{q}\mathbf{v}+\hat{\mathbf{J}}),\tag{4.2}$$

where $\mathbf{J}'=\sigma'\dot{\mathbf{x}}$. The effects of the continuum of free charges are taken into account, through the mean values at $\mathbf{x}$ of density and current on $\hat{\mathcal{P}}_t$, in the right-hand sides of (4.2). Equation (4.1) is formally the same as the law on magnetic flux density and Faraday's law in vacuum. In order to obtain the macroscopic form of Gauss' law and Ampère's law given in (4.2), we evaluate $\sigma'(\mathbf{x},t)$ and $\mathbf{J}'(\mathbf{x},t)$ in terms of macroscopic fields. This can be achieved by expanding these quantities about $\mathbf{x}+\boldsymbol{\xi}$ and averaging over the volume of the continuum particle $\mathcal{P}_t$. Concerning with Gauss' law (see also [9]), by a Taylor expansion, up to order n,

$$\sigma'(\mathbf{x},\boldsymbol{\xi},t)=\sum_{k=1}^{n}\frac{(-1)^{-1}}{(k-1)!}\nabla^{(k-1)}\sigma'\Big|_{\mathbf{x}+\boldsymbol{\xi}}\boldsymbol{\xi}^{(k-1)},$$

where $\nabla^{(k)}$ denotes nth order differentiation with respect to $\mathbf{x}$ and $\boldsymbol{\xi}^{(k)} = \underbrace{\boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \cdots \otimes \boldsymbol{\xi}}_{k}$. Consistently with the approximation of Eq. (3.17), we consider here $n = 2$, and exploiting Eq. (2.9), we obtain

$$\frac{1}{\Delta v'} \int_{\mathcal{P}_t} \sigma'(\mathbf{x}, \boldsymbol{\xi}, t)\, \mathrm{d}v' = q'(\mathbf{x}, t) - \nabla \cdot \left(\mathbf{p}(\mathbf{x}, t) - \frac{1}{2} \nabla \cdot \mathbf{Q}(\mathbf{x}, t) \right).$$

Then, introducing the polarization, up to multipoles of order 2,

$$\mathbf{P} = \mathbf{p} - \frac{1}{2} \nabla \cdot \mathbf{Q}, \tag{4.3}$$

from $(4.2)_1$, we obtain the Gauss' law in the usual form

$$\nabla \cdot \mathbf{D} = q, \tag{4.4}$$

where $\mathbf{D} = \mathbf{E} + \mathbf{P}$ is the electric displacement. A less easy derivation is required for the Ampère's law. Again, by a Taylor expansion

$$\mathbf{J}'(\mathbf{x}, \boldsymbol{\xi}, t) = \sum_{k=1}^{n} \frac{(-1)^{-1}}{(k-1)!} \nabla^{(k-1)} (\sigma' \dot{\mathbf{x}}) \Big|_{\mathbf{x}+\boldsymbol{\xi}} \boldsymbol{\xi}^{(k-1)},$$

and up to terms of order 2 in $\boldsymbol{\xi}$,

$$\frac{1}{\Delta v'} \int_{\mathcal{P}_t} \mathbf{J}'(\mathbf{x}, \boldsymbol{\xi}, t)\, \mathrm{d}v' = q'\mathbf{v} + 2\mathbf{N}\mathbf{p} - (\nabla \cdot \mathbf{p})\mathbf{v} - (\mathbf{p} \cdot \nabla)\mathbf{v}$$
$$- \nabla \cdot (\mathbf{N}\mathbf{Q})^T + \frac{1}{2} \nabla \cdot (\mathbf{Q}\mathbf{L}) + \frac{1}{2} \nabla \cdot [(\nabla \cdot \mathbf{Q})\mathbf{v}].$$

Then, exploiting the evolution Eq. $(3.17)_{1,2}$, accounting for the identity

$$\nabla \cdot (\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}) = -\nabla \times (\mathbf{a} \times \mathbf{b}),$$

after some rearrangements and exploiting the definition (4.3), we obtain

$$\frac{1}{\Delta v'} \int_{\mathcal{P}_t} \mathbf{J}'(\mathbf{x}, \boldsymbol{\xi}, t)\, \mathrm{d}v' = q'\mathbf{v} + \mathbf{N}\mathbf{p} + \frac{\partial \mathbf{P}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{P}) - \frac{1}{2} \nabla \times \boldsymbol{\mu}$$

where the vector $\boldsymbol{\mu}$ is, in components,

$$\mu_i = \epsilon_{ijk} (N_{jp} - L_{jp}) Q_{pk}. \tag{4.5}$$

Substituting into $(4.2)_2$, and using (2.12), we get

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{P}}{\partial t} - \frac{1}{c} \nabla \times (\mathbf{v} \times \mathbf{P}) + \frac{1}{2c} \nabla \times \boldsymbol{\mu}.$$

Finally, posing

$$\boldsymbol{\mathcal{M}} = \frac{1}{2c} \boldsymbol{\mu}, \qquad \mathbf{M} = \boldsymbol{\mathcal{M}} - \frac{1}{c} \mathbf{v} \times \mathbf{P},$$

and accounting for the definition of $\mathbf{D}$, we obtain the usual form of the Ampère's law,

$$\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c} \mathbf{J}, \tag{4.6}$$

where $\mathbf{H}$ is the magnetic field defined by $\mathbf{B} = \mathbf{H} + \mathbf{M}$. The vector $\mathbf{M}$ is the magnetization field and $\boldsymbol{\mathcal{M}}$ turns out to be the magnetization in the co-moving frame (see, e.g., [1]). We remark the essential role of microdeformation in defining the magnetization, via the tensor $\mathbf{N}$ in Eq. (4.5). Concluding the present derivation, we note that the conservation of total charge (3.11) follows from the Ampère's law (4.6), after taking the divergence of both sides and accounting for the Gauss' law (4.4).

