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ORIGINAL ARTICLE

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Finite gradient elasticity and plasticity: a constitutive mechanical framework

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Abstract Following a suggestion by Forest and Sievert (Acta Mech 160:71–111, 2003), a constitutive frame for a general gradient elastoplasticity for finite deformations is established. The basic assumptions are the principle of Euclidean invariance and the isomorphy of the elastic ranges. Both the elastic and the plastic laws include the first and the second deformation gradient. The starting point is an objective expression for the stress power.

Keywords Gradient plasticity · Gradient elasticity · Finite deformations

1 Introduction

In the past decades, there has been a blossoming activity in creating material models with internal length scales induced by higher deformation gradients. Already a century ago this was done for elastic fluids by Korteweg [34], followed by the classical works of Toupin [54] and Mindlin [40] on elastic solids. Afterwards theories of continuum dislocation dynamics claimed the need for considering higher gradients of the plastic strains, perhaps starting with Ashby [3], Triantafyllidis and Aifantis [55] and later Fleck [26] and Gurtin [30] thus initiating a broad amount of research activities.[1] In the sequel appeared also many other fields of applications where higher gradient models for elastic or plastic materials are used, like in geomechanics [37], continuum damage mechanics [49], material growth [17], to mention just a few of them.

Consequently, it would be desirable to find a general format or framework into which all these constitutive models can be imbedded and joint properties common to all of them be demonstrated. This was intended for small deformations by Bertram and Forest [13]. In many applications, however, the deformations and the gradient of the deformations are not small, but can be rather large. For such cases, a generalization of this linear theory into the nonlinear range is needed. Such an inclusion is by no means trivial or straightforward. In particular, many principal questions from finite plasticity arise again, such as the following ones.

- What are appropriate kinematical and dynamical variables for modelling of second order materials?
- How do they transform under Euclidean transformations and changes of the reference placement?
- What are reduced forms for gradient elasticity and plasticity being invariant under Euclidean transformations?
- How can the concept of isomorphisms be translated to a gradient format?
- What are the symmetry transformations for constitutive equations defined on such higher order spaces?

[1] See [31,51] and [29] for literature surveys.

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- How can one introduce elastic state variables and plastic internal variables in a natural way?
- Does there exist a decomposition of the kinematical variables, either in a multiplicative or in an additive form?
- How does the elastic law evolve during yielding?

All these questions will be addressed in the present article.

While there is already a high number of contributions to gradient elasticity and plasticity within the linear format, publications on the same class of materials for large deformations are still limited, and in the few cases, one will mainly find particular cases rather than a general theory. Examples are [1,4,18–20,25,31,33,38,39,44,45,48,52], to mention just a few. In most of them, the internal length scale enters through the gradient of some internal variable, like hardening variables or the plastic deformation. Examples for such procedure are given by Gurtin et al. [32], Luscher et al. [38], and Cleja-Tigoiu [20]. Our intention here, however, is to not limit the theory to such special cases, but instead to allow for a second and a third order plastic variable as being substantially independent of each other (*unconstrained gradient plasticity*). This is the more general case, and there is no rationale known which would exclude this choice. Particular cases, in which this independence is not given, should be nevertheless contained in this general setting.

A nontrivial problem in finite deformation theory is the choice of appropriate and practical third order variables. In contrast to Polizzotto [48], we use material variables and reduced forms. This saves us from introducing objective time derivatives for the stresses and strains. Some authors like Toupin [54], Svendsen et al. [52], and Hwang et al. [33] use the gradient of Green's strain tensor as a higher order material variable. Although principally equivalent to our procedure, it turns out that this choice leads to rather complicated expressions in elastoplasticity even if applied only for the isotropic case.

In [47,53,58] the symmetry properties of gradient materials are also investigated in the context of elastic fluids. These authors, however, apply them to variables which are neither spatial nor material and, thus, inappropriate for material modelling of solids.

The methodology of this paper is oriented along the lines which have been suggested for simple elastoplastic materials in, e.g., [10], following an approach by Forest and Sievert [27].

The outline of this paper is as follows. After introducing some notations necessary for handling higher order tensors, we introduce second and third order deformation variables defined in the reference placement. The starting point for the introduction of the stresses is the internal stress power enlarged by a higher-order term. This is the terminating point of other contributions like Germain [28], Trostel [56][2] giving us the basis with balance laws and boundary conditions for this class of materials, on which the material modelling can be build upon.

Once having defined appropriate material stress and strain variables, we consider gradient elasticity. After working out the effect of a change of the reference placement, the concept of elastic isomorphy allows us to define elastic symmetry transformations. With these concepts, we are able to distinguish isotropic and anisotropic elasticity. As an example, we give a linear form of the gradient elasticity.

The plasticity theory is based on the concept of elastic ranges, in each of which the elastic properties are assumed to be the same (elastic isomorphy). This leads to a decomposition of the elastic and plastic variables in a natural way. The elastic ranges are bounded by appropriate yield limits. The hardening behavior is only worked out in a rather general way in order to leave enough space for modelling individual material behavior.

The entire theory includes both isotropic and fully anisotropic behavior in all of its constituents (elastic laws, yield criterion, flow and hardening rules).

2 Notations

We tried to follow the standard notations in the field of tensor calculus within continuum mechanics. However, by the inclusion of third order tensors (triads), many more operations have to be introduced if one prefers a direct notation. In any case, the author did his best to use a notation which is as customary, direct, and uncomplicated as possible.

We write for vectors small bold letters like $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x} \in \mathscr{V}$, etc. and for dyads or second order tensors $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$, etc. For triads or third order tensors, we use $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, etc. kth-order tensors with $k > 3$ are notated like $\overset{\langle k \rangle}{\boldsymbol{C}}$. $\mathscr{R}$ denotes the real numbers, $\mathscr{R}^+$ the positive reals, $\mathscr{O}rth$ is the orthogonal group within the dyads.

[2] See also [11] and [23].

We denote an arbitrary basis by $\{\mathbf{r}_i\}$ and its dual by $\{\mathbf{r}^i\}$. In particular, such bases occur as the natural bases induced by a coordinate system $\{\varphi^i\}$ and then written as $\{\mathbf{r}_{\varphi i}\}$ and $\{\mathbf{r}^i_\varphi\}$. An orthonormal vector basis is written as $\{\mathbf{e}_i\}$.

The standard scalar products between tensors of equal order are indicated by a dot " $\cdot$ ". The simple contraction of vectors and tensors is written without a product dot like $\mathbf{T}\,\mathbf{v}$, $\mathbf{T}\,\mathbf{U}$, $\mathbf{T}\,\boldsymbol{A}$, etc. The tensor product is written as $\otimes$.

While a second order tensor $\mathbf{T}$ has a unique transpose $\mathbf{T}^T$, a third order tensor has more than one. We will mainly need the right sub-transpose $\boldsymbol{A}^t$ that is defined by the identity $\boldsymbol{A}^t \cdot \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = \boldsymbol{A} \cdot \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b}$ for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. This gives for the components with respect to an orthonormal vector basis $(A^t)_{ijk} = A_{ikj}$. If a triad is symmetric with respect to this particular transposition, we call it *right subsymmetric*.

For two triads, we obtain then

$$\boldsymbol{A}^t \cdot \boldsymbol{B} = \boldsymbol{A} \cdot \boldsymbol{B}^t. \tag{1}$$

Very helpful for higher order tensors is the Rayleigh[3] product. It maps all basis vectors of a tensor simultaneously without changing its components. To be more precise, let $\overset{\langle k\rangle}{\boldsymbol{C}}$ be a tensor of kth-order ($k \geq 1$) and $\mathbf{T}$ a dyad. Then the Rayleigh product between them is defined as

$$\mathbf{T} * \overset{\langle k\rangle}{\boldsymbol{C}} = \mathbf{T} * (C^{ik\ldots l}\mathbf{r}_i \otimes \mathbf{r}_k \otimes \cdots \otimes \mathbf{r}_l) := C^{ik\ldots l}\,(\mathbf{T}\,\mathbf{r}_i) \otimes (\mathbf{T}\,\mathbf{r}_k) \otimes \cdots \otimes (\mathbf{T}\,\mathbf{r}_l)\,. \tag{2}$$

Of course, the result does not depend on the choice of the basis. If $\mathbf{T}$ is orthogonal, then the product is a rotation of $\overset{\langle k\rangle}{\boldsymbol{C}}$. For $k \equiv 1$ the Rayleigh product coincides with a linear mapping

$$\mathbf{T} * \mathbf{c} = \mathbf{T}\mathbf{c}, \tag{3}$$

and for $k \equiv 2$ we obtain

$$\mathbf{T} * \mathbf{C} = \mathbf{T}\,\mathbf{C}\,\mathbf{T}^T. \tag{4}$$

The Rayleigh product acts on a simple triad like

$$\mathbf{T} * \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = (\mathbf{T}\mathbf{a}) \otimes (\mathbf{T}\mathbf{b}) \otimes (\mathbf{T}\mathbf{c}) = (\mathbf{T}\mathbf{a}) \otimes (\mathbf{T}\mathbf{b}) \otimes \mathbf{c}\,\mathbf{T}^T = \mathbf{T}(\mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b}\,\mathbf{T}^T)^t\mathbf{T}^T \tag{5}$$

or analogously for a triad $\boldsymbol{A}$

$$\mathbf{T} * \boldsymbol{A} = \mathbf{T}(\boldsymbol{A}^t\mathbf{T}^T)^t\mathbf{T}^T. \tag{6}$$

The Rayleigh product is associative in the left factor

$$\mathbf{S} * (\mathbf{T} * \overset{\langle k\rangle}{\boldsymbol{C}}) = (\mathbf{S}\,\mathbf{T}) * \overset{\langle k\rangle}{\boldsymbol{C}} \tag{7}$$

and in the right one. In fact, if $\overset{\langle k\rangle}{\boldsymbol{C}}$ and $\overset{\langle m\rangle}{\boldsymbol{D}}$ are tensors of arbitrary order, then we have

$$\mathbf{T} * (\overset{\langle k\rangle}{\boldsymbol{C}} \otimes \overset{\langle m\rangle}{\boldsymbol{D}}) = (\mathbf{T} * \overset{\langle k\rangle}{\boldsymbol{C}}) \otimes (\mathbf{T} * \overset{\langle m\rangle}{\boldsymbol{D}}) \tag{8}$$

for all dyads $\mathbf{T}$. This would not hold, if we replace the tensor product by the composition or an arbitrary contraction, unless $\mathbf{T}$ is orthogonal.

