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SHORT COMMUNICATION

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Expansion/contraction of a spherical elastic/plastic shell revisited

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Abstract A semi-analytic solution for the elastic/plastic distribution of stress and strain in a spherical shell subject to pressure over its inner and outer radii and subsequent unloading is presented. The Bauschinger effect is taken into account. The flow theory of plasticity is adopted in conjunction with quite an arbitrary yield criterion and its associated flow rule. The yield stress is an arbitrary function of the equivalent strain. It is shown that the boundary value problem is significantly simplified if the equivalent strain is used as an independent variable instead of the radial coordinate. In particular, numerical methods are only necessary to evaluate ordinary integrals and solve simple transcendental equations. An illustrative example is provided to demonstrate the distribution of residual stresses and strains.

Keywords Spherical shell · Bauschinger effect · Semi-analytic solution

1 Introduction

The expansion/contraction of a spherical shell is one of the classical problems of solid mechanics. In particular, solutions for elastic perfectly/plastic shells at small strains can be found in many monographs on plasticity theory, for example [1–3]. In general, solutions for strain-hardening material are outlined in [1,2]. However, it has been mentioned that 'The integration can only be effected by a small-arc process; no investigation of this appears to have been published' in [1] and 'A complete solution can now be obtained by an iterative or successive approximation method' in [2]. It is shown in the present paper that a semi-analytic solution for a spherical shell subject to internal and external pressure and subsequent unloading exists for any hardening law. The flow theory of plasticity is adopted in conjunction with quite an arbitrary yield criterion and its associated flow rule. Strains are supposed to be small. Numerical methods are only necessary to solve transcendental equations and evaluate ordinary integrals. A similar boundary value problem for an elastic shell at large strains has been solved in [4]. In this case, the solution is given in terms of special functions. A number of solutions for elastic perfectly/plastic hollow spheres which can be considered as extensions of the solutions [1–3] are

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available. In particular, thick-walled spheres subject to thermal loading are studied in [5,6]. Functionally graded spherical vessels subject to internal pressure and thermo-mechanical loading have been considered in [7] and [8], respectively. Strain (or work)-hardening models have been adopted in [9–12]. An analytic solution for the contraction of a hollow sphere of linear-hardening material has been found in [9]. However, elastic compressibility has been neglected in this solution. Papers [10–12] deal with thermal loading. Numerical methods are used in [10,11]. A semi-analytical solution for a deformation theory of plasticity based on a modified Ramberg-Osgood stress–strain relation has been derived in [12]. An elegant rigid/plastic solution to describe thermoplastic behavior of a thick-walled sphere has been given in [13]. A solution for strain-hardening material at large strains has been provided in [14] for a fully plastic shell. Distinctive features of the present solution are that the material is elastically compressible, the flow theory of plasticity is used, the solution is valid for any hardening law and the Bauschinger effect is taken into account at the stage of unloading. At the stage of loading, the general solution in the plastic zone can be derived from the solution given in [14] assuming that strains are small.

A particular case of the problem considered in the present paper is the expansion of a spherical cavity in an infinite medium. Available solutions to this problem are more advanced that those for a spherical shell. Most of these solutions have been obtained at large strains and include inertia terms. A recent review of such solutions is provided in [15]. Of particular interest for the present paper is the solution given in [16]. It has been shown in this paper as well as in [14] that it is advantageous to use the equivalent stress as an independent variable in the plastic zone. In the present paper, it is shown that it is advantageous to use the equivalent plastic strain to find the solution for an elastic/plastic shell.

2 Statement of the problem

Consider a spherical shell of internal radius a_0 and external radius b_0 subject to pressure $P_a > 0$ over the internal radius and pressure $P_b > 0$ over the external radius. Introduce a spherical coordinate system (r, θ, φ) with its origin coinciding with the center of the shell. Let σ_r, σ_θ and σ_φ be the normal stresses in this coordinate system. Symmetry dictates that these stresses are the principal stresses and, moreover,

$$\sigma_\theta = \sigma_\varphi. \tag{1}$$

The stress boundary conditions are written in the spherical coordinates as

$$\sigma_r = -P_a \tag{2}$$

for $r = a_0$ and

$$\sigma_r = -P_b \tag{3}$$

for $r = b_0$. Strains are supposed to be small. In plastic regions, the strain tensor is assumed to be the sum of an elastic part and a plastic part. In particular, in the spherical coordinates

$$\varepsilon_r = \varepsilon_r^e + \varepsilon_r^p, \quad \varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p, \quad \varepsilon_\varphi = \varepsilon_\varphi^e + \varepsilon_\varphi^p. \tag{4}$$

Here ε_r, ε_θ and ε_φ are the total normal strains, the superscript e denotes the elastic portion of the total strains and the superscript p denotes the plastic portion of the total strains. The elastic strains are related by Hooke's law to the stresses. In particular, using (1)

$$E\varepsilon_r^e = \sigma_r - 2\nu\sigma_\theta, \quad E\varepsilon_\theta^e = E\varepsilon_\varphi^e = (1-\nu)\,\sigma_\theta - \nu\sigma_r \tag{5}$$

where E is Young's modulus and ν is Poisson's ratio. By virtue of the symmetry, it is not necessary to precisely specify the yield criterion for plastically incompressible materials. In particular, any yield criterion for such materials is represented by a curve in the plane Π whose equation is $\sigma_r + \sigma_\theta + \sigma_\varphi = 0$ (Fig. 1). Equation (1) requires that the state of stress in a plastic zone corresponds to point M (or N) throughout the process of deformation. The solution given in the present paper is valid for all yield loci satisfying the requirement that the normal vector at point M (and N) is parallel to the orthogonal projection of the axis σ_r on the plane Π. A consequence of this requirement and the associated flow rule is that

$$\dot{\varepsilon}_r^p = -2\dot{\varepsilon}_\theta^p = -2\dot{\varepsilon}_\varphi^p. \tag{6}$$