5 Micromechanical balance laws

In order to obtain the mechanical balance equations for the problem at hand, we derive force, moment of force, and power densities. These derivations are the same as those shown in [7] apart from the present additional terms due to free charges. We denote, respectively, by $\mathbf{f}'^{\mathrm{em}}$ and $\hat{\mathbf{f}}^{\mathrm{em}}$ the electromagnetic force on a point of the continuum particle $\mathcal{P}_t$ and on the corresponding point in $\hat{\mathcal{P}}_t$. In the following, for brevity, we omit the explicit dependence on time,

$$\mathbf{f}'^{\mathrm{em}}(\mathbf{x}+\boldsymbol{\xi}) = \sigma'(\mathbf{x}+\boldsymbol{\xi})\left[\mathbf{E}(\mathbf{x}+\boldsymbol{\xi}) + \frac{\mathbf{v}+\mathbf{N}\boldsymbol{\xi}}{c}\times\mathbf{B}(\mathbf{x}+\boldsymbol{\xi})\right],$$
$$\hat{\mathbf{f}}^{\mathrm{em}}(\mathbf{x}+\hat{\boldsymbol{\xi}}) = \hat{\sigma}(\mathbf{x}+\hat{\boldsymbol{\xi}})\left[\mathbf{E}(\mathbf{x}+\hat{\boldsymbol{\xi}}) + \frac{\mathbf{v}+\hat{\mathbf{v}}(\mathbf{x}+\hat{\boldsymbol{\xi}})}{c}\times\mathbf{B}(\mathbf{x}+\hat{\boldsymbol{\xi}})\right].$$

The electromagnetic force density at $\mathbf{x}$ is

$$\mathbf{f}^{\mathrm{em}}(\mathbf{x}) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\mathbf{f}'^{\mathrm{em}}(\mathbf{x}+\boldsymbol{\xi})\,\mathrm{d}v' + \frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\hat{\mathbf{f}}^{\mathrm{em}}(\mathbf{x}+\hat{\boldsymbol{\xi}})\,\mathrm{d}v'. \tag{5.1}$$

According to the assumptions adopted in deriving Maxwell's equations, we replace $\mathbf{E}$ and $\mathbf{B}$ with their values at $\hat{\boldsymbol{\xi}} = \mathbf{0}$ in $\hat{\mathcal{P}}_t$. Then, we expand $\mathbf{E}$ and $\mathbf{B}$ about $\boldsymbol{\xi} = \mathbf{0}$ in the first integral of Eq. (5.1), retaining terms up to the second order in $\boldsymbol{\xi}$. Exploiting $(2.6)_2$, (2.9), and (2.13), we obtain

$$\begin{aligned}\mathbf{f}^{\mathrm{em}} &= q\,\mathcal{E} + (\mathbf{p}\cdot\nabla)\,\mathcal{E} + \frac{1}{2}(\mathbf{Q}\cdot\nabla)\nabla\,\mathcal{E} + \frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{p}]\times\mathbf{B}\\ &\quad+\frac{1}{c}[(\mathbf{N}-\mathbf{L})\mathbf{Q}\nabla]\times\mathbf{B} + \frac{1}{2c}\mathbf{B}\times[(\mathbf{Q}\cdot\nabla)\mathbf{L}^T] + \frac{1}{c}\hat{\mathbf{J}}\times\mathbf{B}.\end{aligned} \tag{5.2}$$

where

$$\mathcal{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \frac{1}{c}\mathbf{v}\times\mathbf{B}(\mathbf{x},t),$$

is the electromotive intensity field. Analogous derivations for the moment of force and power densities yield

$$\boldsymbol{\tau}^{\mathrm{em}}(\mathbf{x}) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}(\mathbf{x}+\boldsymbol{\xi})\times\mathbf{f}'^{\mathrm{em}}\,\mathrm{d}v' + \frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\mathbf{x}\times\hat{\mathbf{f}}^{\mathrm{em}}\,\mathrm{d}v' = \mathbf{x}\times\mathbf{f}^{\mathrm{em}}(\mathbf{x}) + \mathbf{c}^{\mathrm{em}}(\mathbf{x}). \tag{5.3}$$

$$\begin{aligned}w^{\mathrm{em}}(\mathbf{x}) &= \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\mathbf{f}'^{\mathrm{em}}\cdot(\mathbf{v}+\mathbf{N}\boldsymbol{\xi})\,\mathrm{d}v' + \frac{1}{\Delta v'}\int_{\hat{\mathcal{P}}_t}\hat{\mathbf{f}}^{\mathrm{em}}\cdot(\mathbf{v}+\hat{\mathbf{v}}),\mathrm{d}v'\\ &= \mathbf{f}^{\mathrm{em}}(\mathbf{x})\cdot\mathbf{v} + \mathrm{tr}(\mathbf{C}^{\mathrm{em}}(\mathbf{x})\mathbf{N}) + \hat{\mathbf{J}}\cdot\,\mathcal{E}(\mathbf{x}).\end{aligned} \tag{5.4}$$

where

$$C^{\mathrm{em}}_{ij} = p_i\mathcal{E}_j + \left[\mathcal{E}_{i,k} + \frac{1}{c}\epsilon_{ipq}(N_{pk}-L_{pk})B_q\right]Q_{kj}, \qquad c^{\mathrm{em}}_i = \epsilon_{ijk}C^{\mathrm{em}}_{jk}.$$