In this product, the second order identity tensor also gives the identity mapping

$$\mathbf{I} * \overset{\langle k\rangle}{\boldsymbol{C}} = \overset{\langle k\rangle}{\boldsymbol{C}} \tag{9}$$

and the inversion is done by

$$\mathbf{T} * (\mathbf{T}^{-1} * \overset{\langle k\rangle}{\boldsymbol{C}}) = \overset{\langle k\rangle}{\boldsymbol{C}}\,. \tag{10}$$

The Rayleigh product commutes with the composition with the inverse in the following sense

$$\mathbf{T}^{-1}(\mathbf{T} * \overset{\langle k\rangle}{\boldsymbol{C}}) = \mathbf{T} * (\mathbf{T}^{-1}\,\overset{\langle k\rangle}{\boldsymbol{C}}). \tag{11}$$

[3] See [42, footnote 3, p. 1661] and [10, p. 44].

For two second order tensors $\mathbf{A}$ (invertible) and $\mathbf{B}$ and a higher order tensor $\overset{\langle k\rangle}{\boldsymbol{C}}$, we obtain the rule

$$\mathbf{B}\,\mathbf{A}^{-1}(\mathbf{A} * \overset{\langle k\rangle}{\boldsymbol{C}}) = \mathbf{A} * (\mathbf{A}^{-1}\mathbf{B}\,\overset{\langle k\rangle}{\boldsymbol{C}}). \tag{12}$$

For the scalar product of arbitrary tensors, we get

$$(\mathbf{T} * \overset{\langle k\rangle}{\boldsymbol{C}}) \cdot \overset{\langle k\rangle}{\boldsymbol{D}} = \overset{\langle k\rangle}{\boldsymbol{C}} \cdot (\mathbf{T}^T * \overset{\langle k\rangle}{\boldsymbol{D}}). \tag{13}$$

Besides the Rayleigh product, we will need another product between an invertible dyad $\mathbf{T}$ and a triad $\boldsymbol{A}$ denoted by

$$\begin{aligned}\mathbf{T} \circ \boldsymbol{A} &:= A_{ijk}(\mathbf{T}^{-T}\mathbf{e}_i) \otimes (\mathbf{T}\,\mathbf{e}_j) \otimes (\mathbf{T}\,\mathbf{e}_k)\\ &= \mathbf{T}^{-T}[\boldsymbol{A}^t\mathbf{T}^T]^t\mathbf{T}^T\\ &= \mathbf{T} * (\mathbf{T}^{-1}\mathbf{T}^{-T}\boldsymbol{A})\\ &= \mathbf{T}^{-T}\mathbf{T}^{-1}(\mathbf{T} * \boldsymbol{A}).\end{aligned} \tag{14}$$

using (6) and (12). The following rules hold for this product.

$$(\mathbf{T} \circ \boldsymbol{A}) \cdot \boldsymbol{B} = \boldsymbol{A} \cdot (\mathbf{T}^T \circ \boldsymbol{B}) \tag{15}$$

for all dyads $\mathbf{T}$ and all triads $\boldsymbol{A}$ and $\boldsymbol{B}$. The second order identity tensor also gives the identity mapping

$$\mathbf{I} \circ \boldsymbol{A} = \boldsymbol{A}, \tag{16}$$

and the inversion is done by

$$\mathbf{T} \circ (\mathbf{T}^{-1} \circ \boldsymbol{A}) = \boldsymbol{A}. \tag{17}$$

Furthermore, the product is associative

$$\mathbf{S} \circ (\mathbf{T} \circ \boldsymbol{A}) = (\mathbf{S}\,\mathbf{T}) \circ \boldsymbol{A} \tag{18}$$

for all dyads $\mathbf{S}$ and $\mathbf{T}$ and triads $\boldsymbol{A}$.
For the case of $\mathbf{T}$ being orthogonal, this transformation coincides with the Rayleigh product.
As an overview over the different sets, spaces, and groups, we give the following list.

$\mathscr{R}$	the real numbers
$\mathscr{R}^+$	the positive reals
$\mathscr{V}$	three-dimensional Euclidean vector space
$\mathscr{O}rth$	orthogonal group within the dyads
$\mathscr{L}in$	$:= \{(\mathbf{T}, \boldsymbol{T}) \mid \mathbf{T}$ dyad, $\boldsymbol{T}$ triad with right subsymmetry$\}$
$\mathscr{C}onf$	$:= \{(\mathbf{C}, \boldsymbol{K}) \in \mathscr{L}in \mid \mathbf{C}$ positive-definite and symmetric dyad, $\boldsymbol{K}$ triad with right subsymmetry$\}$
$\mathscr{I}nv$	$:= \{(\mathbf{P}, \boldsymbol{P}) \in \mathscr{L}in \mid \mathbf{P}$ invertible dyad, $\boldsymbol{P}$ triad with right subsymmetry$\}$
$\mathscr{U}nim$	$:= \{(\mathbf{P}, \boldsymbol{P}) \in \mathscr{L}in \mid \mathbf{P}$ unimodular dyad ($\det \mathbf{P} = 1$), $\boldsymbol{P}$ triad with right subsymmetry$\}$
$\mathscr{B}_0$	region occupied by the body in the reference placement
$\mathscr{B}_t$	region occupied by the body in the current placement

3 Kinematics

We will denote the region occupied by the body in the reference placement by $\mathscr{B}_0$ and the region in the current placement by $\mathscr{B}_t$.

Let $\boldsymbol{\chi}$ be the motion of the body and

$$\mathbf{F} = Grad\, \boldsymbol{\chi} = \boldsymbol{\chi} \otimes \nabla_L \quad \text{with determinant } J = det\, \mathbf{F} \tag{19}$$

the deformation gradient. Here, ∇_L denotes the Lagrangean nabla. With respect to the natural bases of a spatial coordinate system $\{\varphi^i\}$ and a material one $\{\Psi^i\}$ it can be calculated as

$$\mathbf{F} = \frac{\partial \varphi^k}{\partial \Psi^i} \mathbf{r}_{\varphi k} \otimes \mathbf{r}^i_{\Psi}. \tag{20}$$

We will later on need the differential of the inverse of the deformation gradient

$$\begin{aligned} d(\mathbf{F}^{-1}) &= Grad\ \mathbf{F}^{-1}\, d\mathbf{x}_0 = -\mathbf{F}^{-1}\ d\mathbf{F}\, \mathbf{F}^{-1} \\ &= -\mathbf{F}^{-1}(Grad\ \mathbf{F}\ d\mathbf{x}_0)\mathbf{F}^{-1} \\ &= -\mathbf{F}^{-1}[(Grad\ \mathbf{F})\ \mathbf{F}^{-1}]^t\ d\mathbf{x}_0. \end{aligned} \tag{21}$$

Thus,

$$Grad\, \mathbf{F}^{-1} = -\mathbf{F}^{-1}[(Grad\, \mathbf{F})\mathbf{F}^{-1}]^t. \tag{22}$$

The spatial velocity gradient is

$$\mathbf{L} = grad\, \boldsymbol{\chi}^{\bullet} = \mathbf{v} \otimes \nabla_E = \mathbf{D} + \mathbf{W} = \mathbf{F}^{\bullet}\mathbf{F}^{-1} \tag{23}$$

where the suffix E stands for the Eulerean or spatial derivative. The dot denotes the material time derivative. Its symmetric part is $\mathbf{D}$ and its skew one $\mathbf{W}$. We also have the relation with the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$

$$\mathbf{C}^{\bullet} = \mathbf{F}^T * 2\,\mathbf{D}. \tag{24}$$

For two second-order differentiable tensor fields in the Lagrangean description $\mathbf{S}$ and $\mathbf{T}$, we obtain for the gradient of the product

$$Grad\ \ (\mathbf{S}\,\mathbf{T}) = \mathbf{S}\, Grad\, \mathbf{T} + \left[(Grad\, \mathbf{S})^t\ \mathbf{T}\right]^t. \tag{25}$$

This can be verified by the following calculation

$$Grad\, (\mathbf{S}\,\mathbf{T}) = (\mathbf{S}\,\mathbf{T}) \otimes \nabla_L = \mathbf{S}(\mathbf{T} \otimes \nabla_L) + \overset{\downarrow}{\mathbf{S}}\,\mathbf{T} \otimes \nabla_L = \mathbf{S}\, Grad\, \mathbf{T} + \overset{\downarrow}{\mathbf{S}}\,\mathbf{T} \otimes \nabla_L \tag{26}$$

where the arrows indicate the term to which nabla has to be applied. The last term is then with respect to an orthonormal basis

$$\overset{\downarrow}{\mathbf{S}}\,\mathbf{T} \otimes \nabla_L = S_{ij,k} T_{jm} \mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_k \tag{27}$$

while

$$\begin{aligned} \left[(Grad\, \mathbf{S})^t\ \mathbf{T}\right]^t &= [(S_{ij,k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)^t\mathbf{T}]^t \\ &= [S_{ij,k}\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j\mathbf{T}]^t \\ &= [S_{ij,k}T_{jm}\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_m]^t \\ &= S_{ij,k}T_{jm}\mathbf{e}_i \otimes \mathbf{e}_m \otimes \mathbf{e}_k \end{aligned} \tag{28}$$

gives the same. An analogous result holds also for the gradient in the Eulerean description.

For any tensor field ϕ, we have by the chain rule

$$Grad\, \phi = (grad\, \phi)\mathbf{F} \quad \text{and} \quad grad\, \phi = (Grad\, \phi)\mathbf{F}^{-1} \tag{29}$$

where it is understood that ϕ is in the Lagrangean description if *Grad* is applied, and in the Eulerian one if *grad* is.

We will also need the second deformation gradient

$$Grad\, \mathbf{F} = Grad\, Grad\, \boldsymbol{\chi} = \boldsymbol{\chi} \otimes \nabla_L \otimes \nabla_L \tag{30}$$

which is a triad (field) with the right subsymmetry by definition.

