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ORIGINAL ARTICLE

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On a stress-power-based characterization of second-gradient elastic fluids

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Abstract An algebraic characterization of fluidity applicable to second-gradient materials is argued, individuating a collection of deformations from a reference placement that entail no pointwise stress-power expenditure. For simplicity, the characterization in question is developed in the context of elastic materials, within which general representations for the stress response, both Cauchy-like and Piola-like, of elastic second-gradient fluids are derived.

Keywords Fluids · Second gradient · Symmetry group · Stress power

1 Introduction

By *n*th-*gradient* elastic materials, we mean those materials whose mechanical response depends on the present value of the first n deformation gradients. nth-gradient elasticity is a generalization of classical finite elasticity, for which $n = 1$. The case $n = 2$ was treated first in a so-called 'hyperelastic' format, that is to say, by encapsulating the material response into an assignment of a stored-energy functional depending on the first and second deformation gradients. Consideration of higher gradients and inelastic materials came later, leading a rather small group of researchers to sketch the architecture of a general theory of nth-gradient material bodies.

Our present intention is not to give a full account of that theory, which is still in the need of a full-fledged setting, even in the elastic case. Indeed, the posing of initial- and boundary-value problems, let alone a study of their solutions, is a formidable and as today only partially achieved task: the problems to solve are ruled by a system of two evolution equations each of which involves the two stress measures if the theory in a generally inextricable manner (a similar difficulty is encountered with the boundary conditions of Neumann type). Instead, we aim to propose *a mathematical definition of fluidity*, so as to sort second-gradient fluids from second-gradient materials having a different aggregation state. Our definition is based on a requirement of invariance of the *stress power*, a construct that, in our view, plays a central role in the mechanical characterization of any material class.

Given our limited scope, in the following, we focus briefly, with no pretenses of completeness, on those contributions in the literature that concern the description of contact interactions and of the associated stress measures and stress-power expenditure.

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1.1 Some historical notes

In the early years 1960, a pioneering research on second-gradient elasticity was made by Toupin [19,20], who took a variational approach to investigate the equilibrium problem of a material body whose stored energy depends on the first and second deformation gradients.

Some ten years later, a systematic approach to higher-gradient theories by the method of virtual powers was proposed by Germain [9,10]; an issue made clear by the work of Germain was that to account for a body's stress state as many stress measures as deformation gradients were needed, of tensorial order ranging from 2 to $(n+1)$. For a recent discussion of a power-based approach to the characterization of nth-gradient materials, we refer the reader to a paper by Degiovanni et al. [1], where the subject is developed in detail with great mathematical care.

With the work of Toupin, it became clear that unusual contact interactions were in order between adjacent parts of a nth-gradient material body and between such a body and its environment. In particular, as shown in [19] and confirmed in [2,4,17], at the boundary of a second-gradient elastic body a stress field may be found, which, as expected, turns out to be a *surface-tension field* when the material can be thought of as a fluid [22].

The mechanical behavior of second- or higher-gradient materials—be they elastic or not—is further complicated by the unexpected appearence of *edge contact forces*, that is to say, line distribution of forces over the boundary edges. A discussion of this puzzling feature, that Toupin had noticed in [20], is found in [4] and, in much greater mathematical detail, in a complex paper by Noll and Virga [14], where an axiomatic foundation is provided for a theory of mechanical interactions accounting for edge forces distributed over the boundary of the common surface between adjacent body parts; see also [15]. In this connection, it is worth mentioning the work of Dell'Isola and Seppecher [2], who showed how postulating that the stress power satisfies certain physically motivated a-priori inequalities not only makes surface and edge terms emerge, similar to those found by Toupin in his more special context, but also permits to prove the existence of an associated "edge-stress tensor". These matters were also addressed by us in [17], with a different method but equivalent representation results for the stress measures in terms of the surface and edge contact interactions.

1.2 Gradient-sensitive fluids

The above scant historical notes are meant to help the reader to place into context our proposal to come of a mathematical criterion for calling a second-gradient material a fluid. We begin by reviewing the standard notion of a compressible elastic fluid, whose constitutive equation

$$\mathbf{T} = -\pi(\rho)\,\mathbf{I} \tag{1}$$

postulates that the Cauchy stress $\mathbf{T}$ is a multiple of the identity tensor $\mathbf{I}$ through a nonnegative-valued pressure function π of the current mass density ρ. The reason why a material whose mechanical response can be given the form (1) is classified as a *first-gradient* material is that ρ depends on the deformation gradient $\mathbf{F}$ in such a way that, for ρ_0 the referential mass density, the mass-conservation law:

$$\rho \det \mathbf{F} = \rho_0 \tag{2}$$

is satisfied (needless to say, all alternative forms of the response law for these materials feature a dependence on the first deformation gradient). The reason why (1) is said to describe a *fluid* is that it embodies a property considered characteristic of fluids, namely, that by measuring stress it is impossible to detect whether an arbitrary volume-preserving deformation of gradient $\mathbf{H}$, that is, a deformation such that

$$\det \mathbf{H} = 1, \tag{3}$$

was made to precede whatever deformation of gradient $\mathbf{F}$.

Since long, fluids that are sensitive not only to ρ but also to its gradients, in short, *gradient-sensitive fluids*, play a relevant role in the study of certain phase transitions and, more recently, in *microfluidics* (see [5–8], where an extensive list of references can be found). As a glance to (2) makes clear, a gradient-sensitive fluid must be a higher-gradient material. A well-known first proposal for a broad generalization of (1) was advanced by Korteweg in 1901 [13]; in the formulation of Dunn and Serrin [3], it reads:

$$\mathbf{T} = (-\pi(\rho) + \alpha\Delta\rho + \beta|\,\mathrm{grad}\,\rho|^2)\mathbf{I} + \delta\,\mathrm{grad}\,\rho \otimes \mathrm{grad}\,\rho + \gamma\,\mathrm{grad}^2\rho, \tag{4}$$

and is parameterized, in addition to the function π, by the four material moduli α, β, γ and δ (here Δ is the laplacian and grad^2 is the twice-iterated space gradient operator); a Korteweg fluid is a *second-gradient* material if both α and γ are null, and a *third-gradient* material if one of the two is not.[1]

One may ask why Korteweg's materials are called 'fluids': what mathematical character of their constitutive prescription induces us to classify their macroscopic aggregation state as fluid-like, and as such different, say, from that of nth-gradient materials that we would rather call 'solids'?

There is no textbook answer to this question, that can be posed also outside the context of gradient materials, within which one expects any answer to depend on the order n of the highest gradient. We now proceed to introduce and motivate our answer, in the case of second-gradient materials.

1.3 A stress-power-based classification of materials

In the vulgate of continuum mechanics, a material body is termed *simple* if its stress response depends on the history of $\mathbf{F}$ (cf. [21], Sect. 28), and according to a criterion proposed by W. Noll that we employed in the preceding subsection for the constitutive equation (1), in *simple fluids*, the stress state accompanying any given deformation history should be insensitive to all previous volume-preserving deformations.

There is a variety of *complex* (i.e., not simple) material classes, and for none of them, to our knowledge, a classification criterion for aggregation states has been proposed. Note that a material class may be complex not only because it is *multigradient*, as exemplified by Korteweg materials, but also, for example, because it is *multivelocity*, in the sense that one or more kinematical fields, in addition to motion velocity, are needed to describe its changes in shape. Our present paper may be regarded as the first part of a much more ambitious project aiming to sort the aggregation states of materials that are complex in the one or the other of many possible senses: to begin with, we put forward a fluidity notion for second-gradient materials, and we show that, in particular, Korteweg materials with $\alpha = \gamma = 0$ are rightly called fluids.

Instead of characterizing simplicity/complexity and fluidity/solidity on the basis of an inspection of the constitutive law for stress and its group invariance, it was proposed in [16] to define those notions in terms of the stress power and its own group invariance. Precisely, according to [16], a material is classified as *simple* if (its stress response depends on the history of the deformation gradient and) the *stress power* expended per unit current volume in a motion inducing a velocity field $\mathbf{v}$ has the expression

$$\mathbf{T} \cdot \mathrm{grad}\, \mathbf{v}; \tag{5}$$

moreover, in the viscoelastic case when

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}, \dot{\mathbf{F}}), \quad \dot{\mathbf{F}} = (\mathrm{grad}\, \mathbf{v})\mathbf{F},^{2} \tag{6}$$

a material is classified as a *fluid* if the *equilibrium stress power*

$$\mathbf{T}^{eq} \cdot \mathrm{grad}\, \mathbf{v}, \quad \mathbf{T}^{eq} = \widetilde{\mathbf{T}}(\mathbf{F}) := \widehat{\mathbf{T}}(\mathbf{F}, \mathbf{0}), \tag{7}$$

(that is to say, the power expended by the equilibrium, or elastic, stress $\mathbf{T}^{eq}$) is invariant under the group of all volume-preserving deformations. Equivalently, in view of the fact that with the use of $(6)_2$ and (7), the equilibrium stress power takes the following form:

$$\mathbf{T}^{eq} \cdot \mathrm{grad}\, \mathbf{v} = \widetilde{\mathbf{T}}(\mathbf{F})\mathbf{F}^{-T} \cdot \dot{\mathbf{F}},$$

a viscoelastic material is classified as a fluid if the equilibrium stress mapping $\widetilde{\mathbf{T}}$ satisfies the following requirement:

$$\widetilde{\mathbf{T}}(\mathbf{F})\mathbf{F}^{-T} \cdot \dot{\mathbf{F}} = \widetilde{\mathbf{T}}(\mathbf{F}\mathbf{H})(\mathbf{F}\mathbf{H})^{-T} \cdot (\dot{\overline{\mathbf{F}\mathbf{H}}}) \tag{8}$$

for all $\mathbf{H}$ satisfying (3) and for all $\mathbf{F}$ such that $\det \mathbf{F} > 0$. As shown in [16], this requirement implies the following representation result:

$$\widetilde{\mathbf{T}}(\mathbf{F}) = \widetilde{\tau}(\det \mathbf{F})\, \mathbf{I};$$

[1] Korteweg developed his equation in the context of his study of surface tension and capillarity; apparently, he held the view that the stress in a fluid should take a very special expression in regions where steep changes in density are to be expected.

[2] Here and henceforth a superposed dot denotes temporal differentiation.

hence, provided mass is conserved pointwise according to (2), the equilibrium stress of all simple fluids is a pressure and can be given the form (1).