Similar results can be obtained for the mechanical contributions to force per unit mass, couple, and power densities. In this case, no terms arise from the free charge continuum. Actually, the expansion of the force microdensity $\mathbf{f}'(\mathbf{x}+\boldsymbol{\xi}, t)$ about $\boldsymbol{\xi} = \mathbf{0}$, accounting for (2.8), yields, at $\mathbf{x}$,

$$\mathbf{f}(\mathbf{x}) = \frac{1}{\rho\Delta v'}\int_{\mathcal{P}_t}\rho'(\mathbf{x}+\boldsymbol{\xi})\,\mathbf{f}'(\mathbf{x}+\boldsymbol{\xi})\,\mathrm{d}v' = \mathbf{f}'(\mathbf{x}) + \frac{1}{2}\mathcal{I}\,\nabla(\nabla\mathbf{f}'(\mathbf{x})), \tag{5.5}$$

$$\boldsymbol{\tau}(\mathbf{x}) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\rho'(\mathbf{x}+\boldsymbol{\xi})\times\mathbf{f}'(\mathbf{x}+\boldsymbol{\xi})\,\mathrm{d}v' = \rho[\mathbf{x}\times\mathbf{f}(\mathbf{x}) + \mathbf{c}(\mathbf{x})], \tag{5.6}$$

$$w(\mathbf{x}) = \frac{1}{\Delta v'}\int_{\mathcal{P}_t}\rho'(\mathbf{x}+\boldsymbol{\xi})\,\mathbf{f}'(\mathbf{x}+\boldsymbol{\xi})\cdot(\mathbf{v}+\mathbf{N}\boldsymbol{\xi})\,\mathrm{d}v' = \rho[\mathbf{f}(\mathbf{x})\cdot\mathbf{v} + \mathrm{tr}(\mathcal{I}^T(\nabla\mathbf{f}(\mathbf{x}))\mathbf{N})], \tag{5.7}$$

where

$$c_i = \epsilon_{ijk}\, f'_{k,l}\mathcal{I}_{jl}.$$

The balance equations for momentum, moment of momentum, and energy can be derived by a standard procedure introducing the (second order) stress tensors $\mathbf{T}$, $\mathbf{S}$, the (third order) couple stress tensor $\mathbf{m}$, according to the classical micromorphic continuum theory (see [5,7]). We do not repeat here the details of this derivation, but, accounting for the presence of the electric current contributions included in Eqs. (5.2)–(5.4), we give the resulting general balance laws. To this end, we recall the definition of the rate of spin inertia tensor, which in the present notations reads

$$\boldsymbol{\sigma} = \frac{1}{\Delta v'} \int_{\mathcal{P}_t} \rho'(\mathbf{x}+\boldsymbol{\xi})\ddot{\boldsymbol{\xi}} \otimes \boldsymbol{\xi}\, \mathrm{d}v'.$$

Owing to (2.10), we can write

$$\boldsymbol{\sigma} = \dot{\mathbf{N}}\mathcal{I} + \mathbf{N}\mathbf{N}\mathcal{I}\,.$$

Then, denoting by e the internal energy per unit mass, we obtain

$$\rho\dot{\mathbf{v}} = \rho\mathbf{f} + \mathbf{f}^{\mathrm{em}} + \nabla\cdot\mathbf{T}, \tag{5.8}$$
$$\rho\boldsymbol{\sigma} = \rho(\nabla\mathbf{f})^T\mathcal{I} + \mathbf{C}^{\mathrm{em}} + \mathbf{T}^T - \mathbf{S} + \nabla\cdot\mathbf{m}. \tag{5.9}$$
$$\rho\dot{e} = \mathrm{tr}(\mathbf{SN}) - \mathrm{tr}(\mathbf{TN} - \mathbf{TL}) + m_{ijk}N_{jk,i} + \rho h - \nabla\cdot\mathbf{q} + \hat{\mathbf{J}}\cdot\boldsymbol{\mathcal{E}}\,, \tag{5.10}$$

where h and $\mathbf{q}$ are, respectively, the heat supply per unit mass and the heat flux. We observe that Eq. (5.9) is the dual form of the vectorial balance of moment of momentum. Here, $\mathbf{S}$ is a symmetric tensor, and only the skew symmetric parts of $\mathbf{C}^{\mathrm{em}}$ and $\mathbf{T}$ effectively contribute to the right-hand side of (5.9).