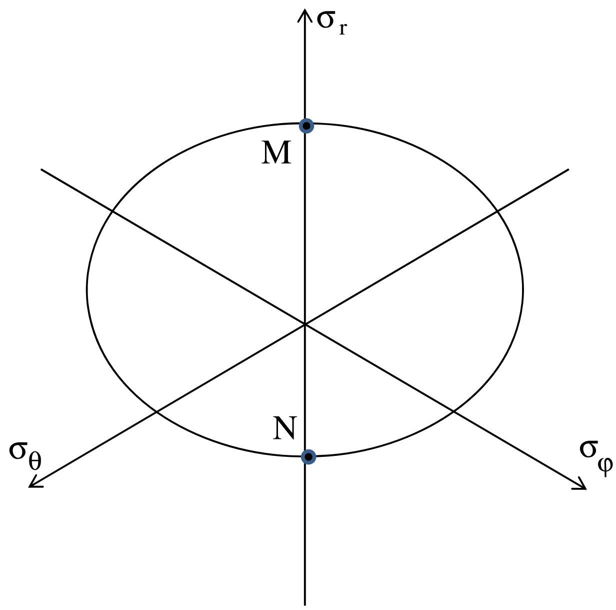

Fig. 1 Geometric representation of the yield criterion

Moreover, $\dot{\varepsilon}_r^p > 0$ at point M and $\dot{\varepsilon}_r^p < 0$ at point N. Here, the superimposed dot denotes the time derivative. Equation (6) can be immediately integrated to give

$$\varepsilon_r^p = -2\varepsilon_\theta^p = -2\varepsilon_\varphi^p. \tag{7}$$

It has been taken into account here that all plastic strains vanish simultaneously. The state of stress at points M (or N) satisfies the following equation

$$|\sigma_r - \sigma_\theta| = \sigma_0 \Phi\left(\varepsilon_{eq}^p\right) \tag{8}$$

where σ_0 is a reference stress, ε_{eq}^p is the equivalent plastic strain and $\Phi\left(\varepsilon_{eq}^p\right)$ is an arbitrary function of its argument satisfying the conditions $\Phi(0) = 1$ and $\Phi'\left(\varepsilon_{eq}^p\right) \equiv d\Phi/d\varepsilon_{eq}^p \geq 0$ for all ε_{eq}^p. Using (6) the equivalent plastic strain rate is represented by the following equation

$$\dot{\varepsilon}_{eq}^p = \sqrt{\frac{2}{3}}\sqrt{\dot{\varepsilon}_r^2 + \dot{\varepsilon}_\theta^2 + \dot{\varepsilon}_\varphi^2} = |\dot{\varepsilon}_r|. \tag{9}$$

The only non-trivial equilibrium equation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\theta)}{r} = 0. \tag{10}$$

3 Solution at loading

The general elastic solution for stress is

$$\frac{\sigma_r}{\sigma_0} = A + \frac{B}{\rho^3}, \quad \frac{\sigma_\theta}{\sigma_0} = A - \frac{B}{2\rho^3} \tag{11}$$

where A and B are constant and $\rho = r/b_0$. It is evident from this solution that in the range $a \leq \rho \leq 1$ the function $|\sigma_r - \sigma_\theta|$ attains its maximum at $\rho = a$ where $a = a_0/b_0$. Therefore, according to (6) the plastic zone starts to develop from the inner radius of the shell. In what follows, it is assumed that the plastic zone exists and its radius increases. Let ρ_c be the dimensionless radius of the elastic/plastic boundary. The solution

(11) is valid in the range $\rho_c \le \rho \le 1$. Then, combining the boundary condition (3) and the solution (11) yields $B = -A - p_b$ and, therefore,

$$\frac{\sigma_r}{\sigma_0} = A\left(1 - \rho^{-3}\right) - p_b\rho^{-3}, \quad \frac{\sigma_\theta}{\sigma_0} = A\left(1 + \frac{1}{2\rho^3}\right) + \frac{p_b}{2\rho^3} \tag{12}$$

in the range $\rho_c \le \rho \le 1$. Here $p_b = P_b/\sigma_0$ (p_a will stand for P_a/σ_0).

Consider the plastic zone. Let τ_r, τ_θ and τ_φ be the deviatoric stress components. Since $\tau_r + \tau_\theta + \tau_\varphi \equiv 0$ and $\tau_\theta = \tau_\varphi$ according to (1), the state of stress corresponding to the points M and N is (Fig. 1)

$$\frac{\tau_r}{\sigma_0} = -\frac{2m}{3}\Phi\left(\varepsilon_{eq}^p\right), \quad \frac{\tau_\theta}{\sigma_0} = \frac{\tau_\varphi}{\sigma_0} = \frac{m}{3}\Phi\left(\varepsilon_{eq}^p\right). \tag{13}$$

Here $m = +1$ for expansion (point N) and $m = -1$ for contraction (point M). Therefore, σ_0 denotes the initial yield stress in tension in the radial direction if $m = -1$ and the initial yield stress in compression in the radial direction if $m = 1$. It is worthy of note that it is not required that these initial yield stresses are of the same magnitude. Substituting (13) into (5) gives

$$\begin{aligned} k^{-1}\varepsilon_r^e &= (1-2\nu)\frac{\sigma}{\sigma_0} - \frac{2m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right), \\ k^{-1}\varepsilon_\theta^e = k^{-1}\varepsilon_\varphi^e &= (1-2\nu)\frac{\sigma}{\sigma_0} + \frac{m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right) \end{aligned} \tag{14}$$

where $k = \sigma_0/E$. It follows from (4) and (14) that

$$\begin{aligned} k^{-1}\varepsilon_r &= k^{-1}\varepsilon_r^p + (1-2\nu)\frac{\sigma}{\sigma_0} - \frac{2m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right), \\ k^{-1}\varepsilon_\theta &= k^{-1}\varepsilon_\theta^p + (1-2\nu)\frac{\sigma}{\sigma_0} + \frac{m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right). \end{aligned} \tag{15}$$