4 Stress power

The starting point for our gradient theory is the stress power after Bertram and Forest [11][4] of a body

$$P = \int_{\mathscr{B}_t} 1/\rho(\mathbf{T} \cdot grad\, \mathbf{v} + \boldsymbol{S} \cdot grad\, grad\, \mathbf{v})dm \tag{31}$$

with Cauchy's stress tensor $\mathbf{T}$ and a spatial hyperstress tensor of third order $\boldsymbol{S}$.

Cauchy's equations (local balances of linear and moment of momentum) are for second gradient materials

$$div(\mathbf{T} - div\boldsymbol{S}) + \rho\mathbf{b} = \rho\mathbf{v}^\bullet \tag{32}$$
$$\mathbf{T} = \mathbf{T}^T. \tag{33}$$

The field of the body forces $\mathbf{b}$ may also contain the divergences of higher order tensorial body forces, as introduced by Germain [28]. The derivation of these balance laws together with higher-order boundary conditions[5] can be already found in Toupin [54], [40,41], and many other papers in the sequel. The consideration of boundary value problems, however, is beyond the scope of the present article.

$\mathbf{T}$ is symmetric because of the balance of moment of momentum, and the first term in (31) can be substituted by $\mathbf{T} \cdot \mathbf{D}$. $grad\, grad\, \mathbf{v}$ has the right subsymmetry by definition. So the same symmetry can be imposed on $\boldsymbol{S}$ without loss of generality within the present frame. The balance of moment of momentum does not impose any restriction on $\boldsymbol{S}$.

We will next bring the stress power in a material form which is invariant under Euclidean transformations. For this purpose, we use (23), (29), (25), (22), and (6)

$$\begin{aligned} grad\, grad\, \mathbf{v} &= Grad(\mathbf{F}^\bullet\ \mathbf{F}^{-1})\ \mathbf{F}^{-1} \\ &= -\mathbf{F}^\bullet\ \mathbf{F}^{-1}[(Grad\ \mathbf{F})\ \mathbf{F}^{-1}]^t\ \mathbf{F}^{-1} + [(Grad\ \mathbf{F}^\bullet)\ \mathbf{F}^{-1}]^t\ \mathbf{F}^{-1} \\ &= \mathbf{F}^{-T} \circ [-\mathbf{F}^{-1}\ \mathbf{F}^\bullet\ \mathbf{F}^{-1} Grad\ \mathbf{F} + \mathbf{F}^{-1} Grad\ \mathbf{F}^\bullet] = \mathbf{F}^{-T} \circ \boldsymbol{K}^\bullet \end{aligned} \tag{34}$$

with

$$\boldsymbol{K} := \mathbf{F}^{-1} Grad\, \mathbf{F}. \tag{35}$$

This triad is the Lagrangean form of the second gradient and has been used by Chambon et al. [16], Forest and Sievert [27], Cleja-Tigoiu [20] and other authors.[6] It is sometimes called *connection* which, however, gives rise for a confusion with nabla. Here, we prefer the name **curvature tensor**, although it has to be distinguished from the well-known Riemannean curvature tensor. The product $\circ$ in (34) can be interpreted as the pushforward from the reference placement to the current placement taking into account the different transformation behavior of tangent and cotangent vectors.

These fields can be calculated with respect to the natural bases of the coordinate systems $\{\varphi^i\}$ and $\{\Psi^i\}$

$$\begin{aligned} Grad\, \mathbf{F} &= \left(\frac{\partial \varphi^k}{\partial \Psi^i} \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi^i\right) \otimes \frac{\partial}{\partial \Psi^j} \mathbf{r}_\Psi^j \\ &= \left[\frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j} + \frac{\partial \varphi^l}{\partial \Psi^i} \frac{\partial \varphi^m}{\partial \Psi^j} \left(\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi^k\right) + \frac{\partial \varphi^k}{\partial \Psi^l} \left(\frac{\partial \mathbf{r}_\Psi^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i}\right)\right] \mathbf{r}_{\varphi k} \otimes \mathbf{r}_\Psi^i \otimes \mathbf{r}_\Psi^j \end{aligned} \tag{36}$$

and

$$\boldsymbol{K} = \frac{\partial \Psi^p}{\partial \varphi^k} \left[\frac{\partial^2 \varphi^k}{\partial \Psi^i \partial \Psi^j} + \frac{\partial \varphi^l}{\partial \Psi^i} \frac{\partial \varphi^m}{\partial \Psi^j} \left(\frac{\partial \mathbf{r}_{\varphi l}}{\partial \varphi^m} \cdot \mathbf{r}_\varphi^k\right) + \frac{\partial \varphi^k}{\partial \Psi^l} \left(\frac{\partial \mathbf{r}_\Psi^l}{\partial \Psi^j} \cdot \mathbf{r}_{\Psi i}\right)\right] \mathbf{r}_{\Psi p} \otimes \mathbf{r}_\Psi^i \otimes \mathbf{r}_\Psi^j. \tag{37}$$

It can be shown ([35], see also [33]) that the curvature tensor $\boldsymbol{K}$ can be determined by the right Cauchy-Green tensor $\mathbf{C}$ and its gradient according to

$$\boldsymbol{K} = \mathbf{C}^{-1}\ Sym\ Grad\, \mathbf{C} \tag{38}$$

[4] See also [56].
[5] See also [5] and [15].
[6] See also [46].

with the following symmetrization of a triad

$$Sym\, T^{ijk} := 1/2 \left(T^{ijk} + T^{ikj} - T^{kji}\right). \tag{39}$$

We obtain for the global stress power (31) with (34), (35), and (15)

$$\begin{aligned} P &= \int_{\mathscr{B}_0} 1/\rho_0 J \{\mathbf{T} \cdot (\mathbf{F}^{-T} * 1/2\ \mathbf{C}^\bullet) + \boldsymbol{S} \cdot (\mathbf{F}^{-T} \circ \boldsymbol{K}^\bullet)\}\ dm \\ &= \int_{\mathscr{B}_0} 1/\rho_0\ \{1/2\ \mathbf{S} \cdot \mathbf{C}^\bullet + \boldsymbol{S}_{\mathrm{K}} \cdot \boldsymbol{K}^\bullet\}\ dm \end{aligned} \tag{40}$$

with two material stress tensors, namely the second Piola–Kirchhoff tensor

$$\mathbf{S} := J\mathbf{F}^{-1}\mathbf{T}\,\mathbf{F}^{-T} = \mathbf{F}^{-1} * J\mathbf{T} \tag{41}$$

and a third-order material hyperstress tensor defined as

$$\boldsymbol{S}_{\mathrm{K}} := \mathbf{F}^{-1} \circ J\boldsymbol{S}. \tag{42}$$

The product $\circ$ in (42) can be interpreted as the pull-back of $\boldsymbol{S}$ from the current placement to the reference placement.

5 Gradient elasticity

Before we start with gradient plasticity, it is necessary to introduce gradient elasticity. This will be done by extending the definition of a simple elastic material to a second order one by enlarging the set of independent variables by the second deformation gradient.

Definition 1 We will call a material a **second order elastic material** if the stress tensors are functions of the motion, the deformation gradient, and the gradient of the deformation gradient:

$$\mathbf{T} = f(\boldsymbol{\chi}, Grad\,\boldsymbol{\chi}, Grad\,Grad\,\boldsymbol{\chi}) = f(\boldsymbol{\chi}, \mathbf{F}, Grad\,\mathbf{F}) \tag{43}$$

$$\boldsymbol{S} = F(\boldsymbol{\chi}, Grad\,\boldsymbol{\chi}, Grad\,Grad\,\boldsymbol{\chi}) = F(\boldsymbol{\chi}, \mathbf{F}, Grad\,\mathbf{F}) \tag{44}$$

where it is understood that all variables are taken at the same material point at the same instant of time.

These constitutive equations can be further reduced by means of the Euclidean invariance principle (see [10], therein called PISM), which we assume in the following form.

Principle of Euclidean Invariance. *The stress power for any body at the end of any motion* $\boldsymbol{\chi}(\mathbf{x}_0, \tau)\big|_{\tau=0}^{t}$ *in some time interval [0,t] equals the stress power after superimposing a rigid body motion upon the original motion*

$$\left\{\mathbf{Q}(\tau)\,\boldsymbol{\chi}(\mathbf{x}_0, \tau) + \mathbf{c}(\tau)\big|_{\tau=0}^{t}\right\} \tag{45}$$

with arbitrary differentiable time functions $\mathbf{Q}(\tau) \in \mathscr{O}rth$ *and* $\mathbf{c}(\tau) \in \mathscr{V}$.

The Euclidean transformation (45) has nothing to do with changes of observers. The invariance under such changes of observers has already been used for the objectivity of the stress power (31), which led us to the objectivity of the stress tensors. The invariance under observer changes does not lead to reduced forms (see [9]), in contrast to the *Principle of Euclidean Invariance* above, as we will show in the sequel.

The action of the group transformation (45) determines the transformation behavior of all kinematic variables. We will further call a tensor $\boldsymbol{T}$ of arbitrary order *invariant* if it is not affected by any modification (45), and *objective* if it is rotated into $\mathbf{Q} * \boldsymbol{T}$. For scalars, the two properties coincide. As a result, $\mathbf{D}$ (but not $\mathbf{L}$) and *grad* $\mathbf{L}$ turn out to be objective.

Then one can easily show that the above assumption on the stress power is generally fulfilled if and only if $\mathbf{T}$ and $\boldsymbol{S}$ are objective. For the Cauchy stresses $\mathbf{T}$ this is known since long.[7] For $\boldsymbol{S}$ it follows from (31) and the fact that *grad grad* $\mathbf{v}$ is objective, using the rule (13).

[7] See, e.g., [10, pp. 155–156].

In contrast to them, the following tensors are invariant: $\mathbf{C}$, $\boldsymbol{K}$ and their duals $\mathbf{S}$, $\boldsymbol{S}_{\mathrm{K}}$, which makes them good candidates for material modelling.

