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ORIGINAL ARTICLE

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Micromorphic continua: non-redundant formulations

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Abstract The kinematics of generalized continua is investigated and key points concerning the definition of overall tangent strain measure are put into evidence. It is shown that classical measures adopted in the literature for micromorphic continua do not obey a constraint qualification requirement, to be fulfilled for well-posedness in optimization theory, and are therefore termed redundant. Redundancy of continua with latent microstructure and of constrained COSSERAT continua is also assessed. A simplest, non-redundant, kinematic model of micromorphic continua, is proposed by dropping the microcurvature field. The equilibrium conditions and the related variational linear elastostatic problem are formulated and briefly discussed. The simplest model involves a reduced number of state variables and of elastic constitutive coefficients, when compared with other models of micromorphic continua, being still capable of enriching the CAUCHY continuum model in a significant way.

Keywords Tangent strain measures · Generalized continua · Micromorphic and micropolar models · Redundant kinematics · Non-redundant formulations · Simplest micromorphic model · Micromechanics

1 Introduction and motivation

A large class of engineering materials, deformable porous solids, composites, polymers, crystals, microcracked solids and biological tissues, such as bones and muscles, are natural candidates to be modeled by means of the theory of micromorphic continua.

Indeed, in considering materials with well-organized microstructures, a CAUCHY continuum theory appears to be unable to provide a satisfactory modeling. This is the case for engineering materials in which sharply contrasting materials properties or highly heterogeneous hierarchical microstructures are present.

The adoption of more complex models is commonly considered as compelling to describe the essential features of the mechanical behavior of complex materials characterized by a microstructure spanning several length scales. For these materials, it is rather unlikely to get, as a suitable macroscopic model, the simple standard CAUCHY continuum.

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Micromorphic mechanics is presently a quite active field of research both from theoretical and applied points of view. For this reason a reflection concerning the foundations of the present state of the art and the verification of essential mechanical well-posedness requirements, can be considered to be advisable.

As a matter of fact, in investigating the formulation of various proposals of micromorphic and micropolar continua, the first author became convinced, some 15 years ago [1], of the fact that most classical models do show an evident redundancy of their kinematical tangent strain descriptors.

This feature has been a main obstacle for the application of such generalized continua to engineering problems, due to the ensuing intractable multitude of constitutive parameters required by the models, even in the simulation of linear elastic behaviors.

The present paper, which is an outcome of the long-term scientific collaboration between the authors, provides a rather detailed illustration of the matter, performed with the adoption of an intrinsic formalism, aimed at emphasizing mechanical concepts rather than analytical developments. No components of the involved tensor fields will appear in the treatment.

The ultimate target is to underline essential critical features of classical models of generalized continua and to present a proposal of a simplest model of micromorphic continuum, free from redundancies but still capable of enriching the CAUCHY continuum model in a significant way. Effectiveness of the model in providing useful simulations for applications will be discussed in further investigations.

2 Generalities

A generalized 3D continuum is conceived as a macro-body, a CAUCHY medium, with 3D microbodies attached at each of its points.

In the micromorphic models of generalized continua considered in [2–4] the microbodies are assumed to undergo an arbitrary act of motion with a microuniform gradient. The overall kinematics is described by the field of spatial macro-velocities and by a field of linear operators intended to describe the homogeneous distortion rate of the microbodies. Investigations on classical models are carried out in [5,6] and a hierarchy of models of generalized continua is considered in [7].

A review of pertinent literature has been provided in [8] and a clear exposition of the kinematics of polar media may be found in [9]. Mathematical tools aimed to assessing existence and uniqueness results for the relaxed micromorphic model proposed in [10] have been contributed in [11–15]. Detailed bibliographic lists are included in the references above.

A main starting point in the revisitation of the theory of generalized continua presented in this paper, consists in showing that the classical micromorphic models proposed by MINDLIN [3], ERINGEN and ŞUHUBI [4], GERMAIN [16] are kinematically redundant. The same applies to subsequent modifications. A preliminary communication on the matter was given by the second author in [17].

The plan is the following.

Essentials of kinematical aspects of CAUCHY 3D continua are briefly recalled in Sect. 3 for comparison sake. Tangent strain measures of the classical generalized models proposed in [3,4,16] are collected in Sect. 4.

According to a basic criterion in the theory of optimization [18], constraint qualification is imposed to the implicit representation of rigid body motions. In Sects. 4 and 5 classical kinematical models for generalized continua are displayed and the redundancy of the involved kinematics is put into evidence.

The overall tangent strain measure adopted in the *relaxed micromorphic model*, proposed in [10,15] is displayed in Sect. 6 and non-redundancy is inferred.

The simplest non-redundant model is proposed in Sect. 7 by dropping the microcurvature term from MINDLIN formulation, and the basic equilibrium equations are displayed.

The variational formulation of the relevant elastostatic problem is derived and briefly discussed in Sects. 8 and 9. The essential innovative contributions brought to the topic are summarized in the concluding Sect. 10 with a synoptical comparison of the complexity of the investigated models.

3 Cauchy continuum: kinematical model

Let us preliminarily recall some kinematical notions for a CAUCHY 3D continuum as reformulated in the context of 4D space–time EUCLID manifold $\mathcal{E}$ in [19–21].

On a placement $\boldsymbol{\Omega}$ in the dynamical trajectory manifold $\mathcal{T}_{\mathcal{E}} \subset \mathcal{E}$,[1] the spatial velocity field is denoted by $\mathbf{v} : \boldsymbol{\Omega} \mapsto T\mathbf{S}_{\boldsymbol{\Omega}}$, with $\mathbf{S}_{\boldsymbol{\Omega}}$ spatial slice containing $\boldsymbol{\Omega}$ and T tangent functor.

In continuum mechanics a peculiar notion concerns the definition of *rigid* velocity fields. These fields, also termed *infinitesimal isometries*, play a primary role in the definition of dynamical equilibrium and in the formulation of consistent constitutive relations [22–24].