It follows from (6) and (9) that $\dot{\varepsilon}_{eq}^p = 2m\dot{\varepsilon}_\theta^p = -m\dot{\varepsilon}_r^p$. Integrating gives

$$\varepsilon_{eq}^p = 2m\varepsilon_\theta^p = -m\varepsilon_r^p. \tag{16}$$

It has been taken into account here that $\varepsilon_{eq}^p = \varepsilon_r^p = \varepsilon_\theta^p = 0$ at the elastic/plastic boundary. Using (16) equation (15) is transformed to

$$\begin{aligned} k^{-1}\varepsilon_r &= -mk^{-1}\varepsilon_{eq}^p + (1-2\nu)\tfrac{\sigma}{\sigma_0} - \tfrac{2m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right), \\ k^{-1}\varepsilon_\theta &= mk^{-1}\tfrac{\varepsilon_{eq}^p}{2} + (1-2\nu)\tfrac{\sigma}{\sigma_0} + \tfrac{m(1+\nu)}{3}\Phi\left(\varepsilon_{eq}^p\right). \end{aligned} \tag{17}$$

The equation of strain compatibility is $\varepsilon_r = \partial(r\varepsilon_\theta)/\partial r$. Substituting (17) into this equation yields

$$\frac{m(1-2\nu)k}{\sigma_0}\frac{\rho\partial\sigma}{\partial\rho} + \left[\frac{1}{2} + \frac{k(1+\nu)}{3}\Phi'\left(\varepsilon_{eq}^p\right)\right]\frac{\rho\partial\varepsilon_{eq}^p}{\partial\rho} + \frac{3}{2}\varepsilon_{eq}^p + k(1+\nu)\Phi\left(\varepsilon_{eq}^p\right) = 0. \tag{18}$$

Substituting (13) into (10) gives

$$m\frac{\rho}{\sigma_0}\frac{\partial\sigma}{\partial\rho} = 2\Phi\left(\varepsilon_{eq}^p\right) + \frac{2\rho}{3}\Phi'\left(\varepsilon_{eq}^p\right)\frac{\partial\varepsilon_{eq}^p}{\partial\rho}. \tag{19}$$

Eliminating the derivative $\partial\sigma/\partial\rho$ in (18) by means of (19) leads to

$$\frac{1}{3}\left[1 + 2k(1-\nu)\Phi'\left(\varepsilon_{eq}^p\right)\right]\rho\frac{\partial\varepsilon_{eq}^p}{\partial\rho} + \varepsilon_{eq}^p + 2k(1-\nu)\Phi\left(\varepsilon_{eq}^p\right) = 0. \tag{20}$$

This equation can be in general derived from the solution given in [14]. Integrating (20) and using the condition $\varepsilon_{eq}^p = 0$ at $\rho = \rho_c$ result in

$$\left(\frac{\rho_c}{\rho}\right)^3 = \frac{\varepsilon_{eq}^p + 2k(1-\nu)\Phi\left(\varepsilon_{eq}^p\right)}{2k(1-\nu)}. \tag{21}$$

This equation determines ε_{eq}^{p} and a function of ρ in implicit form. Using (20) to eliminate the derivative $\partial\varepsilon_{eq}^{p}/\partial\rho$ in (19) and, then, using (20) again to replace differentiation with respect to ρ with differentiation with respect to ε_{eq}^{p} yields

$$\frac{\partial\sigma}{\sigma_0\partial\varepsilon_{eq}^{p}} = -m\frac{2\left[\Phi\left(\varepsilon_{eq}^{p}\right) - \varepsilon_{eq}^{p}\Phi'\left(\varepsilon_{eq}^{p}\right)\right]}{3\left[\varepsilon_{eq}^{p} + 2k\left(1-\nu\right)\Phi\left(\varepsilon_{eq}^{p}\right)\right]}. \tag{22}$$

Let $\beta\sigma_0$ be the value of σ at $\rho = \rho_c$. Since $\varepsilon_{eq}^{p} = 0$ at $\rho = \rho_c$, the solution of Eq. (22) satisfying this condition is

$$\frac{\sigma}{\sigma_0} = -m\frac{2}{3}\int_0^{\varepsilon_{eq}^{p}} \frac{\left[\Phi\left(\chi\right) - \chi\Phi'\left(\chi\right)\right]}{\left[\chi + 2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi + \beta \tag{23}$$

where χ is a dummy variable of integration. The radial stress must be continuous across the elastic/plastic boundary. The material just on the elastic side of the elastic/plastic boundary must satisfy the yield criterion [1]. Therefore, the circumferential stress is also continuous across the elastic/plastic boundary. Using (12), (13) and (23), these two conditions can be written as

$$A\left(1-\rho_c^{-3}\right) - p_b\rho_c^{-3} = \beta - \frac{2m}{3}, \quad A\left(1+\frac{1}{2\rho_c^3}\right) + \frac{p_b}{2\rho_c^3} = \beta + \frac{m}{3}. \tag{24}$$

Solving these equations for A and β gives

$$A = \frac{2m}{3}\rho_c^3 - p_b, \quad \beta = \frac{2m}{3}\rho_c^3 - p_b. \tag{25}$$

Substituting (25) into (12), (13) and (23) leads to

$$\frac{\sigma_r}{\sigma_0} = \frac{2m}{3}\rho_c^3\left(1-\rho^{-3}\right) - p_b, \quad \frac{\sigma_\theta}{\sigma_0} = \frac{2m}{3}\rho_c^3\left(1+\frac{1}{2\rho^3}\right) - p_b \tag{26}$$

in the range $\rho_c \le \rho \le 1$ and

$$\frac{\sigma}{\sigma_0} = -m\frac{2}{3}\int_0^{\varepsilon_{eq}^{p}} \frac{\left[\Phi\left(\chi\right) - \chi\Phi'\left(\chi\right)\right]}{\left[\chi + 2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi + \frac{2m}{3}\rho_c^3 - p_b \tag{27}$$

in the range $a \le \rho \le \rho_c$. Let ε_a be the value of ε_{eq}^{p} at $r = a_0$ (or $\rho = a$). Then, it follows from (21) that

$$\left(\frac{\rho_c}{a}\right)^3 = \frac{\varepsilon_a + 2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)}{2k\left(1-\nu\right)}. \tag{28}$$

The range of validity of the present solution is restricted by the condition $\rho_c = 1$. Substituting this condition into (28) gives the following equation for the maximum possible value of $\varepsilon_a = \varepsilon_m$

$$\frac{a^3\left[\varepsilon_m + 2k\left(1-\nu\right)\Phi\left(\varepsilon_m\right)\right]}{2k\left(1-\nu\right)} = 1. \tag{29}$$

