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# Distinguished rheological models for solids in the framework of a thermodynamical internal variable theory

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**Abstract** We present and analyse a thermodynamical theory of rheology with single internal variable. The universality of the model is ensured as long as the mesoscopic and/or microscopic background processes satisfy the applied thermodynamical principles, which are the second law, the basic balances and the existence of an additional—tensorial—state variable. The resulting model, which we suggest to call the Kluitenberg–Verhás body, is the Poynting–Thomson–Zener body with an additional inertial element or, in other words, is the extension of Jeffreys model to solids. We argue that this Kluitenberg–Verhás body is the natural thermodynamical building block of rheology. An important feature of the presented methodology is that nontrivial inequality-type restrictions arise for the four parameters of the model. We compare these conditions and other aspects to those of other known thermodynamical approaches, like Extended Irreversible Thermodynamics or the original theory of Kluitenberg.

**Keywords** Rheology · Solid · Irreversible thermodynamics · Internal variable

## 1 Introduction

The method of internal variables can be considered as a universal modelling tool for classical, macroscopic field theories. It is universal in the sense that one introduces minimal assumption about the physical mechanism of the modelled phenomena. An additional field variable is the starting point. Its relation and coupling to existing physical quantities and its evolution equation are the key questions that are to be answered. Irreversible thermodynamics proposes a particular method in this respect [1, 2]. The general idea is that one can obtain the form of the evolution equation and also the connection to other processes considering only general principles,

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first of all, the second law [3–5]. Evolution equations derived from any structural, mesoscopic or microscopic realization of the extra field variable must belong to this general form, as long as they are restricted by the same general principles. The first application of thermodynamical ideas for continua is due to Eckart, who also investigated deviations from ideal elastic behaviour, in another wording, anelasticity [6,7]. A thermodynamical framework for anelasticity with internal variables was first treated by Biot [8] and developed by Kluitenberg as a state variable theory [9–17]. Less systematic applications are popular for various phenomena in solids, sometimes without any thermodynamical background [18].

The concept of internal variables has a long history [19,20], and there are numerous different versions and names. The name *internal degrees of freedom* was introduced in thermodynamics as an extension of the configurational space and originally denoted an extension of the deterministic field with statistical and probabilistic aspects [21,22]. This is the origin of the so-called mesoscopic theories [23–25]. This meaning is strictly distinguished from the macroscopic fields named under the same terminology *internal degrees of freedom*, used by Maugin for field theories with Lagrangian dynamics [1,26]. The so-called *internal variables of state* are again different, as their evolution is relaxational and has a thermodynamic origin [1,3]. In spite of the fine details where these notions are different for different authors (controllability, boundary conditions, weak nonlocality, etc.), we can find a sufficiently general framework where these concepts coincide and appear as a powerful modelling tool of modern continuum physics [27,28]. In what follows, we do not address these aspects but demonstrate the constructive modelling power of a seemingly restricted conceptual framework.

A specific version of the concept of internal variable is called *dynamical degree of freedom* [29,30], which variable becomes zero in local thermodynamical equilibrium, and thus quantifies, along a process, the instantaneous deviation from equilibrium, so to say, the amount of irreversibility present. This simple, natural and general assumption, utilized by us here as well, is crucial. Other approaches introduce various additional conditions due to the considered direct interpretation of the state variables; hence, they lead to different classification and different restrictions on the constitutive coefficients. For example, in Extended Irreversible Thermodynamics (EIT) [31,32], the additional state variable is the dissipative stress, while in the original Kluitenberg theory, the state variable is anelastic strain, which additively modifies the elastic strain of the rheological process.

The tensorial order of the internal variable can usually be deduced from the properties of the modelled phenomenon. For example, for a simple description of the damage of solids, a scalar variable may suffice, characterizing the level of degradation of the structure of the solid. For describing heat conduction effects beyond the Fourier theory, a correction to the heat flux is expected; therefore, a vectorial internal variable offers itself [33–35]. For the description of rheological effects, corresponding to the fact that stress and strain are symmetric tensorial quantities, the naturally expected, and here confirmed, internal variable is a symmetric second-order tensor.

As shown below, for linear Onsagerian equations, one can eliminate this internal state variable and obtains a linear relationship between stress, strain and some of its time derivatives: the zeroth and first derivative of stress and the zeroth, first and second derivative of strain. In notation, we will denote this by  $(0, 1 \asymp 0, 1, 2)$ . Consequently, this relationship covers a number of classic rheological models as special subclasses: the Kelvin–Voigt model, which is the  $(0 \asymp 0, 1)$  case, the  $(0, 1 \asymp 1)$  Maxwell model, the  $(0, 1 \asymp 0, 1)$  Poynting–Thomson–Zener model and the  $(0, 1 \asymp 1, 2)$  Jeffreys model. Therefore, the internal variable approach provides a universal framework for discussing these models on a common ground and, in particular, to investigate the thermodynamical properties of these models.

More closely, we find here that one  $(0, 1 \asymp 0, 1, 2)$  model arises for the relationship between the deviatoric part of stress and of strain (and its derivatives), and another independent  $(0, 1 \asymp 0, 1, 2)$  relationship emerges for the spherical (trace) part of these tensors. This means four material coefficients for the deviatoric part and another four for the spherical part. Remarkably, it is just this  $4 + 4 = 8$ -parameter model that is found needed and satisfactory in the Anelastic Strain Recovery method [36–38], an experimental method determining underground 3D in situ stress via measuring the rheological relaxation of borehole rock samples, and which has also been utilized recently for evaluating uniaxial experiments stretching plastic samples ([39], C. Asszonyi, A. Csátár, T. Fülöp: Thermomechanics of elastic, thermal expansion and plastic processes—theory and experiment, in preparation). In uniaxial loading processes, the deviatoric and spherical parts get intertwined, and the resulting relationship between stress and longitudinal strain turns out to be a  $(0, 1, 2, 3 \asymp 0, 1, 2, 3, 4)$  model, as is shown in the Appendix.

In a  $(0, 1 \asymp 0, 1, 2)$  model, the material coefficients cannot be arbitrary because thermodynamics—more closely, thermodynamical stability and non-negative entropy production—reveals conditions on them. A non-trivial finding below is that, even within the remaining parameter region, only half of it can be represented

via rheological networks made of springs and dashpots. For the other half, an additional element, the (0  $\asymp$  2) inertial element—introduced by Verhás [29,40]—is also necessary.

There is an additional advantage of the elimination procedure: it makes it manifest that the problem of boundary conditions does not emerge here. This is actually due to the homogeneous (i.e. free-of-gradients) constitutive equations, and thus, uniqueness of a solution is ensured by the same boundary conditions that determine a unique solution for the corresponding elasticity problem, only the naturally expected further initial conditions have to be added [41]. In addition, in the light of a unified treatment with the Maugin-type internal degrees of freedom and considering also a weakly nonlocal extension, the question of boundary conditions can be handled either by variational or direct thermodynamic prescriptions [27,28].

Our treatment here is different from the ones given by Kluitenberg, by EIT and by Verhás, and also from the approach of classical irreversible thermodynamics, in an important respect. When introducing the classic constitutive solution of the entropy inequality, assuming linear relationships between the thermodynamical fluxes and forces, we do not impose Onsager–Casimir symmetry relations. The principal reasons of this choice are Occam’s razor: without microscopic interpretation of the internal variables, the conditions of Onsager need not apply [42,43]. This way the rightful criticism of Truesdell [44] does not hold for our treatment. This generality has already proved to be crucial for the idea of dual internal variables [27], which turned out to be a powerful unification method for waves in solids [45–49] and also for deriving generalized mechanics [28,50].

In what follows, we develop the thermodynamical theory of rheology of solids in the small strain approximation. First, we present the essential steps in one spatial dimension, starting from the mechanical properties and introducing the thermodynamical requirements. The essential elements of the theory, as well as the most important consequences, are manifest in this presentation. Then we develop the three-dimensional complete version. Finally, we discuss our approach and compare it to the original assumptions and consequences by Kluitenberg and by EIT. Two Appendices are devoted to two special technical aspects: the combined behaviour of the deviatoric and volumetric components in uniaxial loadings, and a variant of the introduction of the internal state variable.

## 2 The procedure in one spatial dimension

### 2.1 The initial system

According to the internal state variable methodology, first we have an initial thermodynamical system, then we assume an additional internal variable, shift the entropy by a concave expression of it, and ensure the positive definiteness of entropy production via Onsagerian equations. For our present purposes, the initial system is a linearly elastic solid. To place the focus on the essential aspects, let us start with the simple setting of the one space dimensional case.