Now, it is well known that all first-gradient materials, whatever their response to deformation histories, behave as elastic materials 'at equilibrium', that is, when the past history is the rest history. Thus, the approach of [16] can be extended to classify first-gradient materials whose response depends on the past history in more general a manner than viscous, provided the stress, and hence the stress power, is split additively into equilibrium and non-equilibrium parts, and that the invariance requirement concerns solely the equilibrium stress power. For this reason, to try and apply a similar approach to sort aggregation states of second-gradient materials, it is enough to restrict attention to those that are, in a sense that must be made precise, elastic; but, the results we obtain apply to a larger constitutive class.

Two choices must be made. Firstly, the notion of stress power must be generalized; as detailed in Sect. 4, we do this in a manner that is, by now, standard, by taking

$$\mathbf{T}\cdot\operatorname{grad}\mathbf{v}+\mathsf{T}\cdot\operatorname{grad}^2\mathbf{v} \tag{9}$$

in the place of expression (5), with the stress measures $(\mathbf{T},\mathsf{T})$, a pair of tensors, the one of the second order and the other of the third order, both depending on the first and second deformation gradients. Secondly, so as to generalize expression (8), a suitable group of power-undetectable transformations of the reference placement must be selected.

In this connection, we remark that stress-power invariance has for complex materials far less well-known consequences than in the simple case. For nth-gradient elastic materials as well as for other material classes, such as various types of microstructured materials, an expression for the stress power is laid down without difficulties; not so for a characterization of the collection of undetectable reference transformations. For the $n=2$ case, since to specify a deformation 'up to the second order' its first and second gradients suffice (Sect. 3.1), to single out fluids we borrow from [12,18], a choice that, roughly speaking, consists in selecting as undetectable those transformations of the reference placement that preserve volume 'up to the second order', in a sense made clear in Sect. 5.2.

1.4 Extended summary

Our paper is organized as follows.

Sections from 2 and 3 have a preparatory role: our notation is detailed in Sect. 2, where certain algebraic results to be used later on are listed, and Sect. 3 contains the kinematical formulae relevant to our developments.

Section 4 is devoted to a presentation of the referential and current forms of the stress power, the related Cauchy- and Piola-like stress measures; the symmetries imposed on these measures to cope with the requirement that the stress power be invariant under observer changes are recapitulated in the "Appendix". The contents of this section, although relevant for our further developments, are generally well known in the continuum mechanics community; we include them, as well as those of the "Appendix", for the reader's convenience.

Sections 5, 6, and 7, are the paper's bulk. The choice of the symmetry group $\mathscr{F}$ that, in our view, characterizes fluidity in the case of second-gradient materials is motivated in Sect. 5; the entailed representation results are arrived at in Sect. 7 by exploiting those features of $\mathscr{F}$'s Lie-group structure that are collected in Sect. 6. As to the Cauchy-like stress measure $\mathbf{T}$, we find:

$$\mathbf{T}=\widehat{\alpha}_1(\rho,|\operatorname{grad}\rho|)\mathbf{I}+\widehat{\alpha}_2(\rho,|\operatorname{grad}\rho|)\operatorname{grad}\rho\otimes\operatorname{grad}\rho, \tag{10}$$

an equation that includes as a special case Korteweg's (4) for α and γ equal to 0. As to T, we find:

$$\mathsf{T}=\widehat{\alpha}_3(\rho,|\operatorname{grad}\rho|)\sum_{i=1}^{3}\big(\mathbf{e}_i\otimes(\operatorname{grad}\rho\otimes\mathbf{e}_i+\mathbf{e}_i\otimes\operatorname{grad}\rho)\big), \tag{11}$$

where the vectors $\mathbf{e}_i$ form an orthonormal triad.[3] Relation (11) indicates that, somehow counterintuitively, elastic second-gradient fluids are generally capable to respond to deformation developing both diffuse surface

[3] Note that, with (10) and (11), the stress power relation (9) yields:

$$\mathbf{T}\cdot\operatorname{grad}\mathbf{v}+\mathsf{T}\cdot\operatorname{grad}^2\mathbf{v}=\widehat{\alpha}_1(\rho,|\operatorname{grad}\rho|)\operatorname{div}\mathbf{v}+\widehat{\alpha}_2(\rho,|\operatorname{grad}\rho|)\operatorname{grad}\rho\cdot(\operatorname{grad}\mathbf{v})\operatorname{grad}\rho$$
$$+2\widehat{\alpha}_3(\rho,|\operatorname{grad}\rho|)\operatorname{grad}(\operatorname{div}\mathbf{v})\cdot\operatorname{grad}\rho.$$

interactions and edge-concentrated tractions [2,15,17]; we plan to further discuss this issue elsewhere. In our opinion, Eqs. (10) and (11) give a rationale to calling gradient-sensitive all elastic second-gradient materials that are fluids according to our definition.

Finally, in Sect. 8, we give further reasons to take the current mass density and its gradient as the state variables of choice for elastic second-gradient fluids, and we derive the restrictions placed by invariance under observer changes on the representation of $\mathbf{T}$ and T in terms of those state variables.

2 Algebraic preliminaries

We use boldface small Latin letters for vectors, capital Latin letters for second-order tensors, a sans-serif font for third-order tensors, and we denote their spaces by, respectively, $\mathscr{V}$, Lin, and Lin; $\mathscr{V}$ is the translation space of the three-dimensional Euclidean point space $\mathscr{E}$.

2.1 Third-order tensors

We find it advisable to present, in some detail, certain algebraic operations involving third-order tensors and, in particular, their actions on vectors and second-order tensors.

(*i*) Any $\mathsf{K} \in \mathsf{Lin}$ may be regarded either as a trilinear map from $\mathscr{V}$ into the reals or as a linear map from Lin into $\mathscr{V}$. In the first case, we write:

$$\mathsf{K}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = K_{ijk} a_i b_j c_k \tag{12}$$

(summation over repeated indices is to be understood here and henceforth) for the action of K on the ordered triplet of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$K_{ijk} = \mathsf{K}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$$

are the Cartesian components of K with respect to an orthonormal triad $\mathbf{e}_i$. In the second case, we write:

$$\mathsf{K}[\mathbf{A}] = K_{ijk} A_{jk} \mathbf{e}_i \qquad (\mathsf{K}[\mathbf{A}])_i = K_{ijk} A_{jk}$$

for the action of K on the second-order tensor $\mathbf{A}$.

(*ii*) The action of $\mathbf{A} \in \text{Lin}$ on $\mathsf{K} \in \mathsf{Lin}$ is defined as follows:

$$(\mathbf{A}\mathsf{K})(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathsf{K}(\mathbf{A}^T\mathbf{a}, \mathbf{b}, \mathbf{c}) \qquad (\mathbf{A}\mathsf{K})_{ijk} = A_{ip} K_{pjk}$$

(here $\mathbf{A}^T$ is the transpose of $\mathbf{A}$).

(*iii*) The *inner product* on Lin is the unique symmetric bilinear real-valued map on $\mathsf{Lin} \times \mathsf{Lin}$ such that

$$(\mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1) \cdot (\mathbf{a}_2 \otimes \mathbf{b}_2 \otimes \mathbf{c}_2) = (\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{b}_1 \cdot \mathbf{b}_2)(\mathbf{c}_1 \cdot \mathbf{c}_2)$$

(here the symbol $\otimes$ denotes tensor product). Accordingly,

$$\mathsf{K} \cdot \mathsf{L} = K_{ijk} L_{ijk}.$$

(*iv*) Recall that, for second-order tensors, the trace is the unique linear real-valued map on Lin such that

$$\text{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}.$$

By a *trace operator* on Lin, we mean the unique linear map into $\mathscr{V}$ such that

$$\mathsf{tr}(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(the use of sans-serif typeface signals that "tr" is different from "tr"). It follows from this definition that

$$\mathsf{tr}\,\mathsf{K} = K_{iik} \mathbf{e}_k \qquad (\mathsf{tr}\,\mathsf{K})_k = K_{iik},$$

and hence that tr involves saturation of the first two indices. We shall need no other saturation operator. Instead, we shall make use of right and left *shift operators*, defined as follows:

$$\overrightarrow{\mathsf{K}}(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathsf{K}(\mathbf{c},\mathbf{a},\mathbf{b}) \qquad \overrightarrow{K}_{ijk} = K_{kij},$$
$$\overleftarrow{\mathsf{K}}(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathsf{K}(\mathbf{b},\mathbf{c},\mathbf{a}) \qquad \overleftarrow{K}_{ijk} = K_{jki}.$$

The superposed arrows are meant to suggest the idea of pushing cyclically by one position to the right or to the left the indices of K. By iterated composition, other shift operators are obtained from the two basic ones:

$$\overleftarrow{\overleftarrow{\mathsf{K}}} = \overrightarrow{\mathsf{K}}, \quad \overrightarrow{\overrightarrow{\mathsf{K}}} = \overleftarrow{\mathsf{K}}, \quad \overleftarrow{\overrightarrow{\mathsf{K}}} = \mathsf{K} = \overrightarrow{\overleftarrow{\mathsf{K}}}, \quad \overleftarrow{\overleftarrow{\overleftarrow{\mathsf{K}}}} = \mathsf{K} = \overrightarrow{\overrightarrow{\overrightarrow{\mathsf{K}}}}.$$

From an abstract point of view, we may look at the set $\{\mathsf{K}, \overleftarrow{\mathsf{K}}, \overrightarrow{\mathsf{K}}\}$ as the orbit in Lin of K under an action of Z_3, the cyclic group of order three.

(*v*) For $\mathsf{K} \in \mathsf{Lin}$ and $\mathbf{B}, \mathbf{C} \in \mathrm{Lin}$, we let $\mathsf{K}[\mathbf{B},\mathbf{C}] \in \mathsf{Lin}$ be defined by

$$\mathsf{K}[\mathbf{B},\mathbf{C}](\mathbf{a},\mathbf{b},\mathbf{c}) = \mathsf{K}(\mathbf{a},\mathbf{B}\mathbf{b},\mathbf{C}\mathbf{c}) \quad (\mathsf{K}[\mathbf{B},\mathbf{C}])_{ijk} = K_{ipq}B_{pj}C_{qk}; \tag{13}$$

in particular, it follows from this definition that

$$\mathsf{K}[\mathbf{I},\mathbf{I}] = \mathsf{K}, \quad (\mathsf{K}[\mathbf{B}_1,\mathbf{C}_1])[\mathbf{B}_2,\mathbf{C}_2] = \mathsf{K}[\mathbf{B}_1\mathbf{B}_2,\mathbf{C}_1\mathbf{C}_2] \tag{14}$$

(here $\mathbf{I}$ denotes the identity of Lin).