The value of the radial stress at $\rho = a$ is determined from (13) and (27) at $\varepsilon_{eq}^{p} = \varepsilon_a$. Then, using the boundary condition (2) results in

$$\frac{2m}{3}\Phi\left(\varepsilon_a\right) + m\frac{2}{3}\int_0^{\varepsilon_a} \frac{\left[\Phi\left(\chi\right) - \chi\Phi'\left(\chi\right)\right]}{\left[\chi + 2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi - \frac{2m}{3}\rho_c^3 = p_a - p_b. \tag{30}$$

Eliminating here ρ_c by means of (28) leads to the following equation for ε_a

$$\frac{2}{3}\Phi(\varepsilon_a) + \frac{2}{3}\int_0^{\varepsilon_a} \frac{\left[\Phi(\chi) - \chi\Phi'(\chi)\right]}{\left[\chi + 2k(1-\nu)\Phi(\chi)\right]}\mathrm{d}\chi - \frac{\left[\varepsilon_a + 2k(1-\nu)\Phi(\varepsilon_a)\right]a^3}{3k(1-\nu)} = m(p_a - p_b). \tag{31}$$

Once this equation has been solved, the dimensionless radius of the elastic/plastic boundary immediately follows from (28). For any loading path, it is now possible to check whether this radius is an increasing function of the time. The distribution of stresses is given by (26) in the elastic zone and by (13), (21) and (27) in the plastic zone. The latter is in parametric form with ε_{eq}^p being the parameter whose range is $0 \le \varepsilon_{eq}^p \le \varepsilon_a$. Having determined the distribution of stresses, the distribution of strains in the elastic zone is found from (5). The distribution of plastic strains in the plastic zone follows from (16) and (21). The distribution of total strains in the plastic zone is found from (17), (21) and (27).

4 Residual stresses and strains

4.1 Purely elastic unloading

If the shell is unloaded from a partly plastic state, residual stresses and strains occur. If there is no reversed plastic zone, then the solution for the increment of stress has the same form as the solution (12) in which p_b should be replaced with $-p_b$. As a result,

$$\frac{\Delta\sigma_r}{\sigma_0} = \Delta A\left(1 - \rho^{-3}\right) + p_b\rho^{-3}, \quad \frac{\Delta\sigma_\theta}{\sigma_0} = \Delta A\left(1 + \frac{1}{2\rho^3}\right) - \frac{p_b}{2\rho^3} \tag{32}$$

where ΔA is constant. This solution should satisfy the condition $\Delta\sigma_r = P_a$ at $\rho = a$. Substituting this condition into (32) gives

$$\Delta A = \frac{a^3 p_a - p_b}{a^3 - 1}. \tag{33}$$

Eliminating ΔA in (32) by means of (33) yields

$$\begin{aligned} &\frac{\Delta\sigma_r}{\sigma_0} = \frac{\left(a^3 p_a - p_b\right)\left(1 - \rho^{-3}\right)}{\left(a^3 - 1\right)} + p_b\rho^{-3}, \quad \frac{\Delta\sigma_\theta}{\sigma_0} = \frac{\left(a^3 p_a - p_b\right)\left(2 + \rho^{-3}\right)}{2\left(a^3 - 1\right)} - \frac{p_b}{2\rho^3}, \\ &\frac{\Delta\tau_r}{\sigma_0} = \frac{(p_a - p_b)\,a^3}{\left(1 - a^3\right)\rho^3}, \quad \frac{\Delta\tau_\theta}{\sigma_0} = \frac{\Delta\tau_\varphi}{\sigma_0} = -\frac{(p_a - p_b)\,a^3}{2\left(1 - a^3\right)\rho^3}, \quad \frac{\Delta\sigma}{\sigma_0} = \frac{p_b - p_a a^3}{1 - a^3}. \end{aligned} \tag{34}$$

Having this solution, the increment of strain is found from Hooke's law. The distribution of residual stresses and strains is obtained by adding the respective increments to the distribution of stresses and strains determined in the previous section.

4.2 Initiation of reversed yielding

The Bauschinger effect in the boundary value problem under consideration can be described in the same manner as in [17]. In particular, by analogy to (13), the state of stress in the reversed plastic zone is given by

$$\frac{\tau_r^{res}}{\sigma_0} = \frac{2m}{3}\Lambda\left(\varepsilon_{eq}^p\right), \quad \frac{\tau_\theta^{res}}{\sigma_0} = \frac{\tau_r^{res}}{\sigma_0} = -\frac{m}{3}\Lambda\left(\varepsilon_{eq}^p\right) \tag{35}$$

where ε_{eq}^p is the forward equivalent strain given by (21) and $\Lambda\left(\varepsilon_{eq}^p\right)$ is an arbitrary function of its argument satisfying the conditions $\Lambda(0) = 1$ and $\mathrm{d}\Lambda/\mathrm{d}\varepsilon_{eq}^p \le 0$ for all ε_{eq}^p. This inequality accounts for a reduction in flow stress accompanied a reversal to the plastic strain. Here and in what follows, the superscript *res* denotes residual stresses and strains after unloading. The validity of the solution (34) is controlled by the condition

$m\left(\tau_r + \Delta\tau_r - \tau_r^{res}\right) \le 0$. Substituting (13), (34) and (35) into this equation and then eliminating ρ^3 by means of (21) yield

$$\Omega\left(\varepsilon_{eq}^p\right) = -\frac{2}{3}\left[\Phi\left(\varepsilon_{eq}^p\right) + \Lambda\left(\varepsilon_{eq}^p\right)\right] + \frac{m\left(p_a - p_b\right)a^3\left[\varepsilon_{eq}^p + 2k\left(1-\nu\right)\Phi\left(\varepsilon_{eq}^p\right)\right]}{2k\left(1-\nu\right)\left(1-a^3\right)\rho_c^3} \le 0 \tag{36}$$

Differentiating (36) gives

$$\frac{d\Omega}{d\varepsilon_{eq}^p} = -\frac{2}{3}\left[\Phi'\left(\varepsilon_{eq}^p\right) + \Lambda'\left(\varepsilon_{eq}^p\right)\right] + \frac{m\left(p_a - p_b\right)a^3\left[1 + 2k\left(1-\nu\right)\Phi'\left(\varepsilon_{eq}^p\right)\right]}{2k\left(1-\nu\right)\left(1-a^3\right)\rho_c^3} \tag{37}$$