Namely, we have a solid, with elastic strain (or deformedness, see [51]) variable  $\varepsilon$ , which is a scalar in one dimension, and with elastic stress

$$\sigma(\varepsilon) = E\varepsilon, \quad (1)$$

where, again for simplicity, the Young modulus  $E$  is treated as constant. Assuming small deformations only, which is actually a fairly good approximation for many situations concerning solids, the velocity gradient is approximately equal to  $\dot{\varepsilon}$ , the time derivative of strain and the mechanical power exerted by  $\sigma$  is simplified to

$$\sigma\dot{\varepsilon} = E\varepsilon\dot{\varepsilon} = \frac{d}{dt}\left(\frac{E}{2}\varepsilon^2\right) = \rho\frac{d}{dt}\left(\frac{E}{2\rho}\varepsilon^2\right) = \rho\dot{e}_{el} \quad (2)$$

with the specific elastic energy

$$e_{el}(\varepsilon) = \frac{E}{2\rho}\varepsilon^2 \quad (3)$$

and the mass density  $\rho$ , which is constant within the range of small deformations. Staying within the small-deformation regime, we also do not have to distinguish among partial time derivative, comoving/substantial time derivative and the various objective time derivatives (see [52] for the finite deformation differences among such derivatives).

As the function of temperature  $T$  and strain  $\varepsilon$ , the specific internal energy  $e$  of the initial system is considered as

$$e(T, \varepsilon) = e_{\text{th}}(T) + e_{\text{el}}(\varepsilon), \quad (4)$$

$e_{\text{th}}(T)$  being related to the constant or nonconstant specific heat [ $e_{\text{th}}(T) = cT$  in the constant case]. The separated variables in (4) indicate that thermal expansion is now also disregarded.

Again if written as a function of temperature, the thermodynamically corresponding specific entropy is  $s = s(T)$ , satisfying the thermodynamical consistency property

$$\frac{ds}{dT} = \frac{1}{T} \frac{de_{\text{th}}}{dT}, \quad (5)$$

which follows—via (4), (3) and (1)—from the Gibbs relation  $\rho de = \rho T ds + \sigma d\varepsilon$ , rearrangeable as

$$\rho ds = \frac{\rho}{T} de - \frac{\sigma}{T} d\varepsilon. \quad (6)$$

For example, in the constant specific heat case,

$$s(T) = c \ln \frac{T}{T_0} + s_0 \quad (7)$$

is found, with auxiliary constants  $T_0, s_0$ .

Taking (2) for the expression of power, the balance of internal energy—i.e. the first law of thermodynamics—is now

$$\rho \dot{e} = -j_e' + \sigma \dot{\varepsilon} \quad (8)$$

where  $j_e$  is the heat current and prime denotes the spatial derivative.

For the entropy current  $j_s$ , we take the standard choice  $j_s = j_e/T$  [22]. Then, utilizing (5) in the balance for entropy,

$$\rho \dot{s} = -j_s' + \pi_s, \quad (9)$$

gives for the entropy production  $\pi_s$ , via (6) and (8),

$$\pi_s = \rho \dot{s} + j_s' = \frac{\rho}{T} \dot{e} - \frac{\sigma}{T} \dot{\varepsilon} + \left(\frac{j_e}{T}\right)' = -\frac{j_e'}{T} + \left(\frac{j_e}{T}\right)' = j_e \left(\frac{1}{T}\right)', \quad (10)$$

which we can ensure to be positive definite by choosing Fourier heat conduction,

$$j_e = \lambda \left(\frac{1}{T}\right)', \quad (11)$$

with a positive heat conduction coefficient  $\lambda$ .

We can also work in the canonical thermodynamical variables  $e, \varepsilon$ . This is achieved by expressing  $T$  from (4) as a function of  $e$  and  $\varepsilon$  and substituting it into the entropy, obtaining  $s(e, \varepsilon)$ . For example, in the constant specific heat case, we have

$$s(e, \varepsilon) = c \ln \frac{e - (E/2)\varepsilon^2}{cT_0} + s_0. \quad (12)$$

In the canonical variables, temperature and stress are accessed [cf. (6)] as

$$\frac{1}{T}(e, \varepsilon) = \left. \frac{\partial s}{\partial e} \right|_{\varepsilon}, \quad \frac{\sigma}{\rho T}(e, \varepsilon) = - \left. \frac{\partial s}{\partial \varepsilon} \right|_e. \quad (13)$$

Entropy can be shown to be concave in the canonical variables under the conditions  $T > 0$ ,  $E > 0$  and  $de_{\text{th}}/dT > 0$ .

We note that the reason we started using the variables  $T, \varepsilon$  instead of the canonical variables  $e, \varepsilon$  is that this enabled us to express the neglect of thermal expansion easily [cf. (4)].

The first stage, the characterization of the initial system has thus been completed.

## 2.2 Introducing the internal variable

Now comes the second step: in addition to the variables  $e, \varepsilon$ , let us assume the existence of an additional variable  $\xi$ , and shift entropy by a  $\xi$  dependent term. This term should be concave and vanishing for  $\xi = 0$  (in equilibrium). Assuming that the second derivative of entropy with respect to  $\xi$  is nonzero, by the Morse lemma, we can simply take the additional term in the form  $-\frac{1}{2}\xi^2$ :

$$\tilde{s}(e, \varepsilon, \xi) = s(e, \varepsilon) - \frac{1}{2}\xi^2; \quad (14)$$

namely, we can *choose*  $\xi$  to be the variable in which the additional term is of this form. Note that this choice can be made only if we have no direct physical knowledge about the origin of the assumed internal variable and have thus a freedom in choosing it. If we had some explicit information about  $\xi$ —for example, a microscopic interpretation—then we should not have enforced this specific form for the additional term but a more general concave function must have been allowed, possibly dependent on  $e$  and  $\varepsilon$  as well [29,30]. In our present case, we do not possess such background knowledge.

In the light of (6), we find the Gibbs relation for the extended entropy to be

$$\rho d\tilde{s} = \frac{\rho}{T} de - \frac{\sigma}{T} d\varepsilon - \rho \xi d\xi. \quad (15)$$

Most significantly, rheological effects manifest themselves in the mechanical behaviour, so, in parallel, we allow an additional source of mechanical stress also:

$$\tilde{\sigma} = \sigma + \hat{\sigma}, \quad (16)$$

$\hat{\sigma}$  being the stress contribution of rheological (nonequilibrium) origin, and  $\tilde{\sigma}$  denoting the total stress. Consequently, in the mechanical power, and in the balance for internal energy, an additional term  $\hat{\sigma}\dot{\varepsilon}$  appears:

$$\rho\dot{e} + j'_e = \tilde{\sigma}\dot{\varepsilon} = \sigma\dot{\varepsilon} + \hat{\sigma}\dot{\varepsilon}. \quad (17)$$

Utilizing this and (15), the entropy production can be calculated, assuming  $j_{\tilde{s}} = j_e/T$ :

$$\pi_{\tilde{s}} = \rho\dot{\tilde{s}} + j'_{\tilde{s}} = \frac{\rho}{T}\dot{e} - \frac{\sigma}{T}\dot{\varepsilon} - \rho\xi\dot{\xi} + \left(\frac{j_e}{T}\right)' \quad (18)$$

$$= -\frac{j'_e}{T} + \frac{\hat{\sigma}}{T}\dot{\varepsilon} - \rho\xi\dot{\xi} + \left(\frac{j_e}{T}\right)' = j_e \cdot \left(\frac{1}{T}\right)' + \frac{\hat{\sigma}}{T}\dot{\varepsilon} - \rho\xi\dot{\xi}. \quad (19)$$

In the rhs, for the first term (heat conduction), let us keep the previous Fourier choice  $j_e = \lambda\left(\frac{1}{T}\right)'$ ; actually, in the three-dimensional treatment, we will see that in isotropic materials, heat conduction cannot couple to the rheological side because the former is vectorial while the latter is a sum of a tensorial (deviatoric) and a scalar (spherical) contribution. Concerning the remaining part, let us ensure its positive definiteness, rewritten as

$$\hat{\sigma}\dot{\varepsilon} - \rho T \xi \dot{\xi} \geq 0, \quad (20)$$

via Onsagerian equations

$$\hat{\sigma} = l_{11}\dot{\varepsilon} + l_{12}(-\rho T \xi), \quad (21)$$

$$\dot{\xi} = l_{21}\dot{\varepsilon} + l_{22}(-\rho T \xi) \quad (22)$$

with appropriate conditions on the coefficients  $l_{ij}$ . Remarkably, these coefficients may depend on the state variables and also on the thermodynamic forces. Therefore, the above constitutive equations may be quasilinear and even nonlinear according to the classification in [53]. The conditions on the  $l_{ij}$ s are defined by the positive definiteness of the quadratic form obtained by substituting (21)–(22) into (20), finding

$$l_{11}\dot{\varepsilon}^2 + (l_{12} + l_{21})\dot{\varepsilon}(-\rho T \xi) + l_{22}(-\rho T \xi)^2 = (\dot{\varepsilon} - \rho T \xi) \begin{pmatrix} l_{11} & l_{12}^S \\ l_{12}^S & l_{22} \end{pmatrix} \begin{pmatrix} \dot{\varepsilon} \\ -\rho T \xi \end{pmatrix} \geq 0 \quad (23)$$

with  $l_{12}^S = \frac{1}{2}(l_{12} + l_{21})$ , the offdiagonal element of the symmetric part of the matrix  $l$ . Hence, the symmetric part  $l^S$  of the matrix  $l$  is required to be positive definite. This necessitates, due to Sylvester’s criterion,

$$l_{11} \geq 0, \quad l_{22} \geq 0, \quad \det l^S \geq 0. \tag{24}$$

(These three conditions are not independent: in fact, either of the first two is implied by the other two.)