We will further-on denote the binary set of elements like $\{\mathbf{T}, \boldsymbol{T}\}$ consisting of all dyads $\mathbf{T}$ and triads $\boldsymbol{T}$ with right subsymmetry by $\mathscr{Lin}$. This space has the dimension 9 + 18 = 27. A subset of this space is formed by all positive-definite and symmetric second order tensors $\mathbf{C}$ and all triads with right subsymmetry $\boldsymbol{K}$, which we call the **space of configurations** $\mathscr{Conf}$. This set is imbedded in a space with dimension 6 + 18 = 24. Another subset of $\mathscr{Lin}$ is formed by all invertible dyads and all triads with right subsymmetry, which we denote by $\mathscr{Inv}$. We can further restrict this subset to those dyads which are unimodular (determinant equal 1) denoted by $\mathscr{Unim}$.

We define reduced forms in exactly the same way as Truesdell and Noll [57, p. 66]. This means that every second order elastic material that obeys the Principle of Euclidean Invariance can be brought into such a reduced form. Reduced forms of the elastic laws (43) and (44) are then

$$\mathbf{S} = k(\mathbf{C}, \boldsymbol{K}) \tag{46}$$

$$\boldsymbol{S}_{\mathrm{K}} = K(\mathbf{C}, \boldsymbol{K}) \tag{47}$$

by two elastic laws which are defined on the space of configurations

$$k : \mathscr{Conf} \to \mathscr{Lin}$$
$$K : \mathscr{Conf} \to \mathscr{Lin}.$$

If, moreover, the elastic material is **hyperelastic** then there exists a specific elastic energy

$$w : \mathscr{Conf} \to \mathscr{R}$$

such that the specific stress power after (40) equals

$$\begin{aligned} p &:= 1/\rho_0 \; [1/2 \; \mathbf{S} \cdot \mathbf{C}^{\bullet} + \boldsymbol{S}_{\mathrm{K}} \cdot \boldsymbol{K}^{\bullet}] \\ &= w(\mathbf{C}, \boldsymbol{K})^{\bullet} = \partial_{\mathbf{C}} w(\mathbf{C}, \boldsymbol{K}) \cdot \mathbf{C}^{\bullet} + \partial_{\boldsymbol{K}} w(\mathbf{C}, \boldsymbol{K}) \cdot \boldsymbol{K}^{\bullet}. \end{aligned} \tag{48}$$

By comparison, we obtain the potential relations for (46) and (47)

$$\mathbf{S} = k(\mathbf{C}, \boldsymbol{K}) = 2\rho_0 \partial_{\mathbf{C}} w(\mathbf{C}, \boldsymbol{K}) \tag{49}$$

$$\boldsymbol{S}_{\mathrm{K}} = K(\mathbf{C}, \boldsymbol{K}) = \rho_0 \partial_{\boldsymbol{K}} w(\mathbf{C}, \boldsymbol{K}) \quad \forall (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{50}$$

In the sequel, we will not distinguish between the elastic and the hyperelastic case, since only an elastic material, which is also hyperelastic, is physically meaningful.

6 Change of reference placement

The reduced forms (46) and (47) and the hyperelastic energy depend on the choice of the reference placement κ. If we want to indicate this dependence, we will write for example $k(\kappa, \cdot)$ and $K(\kappa, \cdot)$ for the elastic laws. Their transformation behavior under change of the reference placement plays an important role for isomorphisms and symmetry transformations.

While the spatial quantities *grad* $\mathbf{v}$, *grad grad* $\mathbf{v}$, $\mathbf{T}$, $\boldsymbol{S}$ do not depend on the reference placement, the material ones like $\mathbf{C}$, $\boldsymbol{K}$, $\mathbf{S}$, $\boldsymbol{S}_{\mathrm{K}}$ do so. We will next investigate their transformation behavior under change of the reference placement. We will therefore consider a second reference placement $\underline{\kappa}$ indicated by underlining.

For an arbitrary differentiable field ϕ, we obtain for the two reference placements by the chain rule

$$\underline{Grad}\, \phi = Grad\, \phi\, \mathbf{A} \tag{51}$$

where $\mathbf{A} := \underline{Grad}\, (\kappa\, \underline{\kappa}^{-1})$ is the gradient of the change of reference placement. It is understood that the field ϕ is defined on the corresponding reference placement. In particular, we find by the aid of (25), (51), and (6), (53)

$$\underline{\mathbf{F}} = \underline{Grad}\, \boldsymbol{\chi} = \mathbf{F}\,\mathbf{A} \quad \text{and} \quad \underline{J} = J\, det(\mathbf{A}) \quad \text{and} \quad \underline{\mathbf{C}} = \mathbf{A}^T \mathbf{C}\, \mathbf{A} = \mathbf{A}^T * \mathbf{C} \tag{52}$$

and

$$\underline{Grad}\,\underline{\mathbf{F}} = \underline{Grad}\,(\mathbf{F}\,\mathbf{A}) = \mathbf{F}\,\underline{Grad}\,\mathbf{A} + [(\underline{Grad}\,\mathbf{F})^t\mathbf{A}]^t$$
$$= \mathbf{F}\,\underline{Grad}\,\mathbf{A} + \left[\{(Grad\,\mathbf{F})\,\mathbf{A}\}^t\,\mathbf{A}\right]^t$$
$$= \mathbf{F}\,\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\,Grad\,\mathbf{F}). \tag{53}$$

Thus,

$$\begin{aligned}\underline{\boldsymbol{K}} &= \underline{\mathbf{F}}^{-1}\underline{Grad}\,\underline{\mathbf{F}} = \mathbf{A}^{-1}\mathbf{F}^{-1}\{\mathbf{F}\,\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\,Grad\,\mathbf{F})\}\\ &= \mathbf{A}^{-1}\underline{Grad}\,\mathbf{A} + \mathbf{A}^{-1}\mathbf{F}^{-1}(\mathbf{A}^T * \mathbf{A}^{-T}\,Grad\,\mathbf{F})\\ &= \mathbf{A}^{-1}\,\underline{Grad}\,\mathbf{A} + \mathbf{A}^T * (\mathbf{A}^{-T}\mathbf{A}^{-1}\mathbf{F}^{-1}\,Grad\,\mathbf{F})\\ &= \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^T \circ \boldsymbol{K}\end{aligned} \tag{54}$$

with (14), and

$$\boldsymbol{K}_{\mathbf{A}} := \mathbf{A}^{-1}\,\underline{Grad}\,\mathbf{A}. \tag{55}$$

For the material stresses, we obtain with (52)

$$\begin{aligned}\underline{\mathbf{S}} &= \underline{J}\,\underline{\mathbf{F}}^{-1}\mathbf{T}\,\underline{\mathbf{F}}^{-T}\\ &= det(\mathbf{A})\;J\,\mathbf{A}^{-1}\,\mathbf{F}^{-1}\,\mathbf{T}\,\mathbf{F}^{-T}\,\mathbf{A}^{-T}\\ &= det(\mathbf{A})\;\mathbf{A}^{-1}\,\mathbf{S}\,\mathbf{A}^{-T}\\ &= \mathbf{A}^{-1} * J_A\mathbf{S} \quad \text{with} \quad J_A := det(\mathbf{A})\end{aligned} \tag{56}$$

and with (42), (52), (7), (18) for the hyperstresses

$$\begin{aligned}\underline{\boldsymbol{S}}_{\mathrm{K}} &= \underline{\mathbf{F}}^{-1} \circ \underline{J}\,\boldsymbol{S}\\ &= (\mathbf{A}^{-1}\mathbf{F}^{-1}) \circ (det(\mathbf{A})\,J\,\boldsymbol{S})\\ &= \mathbf{A}^{-1} \circ [\mathbf{F}^{-1} \circ (det(\mathbf{A})\;J\,\boldsymbol{S})]\\ &= \mathbf{A}^{-1} \circ J_A\,\boldsymbol{S}_{\mathrm{K}}\end{aligned} \tag{57}$$

or inversely

$$\mathbf{S} = \mathbf{A} * J_A^{-1}\underline{\mathbf{S}} \quad \text{and} \quad \boldsymbol{S}_{\mathrm{K}} = \mathbf{A} \circ J_A^{-1}\underline{\boldsymbol{S}}_{\mathrm{K}}. \tag{58}$$

The elastic laws (46) and (47) with respect to two different reference placements are then transformed as

$$k(\kappa, \mathbf{C}, \boldsymbol{K}) = \mathbf{A} * J_A^{-1}k(\underline{\kappa}, \mathbf{A}^T * \mathbf{C}, \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^T \circ \boldsymbol{K}) \tag{59}$$
$$K(\kappa, \mathbf{C}, \boldsymbol{K}) = \mathbf{A} \circ J_A^{-1}\,K(\underline{\kappa}, \mathbf{A}^T * \mathbf{C}, \boldsymbol{K}_{\mathbf{A}} + \mathbf{A}^T \circ \boldsymbol{K}) \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf \tag{60}$$

using (54) and (58).

Remark The change of the reference placement enters into these transformations only through the couple $(\mathbf{A}, \boldsymbol{K}_{\mathbf{A}}) \in \mathscr{I}nv$. Reversely, for any such couple one may surely find a corrensponding reference placement obeing (52) and (54). In fact, there are always infinitely many that do so.

The above formulae hold for arbitrary changes of reference placements. If we particularize these results to rigid rotations of the reference placements, then $\mathbf{A} \subset \mathscr{O}rth$ and we have

$$\begin{array}{ll} J_A = 1 & Grad\,\mathbf{A} \equiv \boldsymbol{0}\\ \underline{\boldsymbol{K}} = \mathbf{A}^T * \boldsymbol{K} & \underline{\boldsymbol{S}}_{\mathrm{K}} = \mathbf{A}^T * \boldsymbol{S}_{\mathrm{K}}.\end{array} \tag{61}$$

7 Elastic isomorphy

This concept has been introduced by Truesdell and Noll [57] in Sect. 27 and plays an important role for the formulation of elasticity and elastoplasticity (see [8,10]). It is used to precisely define the notion that two elastic points show the *same elastic behavior*. The following definition is a straight extension of the original one to gradient materials.

Definition 2 Two elastic material points X and Y are called **elastically isomorphic** if we can find reference placements κ_X for X and κ_Y for Y such that the following two conditions hold.