The product $m\left(p_a - p_b\right)$ is always positive. Therefore, the second term on the right-hand side of (37) is positive. The derivatives $\Phi'\left(\varepsilon_{eq}^p\right)$ and $\Lambda'\left(\varepsilon_{eq}^p\right)$ have the opposite signs. For real metals $\left|\Lambda'\left(\varepsilon_{eq}^p\right)\right| > \Phi'\left(\varepsilon_{eq}^p\right)$ and $\Lambda'\left(\varepsilon_{eq}^p\right) < 0$ [17]. Therefore, the first term on the right-hand side of (37) is also positive. Thus, $d\Omega/d\varepsilon_{eq}^p > 0$ in the range $0 \le \varepsilon_{eq}^p \le \varepsilon_a$. Then, it follows from (36) that the reversed plastic zone starts to develop from the inner surface of the shell where ε_{eq}^p attains its maximum magnitude. Let ε_{cr} be the corresponding value of ε_a. The inequality (36) and Eq. (28) result in

$$\frac{2}{3}\left[\Phi\left(\varepsilon_{cr}\right) + \Lambda\left(\varepsilon_{cr}\right)\right] - \frac{m\left(p_a - p_b\right)}{\left(1-a^3\right)} = 0. \tag{38}$$

Replacing ε_a with ε_{cr} in (31) and eliminating $m\left(p_a - p_b\right)$ by means of (38) give the following equation for ε_{cr}

$$\int_0^{\varepsilon_{cr}} \frac{\left[\Phi\left(\chi\right) - \chi\Phi'\left(\chi\right)\right]}{\left[\chi + 2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}d\chi - \frac{\varepsilon_{cr}a^3}{2k\left(1-\nu\right)} = \left(1-a^3\right)\Lambda\left(\varepsilon_{cr}\right). \tag{39}$$

Once this equation has been solved, the corresponding values of $p_a - p_b$ and ρ_c can be found from (38) and (28), respectively.

4.3 Elastic/plastic unloading

If $\varepsilon_a > \varepsilon_{cr}$, then it is necessary to consider the reversed plastic zone in the vicinity of the inner radius of the shell. Let $\rho = \rho_s$ be the outer radius of this zone. The present solution is restricted to the process of unloading in which ρ_s does not decrease throughout the process. It follows from (21) and (28) that

$$\left(\frac{\rho_s}{a}\right)^3 = \frac{\varepsilon_a + 2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)}{\varepsilon_s + 2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)}. \tag{40}$$

The solution (32) is valid in the range $\rho_s \le \rho \le 1$. However, ΔA is not determined by (33). The distribution of the stresses σ^{res} and τ_r^{res} in the range $\rho_s \le \rho \le \rho_c$ follows from (13), (27) and (32) as

$$\begin{aligned}
\frac{\sigma^{res}}{\sigma_0} &= -m\frac{2}{3}\int_0^{\varepsilon_{eq}^p} \frac{\left[\Phi\left(\chi\right) - \chi\Phi'\left(\chi\right)\right]}{\left[\chi + 2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}d\chi + \frac{2m}{3}\rho_c^3 - p_b + \Delta A,\\
\frac{\tau_r^{res}}{\sigma_0} &= -\frac{2m}{3}\Phi\left(\varepsilon_{eq}^p\right) + \frac{\left(p_b - \Delta A\right)}{\rho^3}.
\end{aligned} \tag{41}$$

Equation (19) in which $\Phi\left(\varepsilon_{eq}^p\right)$ is replaced with $-\Lambda\left(\varepsilon_{eq}^p\right)$ and the hydrostatic stress with the residual hydrostatic stress is valid in the range $a \le \rho \le \rho_s$. In particular,

$$m\frac{\rho}{\sigma_0}\frac{\partial\sigma^{res}}{\partial\rho} = -2\Lambda\left(\varepsilon_{eq}^p\right) - \frac{2\rho}{3}\Lambda'\left(\varepsilon_{eq}^p\right)\frac{\partial\varepsilon_{eq}^p}{\partial\rho}. \tag{42}$$

Replacing in this equation the derivative $\partial \sigma^{res}/\partial \rho$ with the derivative $\partial \sigma^{res}/\partial \varepsilon_{eq}^{p}$ and then eliminating the derivative $\partial \varepsilon_{eq}^{p}/\partial \rho$ by means of (20) result in

$$\frac{m}{\sigma_0}\frac{\partial \sigma^{res}}{\partial \varepsilon_{eq}^{p}} = \frac{2}{3}\frac{\Lambda\left(\varepsilon_{eq}^{p}\right)\left[1+2k\left(1-\nu\right)\Phi'\left(\varepsilon_{eq}^{p}\right)\right]}{\left[\varepsilon_{eq}^{p}+2k\left(1-\nu\right)\Phi\left(\varepsilon_{eq}^{p}\right)\right]} - \frac{2}{3}\Lambda'\left(\varepsilon_{eq}^{p}\right). \tag{43}$$

Integrating leads to

$$\frac{m}{\sigma_0}\sigma^{res} = \frac{2}{3}\int\limits_{\varepsilon_a}^{\varepsilon_{eq}^{p}} \frac{\Lambda\left(\chi\right)\left[1+2k\left(1-\nu\right)\Phi'\left(\chi\right)\right]}{\left[\chi+2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi - \frac{2}{3}\Lambda\left(\varepsilon_{eq}^{p}\right). \tag{44}$$