Note that the antisymmetric part,  $l^A$ , of the coefficient matrix  $l$  does not contribute to the entropy production; it is only the symmetric part that creates irreversibility.

### 2.3 Eliminating the internal variable

As mentioned above, the coefficients  $l_{ij}$  need not be constants, and their dependence in the present case could be on temperature. Now let us assume that  $l_{11}$ ,  $\rho T l_{12}$ ,  $l_{21}$  and  $\rho T l_{22}$  are constant, at least along a process, at least to a good approximation—which is a very frequent situation. In this case, it is easy to eliminate the internal variable  $\xi$  from (21)–(22) (although the elimination is straightforward at the general level, too). Notably, the elimination of internal variables has already been applied by Meixner, in [54].

We start by rewriting (22) in a form where a differential operator acts on  $\xi$ :

$$\left(\frac{d}{dt} + \rho T l_{22}\right)\xi = l_{21}\dot{\varepsilon}. \tag{25}$$

Then, if we operate  $\frac{d}{dt} + \rho T l_{22}$  on (21), utilizing (25) yields

$$\rho T l_{22}\dot{\hat{\sigma}} + \dot{\hat{\sigma}} = \rho T(\det l)\dot{\varepsilon} + l_{11}\ddot{\varepsilon}. \tag{26}$$

Knowing  $\hat{\sigma} = \tilde{\sigma} - E\varepsilon$ , we can bring this into the final form

$$\tilde{\sigma} + \frac{1}{\rho T l_{22}}\dot{\tilde{\sigma}} = E\varepsilon + \left(\frac{\det l}{l_{22}} + \frac{E}{\rho T l_{22}}\right)\dot{\varepsilon} + \frac{l_{11}}{\rho T l_{22}}\ddot{\varepsilon}. \tag{27}$$

That is, we have obtained a (0, 1  $\times$  0, 1, 2) rheological model:

$$\tilde{\sigma} + \tau\dot{\tilde{\sigma}} = E_0\varepsilon + E_1\dot{\varepsilon} + E_2\ddot{\varepsilon}, \tag{28}$$

where

$$\tau = \frac{1}{\rho T l_{22}} > 0, \quad E_0 = E, \quad E_1 = \frac{\det l}{l_{22}} + \frac{E_0}{\rho T l_{22}} \geq \frac{E_0}{\rho T l_{22}} > 0, \quad E_2 = \frac{l_{11}}{\rho T l_{22}} \geq 0, \tag{29}$$

which are the necessary and sufficient conditions coming from thermodynamics.

A minor inconvenience is that  $\tau = 0$  and  $E_1 = 0$  are excluded so the (0  $\times$  0) model  $\tilde{\sigma} = E_0\varepsilon$  is not included directly but can be covered as a  $l_{22} \rightarrow \infty$  limit. We can easily overcome this nuisance. Let us observe that, assuming  $l_{22} > 0$ , it is possible to rearrange (21)–(22) as

$$\hat{\sigma} = \frac{\det l}{l_{22}}\dot{\varepsilon} + \frac{l_{12}}{l_{22}}\dot{\xi} = m_{11}\dot{\varepsilon} + m_{12}\dot{\xi}, \tag{30}$$

$$-\rho T \xi = -\frac{l_{21}}{l_{22}}\dot{\varepsilon} + \frac{1}{l_{22}}\dot{\xi} = m_{21}\dot{\varepsilon} + m_{22}\dot{\xi}. \tag{31}$$

Now, equations of the form

$$\hat{\sigma} = m_{11}\dot{\varepsilon} + m_{12}\dot{\xi}, \tag{32}$$

$$-\rho T \xi = m_{21}\dot{\varepsilon} + m_{22}\dot{\xi} \tag{33}$$

are just another possible valid Onsagerian way to ensure the positive definiteness of (20). Namely, using (32)–(33), (20) has the form

$$m_{11}\dot{\varepsilon}^2 + (m_{12} + m_{21})\dot{\varepsilon}\dot{\xi} + m_{22}\dot{\xi}^2 = (\dot{\varepsilon} \ \dot{\xi}) \begin{pmatrix} m_{11} & m_{12}^S \\ m_{12}^S & m_{22} \end{pmatrix} \begin{pmatrix} \dot{\varepsilon} \\ \dot{\xi} \end{pmatrix}, \tag{34}$$

again a quadratic form which is positive definite if and only if

$$m_{11} \geq 0, \quad m_{22} \geq 0, \quad \det m^S \geq 0. \quad (35)$$

Note again that only the symmetric part of  $m$  is related to the entropy production.

When we eliminate  $\xi$  from (32)–(33)—now observing the differential operator  $m_{22} \frac{d}{dt} + \rho T$  in the time derivative of (33), and applying this operator on (32)—the result is

$$\tilde{\sigma} + \frac{m_{22}}{\rho T} \dot{\tilde{\sigma}} = E\varepsilon + \left( m_{11} + \frac{m_{22}}{\rho T} E \right) \dot{\varepsilon} + \frac{\det m}{\rho T} \ddot{\varepsilon}. \quad (36)$$

The coefficients are thus

$$\tau = \frac{m_{22}}{\rho T} \geq 0, \quad E_0 = E, \quad E_1 = m_{11} + \frac{m_{22}}{\rho T} E \geq 0, \quad E_2 = \frac{\det m}{\rho T} \geq 0. \quad (37)$$

We again obtain essentially the same family of rheological models, differences being only at the boundary of the thermodynamically allowed parameter region. Indeed, the latter way, the  $(0 \asymp 0)$  Hooke model is also incorporated. Moreover, the  $\tau = 0, E_0 = 0, E_1 = 0, E_2 > 0$  model—in other words, the  $(0 \asymp 2)$  body—(on which see more in Sect. 2.4) is also uncovered as a thermodynamically valid case. In parallel, there is a part of the boundary that is missing here but was allowed in the former formulation:  $l_{22} = 0$  would have allowed  $(1 \asymp 1, 2)$  models, too. However, those relationships between  $\sigma, \varepsilon$  and derivatives do not include the  $(0 \asymp 0)$  Hooke case, do not provide information for static processes of solids and are thus incomplete for solids.

Hereafter, for definiteness, the coefficients  $m_{ij}$  will be used but analogous statements will hold for the coefficients  $l_{ij}$  as well.

## 2.4 Special cases and analysis

The obtained  $(0, 1 \asymp 0, 1, 2)$  rheological model (28) covers a number of well-known classic cases. In fact, it includes the  $(0 \asymp 0)$  Hooke model, the  $(0 \asymp 0, 1)$  Kelvin–Voigt model, the  $(0, 1 \asymp 1)$  Maxwell model, the  $(0, 1 \asymp 0, 1)$  Poynting–Thomson–Zener model and the  $(0, 1 \asymp 1, 2)$  Jeffreys model. In this respect, an important contribution of the present internal state variable approach is that it reveals thermodynamical constraints of the coefficients of these models. Based on (35), (37) and the identity

$$\det m = \det m^S + \left( \frac{m_{12} - m_{21}}{2} \right)^2 = \det m^S + (m_{12}^A)^2, \quad (38)$$

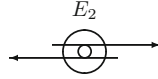
, these constraints read

$$\tau \geq 0, \quad E_0 \geq 0, \quad I_1 := E_1 - \tau E_0 = m_{11} \geq 0, \quad E_2 \geq 0, \quad (39)$$

where the *index of damping*,  $I_1$ , has been introduced. This combination  $I_1$ , together with the similarly defined *index of inertia*,

$$I_2 := E_2 - \tau I_1 = -\frac{m_{12}m_{21}}{\rho T} = \frac{(m_{12}^A)^2 - (m_{12}^S)^2}{\rho T}, \quad (40)$$

are two important characteristics of the  $(0, 1 \asymp 0, 1, 2)$  models.  $I_1 \geq 0$  means a nontrivial combined condition for  $\tau, E_0$  and  $E_1$ , which, for example, rules out the existence of  $(0, 1 \asymp 0)$  and  $(0, 1 \asymp 0, 2)$  models, on thermodynamical ground. In parallel,  $I_2$  is allowed to be positive as well as negative but these two cases indicate remarkably different situations. When  $I_2 < 0$ , in other words, when the symmetric part of the coefficient matrix  $m$  dominates over the antisymmetric part, then if stress is prescribed as a function of time then the characteristic equation of the emerging second-order linear inhomogeneous differential equation  $[\varepsilon_{\text{homog.}}(t) \sim e^{\lambda t} \Rightarrow E_0 + E_1\lambda + E_2\lambda^2 = 0]$  has two negative real roots, meaning that the solution is characterized by two decreasing exponential functions of time. On the other side, when  $I_2 > 0$ , i.e. when the antisymmetric part dominates, the two roots are not real. The real parts are negative, ensuring damping, but the imaginary parts describe oscillations in the solution. This opens the possibility for bringing the material into resonance via a periodic excitation with a frequency near to the self-frequency of the rheological material.