6 Constitutive model for electromagneto-elastic conductors

In view of the next derivations, we rewrite the balance law of energy in a more convenient form. Introducing the following tensors (see [5,7])

$$a_{ij} = L_{ij} - N_{ij}, \quad b_{ijk} = N_{ij,k}, \quad c_{ij} = \frac{1}{2}(N_{ij} + N_{ji}),$$

Eq. (5.10) takes the following form

$$\rho\dot{e} = S_{ij}c_{ji} + T_{ij}a_{ij} + m_{ijk}b_{jki} + \rho h - q_{i,i} + \hat{J}_i\mathcal{E}_i\,. \tag{6.1}$$

Denoting by η the entropy density, we write the second law of thermodynamics as

$$\rho\dot{\eta} \geq \rho\frac{h}{\theta} - \nabla\cdot\frac{\mathbf{q}}{\theta}, \tag{6.2}$$

where θ is the absolute temperature. Introducing the free energy density $\psi = e - \eta\theta$ and eliminating h from (6.1) and (6.2), we obtain the following Clausius–Duhem inequality

$$\rho(\dot{\psi} + \eta\dot{\theta}) \leq S_{ij}c_{ji} + T_{ij}a_{ij} + m_{ijk}b_{jki} - \frac{1}{\theta}q_i\theta_{,i} + \hat{J}_i\mathcal{E}_i\,. \tag{6.3}$$

In order to formulate the constitutive assumptions, compatibly with the second law, we rewrite inequality (6.3) with respect to the reference configuration and introduce the following second Piola–Kirchhoff stress tensors

$$\mathbb{S} = J\mathfrak{X}\mathbf{S}\mathfrak{X}^T, \qquad \mathbb{Y} = J\mathbf{F}^{-1}\mathbf{T}\,\boldsymbol{\chi}\,, \qquad \mathbb{M} = J\mathbf{F}^{-1}\mathbf{m}\mathfrak{X}^T\,\boldsymbol{\chi}\,.$$

Besides, the heat flux, the conduction current, and the electromotive intensity field take the following material form

$$\mathfrak{q} = J\mathbf{F}^{-1}\mathbf{q}, \quad \hat{\mathbb{J}} = J\mathbf{F}^{-1}\hat{\mathbf{J}}, \quad \mathfrak{E} = \boldsymbol{\mathcal{E}}\,\mathbf{F}.$$

Accordingly, exploiting the time rate strains, given by (2.4), the inequality (6.3) can be rewritten in the following form

$$\rho_0(\dot{\psi}+\eta\dot{\theta})-\frac{1}{2}\mathbb{S}_{HK}\dot{\mathcal{C}}_{HK}-\mathbb{Y}_{HK}\dot{\mathfrak{C}}_{HK}-\mathbb{M}_{HKL}\dot{\Gamma}_{KLH}+\frac{1}{\theta}\mathfrak{q}_H G_H-\hat{\mathbb{J}}_H\mathfrak{E}_H\leq 0, \tag{6.4}$$

where $\rho_0=\rho J$ and $\mathbf{G}=\nabla_{\mathbf{X}}\theta$. The last term in the left-hand side of (6.4) can be written more explicitly, using the result (3.19). To this end, by defining the following material fields

$$\hat{\mathbb{P}}=J\mathbf{F}^{-1}\hat{\mathbf{p}},\quad \hat{\mathbb{Q}}=J\mathbf{F}^{-1}\hat{\mathbf{Q}}\mathbf{F},$$

we obtain the form of (3.19) according to the reference configuration

$$\hat{\mathbb{J}}=\dot{\hat{\mathbb{P}}}+\hat{\mathbb{L}}\hat{\mathbb{P}}+(\hat{\mathbb{Q}}\cdot\nabla_{\mathbf{X}})(\nabla\cdot\hat{\mathbf{v}}), \tag{6.5}$$

where

$$\hat{\mathbb{L}}=\mathbf{F}^{-1}[\mathbf{L}+(\nabla\cdot\hat{\mathbf{v}})\mathbf{I}]\mathbf{F}.$$

Then, inequality (6.4) takes the following explicit form

$$\rho_0(\dot{\psi}+\eta\dot{\theta})-\frac{1}{2}\mathbb{S}_{HK}\dot{\mathcal{C}}_{HK}-\mathbb{Y}_{HK}\dot{\mathfrak{C}}_{HK}-\mathbb{M}_{HKL}\dot{\Gamma}_{KLH}+\frac{1}{\theta}\mathfrak{q}_H G_H-\dot{\hat{\mathbb{P}}}_H\mathfrak{E}_H-\hat{\mathcal{J}}_H\mathfrak{E}_H\leq 0 \tag{6.6}$$

where

$$\hat{\mathcal{J}}=\hat{\mathbb{L}}\hat{\mathbb{P}}+(\hat{\mathbb{Q}}\cdot\nabla_{\mathbf{X}})(\nabla\cdot\hat{\mathbf{v}}). \tag{6.7}$$

We are now in a position to assume the pertinent constitutive dependence of the free energy. We firstly observe that the set of constitutive variables encloses the strain measures $\mathcal{C}$, $\mathfrak{C}$, $\mathbf{\Gamma}$, which account for macro- and microdeformations. As to the interaction with the free charge continuum, we may refer to $\hat{q}$, $\hat{\mathbb{P}}$, $\hat{\mathbb{Q}}$, as the pertinent field variables, according to (3.18) and (3.19). As can be easily verified from inequality (6.6), in view of the independence of constitutive variables, ψ does not depend on $\hat{q}$ and $\hat{\mathbb{Q}}$ so, accounting for the dependence on the temperature θ, we pose

$$\psi=\widetilde{\psi}(\mathcal{C},\mathfrak{C},\mathbf{\Gamma},\theta,\hat{\mathbb{P}}).$$