Definition 1 (*Rigid spatial velocity fields*) Spatial velocity fields $\mathbf{v} : \boldsymbol{\Omega} \mapsto T\mathbf{S}_{\boldsymbol{\Omega}}$ are said to be rigid if the length of any line-segment drawn in $\boldsymbol{\Omega}$ does not tend to change by the effect of the motion.[2]

The length measurement tool is the covariant, positive definite and symmetric metric tensor $\mathbf{g} : \boldsymbol{\Omega} \mapsto \mathrm{Cov}(T\boldsymbol{\Omega})$ and the rate of variation of a length during the motion is evaluated by means of the covariant tensor defined by the Lie-derivative:

$$\mathcal{L}_{\mathbf{V}}(\mathbf{g}) : \boldsymbol{\Omega} \mapsto \mathrm{Cov}(T\boldsymbol{\Omega}), \tag{1}$$

where $\mathbf{V} = \partial_{\alpha=0}\, \boldsymbol{\varphi}_\alpha$ is the space–time velocity of the motion $\boldsymbol{\varphi}_\alpha : \mathcal{T}_{\mathcal{E}} \mapsto \mathcal{T}_{\mathcal{E}}$ along the trajectory $\mathcal{T}_{\mathcal{E}}$. The splitting into time and space components gives

$$\mathbf{V} = \mathbf{v} + \mathbf{Z}, \tag{2}$$

with $\mathbf{Z} : \mathcal{E} \mapsto T\mathcal{E}$ is the rigging of time arrows.

Then, if $\mathbf{t} \in T\boldsymbol{\Omega}$ is a versor tangent to a placement $\boldsymbol{\Omega}$, the rate of variation of its length during the motion is expressed by

$$\tfrac{1}{2}\, \mathcal{L}_{\mathbf{V}}(\mathbf{g})(\mathbf{t}\,,\mathbf{t}), \quad \text{being} \quad \mathbf{g}(\mathbf{t}\,,\mathbf{t}) = 1. \tag{3}$$

An implicit description of the linear space of rigid spatial velocities in $\boldsymbol{\Omega}$ is provided by Euler formula for the tangent strain

$$\tfrac{1}{2}\, \mathcal{L}_{\mathbf{V}}(\mathbf{g}) = \mathbf{g} \cdot \mathrm{sym}\nabla(\mathbf{v}) = \mathbf{0}. \tag{4}$$

This is the starting point for introducing the notion of stress field in the Cauchy continuum, as Lagrange multiplier of the rigidity constraint [1,16,26,27].

The next result will be referred to in discussing the kinematics of generalized continua. A proof can be found in [1,28].

Lemma 1 (Euler's kinematical lemma) *The vanishing, at a point* $\mathbf{x} \in \boldsymbol{\Omega}$ *of a 3D connected body, of the gradient of the tangent strain operator* $\mathrm{sym}\nabla(\mathbf{v})$ *implies the vanishing of the second gradient of the spatial velocity field at the same point, i.e.,*

$$\nabla(\mathrm{sym}\nabla(\mathbf{v}))_{\mathbf{x}} = \mathbf{0} \implies (\nabla^2\mathbf{v})_{\mathbf{x}} = \mathbf{0}. \tag{5}$$

Condition Eq. (4) implies that the macro-velocity field is $\mathrm{C}^\infty(\boldsymbol{\Omega})$. The proof of this regularity property follows from the theory of elliptic differential equations [29, p. 384, fn. 21], [30]. The implication in Lemma 1 extends to the nonlinear case of finite displacements [1]. Under the stronger assumption that $\mathrm{sym}(\nabla\mathbf{v}) = \mathbf{0}$, a kinematical result similar to Lemma 1 is well-known [31, §7], and its nonlinear version is named Liouville rigidity Lemma, see [14,32].

4 Micromorphic continua: classical formulations

A 3D *micromorphic* body is geometrically described by a 3D bounded and connected domain $\boldsymbol{\Omega}$, kinematically modeled according to Cauchy theory, and by an overlying microstructure.

At each point $\mathbf{x} \in \boldsymbol{\Omega}$, a 3D linear space $T_{\mathbf{x}}\boldsymbol{\Omega}$ tangent to the domain $\boldsymbol{\Omega}$ and a microbody $\mathcal{M}_{\mathbf{x}}$ are considered. The microbody undergoes an act of motion at the microscale with a homogeneous micrgradient

[1] The dynamical trajectory $\mathcal{T}_{\mathcal{E}}$ is an embedded submanifold of the event manifold $\mathcal{E}$ [25].

[2] Infinitesimal isometries are often characterized in literature by the property that the distance between any pair of material points does not tend to change. This definition is not applicable to wires, that can be heaped up or developed in a straight line without changing their length, or to thin plane sheets, that can be bent around a cylinder without changing the lengths of their material lines.

$\mathbf{G} : T\boldsymbol{\Omega} \mapsto T\boldsymbol{\Omega}$ of the microvelocity field, as described in [3, (1.7) p.5]. The microgradient operator is additively decomposed as[3]

$$\mathbf{G} = \mathrm{sym}(\mathbf{G}) + \mathrm{skew}(\mathbf{G}). \tag{6}$$

The symmetric and the skew-symmetric part, evaluated according to the metric tensor $\mathbf{g} : \boldsymbol{\Omega} \mapsto \mathrm{Cov}(T\boldsymbol{\Omega})$, respectively describe the overall tangent strain and the act of rotation (spin) of the 3D microbody.

Definition 2 (*Generalized rigid kinematics*) The overall rigid body kinematics in a generalized continuum is characterized by the following properties.

1. The underlying CAUCHY continuum undergoes a rigid act of motion, as expressed by vanishing of the tangent macro-strain $\mathrm{sym}\nabla(\mathbf{v}) = \mathbf{0}$, Eq. (4).
2. Each microbody undergoes a uniform rigid act of motion, a requirement expressed by vanishing of the tangent microstrain $\mathrm{sym}(\mathbf{G}) = \mathbf{0}$.
3. The microspin field $\mathrm{skew}(\mathbf{G})$ is spatially uniform, as expressed by the condition $\nabla(\mathrm{skew}(\mathbf{G})) = \mathbf{0}$.
4. The macro-micro relative spin vanishes, a requirement expressed by the condition

$$\mathrm{skew}(\nabla(\mathbf{v})) = \mathrm{skew}(\mathbf{G}). \tag{7}$$

If the tangent microstrain $\mathrm{sym}(\mathbf{G})$ is assumed to vanish identically, then the *micromorphic* model is termed *micropolar*.