- In κ_X and κ_Y the mass densities are equal
$$\rho_{0X} = \rho_{0Y}. \tag{62}$$
- With respect to κ_X and κ_Y the elastic laws are identical
$$k_X(\kappa_X, \cdot) = k_Y(\kappa_Y, \cdot) \tag{63}$$
$$K_X(\kappa_X, \cdot) = K_Y(\kappa_Y, \cdot). \tag{64}$$

Testa and Vianello [53] demand in addition to (62) that also the gradient of the density is equal in the two points, an assumption which probably makes sense in the context of elastic gradient fluids. In the present context of solids, however, we do not see any reason for such a restriction.

If two isomorphic elastic laws are given with respect to arbitrary reference placements $\underline{\kappa}_X$ and $\underline{\kappa}_Y$, then we must probably first transform them to appropriate κ_X and κ_Y using (59) and (60)

$$k_X(\kappa_X, \mathbf{C}_X, \boldsymbol{K}_X) = \mathbf{A}_X * J_X^{-1} k_X(\underline{\kappa}_X, \mathbf{A}_X^T * \mathbf{C}_X, \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^T \circ \boldsymbol{K}_X)$$
$$K_X(\kappa_X, \mathbf{C}_X, \boldsymbol{K}_X) = \mathbf{A}_X \circ J_X^{-1} K_X(\underline{\kappa}_X, \mathbf{A}_X^T * \mathbf{C}_X, \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^T \circ \boldsymbol{K}_X)$$

and

$$k_Y(\kappa_Y, \mathbf{C}_Y, \boldsymbol{K}_Y) = \mathbf{A}_Y * J_Y^{-1} k_Y(\underline{\kappa}_Y, \mathbf{A}_Y^T * \mathbf{C}_Y, \boldsymbol{K}_{\mathbf{A}Y} + \mathbf{A}_Y^T \circ \boldsymbol{K}_Y)$$
$$K_Y(\kappa_Y, \mathbf{C}_Y, \boldsymbol{K}_Y) = \mathbf{A}_Y \circ J_Y^{-1} K_Y(\underline{\kappa}_Y, \mathbf{A}_Y^T * \mathbf{C}_Y, \boldsymbol{K}_{\mathbf{A}Y} + \mathbf{A}_Y^T \circ \boldsymbol{K}_Y)$$

with

$$\mathbf{A}_{X,Y} := Grad(\kappa_{X,Y}\, \underline{\kappa}_{X,Y}^{-1}),\ J_{X,Y} := det(\mathbf{A}_{X,Y}), \quad \text{and} \quad \boldsymbol{K}_{\mathbf{A}X,Y} := \mathbf{A}_{X,Y}^{-1}\, \underline{Grad}\, \mathbf{A}_{X,Y}$$

as well as the mass densities in these reference placements

$$\underline{\rho}_{0X}/\rho_{0X} = J_X \quad \text{and} \quad \underline{\rho}_{0Y}/\rho_{0Y} = J_Y.$$

So the above isomorphy conditions hold if

$$\underline{\rho}_{0X} J_X^{-1} = \underline{\rho}_{0Y} J_Y^{-1}$$
$$k_X(\kappa_X, \mathbf{C}, \boldsymbol{K}) = k_Y(\kappa_Y, \mathbf{C}, \boldsymbol{K})$$
$$K_X(\kappa_X, \mathbf{C}, \boldsymbol{K}) = K_Y(\kappa_Y, \mathbf{C}, \boldsymbol{K}).$$

for all $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf$. Then, they certainly also hold for the inverse transformation

$$(\mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}) \in \mathscr{C}onf$$

so that

$$k_X(\kappa_X, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}) = k_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$
$$K_X(\kappa_X, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}) = K_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$

for all $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf$. If we multiply the first equation by $\mathbf{A}_X^{-1} * J_X$ and the second one by $\mathbf{A}_X^{-1} \circ J_X$, then we see that the left hand sides give the elastic laws in the reference placement $\underline{\kappa}_X$, so that we achive

$$k_X(\underline{\kappa}_X, \mathbf{C}, \boldsymbol{K}) = \mathbf{A}_X^{-1} * J_X\, k_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K})$$
$$K_X(\underline{\kappa}_X, \mathbf{C}, \boldsymbol{K}) = \mathbf{A}_X^{-1} \circ J_X\, K_Y(\kappa_Y, \mathbf{A}_X^{-T} * \mathbf{C}, -\mathbf{A}_X^{-T} \circ \boldsymbol{K}_{\mathbf{A}X} + \mathbf{A}_X^{-T} \circ \boldsymbol{K}).$$

By interpreting the right hand side as a change of the reference placement for Y and keeping in mind the above Remark, we see that the isomorphy conditions hold for this choice of reference placements as well. By the notations $\mathbf{A}_X =: \mathbf{P}$ and $\boldsymbol{K}_{\mathbf{A}} =: \boldsymbol{P}$, the following equivalent, but simpler isomorphy conditions result.[8]

Theorem 1 *Two elastic material points X and Y with elastic laws k_X, K_X and k_Y, K_Y with respect to arbitrary reference placements are elastically isomorphic if and only if there exist two tensors $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{Inv}$ such that*

$$\rho_{0X} = \rho_{0Y}\ det(\mathbf{P}) \tag{65}$$
$$k_Y(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P})\,[\mathbf{P} * k_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P})] \tag{66}$$
$$K_Y(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P})\,[\mathbf{P} \circ K_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P})] \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf} \tag{67}$$

hold with ρ_{0X} and ρ_{0Y} being the mass densities in the reference placements of X and Y, respectively.

The dyad $\mathbf{P}$ can be interpreted as the gradient of the change of the reference placement and the triad $\boldsymbol{P}$ as its second gradient. However, in a local theory, these two tensors can be considered as being independent of each other.

The last two conditions are fulfilled if and only if the specific elastic energy satisfies

$$w_Y(\mathbf{C}, \boldsymbol{K}) = w_X(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) + w_0 \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf} \tag{68}$$

with some constant w_0.

8 Material symmetry

If we particularize the concept of isomorphy to identical points $X \equiv Y$, it defines *automorphy* or **symmetry**. In this case, we consider only one point so that we can drop the point index, and denote the automorphism by $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{Inv}$ to distinguish from the isomorphisms of the previous section. Because of the first isomorphy condition, any automorphism must be proper unimodular in its first entry: $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{Unim}$. This leads us to the following definition using (66) and (67).

Definition 3 For a gradient elastic material with material laws k and K, a **symmetry transformation** is a pair $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{Unim}$ such that

$$k(\mathbf{C}, \boldsymbol{K}) = \mathbf{A} * k(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}) \tag{69}$$
$$K(\mathbf{C}, \boldsymbol{K}) = \mathbf{A} \circ K(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}) \tag{70}$$

holds for all $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$.

For the elastic energy, the symmetry transformation is

$$w(\mathbf{C}, K) = w(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \boldsymbol{K} + \boldsymbol{A}) \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{71}$$

The set of all such symmetry transformations represented by such a couple $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{Unim}$ represents the **symmetry group** of the material. In fact, the transformation is a group under composition in the algebraic sense. Its identity is $(\mathbf{I}, \boldsymbol{O}) \in \mathscr{Unim}$, and the inverse of some $(\mathbf{A}, \boldsymbol{A}) \in \mathscr{Unim}$ is $(\mathbf{A}^{-1}, -\mathbf{A}^{-T} \circ \boldsymbol{A}) \in \mathscr{Unim}$.

This group is used to define isotropy or anisotropy. If the symmetry group is a subgroup of Orth in the first entry and the zero in the second, $(\mathbf{Q}, \boldsymbol{O})$, these transformations can be interpreted as rigid rotations, and we call the respective reference placement an **undistorted state**. If a material allows for such undistorted states, it is a **solid**. If it contains all orthogonal dyads in the first entry, then the material is called **isotropic**. These definitions apply not only to gradient elasticity and hyperelasticity, but also to any inelastic gradient material in an analogous way.

In all of these cases, we obtain after (59) and (60) with respect to undistorted states

$$\mathbf{A} * k\,(\mathbf{C}, \boldsymbol{K}) = k(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}) \tag{72}$$
$$\mathbf{A} * K(\mathbf{C}, \boldsymbol{K}) = K(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}) \tag{73}$$
$$w(\mathbf{C}, \boldsymbol{K}) = w(\mathbf{A} * \mathbf{C}, \mathbf{A} * \boldsymbol{K}) \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{74}$$

[8] This theorem has been used for simple elastic materials in [10]. In a more general format is has been shown already in [6, p. 111] and [7, p. 206].

Thus, for an isotropic material the elastic laws are isotropic tensor functions.

In [43][9] one finds interesting considerations about symmetry of second gradient materials. Murdoch uses other configuration variables in his work, namely $\mathbf{F}$ and $\mathbf{F}^T$ *Grad* $\mathbf{F}$, the latter being a material quantity, in contrast to the first one.

9 Linear elasticity

For many applications, the elastic deformations are rather small, which justifies the linearization of the hyperelastic laws. In this case, one would assume a square form of the configuration tensors for the energy.

In order to avoid the introduction of new notations like a generalized Voigt notation, we use a tensor notation.

The following multiplications for a tensor $\overset{\langle m\rangle}{\boldsymbol{D}}$ and a higher-order tensor $\overset{\langle k\rangle}{\boldsymbol{E}}(k > m)$

$$\overset{\langle k\rangle}{\boldsymbol{E}}[\overset{\langle m\rangle}{\boldsymbol{D}}] \quad \text{and} \quad [\overset{\langle m\rangle}{\boldsymbol{D}}]\overset{\langle k\rangle}{\boldsymbol{E}} \tag{75}$$

denote the m-fold contractions from the respective side. Thus,

$$\overset{\langle k\rangle}{\boldsymbol{E}}[\overset{\langle m\rangle}{\boldsymbol{D}}] = E_{i_1\ldots i_k} D^{i_{k-m+1}\ldots i_k} \mathbf{e}_{i_1} \otimes \cdots \otimes \mathbf{e}_{i_{k-m}} \tag{76}$$

$$[\overset{\langle m\rangle}{\boldsymbol{D}}]\overset{\langle k\rangle}{\boldsymbol{E}} = D^{i_m\ldots i_1} E_{i_1\ldots i_k} \mathbf{e}_{i_{m+1}} \otimes \cdots \otimes \mathbf{e}_{i_k}. \tag{77}$$

Note that in the first application the contraction order is reverse to the second one.