Accordingly, in order that inequality (6.6) be valid, the following constitutive equations must hold

$$\mathbb{S}=\rho_0\widetilde{\psi}_{\mathcal{C}},\quad \mathbb{Y}=\rho_0\widetilde{\psi}_{\mathfrak{C}},\quad \mathbb{M}=\rho_0\widetilde{\psi}_{\mathbf{\Gamma}},\quad \eta=\widetilde{\psi}_{\theta}, \tag{6.8}$$

$$\mathfrak{E}=\rho_0\widetilde{\psi}_{\hat{\mathbb{P}}}, \tag{6.9}$$

where, for any variable A, $\widetilde{\psi}_A$ denotes the partial derivative of $\widetilde{\psi}$ with respect to A. Besides, we get the following residual dissipative inequality

$$\hat{\mathcal{J}}\cdot\mathfrak{E}-\frac{1}{\theta}\mathfrak{q}\cdot\mathbf{G}\geq 0. \tag{6.10}$$

To obtain the complete set of governing equations with respect to the material configuration, we introduce the first Piola–Kirchhoff stress tensor $\mathbb{T}=J\mathbf{F}^{-1}\mathbf{T}$ and the following material fields,

$$\mathfrak{f}=\mathbf{f}\,\boldsymbol{\chi},\quad \mathbf{\Sigma}=\boldsymbol{\chi}\,\boldsymbol{\sigma}\,\boldsymbol{\mathfrak{X}}^T,\quad \mathbb{I}=\boldsymbol{\mathfrak{X}}\boldsymbol{\mathcal{I}}\,\boldsymbol{\mathfrak{X}}^T.$$

Then, Eq. $(3.17)_3$ yields

$$\dot{\mathbb{I}}=\mathbf{0}, \tag{6.11}$$

and the balance Eqs. (5.8) and (5.9) become

$$\rho_0\dot{\mathbf{v}}=\rho_0\mathbf{f}+J\mathbf{f}^{\mathrm{em}}+\nabla_{\mathbf{X}}\cdot\mathbb{T}, \tag{6.12}$$

$$\rho_0\mathbf{\Sigma}=\rho_0[(\nabla_{\mathbf{X}}\mathfrak{f})^T-\mathfrak{f}\mathbf{\Gamma}]\mathfrak{C}^{-T}\mathbb{I}+\mathfrak{C}\mathbb{Y}^T\mathcal{C}^{-1}-\mathcal{C}\mathbb{S}+D_R\cdot\mathbb{M}, \tag{6.13}$$

where

$$(D_R\cdot\mathbb{M})_{HK}=\mathbb{M}_{LHK,L}+\mathbb{M}_{LHM}\Gamma_{KML}-\mathbb{M}_{LMK}\Gamma_{MHL}.$$

Moreover, posing

$$\mathfrak{D} = J\mathbf{F}^{-1}\mathbf{D}, \quad \mathfrak{B} = J\mathbf{F}^{-1}\mathbf{B}, \qquad \mathbb{J} = J\mathbf{F}^{-1}\mathbf{J}, \quad \mathfrak{H} = \left(\mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}\right)\mathbf{F},$$

Maxwell's equations take the following material form

$$\nabla_{\mathbf{X}} \cdot \mathfrak{D} = q_0, \quad \nabla_{\mathbf{X}} \cdot \mathfrak{B} = 0, \tag{6.14}$$

$$\nabla_{\mathbf{X}} \times \mathfrak{E} + \frac{1}{c}\dot{\mathfrak{B}} = 0, \quad \nabla_{\mathbf{X}} \times \mathfrak{H} - \frac{1}{c}\dot{\mathfrak{D}} = \frac{1}{c}\mathbb{J}, \tag{6.15}$$

where $q_0 = Jq$. Finally, the charge conservation (3.11) is replaced by

$$\frac{\partial q_0}{\partial t} = q_0 \nabla \cdot \mathbf{v} - \nabla_{\mathbf{X}} \cdot \mathbb{J}. \tag{6.16}$$

Taking into account the dependence of $\mathbf{f}^{\text{em}}$, $\mathfrak{D}$, and $\mathfrak{H}$ on $\mathbf{p}$ and $\mathbf{Q}$, we conclude that Eqs. (6.6), (6.11)–(6.16), and $(3.17)_{1,2}$ represent the system of nonlinear governing equations for the micromorphic electromagneto-elastic conductor. These equations need the complementary prescription of constitutive Eqs. (6.8) and (6.9). In particular, Eq. (6.9) can be considered, alternatively, as a definite statement on $\mathfrak{E}$ or on $\hat{\mathbb{P}}$, provided $\widetilde{\psi}_{\hat{\mathbb{P}}}$ be invertible. Moreover, additional constitutive equations are required in general for $\mathfrak{q}$ and $\hat{\mathcal{J}}$ according to (6.10) and (6.7). Incidentally, we observe that if contributions of second order in $\hat{\boldsymbol{\xi}}$ are discarded in the conductive current (6.7), the quantity $\hat{\mathbb{Q}}$ can be neglected and inequality (6.10) implies a thermodynamic constraint on $\widetilde{\psi}_{\hat{\mathbb{P}}}$ or its inverse.