Unlike the CAUCHY model, in which the tangent strain is well-characterized by the kinematic differential operator $\mathrm{sym}\nabla$, several choices have been made in the literature in order to provide an implicit description for the rigid body kinematics of a generalized continuum, to fulfill the conditions listed in items 1, 2, 3, 4.

Classical examples of overall tangent strain measures are those proposed by MINDLIN [3], ERINGEN and ŞUHUBI [4], GERMAIN [16], whose essential features are discussed below.

The assumed overall tangent strain are conveniently represented by the block matrix formulations:

1. MINDLIN and GERMAIN overall tangent strain

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \nabla\\ \nabla & -\mathbf{I}\\ \mathrm{sym}\nabla & \mathbf{0}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\nabla(\mathbf{G})\\ \nabla(\mathbf{v}) - \mathbf{G}\\ \mathrm{sym}\nabla(\mathbf{v})\end{bmatrix}, \tag{8}$$

2. ERINGEN and ŞUHUBI overall tangent strain

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \nabla\\ \nabla & -\mathbf{I}\\ \mathbf{0} & \mathrm{sym}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\nabla(\mathbf{G})\\ \nabla(\mathbf{v}) - \mathbf{G}\\ \mathrm{sym}(\mathbf{G})\end{bmatrix}. \tag{9}$$

Note that $27 + 9 + 6 = 42$ parameters are involved in both Eqs. (8) and (9).

The two models are equivalent. This may be seen by splitting the gap $\nabla(\mathbf{v}) - \mathbf{G}$ into symmetric and skew-symmetric parts and observing the linear dependence of the set of tensors

$$\mathrm{sym}\nabla(\mathbf{v}), \quad \mathrm{sym}(\mathbf{G}), \quad \mathrm{sym}(\nabla(\mathbf{v}) - \mathbf{G}). \tag{10}$$

Each of the tangent strain measures in Eqs. (8) and (9) is equivalent to the following one which will be referred to as the MINDLIN–ERINGEN model.

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \nabla\\ \mathbf{0} & \mathrm{sym}\\ \mathrm{sym}\nabla & \mathbf{0}\\ \mathrm{skew}\nabla & -\mathrm{skew}\end{bmatrix} \cdot \begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\nabla(\mathbf{G})\\ \mathrm{sym}(\mathbf{G})\\ \mathrm{sym}\nabla(\mathbf{v})\\ \mathrm{skew}(\nabla(\mathbf{v}) - \mathbf{G})\end{bmatrix}. \tag{11}$$

The microcurvature field $\nabla(\mathbf{G})$, appearing in Eqs. (8), (9) and (11) as component of the overall tangent strain, is considered as responsible for the introduction of a characteristic length scale in the modeling.

[3] In the recent literature the microgradient operator is often denoted by $\mathbf{P}$ and termed *micro (plastic) distortion* [10, p. 645]. We do not append this meaning for sake of freedom in the physical interpretation of micromorphic constitutive models.

This stems out of the fact that the ratio between the physical dimensions of the terms $\nabla(\mathbf{G})$ and $\mathrm{sym}\nabla(\mathbf{v})$ is $[L^{-1}]$, so that, consequently, the ratio between the physical dimensions of the dual microstress and macro-stress terms will be $[L]$. Adopting the standard elastic moduli of CAUCHY model as reference, a factor $[L^2]$ will appear in the elastic constitutive relations.

A fully coupled elastic relation involves $42 \times 43/2 = 903$ elastic coefficients. Uncoupled elasticity requires *only* $27 \times 28/2 + 9 \times 10/2 + 6 \times 7/2 = 378 + 45 + 21 = 444$ elastic coefficients.

4.1 Kinematic redundancy

In investigating constrained problems of optimization theory, it is emphasized that constraint qualification conditions must be fulfilled [18].

A similar requirement applies to the conditions expressing the property of rigidity in the kinematics of generalized continua.

Definition 3 (*Constraint qualification*) A kinematic operator $\mathbf{B}$ which provides an implicit description of a manifold of feasible fields, is said to meet a constraint qualification if no other implicit description can be constructed by extracting a strictly more economical condition from the given operator.

As proven below, although natural and commonly assumed in optimization theory, the kinematic operators for classical micromorphic continua, exposed in Eqs. (8) and (9), do not fulfill the qualification requirement and are therefore termed *redundant*.

Proposition 1 (Redundancy of classical models) *The tangent strain measures proposed by* MINDLIN*,* GERMAIN*,* ERINGEN *and* ŞUHUBI*, here reproduced in Eqs.* (8) *and* (9)*, and in the equivalent formulation* (11)*, are redundant.*

Proof In a connected 3D macro-body at a placement $\boldsymbol{\Omega}$, the vanishing of the lowest three components of the overall tangent strain in Eq. (11) implies the vanishing of the top one. To prove this statement we have to show that

$$\left.\begin{aligned} \mathrm{sym}(\mathbf{G}) &= \mathbf{0} \\ \mathrm{sym}\nabla(\mathbf{v}) &= \mathbf{0} \\ \mathrm{skew}(\nabla(\mathbf{v}) - \mathbf{G}) &= \mathbf{0} \end{aligned}\right\} \implies \nabla(\mathbf{G}) = \mathbf{0}. \tag{12}$$

If $\mathrm{sym}\nabla(\mathbf{v}) = \mathbf{0}$ in $\boldsymbol{\Omega}$ then, by EULER kinematical Lemma 1, we have that

$$\nabla\nabla(\mathbf{v}) = \mathbf{0}. \tag{13}$$

If in addition $\mathrm{sym}(\mathbf{G}) = \mathbf{0}$ and $\mathrm{skew}(\nabla(\mathbf{v}) - \mathbf{G}) = \mathbf{0}$ in $\boldsymbol{\Omega}$, then $\mathbf{G}$ is constant and skew-symmetric and hence the microcurvature component $\nabla(\mathbf{G})$ of the overall tangent strain vanishes identically in $\boldsymbol{\Omega}$. This proves redundancy of the classical models for micromorphic continua. □

We emphasize that redundancy is a purely kinematical notion which is independent of the constitutive modeling. As evident from the implications in Eq. (12), kinematic redundancy stems out of the microcurvature term $\nabla(\mathbf{G})$.