**Fig. 1** The inertial element, corresponding to the elementary  $(0 \asymp 2)$  model,  $\tilde{\sigma} = E_2 \ddot{\epsilon}$

It is important to note that this resonance is completely different from elastic resonance, which occurs when the geometric sizes of a body, the velocity of elastic waves within the body and the frequency of the periodic external force at the boundary are in tune. The rheological resonance is a completely local phenomenon, independent from the geometrical properties of the body, and is a result of the relationship among  $E_0$  and the two rheological coefficients  $E_1$  and  $E_2$ . The inertia-like coefficient refers not to the usual mechanical inertia but to a rheology-related different type of inertia.

That the *overdamped* (or *underinertial*), i.e.  $I_2 < 0$ , models differ remarkably from the *underdamped/overinertial* ( $I_2 > 0$ ) cases can be demonstrated in another way as well. Namely, one can show [55]—via steps analogous to those in Appendix A—that, when two rheological networks are arranged in parallel, the sum of their index of damping equals the index of damping of the resulting system, and the same additive property holds for the index of inertia as long as the resulting model is not beyond  $(0, 1 \asymp 0, 1, 2)$ . Furthermore, when two rheological networks are arranged in series, non-negative indices of damping also prove to lead to non-negative index of damping (and positive ones to positive) and nonpositive indices of inertia lead to nonpositive index of inertia (and negative ones to negative). A reassuring consequence is that networks created by springs and dashpots in a combination of serial and parallel arrangements always have positive index of damping, in conform with the thermodynamical constraint. A nontrivial other consequence, however, is that springs and dashpots—possessing zero index of inertia—can never be combined, in any serial-plus-parallel way, to a rheological model with positive index of inertia. In particular, the generalized Kelvin–Voigt and generalized Maxwell–Wiechert models—which are Kelvin–Voigt or Maxwell models in serial, resp. parallel, arrangement—are not able to reproduce these thermodynamically completely legit models.

This necessitates the introduction of a new rheological element, corresponding to the  $(0 \asymp 2)$  model. The need for this new element was indicated first—to our knowledge—by Verhás [29,40]. Hereafter, we call this element the *inertial* element, and its depicting will follow Verhás’ notation (see Fig. 1).

### 3 The procedure in three spatial dimensions

Now we treat the three-dimensional case, in complete analogy to the one-dimensional situation.

#### 3.1 The initial system

In three dimensions, elastic strain,  $\boldsymbol{\epsilon}$ , and stress,  $\boldsymbol{\sigma}$ , are symmetric tensors. Assuming isotropic Hooke elasticity, stress depends linearly on elastic strain via two scalar elastic coefficients, one connecting the deviatoric tensorial components and the other relating the spherical ones:

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = E^d \boldsymbol{\epsilon}^d + E^s \boldsymbol{\epsilon}^s, \quad \boldsymbol{\epsilon}^s = \frac{1}{3}(\text{tr } \boldsymbol{\epsilon}) \mathbf{1}, \quad \boldsymbol{\epsilon}^d = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^s, \quad E^d = 2G, \quad E^s = 3K. \quad (41)$$

We stay in the small-deformation regime, where (approximately) the density  $\rho$  is constant and the symmetric part of the velocity gradient tensor equals  $\dot{\boldsymbol{\epsilon}}$ . Correspondingly, the mechanical power of  $\boldsymbol{\sigma}$  can be expressed as

$$\text{tr}(\boldsymbol{\sigma} \dot{\boldsymbol{\epsilon}}) = \text{tr}(\boldsymbol{\sigma}^d \dot{\boldsymbol{\epsilon}}^d) + \text{tr}(\boldsymbol{\sigma}^s \dot{\boldsymbol{\epsilon}}^s) = \rho \dot{e}_{\text{el}} \quad (42)$$

with

$$e_{\text{el}}(\boldsymbol{\epsilon}) = \frac{E^d}{2\rho} \text{tr}(\boldsymbol{\epsilon}^d \boldsymbol{\epsilon}^d) + \frac{E^s}{2\rho} \text{tr}(\boldsymbol{\epsilon}^s \boldsymbol{\epsilon}^s). \quad (43)$$

Total specific energy is considered in the form

$$e(T, \boldsymbol{\epsilon}) = e_{\text{th}}(T) + e_{\text{el}}(\boldsymbol{\epsilon}). \quad (44)$$



The Gibbs relation  $\rho de = \rho T ds + \text{tr}(\boldsymbol{\sigma} d\boldsymbol{\epsilon})$ , rearrangeable as

$$\rho ds = \frac{\rho}{T} de - \frac{1}{T} \text{tr}(\boldsymbol{\sigma} d\boldsymbol{\epsilon}), \quad (45)$$

holds with a specific entropy  $s = s(T)$  satisfying

$$\frac{ds}{dT} = \frac{1}{T} \frac{de_{\text{th}}}{dT}. \quad (46)$$

With the expression of power (42), the balance of internal energy (the first law) is

$$\rho \dot{e} = -\nabla \cdot \mathbf{j}_e + \text{tr}(\boldsymbol{\sigma} \dot{\boldsymbol{\epsilon}}). \quad (47)$$

We take the standard choice  $\mathbf{j}_s = \mathbf{j}_e/T$  between the entropy current  $\mathbf{j}_s$  and the heat current  $\mathbf{j}_e$ . Then, in the balance for entropy,

$$\rho \dot{s} = -\nabla \cdot \mathbf{j}_s + \pi_s, \quad (48)$$

the entropy production  $\pi_s$  is calculated, using (45) and (47), as

$$\pi_s = \rho \dot{s} + \nabla \cdot \mathbf{j}_s = \frac{\rho}{T} \dot{e} - \frac{1}{T} \text{tr}(\boldsymbol{\sigma} \dot{\boldsymbol{\epsilon}}) + \nabla \cdot \left( \frac{\mathbf{j}_e}{T} \right) = -\frac{1}{T} \nabla \cdot \mathbf{j}_e + \nabla \cdot \left( \frac{\mathbf{j}_e}{T} \right) = \mathbf{j}_e \cdot \nabla \left( \frac{1}{T} \right), \quad (49)$$

which we set positive definite via choosing Fourier heat conduction,

$$j_e = \lambda \nabla \left( \frac{1}{T} \right), \quad \lambda > 0. \quad (50)$$

The conversion from state variables  $(T, \boldsymbol{\epsilon})$  to the canonical state variables  $(e, \boldsymbol{\epsilon})$  can be achieved by expressing  $T$  from (44), and substituting it into the entropy, obtaining  $s(e, \boldsymbol{\epsilon})$ . Reversely, starting from the canonical variables, temperature is accessed as

$$\frac{1}{T}(e, \boldsymbol{\epsilon}) = \left. \frac{\partial s}{\partial e} \right|_{\boldsymbol{\epsilon}}. \quad (51)$$

Entropy is concave in the canonical variables [under the natural conditions  $T > 0$ ,  $E > 0$  and  $de_{\text{th}}/dT > 0$ ].

Having the initial system in the canonical variables, we are ready for extending it.

### 3.2 Introducing the internal variable

The extended thermodynamical state space is chosen to be spanned by the following variables: internal energy  $e$ , strain  $\boldsymbol{\epsilon}$  and an internal variable  $\boldsymbol{\xi}$ . This latter is taken as a second-order symmetric tensor, based on our purpose to gain an extension of the mechanical aspects (the ‘material law’) of the initial system, to obtain corrections to the relation between stress and strain, which quantities are both symmetric tensors.