It has been taken into account here that $\sigma_r^{res} = \sigma^{res} + \tau_r^{res} = 0$ at $\varepsilon_{eq}^{p} = \varepsilon_a$ and τ_r^{res} is given by (35). A requirement of equilibrium is that the radial stress is continuous across the elastic/plastic boundary. The material just on the elastic side of the elastic/plastic boundary must satisfy the yield criterion [1]. In the problem under consideration, these conditions are equivalent to the requirement that τ_r^{res} and σ^{res} are continuous across the elastic/plastic boundary. Then, it follows from (28), (35), (40), (41) and (44) that

$$\begin{aligned} \Delta A &= p_b - \frac{2ma^3}{3}\frac{\left[\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)\right]\left[\Lambda\left(\varepsilon_s\right)+\Phi\left(\varepsilon_s\right)\right]}{\left[\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)\right]}, \\ &\int\limits_{\varepsilon_a}^{\varepsilon_s}\frac{\Lambda\left(\chi\right)\left[1+2k\left(1-\nu\right)\Phi'\left(\chi\right)\right]}{\left[\chi+2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi + \int\limits_{0}^{\varepsilon_s}\frac{\left[\Phi\left(\chi\right)-\chi\Phi'\left(\chi\right)\right]}{\left[\chi+2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi \\ &= \frac{a^3\left[\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)\right]\left[\varepsilon_s-2k\left(1-\nu\right)\Lambda\left(\varepsilon_s\right)\right]}{2k\left(1-\nu\right)\left[\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)\right]} + \Lambda\left(\varepsilon_s\right). \end{aligned} \tag{45}$$

Equation $(45)^2$ should be solved numerically to find ε_s. Then, ΔA is immediately determined from $(45)^1$. Eliminating ΔA, ρ and ρ_c in (41) by means of $(45)^1$, (21) and (28), respectively, gives

$$\begin{aligned} \frac{\sigma^{res}}{\sigma_0} &= -m\frac{2}{3}\int\limits_{0}^{\varepsilon_{eq}^{p}}\frac{\left[\Phi\left(\chi\right)-\chi\Phi'\left(\chi\right)\right]}{\left[\chi+2k\left(1-\nu\right)\Phi\left(\chi\right)\right]}\mathrm{d}\chi + \frac{a^3 m\left[\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)\right]}{3k\left(1-\nu\right)} \\ &\quad -\frac{2a^3 m}{3}\left[\Lambda\left(\varepsilon_s\right)+\Phi\left(\varepsilon_s\right)\right]\left[\frac{\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)}{\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)}\right], \\ \frac{\tau_r^{res}}{\sigma_0} &= -\frac{2m}{3}\Phi\left(\varepsilon_{eq}^{p}\right) + \frac{2m}{3}\left[\Lambda\left(\varepsilon_s\right)+\Phi\left(\varepsilon_s\right)\right]\left[\frac{\varepsilon_{eq}^{p}+2k\left(1-\nu\right)\Phi\left(\varepsilon_{eq}^{p}\right)}{\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)}\right] \end{aligned} \tag{46}$$

in the range $\varepsilon_s \geq \varepsilon_{eq}^{p} \geq 0$ (or $\rho_s \leq \rho \leq \rho_c$). The distribution of the residual radial and circumferential stresses is determined from (46) and

$$\frac{\sigma_r^{res}}{\sigma_0} = \frac{\sigma^{res}}{\sigma_0} + \frac{\tau_r^{res}}{\sigma_0}, \quad \frac{\sigma_\theta^{res}}{\sigma_0} = \frac{\sigma^{res}}{\sigma_0} - \frac{\tau_r^{res}}{2\sigma_0}. \tag{47}$$

The distribution of the residual radial and circumferential stresses in the range $\varepsilon_a \geq \varepsilon_{eq}^{p} \geq \varepsilon_s$ (or $a \leq \rho \leq \rho_s$) follows from (35), (44) and (47). Substituting ΔA from (45) into (32) yields

$$\begin{aligned} \frac{\Delta\sigma_r}{\sigma_0} &= p_b + \frac{2ma^3}{3}\left[\Lambda\left(\varepsilon_s\right)+\Phi\left(\varepsilon_s\right)\right]\left[\frac{\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)}{\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)}\right]\left(\frac{1}{\rho^3}-1\right), \\ \frac{\Delta\sigma_\theta}{\sigma_0} &= p_b - \frac{2ma^3}{3}\left[\Lambda\left(\varepsilon_s\right)+\Phi\left(\varepsilon_s\right)\right]\left[\frac{\varepsilon_a+2k\left(1-\nu\right)\Phi\left(\varepsilon_a\right)}{\varepsilon_s+2k\left(1-\nu\right)\Phi\left(\varepsilon_s\right)}\right]\left(\frac{1}{2\rho^3}+1\right). \end{aligned} \tag{48}$$

The distribution of the residual radial and circumferential stresses in the range $\rho_c \le \rho \le 1$ is determined from (26) and (48) as

$$\frac{\sigma_r^{res}}{\sigma_0} = \frac{ma^3\left[\varepsilon_a + 2k(1-\nu)\Phi(\varepsilon_a)\right]}{3k(1-\nu)}\left\{1 - \frac{2k(1-\nu)\left[\Lambda(\varepsilon_s) + \Phi(\varepsilon_s)\right]}{\left[\varepsilon_s + 2k(1-\nu)\Phi(\varepsilon_s)\right]}\right\}\left(1 - \frac{1}{\rho^3}\right),$$
$$\frac{\sigma_\theta^{res}}{\sigma_0} = \frac{ma^3\left[\varepsilon_a + 2k(1-\nu)\Phi(\varepsilon_a)\right]}{3k(1-\nu)}\left\{1 - \frac{2k(1-\nu)\left[\Lambda(\varepsilon_s) + \Phi(\varepsilon_s)\right]}{\left[\varepsilon_s + 2k(1-\nu)\Phi(\varepsilon_s)\right]}\right\}\left(1 + \frac{1}{2\rho^3}\right). \quad (49)$$