4.2 Equilibrium

Differential and boundary equilibrium conditions for the classical micromorphic kinematic model formulated by Eq. (11) are deduced from the following abstract expression of GREEN's formula [27]

$$\int_{\boldsymbol{\Omega}} \langle \mathbf{s}, \mathbf{B} \cdot \mathbf{p} \rangle \cdot \boldsymbol{\mu} = \int_{\boldsymbol{\Omega}} \left\langle \mathbf{B}_0' \cdot \mathbf{s}, \mathbf{p} \right\rangle \cdot \boldsymbol{\mu} + \oint_{\partial\boldsymbol{\Omega}} \langle \mathbf{N} \cdot \mathbf{s}, \boldsymbol{\Gamma} \cdot \mathbf{p} \rangle \cdot \partial\boldsymbol{\mu}, \tag{14}$$

where a dot $\cdot$ denotes linear dependence on the subsequent item,

- $\mathbf{p}$ is the overall field of macro-micro kinematic parameters,
- $\mathbf{s}$ is the overall field of stress parameters,
- $\mathbf{B}$ is the kinematic operator,
- $\mathbf{B}_0'$ is the formal adjoint differential equilibrium operator,
- $\mathbf{N}$ and $\boldsymbol{\Gamma}$ are the flux and boundary value operators,
- $\boldsymbol{\mu}$ and $\partial\boldsymbol{\mu}$ are the volume form in $\boldsymbol{\Omega}$ and the area form on the boundary $\partial\boldsymbol{\Omega}$,
- $\langle\, , \rangle$ is the duality pairing.

4.2.1 Equilibrium equations for Mindlin–Eringen model

The macro–micro-kinematical field $\mathbf{p}$ (with $3+9=12$ scalar parameters) and the overall tangent strain $\boldsymbol{\varepsilon}$ and stress field $\mathbf{s}$ (with $27+9+6=42$ scalar parameters) are expressed by the block matrices

$$\mathbf{p} = \begin{bmatrix} \mathbf{v} \\ \mathbf{G} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \nabla(\mathbf{G}) \\ \mathrm{sym}(\mathbf{G}) \\ \mathrm{sym}\nabla(\mathbf{v}) \\ \mathrm{skew}(\nabla(\mathbf{v}) - \mathbf{G}) \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \mathbf{T}_{\mathrm{CURV}} \\ \mathbf{T}_{\mathrm{MICRO}} \\ \mathbf{T}_{\mathrm{MACRO}} \\ \mathbf{T}_{\mathrm{GAP}} \end{bmatrix}. \tag{15}$$

The overall tangent strain field adopted in [3,16] is represented by the kinematic operator $\mathbf{B}$ in Eq. (11). Without loss in generality we may assume that

$$\begin{cases} \mathbf{T}_{\mathrm{MICRO}} = \mathrm{sym}(\mathbf{T}_{\mathrm{MICRO}}), \\ \mathbf{T}_{\mathrm{MACRO}} = \mathrm{sym}(\mathbf{T}_{\mathrm{MACRO}}), \\ \mathbf{T}_{\mathrm{GAP}} = \mathrm{skew}(\mathbf{T}_{\mathrm{GAP}}). \end{cases} \tag{16}$$

Then GREEN's formula writes

$$\int_{\boldsymbol{\Omega}} \left\langle \begin{bmatrix} \mathbf{T}_{\mathrm{CURV}} \\ \mathbf{T}_{\mathrm{MICRO}} \\ \mathbf{T}_{\mathrm{MACRO}} \\ \mathbf{T}_{\mathrm{GAP}} \end{bmatrix}, \begin{bmatrix} \nabla(\mathbf{G}) \\ \mathrm{sym}(\mathbf{G}) \\ \mathrm{sym}\nabla(\mathbf{v}) \\ \mathrm{skew}(\nabla(\mathbf{v}) - \mathbf{G}) \end{bmatrix} \right\rangle \cdot \boldsymbol{\mu} = -\int_{\boldsymbol{\Omega}} \left\langle \begin{bmatrix} \mathrm{Div}(\mathbf{T}_{\mathrm{GAP}} + \mathbf{T}_{\mathrm{MACRO}}) \\ \mathrm{DIV}(\mathbf{T}_{\mathrm{CURV}}) + \mathbf{T}_{\mathrm{GAP}} \end{bmatrix}, \begin{bmatrix} \mathbf{v} \\ \mathbf{G} \end{bmatrix} \right\rangle \cdot \boldsymbol{\mu}$$
$$+ \oint_{\partial \boldsymbol{\Omega}} \left\langle \begin{bmatrix} (\mathbf{T}_{\mathrm{GAP}} + \mathbf{T}_{\mathrm{MACRO}}) \cdot \mathbf{n} \\ \mathbf{T}_{\mathrm{CURV}} \cdot \mathbf{n} \end{bmatrix}, \boldsymbol{\Gamma} \begin{bmatrix} \mathbf{v} \\ \mathbf{G} \end{bmatrix} \right\rangle \cdot \partial \boldsymbol{\mu}. \tag{17}$$