We shift entropy by a concave nonequilibrium term depending—quadratically—on  $\boldsymbol{\xi}$  only. According to the Morse lemma, this new entropy term can be chosen as a pure square term, and hence, the extended entropy function is

$$\tilde{s}(e, \boldsymbol{\epsilon}, \boldsymbol{\xi}) = s(e, \boldsymbol{\epsilon}) - \frac{1}{2} \text{tr}(\boldsymbol{\xi}^2). \quad (52)$$

The Gibbs relation for the extended entropy is

$$\rho d\tilde{s} = \frac{\rho}{T} de - \frac{1}{T} \text{tr}(\boldsymbol{\sigma} d\boldsymbol{\epsilon}) - \rho \text{tr}(\boldsymbol{\xi} d\boldsymbol{\xi}). \quad (53)$$

To obtain some effect on the mechanical aspects, stress is also considered extended by a rheological (nonequilibrium) term:

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \hat{\boldsymbol{\sigma}}. \quad (54)$$

Consequently, the mechanical power (42), and correspondingly the energy balance, gets shifted as

$$\rho \dot{e} + \nabla \cdot \mathbf{j}_e = \text{tr}(\tilde{\boldsymbol{\sigma}} \dot{\boldsymbol{\epsilon}}) = \text{tr}(\boldsymbol{\sigma} \dot{\boldsymbol{\epsilon}}) + \text{tr}(\hat{\boldsymbol{\sigma}} \dot{\boldsymbol{\epsilon}}). \quad (55)$$

With the choice  $\mathbf{j}_{\tilde{s}} = \mathbf{j}_e/T$ , and utilizing (53) and (55), the entropy production is found to be

$$\begin{aligned} \pi_{\tilde{s}} &= \rho \dot{\tilde{s}} + \nabla \cdot \mathbf{j}_{\tilde{s}} = \frac{\rho}{T} \dot{e} - \frac{1}{T} \text{tr}(\boldsymbol{\sigma} \dot{\boldsymbol{\epsilon}}) - \rho \text{tr}(\boldsymbol{\xi} \dot{\boldsymbol{\xi}}) + \nabla \cdot \left( \frac{\mathbf{j}_e}{T} \right) \\ &= -\frac{1}{T} \nabla \cdot \mathbf{j}_e + \frac{1}{T} \text{tr}(\hat{\boldsymbol{\sigma}} \dot{\boldsymbol{\epsilon}}) - \rho \text{tr}(\boldsymbol{\xi} \dot{\boldsymbol{\xi}}) + \nabla \cdot \left( \frac{\mathbf{j}_e}{T} \right) \\ &= \mathbf{j}_e \cdot \nabla \left( \frac{1}{T} \right) + \frac{1}{T} \text{tr}(\hat{\boldsymbol{\sigma}}^d \dot{\boldsymbol{\epsilon}}^d) + \frac{1}{T} \text{tr}(\hat{\boldsymbol{\sigma}}^s \dot{\boldsymbol{\epsilon}}^s) - \rho \text{tr}(\boldsymbol{\xi}^d \dot{\boldsymbol{\xi}}^d) - \rho \text{tr}(\boldsymbol{\xi}^s \dot{\boldsymbol{\xi}}^s). \end{aligned} \quad (56)$$

In the rhs, vectors are present in the first term, scalars in the third and fifth one, and symmetric traceless tensors in the second and fourth term. In an isotropic material, these three types of quantities cannot couple to one another according to the Curie principle, the representation theorem of isotropic functions [29]. Therefore, concerning the term containing vectors, we consider Fourier heat conduction,  $\mathbf{j}_e = \lambda \nabla \left( \frac{1}{T} \right)$ . For the remaining two pairs of terms, the most general Onsagerian solution is

$$\hat{\boldsymbol{\sigma}}^d = l_{11}^d \dot{\boldsymbol{\epsilon}}^d + l_{12}^d (-\rho T \boldsymbol{\xi}^d), \quad \hat{\boldsymbol{\sigma}}^s = l_{11}^s \dot{\boldsymbol{\epsilon}}^s + l_{12}^s (-\rho T \boldsymbol{\xi}^s), \quad (57)$$

$$\dot{\boldsymbol{\xi}}^d = l_{21}^d \dot{\boldsymbol{\epsilon}}^d + l_{22}^d (-\rho T \boldsymbol{\xi}^d), \quad \dot{\boldsymbol{\xi}}^s = l_{21}^s \dot{\boldsymbol{\epsilon}}^s + l_{22}^s (-\rho T \boldsymbol{\xi}^s), \quad (58)$$

or the corresponding version with  $m^d$  and  $m^s$ , generalizing (30)–(31).

Hence, we can see that what we had in one dimension just gets doubled in three dimensions, the two components being independent. The two terms with traceless symmetric tensors have to be positive definite themselves, and the two terms with scalars have to be positive definite independently. The corresponding conditions on the coefficients are the same as for the one-dimensional case.

Eliminating the internal variable in the constant temperature case also leads to two independent  $(0, 1 \asymp 0, 1, 2)$  models,

$$\boldsymbol{\sigma}^d + \tau^d \dot{\boldsymbol{\sigma}}^d = E_0^d \boldsymbol{\epsilon}^d + E_1^d \dot{\boldsymbol{\epsilon}}^d + E_2^d \ddot{\boldsymbol{\epsilon}}^d, \quad \boldsymbol{\sigma}^s + \tau^s \dot{\boldsymbol{\sigma}}^s = E_0^s \boldsymbol{\epsilon}^s + E_1^s \dot{\boldsymbol{\epsilon}}^s + E_2^s \ddot{\boldsymbol{\epsilon}}^s, \quad (59)$$

with thermodynamics-originated inequalities for the altogether eight coefficients.

Remarkably, in a uniaxial process, both the deviatoric and the spherical rheologies are active, independently of each other, and cause a rather complicated resultant uniaxial rheology—see Appendix A for the details.

We close this section with two notes. The first is that the presented thermodynamical approach involves a potential (entropy, and free energy as a consequence); therefore, our treatment is hyperelastic—from the point of view of the complete state space. On the other side, the stress–strain relation obtained by reduction/elimination cannot be obtained from a potential.

The second remark is that more general methods of the exploitation of the entropy principle [56] result, in this classical irreversible thermodynamical case, in the same structure as the simple approach of divergence separation applied here.

#### 4 Comparison of the approaches

Kluitenberg [12] has also obtained a result similar to ours, a pair of  $(0, 1 \asymp 0, 1, 2)$  models, with assumptions different from ours, and one can also derive the  $(0, 1 \asymp 0, 1)$  Poynting–Thomson–Zener body in EIT [31]. These approaches and their different assumptions are discussed in the following section.

#### 4.1 Kluitenberg theory

We have seen that the presented thermodynamical framework distinguishes a particular combination of inertia, relaxation and creep for dissipation, the  $(0, 1 \asymp 0, 1, 2)$  model, as a basic rheological body. Nonequilibrium thermodynamics with internal variables was first systematically applied to linear viscoelasticity by Kluitenberg [9–15], and he obtained also the  $(0, 1 \asymp 0, 1, 2)$  body, as a fundamental building block of thermodynamical modelling.

The Kluitenberg theory is different from our approach. The differences are the following:

1. In the Kluitenberg theory, the internal variable has a fixed physical meaning: it is interpreted as a direct—called anelastic—contribution to the strain: [9]

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\text{el}} + \boldsymbol{\varepsilon}_{\text{anel}}. \quad (60)$$

According to our above approach, this interpretation is not necessary, the coupling of the dissipative terms of the entropy production through linear relations with an arbitrary tensorial internal variable influences the strain.

2. Kluitenberg assumes a relationship between anelastic and elastic stress, as a consequence of the strain interpretation. In particular, for him, the elastic stress is equal to the anelastic one {[11], (4.4)}. He assumes, in particular, that

$$\frac{\partial \hat{s}}{\partial \boldsymbol{\varepsilon}_{\text{anel}}}(\boldsymbol{\varepsilon}_{\text{el}}, \boldsymbol{\varepsilon}_{\text{anel}}) = \frac{\partial s}{\partial \boldsymbol{\varepsilon}_{\text{anel}}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{\text{anel}}) - \frac{\partial s}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{\text{anel}}) = 0. \quad (61)$$

Here,  $\hat{s}(\boldsymbol{\varepsilon}_{\text{el}}, \boldsymbol{\varepsilon}_{\text{anel}}) = \hat{s}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\text{anel}}, \boldsymbol{\varepsilon}_{\text{anel}}) = s(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{\text{anel}})$ .

As we have found above, these assumptions are not necessary for thermodynamical consistency and lead to a different analysis of the rheological coefficients. For example, in the Kluitenberg representation, no restriction is revealed on the sign of  $E_1$ , the coefficient of the strain rate. Moreover, when elastic and anelastic strains of Kluitenberg are quadratic in the entropy, one cannot derive the  $(0, 1 \asymp 0, 1, 2)$  body but obtains the  $(0, 1 \asymp 1, 2)$  Jeffreys model, instead {[11], (4.25)–(4.26)}. However, the Jeffreys body is not a rheological solid (Jeffreys model expresses a viscous fluid for slow processes), and hence, it is unsatisfactory in case of viscoelastic solids. The real basic building block is the  $(0, 1 \asymp 0, 1, 2)$  body.