(*vi*) Symmetry with respect to second-and-third-index exchange will be a property of importance for a tensor $\mathsf{K} \in \mathsf{Lin}$; we set:

$$\mathsf{Sym}^{(2,3)} = \{\mathsf{K} \in \mathsf{Lin} \colon \mathsf{K}(\mathbf{a},\mathbf{b},\mathbf{c}) = \mathsf{K}(\mathbf{a},\mathbf{c},\mathbf{b}),\ \forall\, \mathbf{a},\mathbf{b},\mathbf{c} \in \mathscr{V}\},$$

whence

$$K_{ijk} = K_{ikj} \quad \text{whenever } \mathsf{K} \in \mathsf{Sym}^{(2,3)}.$$

It follows that

$$\mathsf{K} \in \mathsf{Sym}^{(2,3)} \;\Rightarrow\; \mathsf{K}[\mathbf{B},\mathbf{C}] = \mathsf{K}[\mathbf{C},\mathbf{B}].$$

(*vii*) We use the symbol ":" for the unique bilinear map from $\mathsf{Lin} \times \mathsf{Lin}$ into Lin such that

$$(\mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1) : (\mathbf{a}_2 \otimes \mathbf{b}_2 \otimes \mathbf{c}_2) = (\mathbf{b}_1 \cdot \mathbf{a}_2)(\mathbf{c}_1 \cdot \mathbf{b}_2)\, \mathbf{a}_1 \otimes \mathbf{c}_2.$$

Thus,

$$(\mathsf{K} : \mathsf{L})_{ij} = K_{ipq}L_{pqj}$$

(notice that, in general, $\mathsf{K} : \mathsf{L} \neq \mathsf{L} : \mathsf{K}$). We use the symbol ":" also for

$$\mathbf{A} : \mathsf{K} = \mathsf{tr}(\mathbf{A}^T\mathsf{K}) \qquad (\mathbf{A} : \mathsf{K})_k = A_{ij}K_{ijk}.$$

(*viii*) The following identities can be derived from the previous definitions:

$$\mathsf{K} \cdot \mathbf{A}\mathsf{L} = \mathbf{A}^T\mathsf{K} \cdot \mathsf{L},$$
$$\mathsf{K} \cdot \mathbf{A}\mathsf{L} = (\mathsf{K} : \overrightarrow{\mathsf{L}}) \cdot \mathbf{A} = \mathbf{A}^T \cdot (\mathsf{L} : \overrightarrow{\mathsf{K}}), \tag{15}$$
$$\mathsf{L} : \overrightarrow{\mathbf{A}\mathsf{K}} = (\mathsf{L} : \overrightarrow{\mathsf{K}})\mathbf{A}^T, \tag{16}$$
$$\mathsf{K} : \overrightarrow{\mathsf{L}[\mathbf{A},\mathbf{A}]} = \mathsf{K}[\mathbf{A}^T,\mathbf{A}^T] : \overrightarrow{\mathsf{L}}, \tag{17}$$
$$\mathsf{tr}\,\mathsf{K}[\mathbf{B},\mathbf{C}] = \mathbf{C}^T\,\mathsf{tr}(\mathbf{B}\mathsf{K}), \tag{18}$$
$$\mathsf{K}[\mathbf{B},\mathbf{C}] \cdot \mathsf{L} = \mathsf{K} \cdot \mathsf{L}[\mathbf{B}^T,\mathbf{C}^T]; \tag{19}$$

and

$$\mathsf{tr}(\mathbf{F}^{-1}\mathsf{K}[\mathbf{F},\mathbf{F}]) = \mathbf{F}^T\,\mathsf{tr}\,\mathsf{K}, \tag{20}$$

for each invertible $\mathbf{F}$.

2.2 Two orthogonal decompositions

As is well known, each second-order tensor $\mathbf{T}$ can be split uniquely into the sum of a traceless and a 'spherical' element of Lin (spherical ≡ scalar multiple of the identity); accordingly, Lin can be split into the orthogonal direct sum of Lin_0, the subspace of traceless tensors, and Sph, the subspace of spherical tensors:

$$\mathrm{Lin} = \mathrm{Lin}_0 \oplus \mathrm{Sph}.$$

A similar decomposition holds for Lin, provided one introduces the orthogonal subspaces

$$\mathsf{Lin}_0 = \{\mathsf{K} \in \mathsf{Lin} \colon \operatorname{tr} \mathsf{K} = \mathbf{0}\} \quad \text{and} \quad \mathsf{Sph} = \{\mathsf{H} \in \mathsf{Lin} \colon \mathsf{H} = \mathbf{I} \otimes \mathbf{v}, \mathbf{v} \in \mathscr{V}\}.$$

Proposition 1 *For each* $\mathsf{K} \in \mathsf{Lin}$, *there is a unique pair of tensors* $\mathsf{K}_0 \in \mathsf{Lin}_0$ *and* $\mathsf{K}_s \in \mathsf{Sph}$ *such that*

$$\mathsf{K} = \mathsf{K}_0 + \mathsf{K}_s. \tag{21}$$

Thus,

$$\mathsf{Lin} = \mathsf{Lin}_0 \oplus \mathsf{Sph}.$$

Proof If $\mathsf{K} \in \mathsf{Lin}$ admits a decomposition of type (21), there is no other such decomposition with $\bar{\mathsf{K}}_s = \mathbf{I} \otimes \bar{\mathbf{k}}$ and $\bar{\mathbf{k}} \neq \mathbf{k}$: two supposedly different decompositions should satisfy

$$\operatorname{tr} \bar{\mathsf{K}}_s = \operatorname{tr} \mathsf{K} = \operatorname{tr} \mathsf{K}_s,$$

whence $\bar{\mathbf{k}} = \mathbf{k}$. Moreover, for any given $\mathsf{K} \in \mathsf{Lin}$, set

$$\mathbf{k} = \operatorname{tr} \mathsf{K}/3$$

and define K_s and K_0 as follows:

$$\mathsf{K}_s = \mathbf{I} \otimes \mathbf{k}, \qquad \mathsf{K}_0 = \mathsf{K} - \mathsf{K}_s.$$

Since

$$\operatorname{tr} \mathsf{K}_0 = \operatorname{tr} \mathsf{K} - \operatorname{tr} \mathsf{K}_s = \mathbf{0},$$

we conclude that $\mathsf{K}_0 \in \mathsf{Lin}_0$ and that decomposition (21) holds and is unique. □

Later on, we shall need a similar direct-sum decomposition of $\mathsf{Sym}^{(2,3)}$. Let

$$\tilde{\mathsf{S}}\mathsf{ph} = \{\mathsf{V} \in \mathsf{Sym}^{(2,3)} \colon \mathbf{v} \in \mathscr{V},\ \mathsf{V}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \cdot \mathbf{v} + (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \cdot \mathbf{v},\ \ \forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathscr{V}\}, \tag{22}$$

whence, whenever $\mathsf{V} \in \tilde{\mathsf{S}}\mathsf{ph}$,

$$V_{ijk} = I_{ij} v_k + I_{ik} v_j \quad \text{and} \quad \operatorname{tr} \mathsf{V} = 3\mathbf{v} + \mathbf{v} = 4\mathbf{v}. \tag{23}$$

Furthermore, let $\mathsf{Sym}_0^{(2,3)}$ be the subspace of traceless elements of $\mathsf{Sym}^{(2,3)}$:

$$\mathsf{Sym}_0^{(2,3)} = \{\mathsf{S} \in \mathsf{Sym}^{(2,3)} \colon \operatorname{tr} \mathsf{S} = \mathbf{0},\ S_{iik} = 0\}.$$

Notice that, for any choice of $\mathsf{S} \in \mathsf{Sym}_0^{(2,3)}$ and $\mathsf{V} \in \tilde{\mathsf{S}}\mathsf{ph}$,

$$\mathsf{S} \cdot \mathsf{V} = S_{ijk} V_{ijk} = 2 \operatorname{tr} \mathsf{S} \cdot \mathbf{v} = 0,$$

i.e., that $\mathsf{Sym}_0^{(2,3)}$ and $\tilde{\mathsf{S}}\mathsf{ph}$ are orthogonal.

Proposition 2 *For each* $\mathsf{K} \in \mathsf{Sym}^{(2,3)}$, *there is a unique pair of tensors* $\mathsf{S} \in \mathsf{Sym}_0^{(2,3)}$ *and* $\mathsf{V} \in \tilde{\mathsf{S}}\mathsf{ph}$ *such that*

$$\mathsf{K} = \mathsf{S} + \mathsf{V}. \tag{24}$$

Thus,

$$\mathsf{Sym}^{(2,3)} = \mathsf{Sym}_0^{(2,3)} \oplus \tilde{\mathsf{S}}\mathsf{ph}.$$

Proof Assume that decomposition (24) holds. Then, uniqueness is a consequence of the identity:

$$\mathsf{tr}\,\mathsf{K} = \mathsf{tr}\,\mathsf{S} + \mathsf{tr}\,\mathsf{V} = \mathsf{tr}\,\mathsf{V} = 4\mathbf{v},$$

which implies that

$$\mathbf{v} = \mathsf{tr}\,\mathsf{K}/4, \qquad \mathsf{S} = \mathsf{K} - \mathsf{V}.$$

Moreover, let

$$\mathbf{v} = \mathsf{tr}\,\mathsf{K}/4$$

and define $\mathsf{V} \in \tilde{\mathsf{S}}\mathsf{ph}$ by $(23)_1$ and S by setting $\mathsf{S} = \mathsf{K} - \mathsf{V}$. Then, since

$$\mathsf{tr}\,\mathsf{S} = \mathsf{tr}\,\mathsf{K} - \mathsf{tr}\,\mathsf{V} = \mathbf{0},$$

we conclude that $\mathsf{S} \in \mathsf{Sym}_0^{(2,3)}$ and that decomposition (24) holds and is unique. □

3 Kinematic preliminaries

3.1 Pointwise deformations

At a fixed time t, a *deformation* $f(\cdot, t)$ is a smooth, invertible and locally orientation-preserving mapping from a referential region into the region currently occupied by the body of interest. If we write $\mathbf{F}$ and F for the first and second gradients of $f(\cdot, t)$, we have that:

$$\mathbf{F} = \nabla f \in \mathrm{Lin}^+, \qquad \mathsf{F} = \nabla\mathbf{F} = \nabla^2 f \in \mathsf{Sym}^{(2,3)}$$

(here Lin^+ is the group of elements of Lin with positive determinant); the pair $(\mathbf{F}, \mathsf{F})$ embodies all the information needed for a deformation analysis at a point of an elastic second-gradient material body. Each element of the set

$$\mathscr{L} = \{(\mathbf{H}, \mathsf{K}) : \mathbf{H} \in \mathrm{Lin}^+, \mathsf{K} \in \mathsf{Sym}^{(2,3)}\}$$

individuates the common restriction at a given point of an equivalence class of deformations; we call $\mathscr{L}$ the set of (second-order) *pointwise deformations*. Needless to say, a study of nth-gradient materials would entail consideration of deformation gradients up to order n and of collections of nth-order pointwise deformations.