The differential and boundary equilibrium operators $\mathbf{B}_0'$ and $\mathbf{N}$ write thus

$$\mathbf{B}_0' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathrm{Div} & -\mathrm{Div} \\ -\mathrm{DIV} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \cdot\mathbf{n} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{18}$$

and the equilibrium conditions are given by

$$\mathbf{B}_0' \begin{bmatrix} \mathbf{T}_{\mathrm{CURV}} \\ \mathbf{T}_{\mathrm{MICRO}} \\ \mathbf{T}_{\mathrm{MACRO}} \\ \mathbf{T}_{\mathrm{GAP}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{\mathbf{v}} \\ \mathbf{b}_{\mathbf{G}} \end{bmatrix}, \qquad \mathbf{N} \begin{bmatrix} \mathbf{T}_{\mathrm{CURV}} \\ \mathbf{T}_{\mathrm{MICRO}} \\ \mathbf{T}_{\mathrm{MACRO}} \\ \mathbf{T}_{\mathrm{GAP}} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{\mathbf{v}} \\ \mathbf{t}_{\mathbf{G}} \end{bmatrix}, \tag{19}$$

and explicitly

$$\begin{cases} -\mathrm{Div}(\mathbf{T}_{\mathrm{MACRO}} + \mathbf{T}_{\mathrm{GAP}}) = \mathbf{b}_{\mathbf{v}}, & \text{in } \boldsymbol{\Omega}, \\ -\mathrm{DIV}(\mathbf{T}_{\mathrm{CURV}}) - \mathbf{T}_{\mathrm{GAP}} = \mathbf{b}_{\mathbf{G}}, & \text{in } \boldsymbol{\Omega}, \\ (\mathbf{T}_{\mathrm{MACRO}} + \mathbf{T}_{\mathrm{GAP}}) \cdot \mathbf{n} = \mathbf{t}_{\mathbf{v}}, & \text{on } \partial\boldsymbol{\Omega}, \\ \mathbf{T}_{\mathrm{CURV}} \cdot \mathbf{n} = \mathbf{t}_{\mathbf{G}}, & \text{on } \partial\boldsymbol{\Omega}. \end{cases} \tag{20}$$

5 Micropolar continua: classical formulations

A clear exposition of kinematics of micropolar continua can be found in a recent paper by DEL PIERO [9]. The model there proposed is the micropolar analog of the MINDLIN–ERINGEN model of Eq. (11), under the assumption that

$$\mathrm{sym}(\mathbf{G}) = \mathbf{0}, \tag{21}$$

so that, setting

$$\mathbf{w} = \tfrac{1}{2}\mathrm{axial}\,\mathrm{skew}(\mathbf{G}), \tag{22}$$

the overall tangent strain is expressed by

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \nabla\\ \mathbf{0} & \mathrm{sym}\nabla\\ \mathrm{curl} & -\mathbf{I}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\nabla(\mathbf{w})\\ \mathrm{sym}\nabla(\mathbf{v})\\ \mathrm{curl}(\mathbf{v}) - \mathbf{w}\end{bmatrix}. \tag{23}$$

To describe the pair $\{\mathbf{v}, \mathbf{w}\}$ $3+3=6$ scalar kinematic parameters are involved and $9+6+3=18$ scalar parameters are required for the overall tangent strain.

For a coupled linear elastic relation, $18 \times 19/2 = 162$ elastic coefficients are thus needed, while, for an uncoupled elastic relation, as many as $45 + 6 + 21 = 72$ elastic coefficients are required in the anisotropic case.

Contrary to the statement in [9], where the components of the tangent strain in Eq. (23) are qualified as *free generalized deformations*, the model so defined is redundant since

$$\left.\begin{aligned}\mathrm{sym}\nabla(\mathbf{v}) = \mathbf{0}\\ \mathrm{curl}(\mathbf{v}) - \mathbf{w} = \mathbf{0}\end{aligned}\right\} \implies \nabla(\mathbf{w}) = \mathbf{0}. \tag{24}$$

The same model was also adopted in [33].

In the *continuum with latent microstructure* considered by CAPRIZ in [34], and investigated by DEL PIERO in [9] under the name of *constrained* COSSERAT *continuum*, the kinematical constraint $\mathbf{w} = \mathrm{curl}(\mathbf{v})$ in the micropolar model of Eq. (23) is assumed to be identically fulfilled.

The overall kinematical field $\mathbf{p}$ (with $3 + 3 = 6$ scalar parameters) and the overall stress field $\mathbf{s}$ (with $9 + 6 = 15$ scalar parameters) are then expressed in block matrix notation by

$$\mathbf{p} = \begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix}\mathbf{T}_{\mathrm{CURV}}\\ \mathbf{T}_{\mathrm{MACRO}}\end{bmatrix}. \tag{25}$$

The overall tangent strain is represented by the kinematic operator $\mathbf{B}$, expressed as a block matrix by

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \nabla\\ \mathrm{sym}\nabla & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\nabla(\mathbf{w})\\ \mathrm{sym}\nabla(\mathbf{v})\end{bmatrix}, \tag{26}$$

with $\mathbf{T}_{\mathrm{MACRO}} = \mathrm{sym}(\mathbf{T}_{\mathrm{MACRO}})$ and the kinematic constraint

$$\mathbf{w} = \tfrac{1}{2}\,\mathrm{axial}\,\mathrm{skew}\nabla(\mathbf{v}) = \mathrm{curl}(\mathbf{v}). \tag{27}$$

Then GREEN's formula writes

$$\begin{aligned}\int_{\boldsymbol{\Omega}}\left\langle\begin{bmatrix}\mathbf{T}_{\mathrm{CURV}}\\ \mathbf{T}_{\mathrm{MACRO}}\end{bmatrix}, \begin{bmatrix}\nabla(\mathbf{w})\\ \mathrm{sym}\nabla(\mathbf{v})\end{bmatrix}\right\rangle\cdot\boldsymbol{\mu} &= -\int_{\boldsymbol{\Omega}}\mathbf{g}(\mathrm{Div}(\mathbf{T}_{\mathrm{MACRO}}), \mathbf{v}) + \mathbf{g}(\mathrm{Div}(\mathbf{T}_{\mathrm{CURV}}), \mathrm{curl}(\mathbf{v}))\cdot\boldsymbol{\mu}\\ &= -\oint_{\partial\boldsymbol{\Omega}}\mathbf{g}(\mathbf{T}^{A}_{\mathrm{MACRO}}\cdot\mathbf{v}, \mathbf{n}) + \mathbf{g}(\mathbf{T}^{A}_{\mathrm{CURV}}\cdot\mathrm{curl}(\mathbf{v}), \mathbf{n})\cdot\partial\boldsymbol{\mu}.\end{aligned} \tag{28}$$

This leads to complex bulk and boundary equilibrium conditions [9].

The model is still redundant due to the implication in Eq. (24).