Verhás introduced the procedure analogous to ours here, with a single internal variable, for fluids [29,40]. In addition, it was him who emphasized the relevance of the inertial element on thermodynamical grounds. To reflect these precursors, hereafter we will call the  $(0, 1 \asymp 0, 1, 2)$  rheological model the *Kluitenberg–Verhás body*.

#### 4.2 Extended Thermodynamics

In EIT, the dissipative fluxes are considered as state variables [57]. Therefore, the viscous or anelastic strain  $\hat{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}$  is a thermodynamical state variable, instead of being a constitutive quantity (see, e.g., [31]). In our approach, if  $l_{11} = 0$ , then the internal variable  $\boldsymbol{\xi}$  is proportional to the dissipative stress  $\hat{\boldsymbol{\sigma}}$  [see (29)], thus a rescaling of  $\boldsymbol{\xi}$  results in the representation and rheological equations of Extended Thermodynamics. The rheological body obtained this way is the  $(0, 1 \asymp 0, 1)$  Poynting–Thomson–Zener body, a subfamily of the full  $(0, 1 \asymp 0, 1, 2)$  range of possibilities. It is remarkable that the material parameter set to zero here,  $l_{11}$ , is the viscosity of the Kelvin body in the sense of (21), which is normally the main contribution to dissipation. In this sense, the situation is similar to heat conduction, where the representation of the Fourier coefficient is different in case of internal variables and in an Extended Thermodynamical treatment [35]. The extension of the theory with higher order fluxes is different from the extension of an internal variable theory with further internal variables.

### 5 Discussion

In this paper, we have analysed rheological properties of solids as a thermodynamical theory with internal variables. We have obtained a simple nonlinear viscoelastic theory. We have identified the basic thermorheological

body and investigated its properties. The single internal thermodynamical state variable representation distinguishes the standard Poynting–Thomson–Zener body supplemented by an inertial element as the fundamental building block of linear viscoelasticity. We have suggested to name this material model the Kluitenberg–Verhás body.

On one hand, the simplest assumptions were introduced, when the internal variable is a single tensorial dynamical degree of freedom, representing the deviation from the equilibrium state and is subject to thermodynamical restrictions. On the other hand, we have allowed the most general linear Onsagerian equations. The advantage of this is the obtainable *universality*, in the sense that, as long as the tensorial representation and the second law are respected, any particular microscopic and mesoscopic mechanism must lead to the same rheological model family in the linear regime.

After eliminating the internal variable, for isothermal processes, we could formulate and analyse the consequences of the presence of rheology in measurable quantities, stress and strain.

In our approach, the internal variable is not necessarily anelastic strain like in the classical Kluitenberg theory and is not necessarily the dissipative stress like in Extended Thermodynamics. The latter choice is found to be a special case, a certain subfamily within our obtained family of models. The representation of the internal variable as additive deviation from elastic strain would also require a deeper analysis from the point of view of continuum kinematic quantities, including finite strain as well as the requirements of material frame indifference. In this respect, it is remarkable that our recent frame free and objective approach [51], which improves the classic framework of Noll [58,59], leads to an additive decomposition of the deformation rate into an elastic and a plastic (permanent) part. Therefore, the basic question is the clear distinction of the elastic, rheological (recoverable) and the plastic (permanent) parts of the deformation in experiments.

Another characteristic property of our treatment was that we have not assumed symmetry or antisymmetry of the phenomenological coefficient matrix. There is no reason to assume Onsager–Casimir reciprocity without any microscopic interpretation. Moreover, the detailed analysis of Sect. (2.4) has shown that the different possible circuit-like spring–dashpot-based representations are related to different either symmetric or antisymmetric part dominated thermodynamical models. Thermodynamics requires non-negative dashpot and spring coefficients of both representations, and the inertial element is also needed for the underdamped case.

Finally, we emphasize that all the found restrictive properties of our model and the differences from the original Kluitenberg and Extended Thermodynamical approaches can be tested by experiments. The comparison of the coefficient restrictions of the Kluitenberg model with wave propagation experiments has been started by Ciancio et al. in [60–62] with satisfactory results. Direct measurements of creep and relaxation require a careful choice of isotropic materials with a single internal variable and also the separation of spherical and deviatoric parts of the changes. Mixed loading conditions and effective models can be complicated, especially if not only deviatoric but volumetric anelasticity also plays a role (see Appendix A). In this respect, the borehole rock sample experiments of Lin et al. [36–38] support the relevance of both deviatoric and volumetric Kluitenberg–Verhás bodies.

## Appendix A: Uniaxial rheology derived from the deviatoric and spherical components

During uniaxial processes of solids, the stress and strain tensors are, in a suitable Cartesian coordinate system, characterized by matrices

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^{\parallel} & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}^s = \frac{1}{3} \begin{pmatrix} \sigma^{\parallel} & & \\ & \sigma^{\parallel} & \\ & & \sigma^{\parallel} \end{pmatrix}, \quad \boldsymbol{\sigma}^d = \frac{1}{3} \begin{pmatrix} 2\sigma^{\parallel} & & \\ & -\sigma^{\parallel} & \\ & & -\sigma^{\parallel} \end{pmatrix}, \quad (62)$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon^{\parallel} & & \\ & \varepsilon^{\perp} & \\ & & \varepsilon^{\perp} \end{pmatrix}, \quad \boldsymbol{\varepsilon}^s = \frac{1}{3} \begin{pmatrix} \varepsilon^{\parallel} + 2\varepsilon^{\perp} & & \\ & \varepsilon^{\parallel} + 2\varepsilon^{\perp} & \\ & & \varepsilon^{\parallel} + 2\varepsilon^{\perp} \end{pmatrix}, \quad \boldsymbol{\varepsilon}^d = \frac{1}{3} \begin{pmatrix} 2(\varepsilon^{\parallel} - \varepsilon^{\perp}) & & \\ & -(\varepsilon^{\parallel} - \varepsilon^{\perp}) & \\ & & -(\varepsilon^{\parallel} - \varepsilon^{\perp}) \end{pmatrix}, \quad (63)$$

all omitted matrix elements being zero.

Let our solid obey the linear rheological laws

$$\mathcal{S}^d \boldsymbol{\sigma}^d = \mathcal{E}^d \boldsymbol{\varepsilon}^d, \quad (64)$$

$$\mathcal{S}^s \boldsymbol{\sigma}^s = \mathcal{E}^s \boldsymbol{\varepsilon}^s, \quad (65)$$

where the differential operators  $\mathcal{S}^d$ ,  $\mathcal{S}^s$ ,  $\mathcal{E}^d$ ,  $\mathcal{E}^s$  are polynomials of the time derivative operator  $d/dt$ , with constant coefficients. For example, for (59),

$$\mathcal{S}^d = 1 + \tau^d \frac{d}{dt}, \quad \mathcal{E}^d = E_0^d + E_1^d \frac{d}{dt} + E_2^d \frac{d^2}{dt^2}, \quad (66)$$

$$\mathcal{S}^s = 1 + \tau^s \frac{d}{dt}, \quad \mathcal{E}^s = E_0^s + E_1^s \frac{d}{dt} + E_2^s \frac{d^2}{dt^2}. \quad (67)$$

Then, for uniaxial processes, (64) simplifies to the scalar equation

$$\mathcal{S}^d \sigma^{\parallel} = \mathcal{E}^d (\varepsilon^{\parallel} - \varepsilon^{\perp}), \quad (68)$$

and (65) gives

$$\mathcal{S}^s \sigma^{\parallel} = \mathcal{E}^s (\varepsilon^{\parallel} + 2\varepsilon^{\perp}). \quad (69)$$

Our aim is to eliminate  $\varepsilon^{\perp}$  from this pair of equations. Let the operator  $2\mathcal{E}^s$  act on (68), let  $\mathcal{E}^d$  act on (69), and let us consider the sum of the two resulting equations. Observing that

$$\mathcal{E}^s \mathcal{E}^d = \mathcal{E}^d \mathcal{E}^s \quad (70)$$

and, in general, that any two polynomials of  $\frac{d}{dt}$  commute, the obtained sum can be written as

$$(\mathcal{S}^s \mathcal{E}^d + 2\mathcal{S}^d \mathcal{E}^s) \sigma^{\parallel} = 3\mathcal{E}^d \mathcal{E}^s \varepsilon^{\parallel}. \quad (71)$$

In the example of (66)–(67), (71) reads, after normalizing the coefficient of  $\sigma^{\parallel}$  to 1,

$$\begin{aligned} \sigma^{\parallel} + \frac{E_1^d + 2E_1^s + \tau^s E_0^d + 2\tau^d E_0^s}{E_0^d + 2E_0^s} \dot{\sigma}^{\parallel} + \frac{E_2^d + 2E_2^s + \tau^s E_1^d + 2\tau^d E_1^s}{E_0^d + 2E_0^s} \ddot{\sigma}^{\parallel} + \frac{\tau^s E_2^d + 2\tau^d E_2^s}{E_0^d + 2E_0^s} \ddot{\sigma}^{\parallel} \\ = \frac{3E_0^s E_0^d}{E_0^d + 2E_0^s} \varepsilon^{\parallel} + \frac{3(E_0^s E_1^d + E_0^d E_1^s)}{E_0^d + 2E_0^s} \dot{\varepsilon}^{\parallel} + \frac{3(E_0^d E_2^s + E_1^s E_1^d + E_0^s E_2^d)}{E_0^d + 2E_0^s} \ddot{\varepsilon}^{\parallel} \\ + \frac{3(E_1^d E_2^s + E_1^s E_2^d)}{E_0^d + 2E_0^s} \ddot{\varepsilon}^{\parallel} + \frac{3E_2^s E_2^d}{E_0^d + 2E_0^s} \ddot{\varepsilon}^{\parallel}. \end{aligned} \quad (72)$$

Apparently, the emerging uniaxial rheology is a  $(0, 1, 2, 3 \asymp 0, 1, 2, 3, 4)$  model.