For f_1 and f_2 any two deformations, and for $f = f_1 \circ f_2$, we have:

$$\mathbf{F} = \mathbf{F}_1\mathbf{F}_2, \quad \mathsf{F} = \mathbf{F}_1\mathsf{F}_2 + \mathsf{F}_1[\mathbf{F}_2, \mathbf{F}_2], \quad \text{with } \mathsf{F}_\alpha = \nabla\mathbf{F}_\alpha \ (\alpha = 1, 2), \tag{25}$$

two consequences of the chain rule for differentiation. The set $\mathscr{L}$ is made into a group by a composition rule suggested by (25):

$$(\mathbf{H}_1, \mathsf{K}_1) \circ (\mathbf{H}_2, \mathsf{K}_2) = (\mathbf{H}_1\mathbf{H}_2, \mathbf{H}_1\mathsf{K}_2 + \mathsf{K}_1[\mathbf{H}_2, \mathbf{H}_2]); \tag{26}$$

this composition rule makes sense in the all of $\mathrm{Lin} \times \mathsf{Sym}^{(2,3)}$ and will be used to define an action of $\mathcal{L}$ on that space. The unit element of $\mathscr{L}$ is the pair $(\mathbf{I}, \mathsf{O})$, with O the null element of Lin; associativity can be easily checked, and the inverse of $(\mathbf{H}, \mathsf{K})$ is

$$(\mathbf{H}, \mathsf{K})^{-1} = (\mathbf{H}^{-1}, -\mathbf{H}^{-1}\mathsf{K}[\mathbf{H}^{-1}, \mathbf{H}^{-1}]). \tag{27}$$

3.2 Volume-preserving pointwise deformations

A useful subgroup of $\mathscr{L}$ is

$$\mathscr{U} = \{(\mathbf{H}, \mathsf{K}) \in \mathscr{L} : \det \mathbf{H} = 1\},$$

the subset of pairs $(\mathbf{H}, \mathsf{K}) \in \mathscr{L}$ with $\mathbf{H}$ unimodular. Later on, a subgroup of $\mathscr{U}$ itself, namely,

$$\mathscr{F} = \{(\mathbf{H}, \mathsf{K}) \in \mathscr{L} : \det \mathbf{H} = 1, \ \mathrm{tr}(\mathbf{H}^{-1}\mathsf{K}) = \mathbf{0}\}, \tag{28}$$

will be chosen to be the material symmetry group of elastic second-gradient fluids. As anticipated at the end of Sect. 1.3, a physical motivation for this choice is that a pointwise deformation $(\mathbf{H}, \mathsf{K}) \in \mathscr{F}$ preserves volume 'up to the second order'; in that, there is a (n equivalence class of) deformation(s) h, with $\nabla h = \mathbf{H}$ and $\nabla(\nabla h) = \mathsf{K}$, such that

$$\det(\nabla h) = 1 \quad \text{and} \quad \nabla\big(\det(\nabla h)\big) = 0.$$

This assertion holds true because, for each pair $(\mathbf{F}, \mathsf{F})$, we have that

$$\nabla(\det \mathbf{F}) = (\det \mathbf{F})\mathbf{F}^{-T} : \mathsf{F} = (\det \mathbf{F})\, \mathrm{tr}(\mathbf{F}^{-1}\mathsf{F}). \tag{29}$$

To prove that $\mathscr{F}$ is a group, we begin to note that $(\mathbf{I}, \mathsf{O})$ belongs to $\mathscr{F}$. The proof that

$$(\mathbf{H}_1, \mathsf{K}_1), (\mathbf{H}_2, \mathsf{K}_2) \in \mathscr{F} \Rightarrow (\mathbf{H}_1, \mathsf{K}_1) \circ (\mathbf{H}_2, \mathsf{K}_2) \in \mathscr{F}, \tag{30}$$

and that

$$(\mathbf{H}, \mathsf{K}) \in \mathscr{F} \Rightarrow (\mathbf{H}, \mathsf{K})^{-1} \in \mathscr{F} \tag{31}$$

is not completely trivial. Firstly, recall that

$$\det(\mathbf{H}_1\mathbf{H}_2) = (\det \mathbf{H}_1)(\det \mathbf{H}_2), \quad \det(\mathbf{H}^{-1}) = (\det \mathbf{H})^{-1}. \tag{32}$$

Next, in view of (28) and (26), observe that

$$\begin{aligned}
\mathrm{tr}[(\mathbf{H}_1\mathbf{H}_2)^{-1}(\mathbf{H}_1\mathsf{K}_2 + \mathsf{K}_1[\mathbf{H}_2, \mathbf{H}_2])] &= \mathrm{tr}\left(\mathbf{H}_2^{-1}\mathsf{K}_2 + \mathbf{H}_2^{-1}\mathbf{H}_1^{-1}\mathsf{K}_1[\mathbf{H}_2, \mathbf{H}_2]\right)\\
&= \mathrm{tr}\left(\mathbf{H}_2^{-1}\mathsf{K}_2\right) + \mathrm{tr}\left(\mathbf{H}_2^{-1}\mathbf{H}_1^{-1}\mathsf{K}_1\,[\mathbf{H}_2, \mathbf{H}_2]\right)\\
&= \mathrm{tr}\left(\mathbf{H}_2^{-1}\mathsf{K}_2\right) + \mathbf{H}_2^{T}\,\mathrm{tr}\left(\mathbf{H}_1^{-1}\mathsf{K}_1\right)\\
&= \mathbf{0}
\end{aligned}$$

(here identity (20) has been used). Together with (32), this is enough to conclude that (30) holds. Moreover, in view of (28), (27) and (18), write:

$$\mathrm{tr}(-\mathbf{H}\mathbf{H}^{-1}\mathsf{K}[\mathbf{H}^{-1}, \mathbf{H}^{-1}]) = -\,\mathrm{tr}\,\mathsf{K}[\mathbf{H}^{-1}, \mathbf{H}^{-1}] = -\mathbf{H}^{-T}\,\mathrm{tr}(\mathbf{H}^{-1}\mathsf{K});$$

this relation, together with (32), allows to conclude that also (31) holds.

3.3 Pointwise motions

A motion is a differentiable family of deformations $t \mapsto f(\cdot, t)$; the associated velocity field over the referential placement is $\mathbf{v}(\cdot, t) = \partial_t f(\cdot, t)$. By a *pointwise motion* of a second-gradient material body we mean a smooth function of time $t \mapsto \big(\mathbf{F}(t), \mathsf{F}(t)\big) \in \mathscr{L}$.[4] Since deformations are invertible, $\mathbf{v}(\cdot, t)$ can be regarded also as a field over the current placement. When this is the case, we write

$$\operatorname{grad}\mathbf{v}, \quad \operatorname{grad}^2\mathbf{v}$$

for its first and second velocity gradients. Taking the referential gradient of the kinematical relation $(6)_2$ yields, with the use of the chain rule,

$$\dot{\mathsf{F}} = (\operatorname{grad}\mathbf{v})\mathsf{F} + (\operatorname{grad}^2\mathbf{v})[\mathbf{F}, \mathbf{F}]; \tag{33}$$

$(6)_2$ and (33) can be rewritten together in the form:

$$(\dot{\mathbf{F}}, \dot{\mathsf{F}}) = (\operatorname{grad}\mathbf{v}, \operatorname{grad}^2\mathbf{v}) \circ (\mathbf{F}, \mathsf{F}). \tag{34}$$

Notice that the pairs $(\dot{\mathbf{F}}, \dot{\mathsf{F}})$ and $(\operatorname{grad}\mathbf{v}, \operatorname{grad}^2\mathbf{v})$ do not necessarily belong to $\mathscr{L}$, since it is not always true that $\det\dot{\mathbf{F}} > 0$; in (34), the symbol $\circ$ introduced in (26) denotes the right action of the group $\mathscr{L}$ on the linear space $\mathrm{Lin} \times \mathsf{Sym}^{(2,3)}$.

Right multiplication of (34) by the inverse of $(\mathbf{F}, \mathsf{F})$ yields:

$$(\operatorname{grad}\mathbf{v}, \operatorname{grad}^2\mathbf{v}) = (\dot{\mathbf{F}}, \dot{\mathsf{F}}) \circ (\mathbf{F}, \mathsf{F})^{-1},$$

where

$$\begin{gathered} \operatorname{grad}\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \\ \operatorname{grad}^2\mathbf{v} = -\dot{\mathbf{F}}\mathbf{F}^{-1}\mathsf{F}[\mathbf{F}^{-1}, \mathbf{F}^{-1}] + \dot{\mathsf{F}}[\mathbf{F}^{-1}, \mathbf{F}^{-1}] \end{gathered} \tag{35}$$

Equations (34) and (35) express, in a compact and convenient notation, the relations between the pair of first and second velocity gradients and the pair of time derivatives of first and second deformation gradients.

4 Stress power

Stress-power expenditure is what characterizes material behavior in a body motion. As anticipated in the Introduction, this concept is central to our developments. In this section, for both first- and second-gradient materials, we give alternative expressions of its current and referential volume densities, we postulate its invariance under observer changes, and we discuss the consequences of this postulation.