The major transposed of an even order tensor is denoted by $\overset{\langle 2k\rangle}{\boldsymbol{E}}{}^T$ and defined by the identity

$$\overset{\langle 2k\rangle}{\boldsymbol{E}}{}^T[\overset{\langle k\rangle}{\boldsymbol{D}}] := [\overset{\langle k\rangle}{\boldsymbol{D}}]\overset{\langle 2k\rangle}{\boldsymbol{E}} \tag{78}$$

for arbitrary $\overset{\langle k\rangle}{\boldsymbol{D}}$. Such an even order tensor is (major) symmetric if

$$\overset{\langle 2k\rangle}{\boldsymbol{E}} = \overset{\langle 2k\rangle}{\boldsymbol{E}}{}^T \quad \text{or} \quad \overset{\langle 2k\rangle}{\boldsymbol{E}}{}_{i_1\ldots i_{2k}} = \overset{\langle 2k\rangle}{\boldsymbol{E}}{}_{i_{2k}\ldots i_1}. \tag{79}$$

In the physically linear elasticity theory, the elastic energy is assumed to be a square form of the configuration $(\mathbf{C}, \boldsymbol{K})$. Such a form on $\mathscr{Conf}$ would have $24^2/2 + 24/2 = 300$ parameters. In tensor notations it can be represented by

$$w(\mathbf{C}, \boldsymbol{K}) = \frac{1}{4\rho_0}[\mathbf{C} - \mathbf{C}_u]\overset{\langle 4\rangle}{\boldsymbol{E}}[\mathbf{C} - \mathbf{C}_u] + \frac{1}{2\rho_0}[\mathbf{C} - \mathbf{C}_u]\overset{\langle 5\rangle}{\boldsymbol{E}}[\boldsymbol{K} - \boldsymbol{K}_u] + \frac{1}{4\rho_0}[\boldsymbol{K} - \boldsymbol{K}_u]\overset{\langle 6\rangle}{\boldsymbol{E}}[\boldsymbol{K} - \boldsymbol{K}_u] \tag{80}$$

with higher order elasticity tensors $\overset{\langle 4\rangle}{\boldsymbol{E}}, \overset{\langle 5\rangle}{\boldsymbol{E}}, \overset{\langle 6\rangle}{\boldsymbol{E}}$ and some unloaded configuration $(\mathbf{C}_u, \boldsymbol{K}_u) \in \mathscr{Conf}$.

These elasticities can be submitted to the following symmetry conditions:

$\overset{\langle 4\rangle}{\boldsymbol{E}}$:

- left subsymmetry $\{ijkl\} = \{jikl\}$
- right subsymmetry $\{ijkl\} = \{ijlk\}$
- and the major symmetry $\{ijkl\} = \{lkji\}$

with 21 independent constants as customary from classical elasticity

[9] See also [21,24] and [36] for such considerations related to the symmetry group.

$\overset{\langle 5\rangle}{\boldsymbol{E}}$:

- left subsymmetry $\{ijklm\} = \{jiklm\}$
- right subsymmetry $\{ijklm\} = \{ijkml\}$

with 108 independent parameters
$\overset{\langle 6\rangle}{\boldsymbol{E}}$:

- subsymmetry in the 1st and 2nd indices $\{ijklmn\} = \{jiklmn\}$
- right subsymmetry $\{ijklmn\} = \{ijklnm\}$
- and major symmetry $\{ijklmn\} = \{nmlkji\}$

with 171 independent parameters

This gives in total again 300 constants, which can eventually be reduced by the exploitation of symmetry properties. The isotropic versions of the elastic energy can be found in, e.g., [41] and [13] with only seven independent parameters including the two Lamé constants from classical elasticity.[10]

The elastic energy (80) acts as a potential for the stresses with (49) and (50)

$$k(\mathbf{C}, \boldsymbol{K}) = \overset{\langle 4\rangle}{\boldsymbol{E}}[\mathbf{C} - \mathbf{C}_u] + \overset{\langle 5\rangle}{\boldsymbol{E}}[\boldsymbol{K} - \boldsymbol{K}_u] \tag{81}$$

$$K(\mathbf{C}, \boldsymbol{K}) = 1/2[\mathbf{C} - \mathbf{C}_u]\overset{\langle 5\rangle}{\boldsymbol{E}} + 1/2\overset{\langle 6\rangle}{\boldsymbol{E}}[\boldsymbol{K} - \boldsymbol{K}_u] \quad \forall(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{82}$$

These laws are straightforward extensions of the St.-Venant–Kirchhoff law to gradient elasticity. They are physically linear, but geometrically nonlinear, and they fulfill the Euclidean invariance requirement. Note that the linear theory depends on the choice of the stress and configuration variables, in contrast to the preceding nonlinear theory. However, for small deformations, the differences remain negligible.

In the linear case, the isomorphy conditions (66) and (67) become with $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{Inv}$

$$\begin{aligned}&\overset{\langle 4\rangle}{\boldsymbol{E}}_Y[\mathbf{C} - \mathbf{C}_{uY}] + \overset{\langle 5\rangle}{\boldsymbol{E}}_Y[\boldsymbol{K} - \boldsymbol{K}_{uY}]\\ &\quad = \mathbf{P} * det^{-1}(\mathbf{P})\,\{\overset{\langle 4\rangle}{\boldsymbol{E}}_X[\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{uX}] + \overset{\langle 5\rangle}{\boldsymbol{E}}_X[\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{uX}]\}\end{aligned} \tag{83}$$

and

$$\begin{aligned}&[\mathbf{C} - \mathbf{C}_{uY}]\overset{\langle 5\rangle}{\boldsymbol{E}}_Y + \overset{\langle 6\rangle}{\boldsymbol{E}}_Y[\boldsymbol{K} - \boldsymbol{K}_{uY}]\\ &\quad = \mathbf{P} \circ det^{-1}(\mathbf{P})\,\{[\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{uX}]\overset{\langle 5\rangle}{\boldsymbol{E}}_X + \overset{\langle 6\rangle}{\boldsymbol{E}}_X[\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{uX}]\}.\end{aligned} \tag{84}$$

By a comparison in the independent variables $(\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}$ one can determine the transformations of the elasticities and the unloaded configuration like

$$\overset{\langle 4\rangle}{\boldsymbol{E}}_Y = \mathbf{P} * det^{-1}(\mathbf{P})\overset{\langle 4\rangle}{\boldsymbol{E}}_X \quad \text{and} \quad \mathbf{C}_{uY} = \mathbf{P}^{-T} * \mathbf{C}_{uX}. \tag{85}$$

The linearity of the elastic laws will not be assumed in what follows, in order to preserve full generality.

10 Gradient elastoplasticity

For a gradient theory of elastoplasticity, we consider materials for which both the elastic and the plastic behavior are assumed to be of gradient type.

While in an inelastic theory, one would expect that the current stresses depend on the entire deformation process, in elastoplasticity the situation is different. Here, one assumes that after some deformation process the material is within some elastic range for which elastic laws for the stresses are given. Thus, the stresses can be determined by these current elastic laws. And this holds also for any continuation of the deformation process as long as it does not leave the current elastic range. The latter would mean that the material continuously passes through different elastic ranges, a process which characterizes *yielding*. We want to make these concepts more precise.

[10] See [2,22,50].

Definition 4 An **elastic range** is a triple $\{\mathfrak{E}_p, k_p, K_p\}$ consisting of

(1) a non-empty path-connected submanifold with boundary $\mathfrak{E}_p \subset \mathscr{Conf}$ and
(2) elastic laws

$$\mathbf{S} = k_p(\mathbf{C}, \boldsymbol{K}) \tag{86}$$
$$\boldsymbol{S}_{\mathrm{K}} = K_p(\mathbf{C}, \boldsymbol{K}) \tag{87}$$

such that after any continuation process $\{\mathbf{C}(\tau), \boldsymbol{K}(\tau)\}\big|_{t_o}^{t}$, which remains entirely in $\mathfrak{E}_p$

$$\{\mathbf{C}(\tau), \boldsymbol{K}(\tau)\} \in \mathfrak{E}_p \quad \forall \tau \in [t_o, t]$$

the stresses are determined by the final values of the process by elastic laws

$$\mathbf{S}\,(t) = k_p(\mathbf{C}(t), \boldsymbol{K}\,(t)) \tag{88}$$
$$\boldsymbol{S}_{\mathrm{K}}\,(t) = k_p(\mathbf{C}(t), \boldsymbol{K}\,(t)). \tag{89}$$

Note that the two elastic laws are physically determined only for configurations within the specific elastic range $\mathfrak{E}_p$. However, in the sequel we will extend them to the entire space $\mathscr{Conf}$ for simplicity. The next assumption specifies the behavior of a material with elastic ranges.

Assumption At each instant, the elastoplastic material point is associated with an elastic range.

11 Isomorphy of the elastic ranges

During yielding two effects have to be considered. Firstly, the elastic range has to be changed reflecting the hardening or softening of the material. And secondly, the elastic laws associated to these elastic ranges evolve. We will first address this second effect.

For many materials, it is a micro-physically and experimentally well-substantiated fact that during yielding the elastic behavior hardly alters even under very large deformations. This reduces the effort for the identification tremendously, as otherwise one would have to identify the elastic constants at each step of the deformation anew. We now give this assumption a precise form.

Assumption The elastic laws of all elastic ranges are isomorphic.