The increment of strain is determined from (32), (45)[1] and Hooke's law in the range $\rho_s \le \rho \le 1$. Then, the distribution of residual strains is immediately found using the strain solution at the end of loading given in the previous section. The increment of strain in the range $a \le \rho \le \rho_s$ consists of elastic and plastic portions. In order to find the elastic portion, it is necessary to determine the increment of stress. Using (13), (27), (28), (35) and (44)

$$\frac{\Delta\tau_r}{\sigma_0} = \frac{2m}{3}\left[\Lambda\left(\varepsilon_{eq}^p\right) + \Phi\left(\varepsilon_{eq}^p\right)\right],$$
$$\frac{\Delta\sigma}{\sigma_0} = \frac{2m}{3}\int_{\varepsilon_a}^{\varepsilon_{eq}^p} \frac{\Lambda(\chi)\left[1 + 2k(1-\nu)\Phi'(\chi)\right]}{\left[\chi + 2k(1-\nu)\Phi(\chi)\right]}\mathrm{d}\chi - \frac{2m}{3}\Lambda\left(\varepsilon_{eq}^p\right) \quad (50)$$
$$+\frac{2m}{3}\int_0^{\varepsilon_{eq}^p} \frac{\left[\Phi(\chi) - \chi\Phi'(\chi)\right]}{\left[\chi + 2k(1-\nu)\Phi(\chi)\right]}\mathrm{d}\chi - \frac{ma^3\left[\varepsilon_a + 2k(1-\nu)\Phi(\varepsilon_a)\right]}{3k(1-\nu)} + p_b.$$

It follows from Hooke's law that the elastic portion of strain increments is

$$\frac{\Delta\varepsilon_r^e}{k} = (1-2\nu)\frac{\Delta\sigma}{\sigma_0} + (1+\nu)\frac{\Delta\tau_r}{\sigma_0}, \quad \frac{\Delta\varepsilon_\theta^e}{k} = \frac{\Delta\varepsilon_\varphi^e}{k} = (1-2\nu)\frac{\Delta\sigma}{\sigma_0} - \frac{(1+\nu)}{2}\frac{\Delta\tau_r}{\sigma_0}. \quad (51)$$

By analogy to (7), the increments of plastic strain satisfy the equation

$$2\left(\Delta\varepsilon_\theta^p\right) = -\Delta\varepsilon_r^p. \quad (52)$$

Then, using (4) the equation of strain compatibility, $\Delta\varepsilon_r = \partial(\rho\Delta\varepsilon_\theta)/\partial\rho$, transforms to

$$\rho\frac{\partial(\Delta\varepsilon_\theta)}{\partial\rho} + 3\Delta\varepsilon_\theta = \Delta\varepsilon_r^e + 2\left(\Delta\varepsilon_\theta^e\right). \quad (53)$$

The boundary condition to this equation is

$$\Delta\varepsilon_\theta = \Delta\varepsilon_{\theta s} \quad (54)$$

at $\rho = \rho_s$ where $\Delta\varepsilon_{\theta s}$ is the increment of the strain ε_θ on the elastic side of the elastic/plastic boundary $\rho = \rho_s$. Using (40), (48) and Hooke's law yields

$$\frac{\Delta\varepsilon_{\theta s}}{k} = (1-2\nu)p_b - \frac{m}{3}\left[\Lambda(\varepsilon_s) + \Phi(\varepsilon_s)\right]\left\{1 + \nu + 2a^3(1-2\nu)\left[\frac{\varepsilon_a + 2k(1-\nu)\Phi(\varepsilon_a)}{\varepsilon_s + 2k(1-\nu)\Phi(\varepsilon_s)}\right]\right\}. \quad (55)$$

The solution of Eq. (53) satisfying the boundary condition (54) is

$$\Delta\varepsilon_\theta = \frac{1}{\rho^3}\int_{\rho_s}^{\rho} \chi^2\left[\Delta\varepsilon_r^e(\chi) + 2\Delta\varepsilon_\theta^e(\chi)\right]\mathrm{d}\chi + \Delta\varepsilon_{\theta s}\frac{\rho_s^3}{\rho^3}. \quad (56)$$

Here $\Delta\varepsilon_\theta^e$ and $\Delta\varepsilon_r^e$ are understood to be functions of ρ. These functions are readily determined from (21), (50) and (51). Using (52) and (56), the increment of the radial strain is found as

$$\Delta\varepsilon_r = 2\left(\Delta\varepsilon_\theta^e - \Delta\varepsilon_\theta\right) + \Delta\varepsilon_r^e. \quad (57)$$

The distribution of residual strains in the range $a \le \rho \le \rho_s$ is immediately determined using (55), (56), (57) and the strain solution at the end of loading given in the previous section.

5 Illustrative example

The case of linear hardening with different hardening rates under forward and reversed deformation is represented by $\Phi\left(\varepsilon_{eq}^{p}\right) = 1 + C_f \varepsilon_{eq}^{p}$ and $\Lambda\left(\varepsilon_{eq}^{p}\right) = 1 - C_r \varepsilon_{eq}^{p}$ where C_f and C_r are constant. Assume that $m = 1$, $p_b = 0$, $k = 10^{-3}$, $C_f = (9k)^{-1}$, $C_r = 2C_f$, $\nu = 0.3$ and $a = 0.3$. The solution of Eq. (39) is $\varepsilon_{cr} \approx 0.00146$. The distribution of residual stresses and strains has been found for several values of $\varepsilon_a > \varepsilon_{cr}$ using the solution given in Sect. 4.3. In particular, the distribution of the residual radial and circumferential stresses is depicted in Figs. 2 and 3, respectively. The values of ε_a have been chosen such that $\rho_c = 0.4$, $\rho_c = 0.5$ and $\rho_c = 0.6$ at the end of loading. It is seen from Fig. 2 that the qualitative behavior of the residual radial stress depends on this value of ρ_c. In particular, this stress may be either positive or negative in the vicinity of the inner surface of the shell. The distribution of the residual radial and circumferential strains is shown in Figs. 4 and 5, respectively, at the same values of ρ_c at the end of loading. It is seen from Fig. 4 that the residual radial strain may attain a local maximum within the reversed plastic zone.