7 Perfect conductors and Ohm's law for conductors

In this section, we restrict our attention to the conductive properties of the electromagneto-elastic continuum deriving some consequence of Eqs. (6.5), (6.9), and (6.10).

As a first step, we look for a consistent condition which characterizes the ideal case of *perfect conductors*. Commonly, this condition relies on the assumption that a special constitutive equation holds for the conductive current, i.e., the elementary form of the Ohm's law, where $\hat{\mathbf{J}}$ is proportional to $\boldsymbol{\mathcal{E}}$. By definition, perfect conductors are electric conductors with null conductive dissipation for arbitrary $\mathbf{J}$. Usually, they are characterized by the requirement that a finite electromotive intensity $\boldsymbol{\mathcal{E}}$ produces an unbounded magnitude of $\hat{\mathbf{J}}$ or, in other words, the electric conductivity is infinite. In order that this assumption be compatible with a finite dissipation, expressed by the thermodynamic inequality (6.10) or its equivalent forms, $\boldsymbol{\mathcal{E}}$ is required to vanish.

The constitutive model described in the previous section suggests a different characterization of perfect conductors, which is independent on the constitutive equation introduced for the electric current. From the above definition, we immediately obtain the following statement.

Proposition *Let $\mathcal{B}$ be an electromagneto-elastic conductor described according to the constitutive model of Sect. 6. A necessary and sufficient condition for $\mathcal{B}$ to be a perfect conductor is that the free energy $\widetilde{\psi}$ do not depend on $\hat{\mathbb{P}}$.*

Owing to (6.9) and (6.10), this condition implies $\mathfrak{E} = \mathbf{0}$ with a vanishing dissipation due to the electric current. In this case, we obtain $\boldsymbol{\mathcal{E}} = \mathbf{0}$ in the spatial configuration of $\mathcal{B}$ and the Maxwell's equations, corresponding to $(6.14)_1$, and (6.15) reduce to

$$\begin{aligned} &\nabla \cdot \left(\mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = q, \\ &\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}, \\ &\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial}{\partial t}\left(\mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = \frac{1}{c}(q\mathbf{v} + \mathbf{N}\mathbf{p} + \hat{\mathbf{J}}). \end{aligned} \tag{7.1}$$

These last equations, under the additional hypothesis of null charge q and null dipole and quadrupole densities (no polarization and magnetization), correspond to the well-known results of ideal elastic conductors (see [1]). We remark that the independence of the previous characterization on the choice of Ohm's law is a natural and expected fact. Really, differently from the nonideal case, in the governing Eq. (7.1), which are coupled with the mechanical balance laws, the conduction current $\hat{\mathbf{J}}$ is one of the unknown variables and no constitutive assumptions are needed for it.

Now, we consider a conductor with not null dissipative current and assume that the function $\widetilde{\psi}_{\hat{\mathbb{P}}}$ be invertible with respect to $\hat{\mathbb{P}}$. From Eq. (6.9), we obtain

$$\hat{\mathbb{P}} = \Pi(\mathcal{C}, \mathfrak{C}, \boldsymbol{\Gamma}, \theta; \mathfrak{E}) \equiv \Pi(\Lambda; \mathfrak{E}),$$

where we have introduced the set $\Lambda = \{\mathcal{C}, \mathfrak{C}, \boldsymbol{\Gamma} s, \theta\}$. Denoting by $\Pi_{\mathfrak{E}}$ the partial derivative of Π with respect to $\mathfrak{E}$, we get

$$\dot{\hat{\mathbb{P}}} = \Pi_{\mathfrak{E}}(\Lambda; \mathfrak{E})\dot{\mathfrak{E}} + \Pi^*(\dot{\Lambda}; \mathfrak{E}), \tag{7.2}$$

where $\dot{\Lambda} = \{\dot{\mathcal{C}}, \dot{\mathfrak{C}}, \dot{\boldsymbol{\Gamma}}, \dot{\theta}\}$ and Π^* is a homogeneous linear function of the set $\dot{\Lambda}$. In the following, we neglect the term of second order in $\hat{\boldsymbol{\xi}}$ in Eq. (6.7) and write

$$\hat{\mathcal{J}} = \hat{\mathbb{L}}\Pi(\Lambda; \mathfrak{E}).$$

It is worth remarking that, although Π does not depend on $\mathbf{G}$, the quantity $\hat{\mathbb{L}}$, in general, depends on $\mathfrak{E}$ and $\mathbf{G}$ via the divergence of the conduction velocity $\hat{\mathbf{v}}$, which is an undefined quantity. Then, inequality (6.10) can be written as

$$(\hat{\mathbb{L}}\Pi)(\Lambda; \mathfrak{E}, \mathbf{G}) \cdot \mathfrak{E} - \frac{1}{\theta}\mathfrak{q}(\Lambda; \mathfrak{E}, \mathbf{G}) \cdot \mathbf{G} \geq 0. \tag{7.3}$$

Besides, the conduction current becomes

$$\hat{\mathbb{J}} = \Pi_{\mathfrak{E}}(\Lambda; \mathfrak{E})\dot{\mathfrak{E}} + \Pi^*(\dot{\Lambda}; \mathfrak{E}) + (\hat{\mathbb{L}}\Pi)(\Lambda; \mathfrak{E}, \mathbf{G}). \tag{7.4}$$

Equation (7.4) represents the generalized nonlinear Ohm's law for the present electromagneto-elastic model. It accounts for relaxation effects via the first two terms in the right-hand side. The term proportional to $\dot{\mathfrak{E}}$ is due to the interaction of free charges with the microcontinuum, and the term Π^* arises from the inertial effects of macro- and microdeformations in the continuum. We note that, differently from the common generalized Ohm's law in rigid conductors (see, e.g., [14] or [15]), here the relaxation does not depend on the free charge inertia, which is ignored. As a consequence, proper frequencies at which the present relaxation effects become noticeable are much lower than those at which the inertia of free charges becomes relevant.