As a consequence, if $\{\mathfrak{E}_1, k_1, K_1\}$ and $\{\mathfrak{E}_2, k_2, K_2\}$ are two elastic ranges, then according to (65), (66) and (67) there exist two tensors $(\mathbf{P}_{12}, \boldsymbol{P}_{12}) \in \mathscr{Inv}$ such that

- for the mass densities in the reference placements ρ_{01} and ρ_{02} holds

$$\rho_{01} = \rho_{02}\, det\, \mathbf{P}_{12} \tag{90}$$

- and for the elastic laws we have the equalities

$$k_2(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P}_{12})[\mathbf{P}_{12} * k_1(\mathbf{P}_{12}^T * \mathbf{C}, \mathbf{P}_{12}^T \circ \boldsymbol{K} + \boldsymbol{P}_{12})] \tag{91}$$
$$K_2(\mathbf{C}, \boldsymbol{K}) = det^{-1}(\mathbf{P}_{12})[\mathbf{P}_{12} \circ K_1(\mathbf{P}_{12}^T * \mathbf{C}, \mathbf{P}_{12}^T \circ \boldsymbol{K} + \boldsymbol{P}_{12})] \quad \forall (\mathbf{C}, \boldsymbol{K}) \in \mathscr{Conf}. \tag{92}$$

As we have chosen a joint reference placement for all elastic laws of one particular material point (this is, however, not compulsory), we already have $\rho_{01} \equiv \rho_{02}$ and therefore $\mathbf{P}_{12}$ must be proper unimodular, so that the first isomorphy condition (90) is always fulfilled.

Note that in the above Assumption nothing is said about the form or size of $\mathfrak{E}_1$ and $\mathfrak{E}_2$. So, the hardening behavior is not at all restricted by it.

If all elastic laws belonging to different elastic ranges are mutually isomorphic, then because of the group property of isomorphy transformations, they all are isomorphic to some freely chosen **elastic reference laws** k_0 and K_0. While the current elastic laws k_p and K_p vary with time during yielding, these reference laws can always be chosen as constant in time. We thus have the isomorphy condition in the following form.

Theorem 2 *Let k_0 and K_0 be the elastic reference laws for an elastoplastic material. Then, for each elastic range $\{\mathfrak{E}_p, k_p, K_p\}$ there are two tensors $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim$ such that*

$$\mathbf{S} = k_p(\mathbf{C}, \boldsymbol{K}) = \mathbf{P} * k_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \tag{93}$$

$$\boldsymbol{S}_K = K_p(\mathbf{C}, \boldsymbol{K}) = \mathbf{P} \circ K_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \quad \forall (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf. \tag{94}$$

Instead of the two elastic reference laws, we can also introduce an **elastic reference energy** w_0 after (68) such that

$$w_p(\mathbf{C}, \boldsymbol{K}) = w_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \quad \forall (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf \tag{95}$$

and the potentials hold in the form of (49) and (50), which again gives (93) and (94).[11]

If one linearizes these elastic laws, then we obtain after (83) and (84)

$$\begin{aligned} &\overset{\langle 4 \rangle}{\boldsymbol{E}}_p[\mathbf{C} - \mathbf{C}_{up}] + \overset{\langle 5 \rangle}{\boldsymbol{E}}_p[\boldsymbol{K} - \boldsymbol{K}_{up}] \\ &\quad = \mathbf{P} * \{\overset{\langle 4 \rangle}{\boldsymbol{E}}_0[\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{u0}] + \overset{\langle 5 \rangle}{\boldsymbol{E}}_0[\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{u0}]\} \end{aligned} \tag{96}$$

and

$$\begin{aligned} &[\mathbf{C} - \mathbf{C}_{up}]\, \overset{\langle 5 \rangle}{\boldsymbol{E}}_p + \overset{\langle 6 \rangle}{\boldsymbol{E}}_p[\boldsymbol{K} - \boldsymbol{K}_{up}] \\ &\quad = \mathbf{P} \circ \{[\mathbf{P}^T * \mathbf{C} - \mathbf{C}_{u0}]\, \overset{\langle 5 \rangle}{\boldsymbol{E}}_0 + \overset{\langle 6 \rangle}{\boldsymbol{E}}_0[\mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P} - \boldsymbol{K}_{u0}]\} \end{aligned} \tag{97}$$

where the suffix p indicates the (time-dependent) quantities related to the linear forms of k_p and K_p, and the suffix 0 to the (time-independent) ones of the linear forms of the elastic reference laws k_0 and K_0. Again, one can determine the transformations of the elasticities and the unloaded configuration as in (85)

$$\overset{\langle 4 \rangle}{\boldsymbol{E}}_p = \mathbf{P} * \overset{\langle 4 \rangle}{\boldsymbol{E}}_0 \quad \text{and} \quad \mathbf{C}_{up} = \mathbf{P}^{-T} * \mathbf{C}_{u0}. \tag{98}$$

In the present theory, the two variables $(\mathbf{P}, \boldsymbol{P}) \in \mathscr{U}nim$ are chosen as the plastic internal variables. One might be tempted to interpret (93) and (94) as both an additive *and* a multiplicative decomposition of the kinematical variables.[12] They are, however, not introduced as deformations but rather as a transformation of the current elastic law (not of a placement!) to a time-independent reference law, which results in a natural way from the isomorphy condition. We avoid the introduction of an intermediate configuration or a split of some deformation into elastic and plastic parts since it is misleading in a finite deformation theory.[13]

12 Yield criteria

Let us first consider one particular elastic range $\{\mathfrak{E}_p, k_p, K_p\}$. We decompose the $\mathfrak{E}_p$ topologically into its interior $\mathfrak{E}_p^o$ and its boundary $\partial\mathfrak{E}_p$. The latter is called **yield surface** (in the configuration space). In order to describe it more easily, we introduce a real-valued tensor-function in the configuration space

$$\Phi_p : \mathscr{C}onf \to \mathscr{R} \;\; | \;\; (\mathbf{C}, \boldsymbol{K}) \mapsto \Phi_p(\mathbf{C}, \boldsymbol{K})$$

the kernel of which coincides with the yield limit

$$\Phi_p(\mathbf{C}, \boldsymbol{K}) = 0 \Leftrightarrow (\mathbf{C}, \boldsymbol{K}) \in \partial\mathfrak{E}_p. \tag{99}$$

For distinguishing points in the interior and in the exterior of the elastic ranges, we postulate

$$\Phi_p(\mathbf{C}, \boldsymbol{K}) < 0 \Leftrightarrow (\mathbf{C}, \boldsymbol{K}) \in \mathfrak{E}_p^o \tag{100}$$

and, consequently,

$$\Phi_p(\mathbf{C}, \boldsymbol{K}) > 0 \Leftrightarrow (\mathbf{C}, \boldsymbol{K}) \in \mathscr{C}onf \backslash \mathfrak{E}_p. \tag{101}$$

We call such an indicator function or level set function a **yield criterion**, and assume further-on that Φ_p is at least piecewise differentiable.

Instants of **yielding** are characterized by two facts.

---

[11] See the *alternative decomposition* of Forest and Sievert [27].

[12] In [16] such an interpretation is given.

[13] See the comments in [10] on p. 291.

(1) The configuration is currently on the yield limit and, thus, fulfils its **yield condition**

$$\Phi_p(\mathbf{C}, \boldsymbol{K}) = 0. \tag{102}$$

(2) It is about to leave the current elastic range, i.e., the **loading condition**

$$\Phi_p^\bullet = \partial_{\mathbf{C}}\Phi_p \cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\Phi_p \cdot \boldsymbol{K}^\bullet > 0 \tag{103}$$

is fulfilled.

Such a yield criterion is associated with some particular elastic range. In order to obtain a general yield criterion which holds for all elastic ranges in the same form, we have to introduce additional internal variables $\mathbf{Z}$ called **hardening variables** (although they could also describe softening). These can be tensors of arbitrary order or even a vector of such tensors and, thus, form elements of some finite dimensional linear space, the specification of which depends on the particular hardening model. The general form of the yield criterion is assumed to be like

$$\varphi(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}). \tag{104}$$

With this extension, we obtain for the yield condition (102)

$$\varphi(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}) = 0 \tag{105}$$

and for the loading condition (103)

$$\partial_{\mathbf{C}}\varphi \cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\varphi \cdot \boldsymbol{K}^\bullet > 0 \tag{106}$$

where the plastic and the hardening variables are kept constant. As an example, a generalization of the v. Mises yield criterion for gradient materials can be found in [13].

13 Decomposition of the stress power

We will next consider the stress power (40) again and specify it for our elastoplastic material. The specific stress power is with (86), (87), (93), (94), (15)

$$\begin{aligned} p &= 1/\rho_0\ [1/2\ k_p(\mathbf{C}, \boldsymbol{K}) \cdot \mathbf{C}^\bullet + K_p(\mathbf{C}, \boldsymbol{K}) \cdot \boldsymbol{K}^\bullet] \\ &= 1/\rho_0\ [1/2\ \mathbf{P} * k_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \cdot \mathbf{C}^\bullet + \mathbf{P} \circ K_0(\mathbf{P}^T * \mathbf{C}, \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}) \cdot \boldsymbol{K}^\bullet] \\ &= 1/\rho_0\ [1/2\ k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot (\mathbf{P}^T * \mathbf{C}^\bullet) + K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot (\mathbf{P}^T \circ \boldsymbol{K}^\bullet)] \end{aligned} \tag{107}$$

with the abbreviations

$$\mathbf{C}_e := \mathbf{P}^T\mathbf{C}\,\mathbf{P} = \mathbf{P}^T * \mathbf{C} \tag{108}$$

$$\boldsymbol{K}_e := \mathbf{P}^T \circ \boldsymbol{K} + \boldsymbol{P}. \tag{109}$$