For a coupled linear elastic relation, $15 \times 16/2 = 120$ elastic coefficients are needed, while, for an uncoupled elastic relation, as many as $45 + 21 = 66$ elastic coefficients are required in the anisotropic case.

6 Micromorphic continua: the relaxed model

In recent papers [10,15] NEFF and coworkers have proposed a relaxed model of micromorphic continuum with a significant reduction of the number of independent parameters defining the overall tangent strain measure.

In the relaxed model, the overall tangent strain, expressed in our notations, is given by

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathrm{Curl}\\ \mathbf{0} & \mathrm{sym}\\ \mathrm{sym}\nabla & -\mathrm{sym}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathrm{Curl}(\mathbf{G})\\ \mathrm{sym}(\mathbf{G})\\ \mathrm{sym}(\nabla(\mathbf{v})-\mathbf{G})\end{bmatrix}. \tag{29}$$

The number of involved kinematical parameters is halved from 42 (for the classical models in Sect. 4) to $9+6+6=21$.

In the anisotropic case, with a coupled elastic relation, $21\times 22/2=231$ elastic coefficients are needed, while uncoupled elasticity involves as many as $45+21+21=87$ elastic coefficients.

These are to be compared respectively with the huge number of 903 and 444 parameters needed by the classical models described in Sect. 4.

The overall tangent strain measure adopted in the relaxed micromorphic continuum model is non-redundant since

$$\left.\begin{aligned}\mathrm{sym}(\mathbf{G})=\mathbf{0}\\ \mathrm{sym}(\nabla(\mathbf{v})-\mathbf{G})=\mathbf{0}\end{aligned}\right\} \;\not\Longrightarrow\; \mathrm{Curl}(\mathbf{G})=\mathbf{0}. \tag{30}$$

Fulfillment of the requirements in Definition 2 and existence and uniqueness of the relevant elastostatic problem are discussed in [6,11–14] by assessment of inequalities in involved functional spaces.

A reasoning analogous to the one leading to Eq. (11) shows that the relaxed model in Eq. (29) is equivalent to the following one which allows for a more direct comparison with the classical model in Eq. (11).

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathrm{Curl}\\ \mathbf{0} & \mathrm{sym}\\ \mathrm{sym}\nabla & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathrm{Curl}(\mathbf{G})\\ \mathrm{sym}(\mathbf{G})\\ \mathrm{sym}\nabla(\mathbf{v})\end{bmatrix}. \tag{31}$$

7 Micromorphic continua: the simplest model

The simplest model for a micromorphic 3D body can be formulated by dropping the microcurvature term $\nabla(\mathbf{G})$, in the MINDLIN–GERMAIN model Eq. (8). As a consequence the model is non-redundant.

The introduction, in the usual way, of a characteristic length scale has no more room and so also the issues evidenced in [35].

The resulting tangent strain measure is expressed by the kinematic operator

$$\mathbf{B}\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathrm{sym}\\ \mathrm{sym}\nabla & \mathbf{0}\\ \mathrm{skew}\nabla & -\mathrm{skew}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix} = \begin{bmatrix}\mathrm{sym}(\mathbf{G})\\ \mathrm{sym}\nabla(\mathbf{v})\\ \mathrm{skew}(\nabla(\mathbf{v})-\mathbf{G})\end{bmatrix}. \tag{32}$$

We remark a drastic reduction of involved kinematical parameters with respect to the classical micromorphic continua discussed in Sect. 4, from 42 to $6+3+6=15$. By setting

$$\mathbf{p}=\begin{bmatrix}\mathbf{v}\\ \mathbf{G}\end{bmatrix},\quad \mathbf{s}=\begin{bmatrix}\mathbf{T}_{\mathrm{MICRO}}\\ \mathbf{T}_{\mathrm{MACRO}}\\ \mathbf{T}_{\mathrm{GAP}}\end{bmatrix}, \tag{33}$$

with the properties in Eq. (16). The operators of differential and boundary equilibrium $\mathbf{B}_0'$ and $\mathbf{N}$ write as

$$\mathbf{B}_0'=\begin{bmatrix}\mathbf{0} & -\mathrm{div} & -\mathrm{div}\\ \mathbf{I} & \mathbf{0} & -\mathbf{I}\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}\mathbf{0} & \cdot\mathbf{n} & \cdot\mathbf{n}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}, \tag{34}$$

and the equilibrium conditions take the form

$$\mathbf{B}_0' \begin{bmatrix} \mathbf{T}_{\text{MICRO}} \\ \mathbf{T}_{\text{MACRO}} \\ \mathbf{T}_{\text{GAP}} \end{bmatrix} = \begin{bmatrix} \mathbf{b_v} \\ \mathbf{b_G} \end{bmatrix}, \qquad \mathbf{N} \begin{bmatrix} \mathbf{T}_{\text{MICRO}} \\ \mathbf{T}_{\text{MACRO}} \\ \mathbf{T}_{\text{GAP}} \end{bmatrix} = \begin{bmatrix} \mathbf{t_v} \\ \mathbf{t_G} \end{bmatrix}. \tag{35}$$

and explicitly

$$\begin{cases} -\text{div}(\mathbf{T}_{\text{MACRO}} + \mathbf{T}_{\text{GAP}}) = \mathbf{b_v}, & \text{in } \boldsymbol{\Omega}, \\ \mathbf{T}_{\text{MICRO}} - \mathbf{T}_{\text{GAP}} = \mathbf{b_G}, & \text{in } \boldsymbol{\Omega}, \\ (\mathbf{T}_{\text{MACRO}} + \mathbf{T}_{\text{GAP}}) \cdot \mathbf{n} = \mathbf{t_v}, & \text{on } \partial\boldsymbol{\Omega}, \\ \mathbf{0} = \mathbf{t_G}, & \text{on } \partial\boldsymbol{\Omega}. \end{cases} \tag{36}$$

From Eqs. (16) and (36) we infer that

$$\begin{cases} \mathbf{T}_{\text{MICRO}} = \text{sym}(\mathbf{b_G}), \\ -\mathbf{T}_{\text{GAP}} = \text{skew}(\mathbf{b_G}). \end{cases} \tag{37}$$