Let x and $y = f(x, t)$ be the typical points of, respectively, the space regions $\mathscr{P}$ and $\mathscr{P}_t$, a given body part in motion occupies in its reference and current placements. When $\mathcal{B}$ is comprised of a first-gradient material, the stress power P expended over that part in that motion can be defined in two alternative ways:

$$\int_{\mathscr{P}_t} \mathbf{T} \cdot \operatorname{grad}\mathbf{v}\, \mathrm{d}V_y =: P_1(\mathscr{P}, f) := \int_{\mathscr{P}} \mathbf{S} \cdot \dot{\mathbf{F}}\, \mathrm{d}V_x$$

where $\mathbf{T}$ and $\mathbf{S}$ are the *Cauchy* and *Piola stress measures*. With the use of $(6)_2$, this twofold definition yields the well-known relations between $\mathbf{T}$ and $\mathbf{S}$:

$$J\,\mathbf{T} = \mathbf{S}\mathbf{F}^T \quad \text{and} \quad \mathbf{S} = J\,\mathbf{T}\mathbf{F}^{-T},$$

where

$$J = \det\mathbf{F}. \tag{36}$$

[4] Our notion of pointwise motion does not include the motion itself because, in view of its role when constitutive assumptions are specified, we prefer to have it translation-invariant.

This procedure is easily generalized to the case of second-gradient materials, when each of the alternative definitions of the stress power involves a pair of stress measures, namely, $(\mathbf{T}, \mathsf{T})$ of Cauchy's type and $(\mathbf{S}, \mathsf{S})$ of Piola's:

$$\int_{\mathscr{P}_t} \{\mathbf{T} \cdot \operatorname{grad} \mathbf{v} + \mathsf{T} \cdot \operatorname{grad}^2 \mathbf{v}\}\, dV_y =: P_2(\mathscr{P}, f) := \int_{\mathscr{P}} \{\mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}}\}\, dV_x, \tag{37}$$

where $\mathbf{T}, \mathbf{S} \in \mathrm{Lin}$ and $\mathsf{T}, \mathsf{S} \in \mathsf{Sym}^{(2,3)}$, consistently with the fact that both fields $\operatorname{grad}^2 \mathbf{v}$ and $\dot{\mathsf{F}}$ take their values in $\mathsf{Sym}^{(2,3)}$.

Proposition 3 *The equality in (37) is satisfied for all body parts and all velocity fields if and only if the following two equivalent sets of relations hold:*

$$\begin{cases} J\,\mathbf{T} = \mathbf{S}\mathbf{F}^T + \mathsf{S} : \overrightarrow{\mathsf{F}} \\ J\,\mathsf{T} = \mathsf{S}[\mathbf{F}^T, \mathbf{F}^T] \end{cases} \qquad \begin{cases} \mathbf{S} = J\,\mathbf{T}\mathbf{F}^{-T} - (\mathsf{S} : \overrightarrow{\mathsf{F}})\mathbf{F}^{-T} \\ \mathsf{S} = J\,\mathsf{T}[\mathbf{F}^{-T}, \mathbf{F}^{-T}] \end{cases} \tag{38}$$

Proof On the one hand, in view of $(6)_2$, (33), (19), and (15), we have that:

$$\begin{aligned} \int_{\mathscr{P}} \{\mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}}\}\, dV_x &= \int_{\mathscr{P}} \{\mathbf{S} \cdot (\operatorname{grad} \mathbf{v})\mathbf{F} + \mathsf{S} \cdot (\operatorname{grad} \mathbf{v})\mathsf{F} + \mathsf{S} \cdot \operatorname{grad}^2 \mathbf{v}[\mathbf{F}, \mathbf{F}]\}\, dV_x \\ &= \int_{\mathscr{P}} \{[\mathbf{S}\mathbf{F}^T + (\mathsf{S} : \overrightarrow{\mathsf{F}})] \cdot \operatorname{grad} \mathbf{v} + \mathsf{S}[\mathbf{F}^T, \mathbf{F}^T] \cdot \operatorname{grad}^2 \mathbf{v}\}\, dV_x; \end{aligned} \tag{39}$$

on the other hand, in view of (36), we also have that:

$$\int_{\mathscr{P}_t} \{\mathbf{T} \cdot \operatorname{grad} \mathbf{v} + \mathsf{T} \cdot \operatorname{grad}^2 \mathbf{v}\}\, dV_y = \int_{\mathscr{P}} \{\mathbf{T} \cdot \operatorname{grad} \mathbf{v} + \mathsf{T} \cdot \operatorname{grad}^2 \mathbf{v}\} J\, dV_x; \tag{40}$$

a comparison of the last lines of (39) with (40) yields the pair of relations on the left of (38). Next, from the second of those relations, we obtain that

$$J\,\mathsf{T}[\mathbf{F}^{-T}, \mathbf{F}^{-T}] = (\mathsf{S}[\mathbf{F}^T, \mathbf{F}^T])[\mathbf{F}^{-T}, \mathbf{F}^{-T}],$$

that is, in view of (14), the second relation on the right in (38). Finally, to see that the first relations on the right and on the left are equivalent, we compose the former on the right with $\mathbf{F}^{-\mathrm{T}}$. □

5 Material symmetry groups

The main goal of this section is to propose a notion of symmetry group for elastic second-gradient materials. We begin by reviewing the concept as is customarily stated for simple materials.

5.1 First-gradient materials

Classic elasticity is based on the one or the other of the constitutive mappings

$$\mathrm{Lin}^+ \ni \mathbf{F} \mapsto \widehat{\mathbf{T}}(\mathbf{F}) = \mathbf{T} \in \mathrm{Sym} \quad \text{and} \quad \mathrm{Lin}^+ \ni \mathbf{F} \mapsto \widehat{\mathbf{S}}(\mathbf{F}) = \mathbf{S} \in \mathrm{Lin};$$

needless to say, the information carried by the Cauchy's and Piola's mappings $\widehat{\mathbf{T}}$ and $\widehat{\mathbf{S}}$ is the same, provided that

$$\widehat{\mathbf{S}}(\mathbf{F}) = (\det \mathbf{F})\widehat{\mathbf{T}}(\mathbf{F})\mathbf{F}^{-T}. \tag{41}$$

Given $\widehat{\mathbf{T}}$, say, the relative *material symmetry group* is the collection $\mathscr{G}_1$ of all deformations of the reference configuration that are not 'Cauchy-stress detectable', because they leave the Cauchy-stress response to all further deformations unchanged:

$$\mathscr{G}_1 = \{\mathbf{H} \in \mathrm{Lin}\colon\ \det\mathbf{H} = 1,\ \widehat{\mathbf{T}}(\mathbf{F}\mathbf{H}) = \widehat{\mathbf{T}}(\mathbf{F}),\quad \forall\mathbf{F} \in \mathrm{Lin}^+\}; \tag{42}$$

alternatively, and equivalently if (41) holds,

$$\mathscr{G}_1 = \{\mathbf{H} \in \mathrm{Lin}\colon\ \det\mathbf{H} = 1,\ \widehat{\mathbf{S}}(\mathbf{F}\mathbf{H})\mathbf{H}^T = \widehat{\mathbf{S}}(\mathbf{F}),\quad \forall\mathbf{F} \in \mathrm{Lin}^+\}.$$

As mentioned in the Introduction, Noll's criterion sorts elastic materials according to the 'richness' of their symmetry group: they are classified as *solids* if their symmetry group $\mathscr{G}_1$ is a rotation subgroup, *fluids* if the group

$$\mathrm{Uni} := \{\mathbf{H} \in \mathrm{Lin}\colon\ \det\mathbf{H} = 1\}$$

of all not-necessarily-rigid volume-preserving deformations is a subgroup of $\mathscr{G}_1$.[5]

5.2 Second-gradient materials

Consistent with the stress-power-based approach that we take from [16], our proposal is motivated by an invariance argument of the stress power over a process class that we now introduce.

Firstly, let us agree to call a second-gradient material elastic if its pairs of constitutive mappings are

$$\text{either}\quad \begin{cases}\mathbf{T} = \widetilde{\mathbf{T}}(\mathbf{F}, \mathsf{F})\\ \mathsf{T} = \widetilde{\mathsf{T}}(\mathbf{F}, \mathsf{F})\end{cases} \quad\text{or}\quad \begin{cases}\mathbf{S} = \widetilde{\mathbf{S}}(\mathbf{F}, \mathsf{F})\\ \mathsf{S} = \widetilde{\mathsf{S}}(\mathbf{F}, \mathsf{F})\end{cases}.$$

Secondly, for any given pointwise motion $t \mapsto \big(\mathbf{F}(t), \mathsf{F}(t)\big)$ and any given pair $(\mathbf{H}, \mathsf{K}) \in \mathscr{U}$, let us consider the kinematical process

$$t \mapsto (\mathbf{F}^*(t), \mathsf{F}^*(t)) = (\mathbf{F}(t), \mathsf{F}(t)) \circ (\mathbf{H}, \mathsf{K}) = (\mathbf{F}(t)\mathbf{H}, \mathbf{F}(t)\mathsf{K} + \mathsf{F}(t)[\mathbf{H}, \mathbf{H}]),$$

for which

$$\mathbf{F}^* = \mathbf{F}\mathbf{H}, \quad \mathsf{F}^* = \mathbf{F}\mathsf{K} + \mathsf{F}[\mathbf{H}, \mathbf{H}]$$

and

$$\dot{\mathbf{F}}^* = \dot{\mathbf{F}}\mathbf{H}, \quad \dot{\mathsf{F}}^* = \dot{\mathbf{F}}\mathsf{K} + \dot{\mathsf{F}}[\mathbf{H}, \mathbf{H}]. \tag{43}$$

Finally, let us set:

$$\mathbf{S}^* = \tilde{\mathbf{S}}(\mathbf{F}^*, \mathsf{F}^*), \quad \mathsf{S}^* = \tilde{\mathsf{S}}(\mathbf{F}^*, \mathsf{F}^*)$$

and

$$\mathbf{T}^* = \widehat{\mathbf{T}}(\mathbf{F}^*, \mathsf{F}^*), \quad \mathsf{T}^* = \widehat{\mathsf{T}}(\mathbf{F}^*, \mathsf{F}^*).$$

We are now in a position to state

Proposition 4 *The invariance requirement*

$$\mathbf{S}^* \cdot \dot{\mathbf{F}}^* + \mathsf{S}^* \cdot \dot{\mathsf{F}}^* = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}} \tag{44}$$

is satisfied for all pointwise motions if and only if

$$\begin{cases}\mathbf{S}^* = \mathbf{S}\mathbf{H}^{-T} - (\mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}] : \overrightarrow{\mathsf{K}})\mathbf{H}^{-T}\\ \mathsf{S}^* = \mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}]\end{cases} \tag{45}$$

or, equivalently, if and only if

$$\mathbf{T}^* = \mathbf{T}, \quad \mathsf{T}^* = \mathsf{T}. \tag{46}$$

[5] As is well-known (see e.g. Exercise 4, p. 172 of [11]), were not the condition $\det\mathbf{H} = 1$ included in Definition (42), one would implicitly admit that iteration of some reference changes might lead to stress-undetectable 'explosions' (if $\det\mathbf{H} > 1$, the volume measure would diverge) or 'implosions' (if $\det\mathbf{H} < 1$, the volume measure would tend to null).