In general, inequality (7.3) allows for the coupling of heat conduction and conductive current via the constitutive dependence of the heat flux on $\mathfrak{E}$ and $\mathbf{G}$. We are not interested here in such thermo-electromagneto-elastic effects, but we observe that, in comparison with some analysis in the literature, the present model yields similar, although not equivalent, modified Ohm's and Fourier's laws (see [16,17]).

In order to explain the consequence of the Ohm's law (7.4) in comparison with the classical theory of elastic conductors, we derive a modified version of the evolution equation for the magnetic field which represents a standard result on electromagnetic deformable media. To this end, we simplify Eq. (7.4) neglecting the relaxation contribution Π^* and consider a not polarizable and not magnetizable conductor with $q = 0$. Moreover, we discard thermal effects assuming $\hat{\mathbb{L}}$ to be independent on $\mathbf{G}$. Consequently, (7.3) reduces to

$$(\hat{\mathbb{L}}\Pi)(\Lambda; \mathfrak{E}) \cdot \mathfrak{E} \geq 0. \tag{7.5}$$

We rewrite Eq. (7.4) in the spatial configuration as

$$\hat{\mathbf{J}} = \hat{\boldsymbol{\pi}}^{\,\mathrm{e}}(\dot{\boldsymbol{\mathcal{E}}} + \mathbf{L}^T \boldsymbol{\mathcal{E}}) + \frac{1}{J}[\mathbf{L} + (\nabla \cdot \hat{\mathbf{v}})\mathbf{I}]\mathbf{F}\Pi \tag{7.6}$$

where $\hat{\boldsymbol{\pi}}^{\,\mathrm{e}} = \frac{1}{J}\mathbf{F}\Pi_{\mathfrak{E}}\mathbf{F}^T$. From inequality (7.5), we also have

$$[L_{hp} + (\nabla \cdot \hat{\mathbf{v}})\delta_{hp}]F_{pL}\Pi_L\mathcal{E}_h \geq 0, \tag{7.7}$$

and assuming the following linear dependence on $\boldsymbol{\mathcal{E}}$,

$$\nabla \cdot \hat{\mathbf{v}} = \alpha + \beta_p\mathcal{E}_p, \quad F_{pL}\Pi_L = \gamma_p + \hat{\eta}_{ph}\mathcal{E}_h,$$

from (7.7), we obtain the following constraints

$$\beta_p = 0, \quad \gamma_p = 0, \quad \alpha \hat{\eta}_{ph} \equiv \eta_{ph} \text{ positive definite.}$$

Then, the linearized form of (7.6) reads

$$\hat{J}_h = \pi^{\mathrm{e}}_{hk} \dot{\mathcal{E}}_k + \eta_{hk} \mathcal{E}_k, \tag{7.8}$$

where $\boldsymbol{\pi}^{\mathrm{e}}$ and $\boldsymbol{\eta}$ are constant tensors. Under the previous assumptions, the Maxwell's Eqs. (4.1) and (4.6) become

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}, \tag{7.9}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c}(\boldsymbol{\pi}^{\mathrm{e}} \dot{\boldsymbol{\mathcal{E}}} + \boldsymbol{\eta}\, \boldsymbol{\mathcal{E}}) \tag{7.10}$$

where $\boldsymbol{\mathcal{E}} = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{H}$. For isotropic media, $\pi^{\mathrm{e}}_{hk} = \pi^{\mathrm{e}}\delta_{hk}$ and $\eta_{hk} = \eta\delta_{hk}$ with $\eta > 0$. In this case, taking the curl of both sides of (7.10) and using (7.9), we arrive at the following result

$$\frac{1}{c^2}\left[(1+\pi^{\mathrm{e}})\frac{\partial^2 \mathbf{H}}{\partial t^2} + (\pi^{\mathrm{e}}\nabla \cdot \mathbf{v} + \eta)\frac{\partial \mathbf{H}}{\partial t} + (\pi^{\mathrm{e}}\nabla \cdot \dot{\mathbf{v}} + \eta\nabla \cdot \mathbf{v})\mathbf{H}\right] = \Delta \mathbf{H} + \frac{1}{c}\pi^{\mathrm{e}}\nabla \times (\mathbf{v} \cdot \nabla \mathbf{E}). \tag{7.11}$$

If $\pi^{\mathrm{e}} = 0$, Eq. (7.8) yields the usual Ohm's law and (7.11) reduces to the known magnetic transport equation for $\mathbf{H}$ in deformable conductors with conductivity η (see [1,18]). If, in addition, very large values of η are considered, this equation can be approximated by $(7.1)_2$, valid for perfect conductors.