Introducing the effective body force and contact tractions as

$$\begin{cases} \mathbf{b} := \mathbf{b_v} - \text{Div}(\text{skew}(\mathbf{b_G})), \\ \mathbf{t} := \mathbf{t_v} + \text{skew}(\mathbf{b_G}) \cdot \mathbf{n}, \end{cases} \tag{38}$$

the equilibrium conditions are expressed by

$$\begin{cases} -\text{Div}(\mathbf{T}_{\text{MACRO}}) = \mathbf{b}, & \text{in } \boldsymbol{\Omega}, \\ \mathbf{T}_{\text{MACRO}} \cdot \mathbf{n} = \mathbf{t}, & \text{on } \partial\boldsymbol{\Omega}. \end{cases} \tag{39}$$

In the simplest model the symmetric macro-stress $\mathbf{T}_{\text{MACRO}}$ is subject to the standard CAUCHY equilibrium conditions with bulk and boundary force distributions given by Eq. (38), and the microstress is expressed by $\mathbf{T}_{\text{MICRO}} = \text{sym}(\mathbf{b_G})$.

The model can be equivalently formulated by assuming as overall kinematical and stress fields the triplets

$$\mathbf{p} = \begin{bmatrix} \mathbf{v} \\ \text{sym}(\mathbf{G}) \\ \mathbf{w} \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \mathbf{T}_{\text{MICRO}} \\ \mathbf{T}_{\text{MACRO}} \\ \mathbf{T}_{\text{GAP}} \end{bmatrix}, \tag{40}$$

with

$$\mathbf{w} := \tfrac{1}{2}\text{axial}\,\text{skew}(\mathbf{G}). \tag{41}$$

The overall kinematic operator $\mathbf{B}$ is thus provided by

$$\mathbf{B} \begin{bmatrix} \mathbf{v} \\ \text{sym}(\mathbf{G}) \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \text{sym}\nabla & \mathbf{0} & \mathbf{0} \\ \text{curl} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ \text{sym}(\mathbf{G}) \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \text{sym}(\mathbf{G}) \\ \text{sym}(\nabla\mathbf{v}) \\ \text{curl}(\mathbf{v}) - \mathbf{w} \end{bmatrix}, \tag{42}$$

and the operators of differential and boundary equilibrium $\mathbf{B}_0'$ and $\mathbf{N}$ write as

$$\mathbf{B}_0' = \begin{bmatrix} \mathbf{0} & -\text{div} & -\text{div} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \cdot\mathbf{n} & \cdot\mathbf{n} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{43}$$

A non-redundant model for a micropolar 3D body is deduced from Eq. (32) by setting

$$\text{sym}(\mathbf{G}) = \mathbf{0}, \quad \mathbf{T}_{\text{MICRO}} = \mathbf{0}. \tag{44}$$

Dropping the leading terms, the overall tangent strain and stress become

$$\mathbf{B} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \text{curl}(\mathbf{v}) - \mathbf{w} \\ \text{sym}\nabla(\mathbf{v}) \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \mathbf{T}_{\text{MACRO}} \\ \mathbf{T}_{\text{GAP}} \end{bmatrix}. \tag{45}$$

and the equilibrium equations are formally given by Eq. (39).

8 Linear elastostatics

In the simplest non-redundant micromorphic model described by the kinematics in Eq. (32), there are 15 scalar components of the tangent strain measure and hence as many as $15 \times 16/2 = 120$ linear elastic coefficients for a fully coupled linear anisotropic elastic behavior with GREEN potentials.

This multitude is to be compared with the huge number of 903 elastic coefficients for MINDLIN's model in Eq. (8).[4]

The geometrically linearized variational elastostatic problem, will be displayed with reference to the simplest micropolar model described by Eq. (45).

Let us denote by $\mathcal{L}$ the linear space of macro-velocity fields in $\boldsymbol{\Omega}$ conforming with linear boundary conditions, by $\Lambda = \mathcal{L}^*$ the dual space of load distributions and by H the space of square integrable tangent vector fields in $\boldsymbol{\Omega}$.

Assuming a linear elastic relation between the stress and the overall tangent strain

$$\mathbf{s} = \mathbf{E} \cdot \mathbf{B}(\mathbf{p}), \quad \mathbf{p} = \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \tag{46}$$

with $\mathbf{v} \in \mathcal{L}$ macro-velocity field, $\mathbf{w} \in H$ microspin field, $\mathbf{E}$ symmetric and positive definite constitutive operator, the elastostatic problem is expressed by the following variational condition involving the bilinear form $\mathbf{a}$ of elastic energy and the overall force distribution $\mathbf{f}$ acting over the body

$$\mathbf{a}(\mathbf{E} \cdot \mathbf{B}(\mathbf{p}) \,, \mathbf{B}(\delta\mathbf{p})) = \langle \mathbf{f}, \delta\mathbf{p} \rangle. \tag{47}$$

The overall force distribution is composed by the effective macro-loading $\ell \in \Lambda$ and by the microloading $\mathbf{c} \in H$, so that

$$\langle \mathbf{f}, \delta\mathbf{p} \rangle := \langle \ell, \delta\mathbf{v} \rangle + \langle \mathbf{c}, \delta\mathbf{w} \rangle. \tag{48}$$

In the spirit of geometric linearization, velocities and microspin will be treated as small displacements and small rotations. In the formalism of block matrices the linear elastic relation is then expressed by

$$\begin{bmatrix} \mathbf{T}_{\text{MACRO}} \\ \mathbf{T}_{\text{GAP}} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_{22} \end{bmatrix} \cdot \begin{bmatrix} \operatorname{sym}\nabla(\mathbf{v}) \\ \operatorname{curl}(\mathbf{v}) - \mathbf{w} \end{bmatrix}, \tag{49}$$

with $\mathbf{E}_{11}, \mathbf{E}_{22}$ positive definite and symmetric and $\mathbf{E}_{21} = \mathbf{E}_{12}^A$ such that the SCHUR complements $\mathbf{E}_{11} - \mathbf{E}_{12}\mathbf{E}_{22}^{-1}\mathbf{E}_{21}$ and $\mathbf{E}_{22} - \mathbf{E}_{21}\mathbf{E}_{11}^{-1}\mathbf{E}_{12}$ are positive definite.