Proof With the use of (43), (44) yields:

$$\mathbf{S}^* \cdot \dot{\mathbf{F}}\mathbf{H} + \mathsf{S}^* \cdot \dot{\mathbf{F}}\mathsf{K} + \mathsf{S}^* \cdot \dot{\mathsf{F}}[\mathbf{H}, \mathbf{H}] = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}},$$

which, with the help of identities (15) and (19), can be written as

$$\mathbf{S}^*\mathbf{H}^T \cdot \dot{\mathbf{F}} + (\mathsf{S}^* : \overrightarrow{\mathsf{K}}) \cdot \dot{\mathbf{F}} + \mathsf{S}^*[\mathbf{H}^T, \mathbf{H}^T] \cdot \dot{\mathsf{F}} = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}},$$

where both $\dot{\mathbf{F}}$ and $\dot{\mathsf{F}}$ can be chosen independently and arbitrarily. Hence,

$$\begin{cases} \mathbf{S} = \mathbf{S}^*\mathbf{H}^T + \mathsf{S}^* : \overrightarrow{\mathsf{K}} \\ \mathsf{S} = \mathsf{S}^*[\mathbf{H}^T, \mathbf{H}^T] \end{cases}.$$

Some easy manipulations put this result in the desired form (45).

To arrive at (46), we recall the first two of (38) and write:

$$\begin{cases} J\,\mathbf{T}^* = \mathbf{S}^*(\mathbf{F}^*)^T + \mathsf{S}^* : \overrightarrow{\mathsf{F}^*} \\ J\mathsf{T}^* = \mathsf{S}^*[(\mathbf{F}^*)^T, (\mathbf{F}^*)^T] \end{cases},$$

where, after substitution of (45) and with the help of identities (14), (16) and (17), we deduce:

$$\begin{aligned} J\,\mathbf{T}^* &= \mathbf{S}^*(\mathbf{F}\mathbf{H})^T + \mathsf{S}^* : \overrightarrow{\mathbf{F}\mathsf{K}} + \mathsf{S}^* : \overrightarrow{\mathsf{F}[\mathbf{H}, \mathbf{H}]} \\ &= \mathbf{S}\mathbf{H}^{-T}\mathbf{H}^T\mathbf{F}^T - (\mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}] : \overrightarrow{\mathsf{K}})\mathbf{H}^{-T}\mathbf{H}^T\mathbf{F}^T \\ &\quad + \mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}] : \overrightarrow{\mathbf{F}\mathsf{K}} + \mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}] : \overrightarrow{\mathsf{F}[\mathbf{H}, \mathbf{H}]} \\ &= \mathbf{S}\mathbf{F}^T + \mathsf{S} : \overrightarrow{\mathsf{F}} \\ &= J\,\mathbf{T} \end{aligned}$$

and

$$\begin{aligned} J\,\mathsf{T}^* &= \mathsf{S}^*[(\mathbf{F}\mathbf{H})^T, (\mathbf{F}\mathbf{H})^T] \\ &= \mathsf{S}^*[\mathbf{H}^T\mathbf{F}^T, \mathbf{H}^T\mathbf{F}^T] \\ &= (\mathsf{S}[\mathbf{H}^{-T}, \mathbf{H}^{-T}])[\mathbf{H}^T\mathbf{F}^T, \mathbf{H}^T\mathbf{F}^T] \\ &= \mathsf{S}[\mathbf{F}^T, \mathbf{F}^T] \\ &= J\,\mathsf{T}. \end{aligned}$$

□

On the basis of this proposition, we define as follows the symmetry group of an elastic second-gradient material whose stress response to pointwise deformations is described by the pair $(\widetilde{\mathbf{T}}, \widetilde{\mathsf{T}})$:

$$\begin{aligned} \mathscr{G}_2 = \{(\mathbf{H}, \mathsf{K}) \in \mathscr{U} : \widetilde{\mathbf{T}}((\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})) = \widetilde{\mathbf{T}}(\mathbf{F}, \mathsf{F}), \\ \widetilde{\mathsf{T}}((\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})) = \widetilde{\mathsf{T}}(\mathbf{F}, \mathsf{F}),\ \forall (\mathbf{F}, \mathsf{F}) \in \mathrm{Lin}^+ \times \mathsf{Sym}^{(2,3)}\}, \end{aligned} \quad (47)$$

a coherent generalization of (42); each $(\mathbf{H}, \mathsf{K}) \in \mathscr{G}_2$ is a pointwise deformation that cannot be detected by measuring stress (notice that only at this stage we can easily prove, with a classical argument, that $\mathscr{G}_2$ is indeed a group).

5.3 Second-gradient fluids

Definition (47) is applicable to all elastic second-gradient materials, irrespective of their aggregation state. We now try to argue that replacing $\mathscr{U}$ with $\mathscr{F}$ in (47) singles out *fluids*.

In Sect. 3.2, we have interpreted the group $\mathscr{F}$ as the collection of all pointwise transformations that preserve volume 'up to the second order': accordingly, just as is done with the group Uni for first-gradient fluids, it seems reasonable to characterize second-gradient fluids by choosing $\mathscr{F}$ as their material symmetry group. A further motivation for this choice is based on a result established in [12,18] and reproduced with slight variants in our final Sect. 8: for elastic second-gradient materials, a pointwise deformation pair $(\mathbf{F}, \mathsf{F})$ and the corresponding pair $(\rho, \operatorname{grad}\rho)$ of the current mass density and its gradient capture the same constitutively relevant information. With this in mind, consider a matter chunk in its reference configuration, and insist that admissible deformations conserve mass point by point as specified by (2), repeated here for the reader's convenience:

$$\rho \det \mathbf{F} = \rho_0.$$

Take the current gradient of (2), on making use of two facts: that, for $\varphi = \widehat{\varphi}(f(x,t))$ a smooth scalar field, the current and referential gradients of φ satisfy:

$$\operatorname{grad}\varphi = \mathbf{F}^{-T}\nabla\varphi, \tag{48}$$

and that the differential identity (29) holds; the result of this computation is:

$$\operatorname{grad}\rho = \mathbf{F}^{-T}\big((\det\mathbf{F})^{-1}\nabla\rho_0 - \rho\ \operatorname{tr}(\mathbf{F}^{-1}\mathsf{F})\big). \tag{49}$$

Thus, while a pointwise deformation $(\mathbf{F}, \mathsf{F}) \in \mathscr{U}$ preserves mass density, it does change in general the current density gradient (even when $\nabla\rho_0$ is null) and is therefore likely to be stress-detectable if the material in question is gradient-sensitive. However, if $(\mathbf{F}, \mathsf{F}) \in \mathscr{F}$, then (49) reduces to

$$\operatorname{grad}\rho = \mathbf{F}^{-T}\nabla\rho,$$

a realization of identity (48), and we conclude that pointwise deformations in $\mathscr{F}$ preserve also the mass-density gradient.

In conclusion, when it comes to generalize the standard request that the stress response of first-gradient fluids is unaffected by whatever volume-preserving deformation, we find it reasonable to do it in such a way that the stress response of second-gradient fluids filters out all pointwise deformations in $\mathscr{F}$, in the sense that, *whenever the referential density gradient is null, whatever pointwise deformation path in* $\mathscr{F}$ *should be undetectable*. To express this requirement unambiguously in terms of stress-power invariance, we pause to review the manifold structure of ($\mathscr{L}$ and) $\mathcal{F}$.

6 Lie-group structure of $\mathscr{L}$ and $\mathscr{F}$

The reader is referred to [23] for general ideas about Lie groups. We write $\mathscr{T}_g\mathscr{F}$ for the tangent space to $\mathscr{F}$ at some point $g = (\mathbf{H}, \mathsf{K})$; in particular, $\mathscr{T}_e\mathscr{F}$ denotes the tangent space to $\mathscr{F}$ at the unit element $e = (\mathbf{I}, \mathsf{O})$. With a view toward characterizing $\mathscr{T}_e\mathscr{F}$, we consider a curve $(\mathbf{H}(t), \mathsf{K}(t)) \in \mathscr{F}$ such that $(\mathbf{H}(0), \mathsf{K}(0)) = e$, *i.e.*, a curve on $\mathscr{F}$ going through the unit element at $t = 0$. From condition $\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K}) = \mathbf{0}$, by making use of

$$(\mathbf{H}^{-1})^{\cdot} = -\mathbf{H}^{-1}\dot{\mathbf{H}}\mathbf{H}^{-1},$$

we deduce that

$$\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K})^{\cdot} = \operatorname{tr}(-\mathbf{H}^{-1}\dot{\mathbf{H}}\mathbf{H}^{-1}\mathsf{K}) + \operatorname{tr}(\mathbf{H}^{-1}\dot{\mathsf{K}}) = \mathbf{0}, \tag{50}$$

a relation that reduces to

$$\operatorname{tr}\dot{\mathsf{K}} = \mathbf{0} \quad \text{at } t = 0.$$

Moreover, differentiation of condition $\det\mathbf{H}(t) = 1$ yields:

$$\operatorname{tr}(\dot{\mathbf{H}}\mathbf{H}^{-1}) = 0, \tag{51}$$

whence

$$\operatorname{tr}\dot{\mathbf{H}} = 0 \quad \text{at } t = 0.$$

Thus, the tangent space $\mathscr{T}_e\mathscr{F}$ is given by

$$\mathscr{T}_e\mathscr{F} = \{(\mathbf{A}, \mathsf{A}) \in \mathrm{Lin} \times \mathsf{Sym}^{(2,3)} : \ \operatorname{tr}\mathbf{A} = 0, \ \operatorname{tr}\mathsf{A} = \mathbf{0}\} \tag{52}$$

(notice that $\dim \mathscr{F} = \dim \mathscr{T}_e\mathscr{F} = 8 + (3 \times 6 - 3) = 23$, while $\dim \mathscr{L} = \dim \mathscr{T}_e\mathscr{L} = 9 + 3 \times 6 = 27$).