The main difference in Eq. (7.11) with respect to the classical result is the presence of additional convective and time rate convective terms proportional, respectively, to $\nabla \cdot \mathbf{v}$ and $\nabla \cdot \dot{\mathbf{v}}$. These terms modify the balance between diffusive and convective behavior of the magnetic field in the conductor. Finally, for rigid conductors at rest, the presence of π^{e} in the coefficient of the second-order time derivative of $\mathbf{H}$ implies a modification of both velocity and attenuation of magnetic waves.

8 Concluding remarks

The micromorphic theory on the basis of this work accounts for a direct connection between microstructure and electric charge distribution. As shown in Sect. 4, this allows us to derive the macroscopic Maxwell's equations starting from dipole and quadrupole densities and their evolution equations. Polarization and magnetization are thus obtained consistently, avoiding the introduction of additional constitutive assumptions as occurs in the classical micromorphic electromagnetic continuum approaches (see [5,6]).

The present description of electric conduction shows the essential role of microstructure, via the quantities (3.18), in taking into account the interaction of charge carriers with the elastic microcontinuum. On the one hand, this model gives rise to an explicit expression for the conductive electric current $\hat{\mathbf{J}}$ (see Eq. (3.19)), and on the other hand, it allows for the dependence of the free energy on $\hat{\mathbf{p}}$, thus achieving a constitutive characterization of ideal and not ideal conductors. Relaxation effects on electric current arise naturally from this approach and can be compared with those described by the evolution equation for $\hat{\mathbf{J}}$, often proposed according to thermodynamic considerations, to account for inertial effects [14,15].

As it can be easily recognized, the analysis of electric conduction developed in Sect. 7 for not polarized and not magnetized conductors yields an electromagnetic formulation which holds independently on accounting for the mechanical microstructure. We finally remark that if, otherwise, dipole and quadrupole densities are not discarded in the analysis of Sects. 6 and 7, it is possible, in principle, to develop pertinent continuum models for polarizable conductors and, in particular, for piezoelectric semiconductors. As shown in the literature, the analysis of such media often requires the introduction of a noticeable number of phenomenological variables accounting for different types of charge carriers (see [19,20]).

References

1. Maugin, G.A.: Continuum Mechanics of Electromagnetic Solids. North Holland, Amsterdam (1988)
2. Eringen, A.C., Maugin, G.A.: Electrodynamics of Continua I. Springer, New York (1990)
3. Eringen, A.C., Şuhubi, E.S.: Nonlinear theory of simple microelastic solids I. Int. J. Eng. Sci. **2**, 189–203 (1964)
4. Eringen, A.C., Şuhubi, E.S.: Nonlinear theory of simple microelastic solids II. Int. J. Eng. Sci. **2**, 389–404 (1964)
5. Eringen, A.C.: Microcontinuum Field Theories I—Foundations and Solids. Springer, New York (1999)
6. Lee, J.D., Chen, Y., Eskandarian, A.: A micromorphic electromagnetic theory. Int. J. Solids Struct. **41**, 2099–2110 (2004)
7. Romeo, M.: Micromorphic continuum model for electromagnetoelastic solids. ZAMP **62**, 513–527 (2011)
8. Romeo, M.: Electomagnetoelastic equations for ferroelectrics based on a micromorphic dielectric model. Acta Mech. **223**, 205–219 (2012)
9. Romeo, M.: Polarization in dielectrics modeled as micromorphic continua. ZAMP **66**, 1233–1247 (2015)
10. Mindlin, R.D.: Polarization gradient in elastic dielectrics. Int. J. Solids Struct. **4**, 637–642 (1968)
11. Romeo, M.: Electroelastic waves in dielectrics modeled as polarizable continua. Wave Motion **60**, 121–134 (2016)
12. Mazur, P., Nijboer, B.R.A.: On the statistica mechanics of matter in an electromagnetic field. I. Physica **19**, 971–986 (1953)
13. Rosenfeld, L.: Theory of Electrons. North Holland, Amsterdam (1951)
14. Cuevas, S., del Río, J.A., de Haro, M.L.: Consequences of a generalized Ohm's law for magnetic transport in conducting media. J. Phys. D Appl. Phys. **32**, 639–643 (1999)
15. Silveira, F.E.M., Kurcbart, S.M.: Inertial effects on good conductors: resistive heating and magnetic diffusion. Int. J. Eng. Sci. **52**, 22–29 (2012)
16. Ezzat, M.A., Awad, E.S.: Micropolar generalized magneto-thermoelasticity with modified Ohm's and Fourier's laws. J. Math. Anal. Appl. **353**, 99–113 (2009)
17. Ezzat, M.A., Zakaria, M., El-Karamany, A.S.: Effects of modified Ohm's and Fourier's laws on generalized magnetoviscoelastic thermoelasticity with relaxation volume properties. Int. J. Eng Sci. **48**, 460–472 (2010)
18. Landau, L.D., Lifšits, E.M., Pitaevskii, L.P.: Electrodynamics of Continuous Media. Pergamon Press, New York (1984)
19. De Lorenzi, H., Tiersten, H.F.: On the interaction of the electromagnetic field with heat conducting deformable conductors. J. Math. Phys. **16**, 938–957 (1975)
20. Maugin, G.A., Daher, N.: Phenomenological theory of elastic semiconductors. Int. J. Eng. Sci. **24**, 703–731 (1986)