As a simple but remarkable special case, if we have a  $(0 \asymp 0, 1)$  Kelvin–Voigt model deviatorically and a  $(0 \asymp 0)$  Hooke one spherically, the uniaxial rheology proves to be a Poynting–Thomson–Zener model, exhibiting creep and relaxation. Consequently, *deviatoric creep is enough to produce uniaxial relaxation*.

Naturally, it is also possible to eliminate  $\varepsilon^{\parallel}$ , in an analogous way, to obtain a formula containing  $\sigma^{\parallel}$  and its derivatives and  $\varepsilon^{\perp}$  and its derivatives. Or,  $\sigma^{\parallel}$  can be eliminated in favour of  $\varepsilon^{\parallel}$ ,  $\varepsilon^{\perp}$  and derivatives. The former formula can be useful in a stress governed process to show how transversal strain behaves, and the latter to see how much more intricate the relationship between  $\varepsilon^{\perp}$  and  $\varepsilon^{\parallel}$  is than what the Hookean expectation of a constant Poisson ratio would suggest.

The technique used here to derive the uniaxial resultant rheology is similar to how serial and parallel arrangements of rheological models can be calculated [55]. In fact, this is more than a simple similarity: the above steps are *exactly* those how one can determine the rheology of the arrangement

$$(D \sim D \sim S) \parallel (D \sim D \sim S) \parallel (D \sim D \sim S), \quad (73)$$

where  $D$  stands for the deviatoric model (62),  $S$  for the spherical one (65), and  $\sim$  denotes the serial and  $\parallel$  the parallel connection. The uniaxial rheology is thus a certain serial-plus-parallel combination of the deviatoric and the spherical rheology.

## Appendix B: Shifting internal energy

Here, another approach to introduce an internal variable for rheology is shown, which approach leads to the same  $(0, 1 \asymp 0, 1, 2)$  model, after elimination, for constant temperature processes. For simplicity, we exploit only the one-dimensional version.

In this variant, we shift internal energy, rather than entropy, with a quadratic term (see also [11, 12, 15]) of an internal variable  $\zeta$ , which term preserves the convexity property:

$$\tilde{e} = e + \frac{1}{2}\zeta^2, \quad (74)$$

which, in terms of the canonical variables, means the shifting of the internal energy variable in any constitutive function like entropy:

$$\tilde{s}(\tilde{e}, \varepsilon, \zeta) = s\left(\tilde{e} - \frac{1}{2}\zeta^2, \varepsilon\right). \quad (75)$$

As before [see Eq. (54)], the introduction of the internal variable is assumed to be accompanied by a new stress term. Taking the time derivative of this  $\tilde{s}$ , the entropy production gets shifted now by

$$\frac{1}{T}(\hat{\sigma}\dot{\varepsilon} - \rho\zeta\dot{\zeta}). \quad (76)$$

Consequently, apart from a factor  $T$ , the Onsagerian solution happens similarly as done previously. The constant- $T$  elimination of  $\zeta$  also leads to the linear  $(0, 1 \asymp 0, 1, 2)$  rheology, with the same conditions on the coefficients.

What is remarkably different in the two approaches is the balance of internal energy, since now there is an additional, potential-like, internal energy term  $\frac{1}{2}\zeta^2$  so a part of the mechanical power (and of the incoming heat flux) now changes this extra term and only the remaining part changes the specific heat related original internal energy term  $e_{\text{th}}$  and the elastic energy term  $e_{\text{el}}$ . Correspondingly, temperature changes differently than in the entropy shifting approach. This means an experimental possibility to distinguish between the two scenarios.

At last, it is possible to combine the two methodologies: we can permit a quadratic shift of both entropy and internal energy. Then an extra coefficient must be allowed in at least one of the two quadratic expressions: the general case can be written as

$$\tilde{e} = e + \frac{a}{2}\xi^2, \quad \tilde{s}(\tilde{e}, \varepsilon, \xi) = s\left(\tilde{e} - \frac{a}{2}\xi^2, \varepsilon\right) + \frac{b}{2}\xi^2. \quad (77)$$

The entropy production still gets shifted by a term proportional to  $-\xi\dot{\xi}$ , now with a coefficient that contains both  $a$  and  $b$ . The Onsagerian solution also has the same form as in the two previous cases, and the elimination also goes through analogously. The coefficients  $a, b$  are distinguished by their role in the balance of internal energy, hence, in the rate equation for temperature. This can be the basis for determining them experimentally.

## References

1. Maugin, G.A., Muschik, W.: Thermodynamics with internal variables. Part I. General concepts. *J. Non-Equilib. Thermodyn.* **19**, 217–249 (1994)
2. Maugin, G.A., Muschik, W.: Thermodynamics with internal variables. Part II. Applications. *J. Non-Equilib. Thermodyn.* **19**, 250–289 (1994)
3. Coleman, B.D., Gurtin, M.E.: Thermodynamics with internal state variables. *J. Chem. Phys.* **47**(2), 597–613 (1967)
4. Kestin, J.: Internal variables in the local-equilibrium approximation. *J. Non-Equilib. Thermodyn.* **18**, 360–379 (1993)
5. Muschik, W.: Comment to J. Kestin: internal variables in the local-equilibrium approximation. *J. Non-Equilib. Thermodyn.* **18**, 380–388 (1993)
6. Eckart, C.: The thermodynamics of irreversible processes, I. The simple fluid. *Phys. Rev.* **58**, 267–269 (1940)
7. Eckart, C.: The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity. *Phys. Rev.* **73**(4), 373–382 (1948)
8. Biot, M.A.: Theory of stress–strain relations in anisotropic viscoelasticity and relaxation phenomena. *J. Appl. Phys.* **25**(11), 1385–1391 (1954)
9. Kluitenberg, G.A.: Thermodynamical theory of elasticity and plasticity. *Physica* **28**, 217–232 (1962)
10. Kluitenberg, G.A.: A note on the thermodynamics of Maxwell bodies, Kelvin bodies (Voigt bodies), and fluids. *Physica* **28**, 561–568 (1962)