The representative matrices of $\mathbf{E}_{11}$, $\mathbf{E}_{22}$ and $\mathbf{E}_{12}, \mathbf{E}_{21}$ do respectively have $3 \times 4/2 = 6$, $6 \times 7/2 = 21$ and $3 \times 6 = 18$ elastic coefficients, for a total of $9 \times 10/2 = 45$ elastic coefficients.

The bilinear form of the elastic energy functional is then expressed by

$$\begin{aligned} \mathbf{a}(\mathbf{p} \,, \delta\mathbf{p}) := \int_{\boldsymbol{\Omega}} \Big(& \langle \mathbf{E}_{11} \cdot \operatorname{sym}\nabla(\mathbf{v}), \operatorname{sym}\nabla(\delta\mathbf{v}) \rangle + \langle \mathbf{E}_{22} \cdot (\operatorname{curl}(\mathbf{v}) - \mathbf{w}), \operatorname{curl}(\delta\mathbf{v}) - \delta\mathbf{w} \rangle \\ & + \langle \mathbf{E}_{12} \cdot (\operatorname{curl}(\mathbf{v}) - \mathbf{w}), \operatorname{sym}(\nabla(\delta\mathbf{v})) \rangle + \langle \mathbf{E}_{21} \cdot (\operatorname{sym}\nabla(\mathbf{v}), \operatorname{curl}(\delta\mathbf{v}) - \delta\mathbf{w} \rangle \Big) \cdot \boldsymbol{\mu}, \end{aligned} \tag{50}$$

with the related quadratic energy functional:

$$\begin{aligned} \tfrac{1}{2}\,\mathbf{a}(\mathbf{p} \,, \mathbf{p}) := \int_{\boldsymbol{\Omega}} \tfrac{1}{2} \Big(& \langle \mathbf{E}_{11} \cdot \operatorname{sym}\nabla(\mathbf{v}), \operatorname{sym}\nabla(\mathbf{v}) \rangle + \langle \mathbf{E}_{22} \cdot (\operatorname{curl}(\mathbf{v}) - \mathbf{w}), \operatorname{curl}(\mathbf{v}) - \mathbf{w} \rangle \\ & + 2\, \langle \mathbf{E}_{12} \cdot (\operatorname{sym}\nabla(\mathbf{v}), \operatorname{curl}(\mathbf{v}) - \mathbf{w} \rangle \Big) \cdot \boldsymbol{\mu}. \end{aligned} \tag{51}$$

[4] In [3, p. 14], with reference to the redundant micromorphic model, MINDLIN observes that: *Only* $42 \times 43/2 = 903$ *of the* $42 \times 42 = 1764$ *coefficients are independent*. That this feature *makes the general micromorphic model suitable for anything and nothing and has severely hindered the application of micromorphic models*, is a comment reported in [10].

9 Coupled versus uncoupled elasticity and characteristic length

From the variational formulation Eq. (47) and the expression Eq. (50) of the elastic energy functional, performing the variation $\delta\mathbf{w} \in H$ while holding $\delta\mathbf{v} = \mathbf{0}$, we get the condition

$$\mathbf{E}_{22} \cdot (\text{curl}(\mathbf{v}) - \mathbf{w}) + \mathbf{E}_{21} \cdot (\text{sym}\nabla(\mathbf{v})) = -\mathbf{c}. \tag{52}$$

Therefore, under the special assumptions of vanishing microloading ($\mathbf{c} = \mathbf{0}$) and uncoupled elasticity ($\mathbf{E}_{12} = \mathbf{0}$) it follows that $\mathbf{w} = \text{curl}(\mathbf{v})$ and the micropolar simplest elastic model collapses into the one of a standard elastic continuum.

This is not the case when a coupled linear elasticity is considered or when non-elastic contributions to the overall tangent strain are included in the constitutive relation.

The absence of a curvature term makes the usual procedure, leading to the introduction of a (squared) characteristic length on the basis of a special constitutive assumption, unfeasible. A characteristic length may however be introduced in the new micropolar model by a geometric argument which does not involve constitutive aspects [36]. The simplest non-redundant micromorphic model proposed in this paper appears to be a proper candidate for the simulation of complex material behaviors.

10 Closing remarks

The outcomes of the present paper and the essential feature of the proposed simplest micromorphic and micropolar models may be summarized as follows.

1. Classical kinematical models of generalized continua are shown to be based on tangent strain measures that fail to meet the constraint qualification prescription of optimization theory.
2. The micromorphic models proposed by Mindlin [3], Germain [16], Eringen and Şuhubi [4], are shown to be kinematically redundant. Also redundant are the micropolar models considered by Capriz [34] and Del Piero [9].
3. The relaxed model of micromorphic continuum, formulated by Neff and coworkers [10,15], provides a drastic reduction of the number of kinematical parameters, The relevant differential expression of the overall tangent strain measure is non-redundant.
4. The simplest non-redundant model, here formulated by dropping the redundant microcurvature term, is characterized by a further reduction of the number of kinematical parameters, as depicted, for coupled and uncoupled linear elasticity, in Tables 1 and 2, respectively for micromorphic and micropolar models. A characteristic length scale may be introduced in the simplest micropolar model by purely geometric considerations [36].
5. Differential and boundary equilibrium conditions are displayed, and the variational formulation of the linearized elastostatic problem for the simplest non-redundant model is developed and briefly discussed.

The simplest model is free from the inherent conceptual weakness of most redundant models proposed in literature and could contribute to open the way for an improved applicability of the theory of micromorphic and micropolar continua to engineering problems.

Effectiveness of the proposed model will be tested in a further research activity devoted to applications to complex material problems of engineering interest.

Table 1 Elastic parameters: micromorphic model

	Coupled elasticity	Uncoupled elasticity
Mindlin–Eringen	903	444
Neff et al.	231	87
Romano–Barretta	120	48

Table 2 Elastic parameters: micropolar model

	Coupled elasticity	Uncoupled elasticity
Del Piero	162	72
Capriz	120	66
Romano–Barretta	45	27

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