In a Lie group, the tangent spaces at the unit element e and at a typical point g are related by right translation. Indeed, any curve $g(t) = (\mathbf{H}(t), \mathsf{K}(t)) \in \mathscr{F}$ such that

$$g(0) = (\mathbf{H}(0), \mathsf{K}(0)) = (\mathbf{H}, \mathsf{K}) = g$$

can be represented as a right translation (i.e., a right multiplication by g) of a curve $\widehat{g}(t)$ going through the unit element e at $t = 0$:

$$g(t) = \widehat{g}(t) \circ g.$$

Symbolically,

$$\mathscr{T}_g\mathscr{F} = \mathscr{T}_e\mathscr{F} \circ g.$$

More to the point, one can check that, for $g = (\mathbf{H}, \mathsf{K}) \in \mathscr{F}$,

$$\mathscr{T}_g\mathscr{F} = \{(\mathbf{B}, \mathsf{B}) \in \mathrm{Lin} \times \mathsf{Sym}^{(2,3)} : (\mathbf{B}, \mathsf{B}) = (\mathbf{A}, \mathsf{A}) \circ (\mathbf{H}, \mathsf{K}), \ (\mathbf{A}, \mathsf{A}) \in \mathscr{T}_e\mathscr{F}\}$$

or, more explicitly,

$$\mathscr{T}_g\mathscr{F} = \{(\mathbf{B}, \mathsf{B}) \in \mathrm{Lin} \times \mathsf{Sym}^{(2,3)} : \mathbf{B} = \mathbf{A}\mathbf{H}, \mathsf{B} = \mathbf{A}\mathsf{K} + \mathsf{A}[\mathbf{H}, \mathbf{H}], \ (\mathbf{A}, \mathsf{A}) \in \mathscr{T}_e\mathscr{F}\}.^{6} \tag{53}$$

In the following, a major role shall be played by the orthogonal complement of $\mathscr{T}_e\mathscr{F}$ in $\mathrm{Lin} \times \mathsf{Sym}^{(2,3)}$, whose structure can be deduced from (52) with the help of Proposition 2.

Proposition 5 *The orthogonal complement of* $\mathscr{T}_e\mathscr{F}$ *with respect to the linear space* $\mathrm{Lin} \times \mathsf{Sym}^{(2,3)}$ *is*

$$(\mathscr{T}_e\mathscr{F})^{\perp} = \mathrm{Sph} \oplus \widetilde{\mathsf{S}}\mathsf{ph}.^{7}$$

Thus, and this is important for some developments in the next section, each element orthogonal to $\mathscr{T}_e\mathscr{F}$ is uniquely parameterized by *one scalar* and *one vector*.

[6] For $\dot{\mathbf{H}} = \mathbf{A}\mathbf{H}$ and $\dot{\mathsf{K}} = \mathbf{A}\mathsf{K} + \mathsf{A}[\mathbf{H}, \mathbf{H}]$ (with $\operatorname{tr}\mathbf{A} = 0$ and $\operatorname{tr}\mathsf{A} = \mathbf{0}$), by substitution into conditions (51) and (50) we have that

$$\operatorname{tr}(\mathbf{A}\mathbf{H}\mathbf{H}^{-1}) = \operatorname{tr}\mathbf{A} = 0$$

and that

$$\operatorname{tr}(-\mathbf{H}^{-1}\mathbf{A}\mathbf{H}\mathbf{H}^{-1}\mathsf{K} + \mathbf{H}^{-1}\mathbf{A}\mathsf{K} + \mathbf{H}^{-1}\mathsf{A}[\mathbf{H}, \mathbf{H}]) = \operatorname{tr}(-\mathbf{H}^{-1}\mathbf{A}\mathsf{K} + \mathbf{H}^{-1}\mathbf{A}\mathsf{K} + \mathbf{H}^{-1}\mathsf{A}[\mathbf{H}, \mathbf{H}]) = \operatorname{tr}(\mathbf{H}^{-1}\mathsf{A}[\mathbf{H}, \mathbf{H}]).$$

In view of identity (20), we conclude that

$$\operatorname{tr}(\mathbf{H}^{-1}\mathsf{A}[\mathbf{H}, \mathbf{H}]) = \mathbf{H}^{T} \operatorname{tr}\mathsf{A} = \mathbf{0},$$

which confirms that (53) describes the tangent space to $\mathscr{F}$ at $g = (\mathbf{H}, \mathsf{K})$.

[7] Here, we recall, Sph is the space of spherical second-order tensors and $\widetilde{\mathsf{S}}\mathsf{ph}$ the space of third-order tensors characterized by (22).

7 Group invariance of the stress power

In this section, we establish our main result, a representation for the constitutive law of elastic second-gradient fluids in terms of Cauchy-type stress measures.

For us, a *second-gradient material is a fluid if its symmetry group is* $\mathscr{F}$. By a *symmetry motion* we mean a pointwise motion $t \mapsto \big(\mathbf{H}(t), \mathsf{K}(t)\big)$, whose image is a curve in $\mathscr{F}$, such that $(\mathbf{H}(0), \mathsf{K}(0)) = (\mathbf{I}, \mathsf{O})$; in view of our preceding discussion of the Lie-group structure of $\mathscr{F}$, $(\dot{\mathbf{H}}(0), \dot{\mathsf{K}}(0)) \in \mathscr{T}_e\mathscr{F}$, and hence, according to (52), $\operatorname{tr}\dot{\mathbf{H}}(0) = 0$ and $\operatorname{tr}\dot{\mathsf{K}}(0) = \mathbf{0}$. The representation result we are after is going to be a consequence of the postulate that, *for second-gradient fluids, symmetry motions cannot be detected by measuring stress-power expenditures*.

With a view toward making this requirement precise, we consider composition mappings of a symmetry motion and a concomitant pointwise motion:

$$t \mapsto (\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})(t) = \big(\mathbf{F}(t)\mathbf{H}(t), \mathbf{F}(t)\mathsf{K}(t) + \mathsf{F}[\mathbf{H}(t), \mathbf{H}(t)]\big), \tag{54}$$

and we note that time differentiation yields:

$$(\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})^{\cdot} = (\dot{\mathbf{F}}\mathbf{H} + \mathbf{F}\dot{\mathbf{H}}, \dot{\mathbf{F}}\mathsf{K} + \mathbf{F}\dot{\mathsf{K}} + \dot{\mathsf{F}}[\mathbf{H}, \mathbf{H}] + 2\mathsf{F}[\dot{\mathbf{H}}, \mathbf{H}],$$

so that, for $t = 0$,

$$\begin{aligned}(\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})(0) &= \big(\mathbf{F}(0), \mathsf{F}[\mathbf{I}, \mathbf{I}]\big),\\ (\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})^{\cdot}(0) &= (\dot{\mathbf{F}}(0) + \mathbf{F}(0)\dot{\mathbf{H}}(0), \mathbf{F}(0)\dot{\mathsf{K}}(0) + \dot{\mathsf{F}}(0)[\mathbf{I}, \mathbf{I}] + 2\mathsf{F}(0)[\dot{\mathbf{H}}, \mathbf{I}].\end{aligned}$$

We then stipulate that, *at time* $t = 0$*, the stress power expended over any elastic body part in any given pointwise motion be the same as the stress power expended in a composition of type* (54) *of that pointwise motion with whatever symmetry motion*, a stipulation that we find convenient to lay down in the following form:

$$\big(\mathbf{S}(\mathbf{F}, \mathsf{F}) \cdot \dot{\mathbf{F}} + \mathsf{S}(\mathbf{F}, \mathsf{F}) \cdot \dot{\mathsf{F}}\big)_{t=0} = \big(\mathbf{S}((\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})) \cdot (\mathbf{F}\mathbf{H})^{\cdot} + \mathsf{S}((\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})) \cdot (\mathbf{F}\mathsf{K} + \mathsf{F}[\mathbf{H}, \mathbf{H}])^{\cdot}\big)_{t=0}. \tag{55}$$

A necessary and sufficient condition for (55) to be satisfied is that

$$\mathbf{S}(\mathbf{F}, \mathsf{F}) \cdot \mathbf{F}\dot{\mathbf{H}} + \mathsf{S}(\mathbf{F}, \mathsf{F}) \cdot \mathbf{F}\dot{\mathsf{K}} + 2\,\mathsf{S}(\mathbf{F}, \mathsf{F}) \cdot \mathsf{F}[\dot{\mathbf{H}}, \mathbf{I}] = 0 \quad \forall\,(\dot{\mathbf{H}}, \dot{\mathsf{K}}) \in \mathscr{T}_e\mathscr{F},$$

where for short we have written $\mathbf{F}, \mathsf{F}$ for $\mathbf{F}(0), \mathsf{F}(0)$ and $\dot{\mathbf{H}}, \dot{\mathsf{K}}$ for $\dot{\mathbf{H}}(0), \dot{\mathsf{K}}(0)$; since $\dot{\mathbf{H}}$ and $\dot{\mathsf{K}}$ can be chosen independently, this condition can be split into

$$\mathbf{F}^T\mathsf{S} \cdot \dot{\mathsf{K}} = 0 \quad \forall\,\dot{\mathsf{K}} \text{ such that } \operatorname{tr}\dot{\mathsf{K}} = \mathbf{0}, \tag{56}$$

$$\mathbf{F}^T\mathbf{S} \cdot \dot{\mathbf{H}} + 2\,\mathsf{S} \cdot \mathsf{F}[\dot{\mathbf{H}}, \mathbf{I}] = 0 \quad \forall\,\dot{\mathbf{H}} \text{ such that } \operatorname{tr}\dot{\mathbf{H}} = 0, \tag{57}$$

where the characterization (52) of $\mathscr{T}_e\mathscr{F}$ has been used and, again for short, the arguments $(\mathbf{F}, \mathsf{F})$ of the stress functions have been omitted.