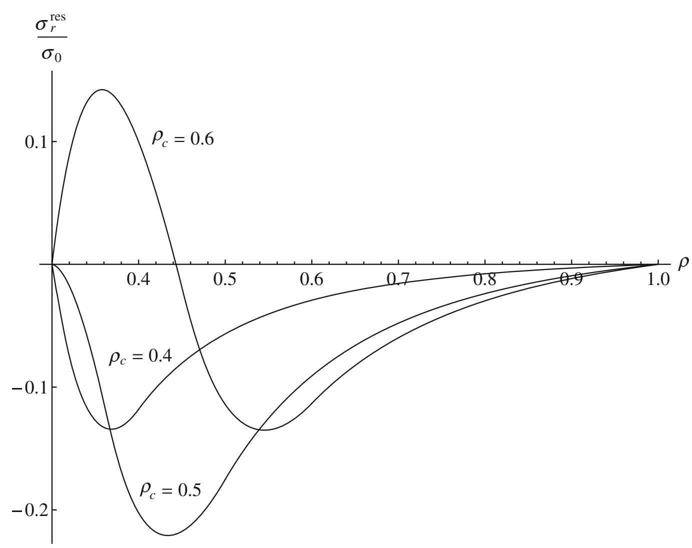

Fig. 2 Distribution of the residual radial stress at different values of ρ_c at the end of loading

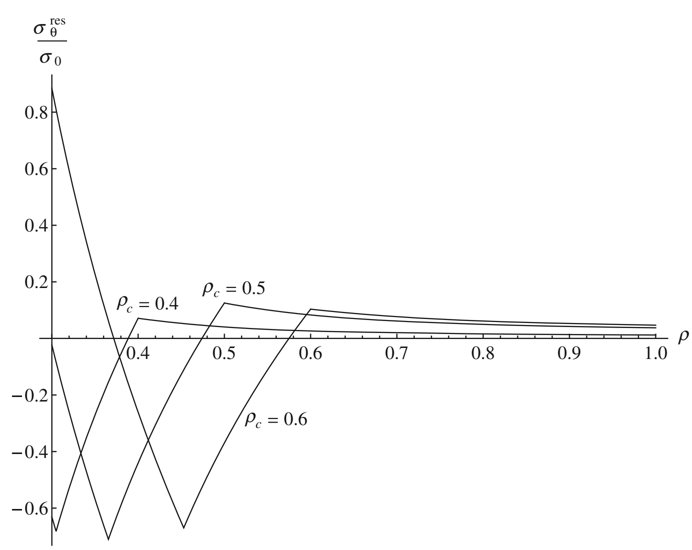

Fig. 3 Distribution of the residual circumferential stress at different values of ρ_c at the end of loading

6 Conclusions

A semi-analytic solution for the expansion/contraction of an elastic/plastic hollow sphere of strain-hardening material and subsequent unloading has been derived. The flow theory of plasticity has been adopted in conjunction with quite an arbitrary yield criterion and its associated flow rule. The Bauschinger effect has been taken into account. Numerical methods are necessary to evaluate ordinary integrals and solve transcendental equations. Pressure release is purely elastic if $\varepsilon_a < \varepsilon_{cr}$ where ε_{cr} is determined from (39). The value of ε_a is related to the pressures applied by Eq. (31). A reversed plastic zone appears if $\varepsilon_a > \varepsilon_{cr}$. In this case, it is convenient to distinguish three domains, namely (*i*) both loading and unloading are purely elastic in the range $\rho_c \le \rho \le 1$, (*ii*) loading is plastic and unloading is elastic in the range $\rho_s \le \rho \le \rho_c$, (*iii*) both loading and unloading are plastic in the range $a \le \rho \le \rho_s$. These three domains are clearly seen in Figs. 3 and 4. The values of ρ_c and ρ_s are found from (28) and (40), respectively. The limitations of the solution given are that both ρ_c and ρ_s are by assumption monotonically increasing functions of the time. In specific calculation, these conditions can be verified by means of Eqs. (28) and (40).

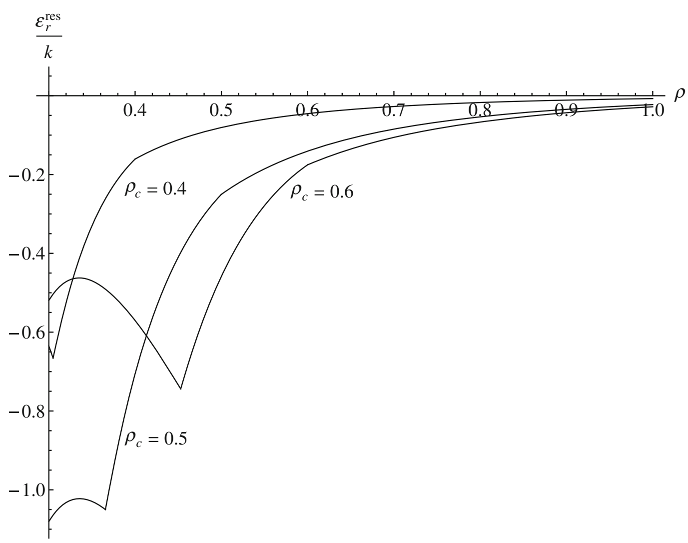

Fig. 4 Distribution of the residual radial strain at different values of ρ_c at the end of loading

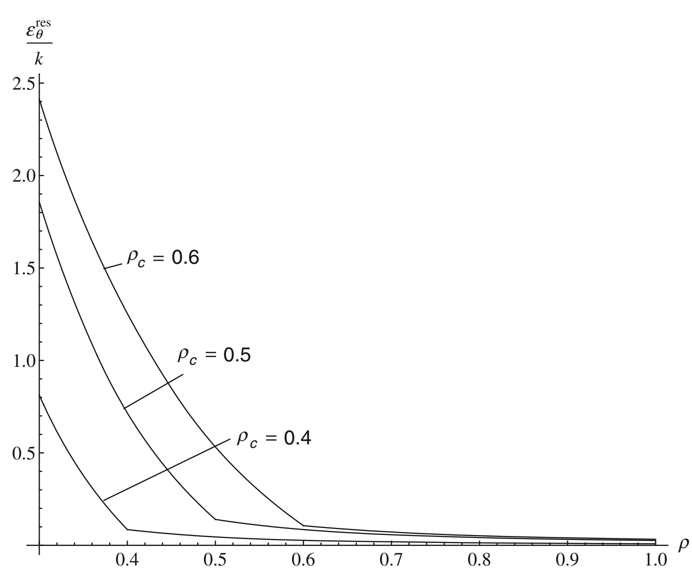

Fig. 5 Distribution of the residual circumferential strain at different values of ρ_c at the end of loading

The key point of the approach developed is to use the equivalent plastic strain as an independent variable instead of the radius. This approach is in line with that used in [14,16]. In these works, the equivalent stress has been used as an independent variable.

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