11. Kluitenberg, G.A.: On rheology and thermodynamics of irreversible processes. *Physica* **28**, 1173–1183 (1962)
12. Kluitenberg, G.A.: On the thermodynamics of viscosity and plasticity. *Physica* **29**, 633–652 (1963)
13. Kluitenberg, G.A.: A unified thermodynamic theory for large deformations in elastic media and in Kelvin (Voigt) media, and for viscous fluid flow. *Physica* **30**, 1945–1972 (1964)
14. Kluitenberg, G.A.: A thermodynamic derivation of the stress–strain relations for Burgers media and related substances. *Physica* **38**, 513–548 (1968)
15. Kluitenberg, G.A., Ciancio, V.: On linear dynamical equations of state for isotropic media. *Phys. A* **93**, 273–286 (1978)
16. Ciancio, V., Kluitenberg, G.A.: On linear dynamical equations of state for isotropic media II: some cases of special interest. *Phys. A* **99**, 592 (1979)
17. Kuiken, G.D.C.: *Thermodynamics of Irreversible Processes. Applications to Diffusion and Rheology*. Wiley, Chichester (1994)
18. Horstmeyer, M.F., Bamman, D.J.: Historical review of internal state variable theory for inelasticity. *Int. J. Plast.* **26**, 1310–1334 (2010)
19. Bridgman, P.W.: *The Nature of Thermodynamics*. Harvard University Press, Cambridge (1943)
20. Maugin, G.: *The Thermomechanics of Nonlinear Irreversible Behaviors (An Introduction)*. World Scientific, Singapore (1999)
21. Prigogine, I., Mazur, P.: Sur l’extension de la thermodynamique aux phénomènes irréversibles liés aux degrés de liberté internes. *Physica* **19**(1), 241–254 (1953)
22. Groot, S.R.de , Mazur, P.: *Non-equilibrium Thermodynamics*. North-Holland Publishing Company, Amsterdam (1962)
23. Blenk, S., Ehrentraut, H., Muschik, W.: Statistical foundation of macroscopic balances for liquid crystals in alignment tensor formulation. *Phys. A* **174**, 119–138 (1991)
24. Reguera, D., Rubí, J.M., Vilar, J.M.: The mesoscopic dynamics of thermodynamic systems. *Journal of Physical Chemistry B* **109**, 21502–21515 (2005); [arXiv:cond-mat/0511651](https://arxiv.org/abs/cond-mat/0511651)
25. Papenfuss, C. Dynamics of internal variables from the mesoscopic background. In: Quak, E., Soomere, T. *Applied Wave Mathematics (Selected Topics in Solids, Fluids, and Mathematical Methods)*, Chapter II., pp. 89–125. Springer, Berlin (2009)
26. Maugin, G.A.: Internal variables and dissipative structures. *J. Non-Equilib. Thermodyn.* **15**, 173–192 (1990)
27. Ván, P., Berezovski, A., Engelbrecht, J.: Internal variables and dynamic degrees of freedom. *J. Non-Equilib. Thermodyn.* **33**(3), 235–254 (2008); [arXiv:cond-mat/0612491](https://arxiv.org/abs/cond-mat/0612491)
28. Ván, P., Papenfuss, C., Berezovski, A.: Thermodynamic approach to generalized continua. *Contin. Mech. Thermodyn.* **25**(3), 403–420 (2014). Erratum: 421–422; [arXiv:1304.4977](https://arxiv.org/abs/1304.4977)
29. Verhás, J.: *Thermodynamics and Rheology*. Akadémiai Kiadó and Kluwer Academic Publisher, Budapest (1997)
30. Ciancio, V., Verhás, J.: On the representation of dynamic degrees of freedom. *J. Non-Equilib. Thermodyn.* **18**(1), 39–50 (1993)
31. Dauby, P.C., Lebon, G.: Constitutive equations of rheological materials: towards a thermodynamic unified approach. *Appl. Math. Lett.* **3**(4), 45–48 (1990)
32. Lebon, G., Jou, D., Casas-Vázquez, J.: *Understanding Non-equilibrium Thermodynamics*. Springer, Berlin (2008)
33. Ciancio, V., Verhás, J.: A thermodynamic theory for radiating heat transfer. *J. Non-Equilib. Thermodyn.* **15**(1), 33–43 (1990)
34. Ciancio, V., Verhás, J.: A thermodynamic theory for heat radiation through the atmosphere. *J. Non-Equilib. Thermodyn.* **16**(1), 57–65 (1991)
35. Ván, P., Fülöp, T.: Universality in heat conduction theory: weakly nonlocal thermodynamics. *Annalen der Physik* **524**(8), 470–478 (2012); [arXiv:1108.5589](https://arxiv.org/abs/1108.5589)
36. Matsuki, K., Takeuchi, K.: Three-dimensional in situ stress determination by anelastic strain recovery of a rock core. *Int. J. Rock Mech. Min. Sci. Geomech. Abstr.* **30**, 1019–1022 (1993)
37. Matsuki, K.: Anelastic strain recovery compliance of rocks and its application to in situ stress measurement. *Int. J. Rock Mech. Min. Sci.* **45**, 952–965 (2008)
38. Lin, W., Kuwahara, Y., Satoh, T., Shigematsu, N., Kitagawa, Y., Kiguchi, T., Koizumi, N.: A case study of 3D stress orientation determination in Shikoku Island and Kii Peninsula, Japan. In: Vrkljan, I. (ed.) *Rock Engineering in Difficult Ground Conditions (Soft Rock and Karst)*. Proceedings of Eurock’09, Cavtat, Croatia, 28–29 Oct. 2009, pp. 277–282. Balkema, London (2010)
39. Kocsis, D., Horváth, R.: Experience acquired in tensile tests of plastics. *Sci. Bull. Ser. C: Fasc. Mech. Tribol. Mach. Manuf. Technol.* **27**, 79–82 (2013)
40. Verhás, J.: *Thermodynamics and Rheology*. Műszaki Könyvkiadó, Budapest (1985) (in Hungarian)
41. Fülöp, T.: Linear problems of elasticity and rheology. In: Asszonyi, C. *On the Solution of Problems in Continuum Mechanics*, Vol. 9 of the Book Series *Mérnökgeológia-Kőzetmechanika Kiskönyvtár*, Chapter I, pp. 11–52. Műegyetemi Kiadó, Budapest (in Hungarian) (2009)
42. Onsager, L.: Reciprocal relations of irreversible processes I. *Phys. Rev.* **37**, 405–426 (1931)
43. Onsager, L.: Reciprocal relations of irreversible processes II. *Phys. Rev.* **38**, 2265–2279 (1931)
44. Truesdell, C.: *Rational Thermodynamics*. 2nd enlarged edition. Springer, New York (1984)
45. Berezovski, A., Engelbrecht, J., Berezovski, M.: Waves in microstructured solids: a unified viewpoint of modelling. *Acta Mech.* **220**, 349–363 (2011)
46. Engelbrecht, J.: Nonlinear wave motion and complexity. *Proc. Estonian Acad. Sci.* **59**(2), 66–71 (2010)
47. Engelbrecht, J., Berezovski, A.: Internal structures and internal variables in solids. *J. Mech. Mater. Struct.* **7**(10), 983–996 (2012)
48. Berezovski, A., Engelbrecht, J.: Thermoelastic waves in microstructured solids: dual internal variables approach. *J. Coupled Syst. Multiscale Dyn.* **1**(1), 112–119 (2013)
49. Engelbrecht, J., Berezovski, A.: On modelling of wave propagation in microstructured solids. *Estonian J. Eng.* **19**(3), 171–182 (2013)
50. Berezovski, A., Engelbrecht, J., Maugin, G.A.: Generalized thermomechanics with dual internal variables. *Arch. Appl. Mech.* **81**(2), 229–240 (2011)

51. Fülöp, T., Ván, P.: Kinematic quantities of finite elastic and plastic deformations. *Math. Methods Appl. Sci.* **35**, 1825–1841 (2012); [arXiv:1007.2892](https://arxiv.org/abs/1007.2892)
52. Ván, P.: Objective time derivatives in non-equilibrium thermodynamics. *Proc. Estonian Acad. Sci.* **57**(3), 127–131 (2008). Lecture held at CPEA'07
53. Gyarmati, I.: The wave approach of thermodynamics and some problems of non-linear theories. *J. Non-Equilib. Thermodyn.* **2**, 233–260 (1977)
54. Meixner, J.: Thermodynamische theorie der elastischen relaxation. *Zeitschrift für Naturforschung* **9a**, 654–663 (1954)
55. Fülöp, T.: Rheological circuits. In: Asszonyi, C. (ed.) *Material Properties of Isotropic Continua*, Vol. 6 of the Book Series *Mérnökgeológia–Kőzetmechanika Kiskönyvtár*, Chapter III, pp. 93–120. Műegyetemi Kiadó, Budapest (in Hungarian) (2008)
56. Ván, P.: Weakly nonlocal irreversible thermodynamics. *Annalen der Physik* **12**(3), 146–173 (2003); [arXiv:cond-mat/0112214](https://arxiv.org/abs/cond-mat/0112214)
57. Jou, D., Casas-Vázquez, J., Lebon, G.: *Extended Irreversible Thermodynamics*. Springer, Berlin (1992); 3rd revised edition (2001)
58. Noll, W.: Space-time structures in classical mechanics. In: *The Foundations of Mechanics and Thermodynamics (Selected papers by Walter Noll)*, pp. 204–210. Springer, Berlin (1974). Originally: Delaware Seminar in the Foundations of Physics, pp. 28–34. Springer, Berlin (1967)
59. Noll, W.: A frame free formulation of elasticity. *J. Elast.* **83**, 291–307 (2006)
60. Ciancio, V., Bartolotta, A., Farsaci, F.: Experimental confirmations on a thermodynamical theory for viscoelastic media with memory. *Phys. B* **394**(1), 8–13 (2007)
61. Ciancio, V.: Non equilibrium thermodynamics with internal variables in Kluitenberg's theory. *AAPP—Phys. Math. Nat. Sci.* **86**(S1), A1(1–14) (2008)
62. Ciancio, V., Ciancio, A., Farsaci, F.: On general properties of phenomenological and state coefficients for isotropic viscoelastic media. *Phys. B* **403**(18), 3221–3227 (2008)