This gives for the rates

$$\mathbf{C}_e^\bullet = (\mathbf{P}^T\mathbf{C}\,\mathbf{P})^\bullet = \mathbf{P}^T\mathbf{C}^\bullet\mathbf{P} + 2\,sym(\mathbf{P}^T\mathbf{C}\,\mathbf{P}^\bullet) = \mathbf{P}^T * \mathbf{C}^\bullet + 2\,sym(\mathbf{C}_e\mathbf{P}^{-1}\mathbf{P}^\bullet) \tag{110}$$

where *sym* stands for the symmetric part, and

$$\begin{aligned} \boldsymbol{K}_e^\bullet &= \mathbf{P}^T \circ \boldsymbol{K}^\bullet + \boldsymbol{P}^\bullet \\ &\quad + K_{ijk}\{(\mathbf{P}^{-1\bullet}\mathbf{e}_i) \otimes (\mathbf{P}^T\mathbf{e}_j) \otimes (\mathbf{P}^T\mathbf{e}_k) + (\mathbf{P}^{-1}\mathbf{e}_i) \otimes (\mathbf{P}^{T\bullet}\mathbf{e}_j) \otimes (\mathbf{P}^T\mathbf{e}_k) + (\mathbf{P}^{-1}\mathbf{e}_i) \otimes (\mathbf{P}^T\mathbf{e}_j) \otimes (\mathbf{P}^{T\bullet}\mathbf{e}_k)\} \\ &= \mathbf{P}^T \circ \boldsymbol{K}^\bullet + \boldsymbol{P}^\bullet - \mathbf{P}^{-1}\mathbf{P}^\bullet(\boldsymbol{K}_e - \boldsymbol{P}) + 2\,subsym[(\boldsymbol{K}_e - \boldsymbol{P})\mathbf{P}^{-1}\mathbf{P}^\bullet] \end{aligned} \tag{111}$$

the term with *subsym* being the symmetric part with respect to the right subsymmetry. We substitute this into (107)

$$\begin{aligned} p = 1/\rho_0\{&1/2\,k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot [\mathbf{C}_e^\bullet - 2\,sym(\mathbf{C}_e\mathbf{P}^{-1}\mathbf{P}^\bullet)] \\ &+ K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot [\boldsymbol{K}_e^\bullet - \boldsymbol{P}^\bullet + \mathbf{P}^{-1}\mathbf{P}^\bullet(\boldsymbol{K}_e - \boldsymbol{P}) - 2\,subsym[(\boldsymbol{K}_e - \boldsymbol{P})\mathbf{P}^{-1}\mathbf{P}^\bullet]\} \end{aligned}$$

and because of the symmetries of the stress tensors

$$
\begin{aligned}
&= 1/\rho_0\{1/2\, k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot \mathbf{C}_e^\bullet + K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot \mathbf{K}_e^\bullet \\
&\quad - 1/2\, k_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot (2\mathbf{C}_e\mathbf{P}^{-1}\mathbf{P}^\bullet) - K_0(\mathbf{C}_e, \boldsymbol{K}_e) \cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1}\mathbf{P}^\bullet(\boldsymbol{K}_e - \boldsymbol{P}) + 2(\boldsymbol{K}_e - \boldsymbol{P})\mathbf{P}^{-1}\mathbf{P}^\bullet]\} \\
&= w_0(\mathbf{C}_e, \boldsymbol{K}_e)^\bullet + \mathbf{S}_p \cdot \mathbf{P}^\bullet + \boldsymbol{S}_p \cdot [\boldsymbol{P}^\bullet - \mathbf{P}^{-1}\mathbf{P}^\bullet(\boldsymbol{K}_e - \boldsymbol{P}) + 2(\boldsymbol{K}_e - \boldsymbol{P})\mathbf{P}^{-1}\mathbf{P}^\bullet]
\end{aligned} \tag{112}
$$

with (95) and the *plastic stress* and *plastic hyper-stress tensor* defined as

$$\mathbf{S}_p := -\mathbf{P}^{-T}\mathbf{C}_e k_0(\mathbf{C}_e, \boldsymbol{K}_e) = -\mathbf{C}\,\mathbf{S}\,\mathbf{P}^{-T} \tag{113}$$

$$\boldsymbol{S}_p := -K_0(\mathbf{C}_e, \boldsymbol{K}_e) = -\mathbf{P}^{-1} \circ \boldsymbol{S}_{\mathrm{K}}. \tag{114}$$

According to (112) the stress power goes into a change of the elastic reference energy and a dissipative part that is only active during yielding and working on the configuration rates $\mathbf{P}^\bullet$ and $\boldsymbol{P}^\bullet$.

14 Flow and hardening rules

For the evolution of the internal plastic variables $\mathbf{P}$, $\boldsymbol{P}$, $\mathbf{Z}$ evolution equations are needed, namely two **flow rules**

$$\mathbf{P}^\bullet = f(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet) \tag{115}$$

$$\boldsymbol{P}^\bullet = F(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet) \tag{116}$$

and a **hardening rule**

$$\mathbf{Z}^\bullet = h(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\bullet, \boldsymbol{K}^\bullet) \tag{117}$$

all assumed to be rate-independent as customary in plasticity. As a special case, one may also choose $\boldsymbol{P}$ as the gradient of $\mathbf{P}$.[14] In this case, only one flow rule (115) is needed. In the general case, the rate-independence can be assured in the usual way by the introduction of a **plastic consistency parameter** $\lambda \geq 0$

$$\mathbf{P}^\bullet = \lambda f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \tag{118}$$

$$\boldsymbol{P}^\bullet = \lambda F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \tag{119}$$

$$\mathbf{Z}^\bullet = \lambda h^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \tag{120}$$

where we normed the increments of the kinematic variables

$$\mathbf{C}^\circ := \mathbf{C}^\bullet/\mu \quad \text{and} \quad \boldsymbol{K}^\circ := \boldsymbol{K}^\bullet/\mu \tag{121}$$

by a factor

$$\mu := \surd(\mid \mathbf{C}^\bullet \mid^2 + L^2 \mid \boldsymbol{K}^\bullet \mid^2) \tag{122}$$

which is (only) positive during yielding. The positive constant L with the dimension of a length is necessary for dimensional reasons and controls the ratio of yielding due to $\mathbf{C}^\bullet$ and $\boldsymbol{K}^\bullet$. We introduced three functions f°, F°, h°, which give the directions of the flow and hardening, while the amount is finally determined by the consistency parameter. The consistency parameter is zero during elastic processes. During yielding, it can be calculated by the yield condition (105) using (118)–(120)

$$
\begin{aligned}
0 &= \varphi(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z})^\bullet \\
&= \partial_{\mathbf{P}}\varphi \cdot \mathbf{P}^\bullet + \partial_{\boldsymbol{P}}\varphi \cdot \mathbf{P}^\bullet + \partial_{\mathbf{C}}\varphi \cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\varphi \cdot \boldsymbol{K}^\bullet + \partial_{\mathbf{Z}}\varphi \cdot \mathbf{Z}^\bullet \\
&= \partial_{\mathbf{P}}\varphi \cdot \lambda f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) + \partial_{\boldsymbol{P}}\varphi \cdot \lambda F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \\
&\quad + \partial_{\mathbf{C}}\varphi \cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\varphi \cdot \mathbf{K}^\bullet + \partial_{\mathbf{Z}}\varphi \cdot \lambda h^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ)
\end{aligned} \tag{123}
$$

which gives

$$
\begin{aligned}
\lambda = &-[\partial_{\mathbf{C}}\varphi \cdot \mathbf{C}^\bullet + \partial_{\boldsymbol{K}}\varphi \cdot \boldsymbol{K}^\bullet]/[\partial_{\mathbf{P}}\varphi \cdot f^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) + \partial_{\boldsymbol{P}}\varphi \cdot F^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ) \\
&+ \partial_{\mathbf{Z}}\varphi \cdot h^\circ(\mathbf{P}, \boldsymbol{P}, \mathbf{C}, \boldsymbol{K}, \mathbf{Z}, \mathbf{C}^\circ, \boldsymbol{K}^\circ)].
\end{aligned} \tag{124}
$$

[14] This applies in particular if one uses the dislocations tensor $\mathbf{F}_{\mathrm{p}}\, Curl\, \mathbf{F}_{\mathrm{p}}$ where $\mathbf{F}_{\mathrm{p}}$ can be identified with $\mathbf{P}^{-1}$, see [32].

As a consequence of the loading condition (106), λ is positive during yielding, and zero otherwise. If we substitute this value of λ in (118)–(120), we obtain the **consistent flow** and **hardening rules**. In all cases, (elastic *and* plastic), the Kuhn–Tucker condition

$$\lambda\varphi = 0 \quad \text{with} \quad \lambda \geq 0 \quad \text{and} \quad \varphi \leq 0 \tag{125}$$

holds since at any time one of the two factors is zero.

15 Conclusions

The present text can be considered as a continuation of the work of Forest and Sievert [27] and Sievert [49]. A finite gradient plasticity theory is there envisaged only in its basics. Unfortunately, Rainer Sievert passed away too early to finish his promising concepts. So, we tried to continue with this task in the spirit of his ideas.

The starting point there and here is a higher order expression of the stress power.[15] The additional variables for extending the theory of simple materials are the third-order curvature tensor and a work-conjugate hyper-stress tensor. By giving both of them a material form which is invariant under Euclidean transformations, we obtain reduced forms for the elastic law and the elastoplastic model. The choice of these variables, however, is immaterial for the results. Any other choice would describe exactly the same material behavior. So far, we followed the lines of the above mentioned articles.

While the introduction of finite gradient elasticity appears to be rather straightforward, for finite plasticity this is certainly not the case. Here, we are again confronted with conceptual questions, which already have been discussed during the last 50 years for finite plasticity of simple materials. We tried to solve these problems by the same concept of *isomorphic elastic ranges* which gave good results already in the much simpler context of non-gradient plasticity (see [8,10]). It turns out that also in the case of gradient plasticity, this concept leads to a natural introduction of additional plastic variables with clear mathematical properties, thus avoiding all the problems and controversies associated with the concept of *unloaded intermediate configurations*.

The intention of this paper is to give a general framework for material models of gradient elasticity and plasticity which is based on clear physical assumptions like

- Euclidean invariance
- isomorphy of the elastic ranges

trying to avoid hidden or tacit assumptions and those lacking a clear physical substantiation. Naturally, the span of this theory turns out to be rather large. This makes the theory, at least in some parts, also more complicated. The latter, however, can be reduced in many cases by restricting to isotropy, linearity, etc.

Commonly, plasticity is understood as rate-independent behavior. Accordingly, we restricted this format to this case. However, it can easily extended to rate-dependent behavior by introducing rate-dependent evolution equations for the flow and the hardening rules.

In a forthcoming article,[16] this theory will be enlarged to include the thermodynamics of finite gradient plasticity following the method of Bertram and Krawietz [12] and Bertram and Forest [13]. In the latter paper, an example for a gradient elastoplastic material model is given.

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[15] See [13].
[16] See [14].

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