Firstly, we take up condition (56), and note that, in view of Proposition 5, it is equivalent to

$$\mathbf{F}^T\mathsf{S} \in \widetilde{\mathsf{S}}\mathsf{ph}$$

or rather, due to (23), to

$$(\mathbf{F}^T\mathsf{S})_{rjk} = I_{rj}m_k + I_{rk}m_j \quad \Leftrightarrow \quad S_{rjk} = F^{-1}_{jr}m_k + F^{-1}_{kr}m_j, \tag{58}$$

for some vector $\mathbf{m}$. Secondly, we turn our attention to the more complex condition (57) and write it componentwise:

$$(S_{lp}F_{lr} + 2S_{ipq}F_{irq})\dot{H}_{rp} = 0,$$

or rather, equivalently because $\dot{\mathbf{H}} \in \mathscr{T}_e\mathscr{F}$ is an arbitrary traceless tensor,

$$S_{lp}F_{lr} + 2S_{ipq}F_{irq} = \pi I_{rp},$$

for some scalar π. Upon multiplication on the left by $\mathbf{F}^{-T}$, we can express $\mathbf{S}$ as

$$S_{sp} = \pi F^{-1}_{ps} - 2 S_{ipq} F_{irq} F^{-1}_{rs},$$

which, in view of the second of (58), finally yields:

$$S_{lj} = \pi F^{-1}_{jl} - 2 \left(F^{-1}_{jp} m_k + F^{-1}_{kp} m_j \right) F_{pik} F^{-1}_{il}. \tag{59}$$

We conclude that, for second-gradient materials, the representation of the Piola-type stress measures $\mathbf{S}$ and S depends on two mappings, the one scalar-valued the other vector-valued, defined over the set of admissible pointwise deformations. Our next goal is to deduce the corresponding representations for the Cauchy-type stress measures $\mathbf{T}$ and T.

Substitutions of (58) and (59) into the first of (38) yields:

$$J T_{lh} = S_{lj} F_{hj} + S_{lrs} F_{hrs} = \pi F^{-1}_{jl} F_{hj} - 2 \left(F^{-1}_{jp} m_k + F^{-1}_{kp} m_j \right) F_{pik} F^{-1}_{il} F_{hj} + \left(F^{-1}_{rl} m_s + F^{-1}_{sl} m_r \right) F_{hrs}. \tag{60}$$

Now,

$$\begin{aligned} F^{-1}_{jl} F_{hj} &= I_{lh}, \\ -2 F^{-1}_{jp} m_k F_{pik} F^{-1}_{il} F_{hj} &= -2 F^{-1}_{il} m_k F_{hik}, \\ \left(F^{-1}_{rl} m_s + F^{-1}_{sl} m_r \right) F_{hrs} &= 2 F^{-1}_{rl} m_s F_{hrs}, \end{aligned}$$

because $F_{hrs} = F_{hsr}$; and,

$$\begin{aligned} F_{hj} m_j &= (\mathbf{Fm})_h, \\ F_{pik} F^{-1}_{il} F^{-1}_{kp} = (\mathsf{F}[\mathbf{F}^{-1}, \mathbf{F}^{-1}])_{ppl} &= \left(\mathrm{tr}(\mathsf{F}[\mathbf{F}^{-1}, \mathbf{F}^{-1}]) \right)_l = \left(\mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}) \right)_l . \end{aligned}$$

Consequently, Eq. (60) can be provisionally rewritten as

$$J \mathbf{T} = \pi \mathbf{I} - 2 \, \mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}) \otimes \mathbf{Fm};$$

this, on exploiting the arbitrariness inherent to $\mathbf{Fm}$ to guarantee the symmetry of $\mathbf{T}$:

$$(\mathbf{Fm}) \times \left(\mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}) \right) = \mathbf{0}, \tag{61}$$

becomes:

$$J \mathbf{T} = \pi \, \mathbf{I} + \tilde{\pi} \, \mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}) \otimes \mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}).$$

We return now to Eq. (49) and assume for the sake of simplicity that $\nabla \rho_0 = \mathbf{0}$, so that

$$\rho^{-1} \, \mathrm{grad} \, \rho = -\mathbf{F}^{-T} \, \mathrm{tr}(\mathbf{F}^{-1} \mathsf{F}). \tag{62}$$

With this, we arrive at

$$\mathbf{T} = \alpha_1 \mathbf{I} + \alpha_2 \, \mathrm{grad} \, \rho \otimes \mathrm{grad} \, \rho \quad T_{ij} = \alpha_1 I_{ij} + \alpha_2 \rho_{,i} \rho_{,j}, \tag{63}$$

a representation for $\mathbf{T}$ parameterized by the scalar-valued constitutive mappings α_1 and α_2.

As to T, we begin by writing the second of (38) in components:

$$J T_{rpq} = S_{rjk} F_{qj} F_{pk};$$

substitution of (58) yields:

$$J \, T_{rqp} = \left(F^{-1}_{jr} m_k + F^{-1}_{kr} m_j \right) F_{qj} F_{pk} = I_{qr} (\mathbf{Fm})_p + I_{pr} (\mathbf{Fm})_q, \tag{64}$$

or rather, on combining (61) and (62) as we just did,

$$T_{rqp} = \alpha_3\big(I_{qr}\rho_{,p} + I_{pr}\rho_{,q}\big),$$

a representation parameterized by another scalar-valued constitutive mapping, α_3.

The case of paramount interest is when $\nabla\rho_0 \equiv \mathbf{0}$, that is, the referential mass density is spatially uniform, and, moreover,

$$\alpha_i = \widehat{\alpha}_i(\rho, |\,\mathrm{grad}\,\rho|) \ \ (i = 1, 2, 3),$$

when (63) and (64) take the following forms:

$$T_{ij} = \widehat{\alpha}_1(\rho, |\,\mathrm{grad}\,\rho|)I_{ij} + \widehat{\alpha}_2(\rho, |\,\mathrm{grad}\,\rho|)\rho_{,i}\rho_{,j}, \tag{65}$$

$$\mathsf{T}_{rqp} = \widehat{\alpha}_3(\rho, |\,\mathrm{grad}\,\rho|)(I_{rq}\rho_{,p} + I_{rp}\rho_{,q}). \tag{66}$$

We find it remarkable that (65), when written as

$$\mathbf{T} = \widehat{\alpha}_1(\rho, |\,\mathrm{grad}\,\rho|)\mathbf{I} + \widehat{\alpha}_2(\rho, |\,\mathrm{grad}\,\rho|)\,\mathrm{grad}\,\rho \otimes \mathrm{grad}\,\rho, \tag{67}$$

has the same form as Korteweg's (4), when the latter is restricted to second-gradient fluids by taking the material moduli α and γ equal to null. We conjecture that an extension of our line of arguments to third-gradient fluids (a quite cumbersome task, perhaps) could provide a similar rationale for the complete expression of Korteweg stress (4). Interestingly, (67) is also arrived at by an observer-invariance argument we will offer in the next section (see (77)), an argument that also identifies the prescription (66) with the middle term of the more general representation (80) for T.

8 More on the constitutive characterization of elastic second-gradient fluids

In this final section, we expose some results which, albeit not essential for our discussion and partly not new, are important for a better understanding of elastic second-gradient fluids and for making this work reasonably self-contained. Precisely, we give an argument that validate the choice of the current mass density and its gradient as state variables, we discuss which restrictions are placed by invariance under observer changes on the Cauchy-type stress measures $\mathbf{T}$ and T, and we prove the ensuing representation results.

8.1 State variables

A response mapping for an elastic second-gradient material has the generic form:

$$\psi = \widehat{\psi}(\mathbf{F}, \mathsf{F}); \tag{68}$$

we interpret $\widehat{\psi}$ as the specific stored energy when it is scalar-valued, as a mapping delivering one or another of the relevant stress measures when it is tensor-valued. We now show that invariance of $\widehat{\psi}$ under the material symmetry group $\mathscr{F}$ we have chosen to characterize fluidity implies that the constitutive information deducible from a pointwise deformation pair $(\mathbf{F}, \mathsf{F})$ and from the corresponding pair $(\rho, \mathrm{grad}\,\rho)$ of the current mass density and its gradient is the same. This issue was addressed in [12,18], where other relevant references can be found, and where the group $\mathscr{F}$ was defined and studied. We summarize the main result of that study under form of our

Proposition 6 *Let $\widehat{\psi}$ as in (68) be a response function for an elastic second-gradient material. Invariance of $\widehat{\psi}$ under $\mathscr{F}$:*

$$\widehat{\psi}(\mathbf{F}, \mathsf{F}) = \widehat{\psi}((\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})) \quad \forall(\mathbf{F}, \mathsf{F}) \in \mathscr{L}, \ \forall(\mathbf{H}, \mathsf{K}) \in \mathscr{F} \tag{69}$$

is a necessary and sufficient condition for the existence of a function $\widetilde{\psi}$ such that

$$\widehat{\psi}(\mathbf{F}, \mathsf{F}) = \widetilde{\psi}(\rho, \mathrm{grad}\,\rho). \tag{70}$$

Proof Fix a material point at which $\rho_0 > 0$ and $\nabla\rho_0 = \mathbf{0}$, and observe that at such a point (49) can be written as

$$\mathbf{F}^{-T}\,\mathsf{tr}(\mathbf{F}^{-1}\mathsf{F}) = -\frac{\operatorname{grad}\rho}{\rho}. \tag{71}$$

Now, for $(\mathbf{F}_1, \mathsf{F}_1)$ and $(\mathbf{F}_2, \mathsf{F}_2)$ two pointwise deformations, Eqs. (2) and (71), via some algebraic manipulations and a final recourse to (20), yield the following chain of equivalences:

$$\begin{cases} \rho_1 = \rho_2 \\ \operatorname{grad}\rho_1 = \operatorname{grad}\rho_2 \end{cases} \Leftrightarrow \begin{cases} \det\mathbf{F}_1 = \det\mathbf{F}_2 \\ \mathbf{F}_1^{-T}\,\mathsf{tr}(\mathbf{F}_1^{-1}\mathsf{F}_1) = \mathbf{F}_2^{-T}\,\mathsf{tr}(\mathbf{F}_2^{-1}\mathsf{F}_2) \end{cases}$$
$$\Leftrightarrow \begin{cases} \det(\mathbf{F}_2^{-1}\mathbf{F}_1) = 1 \\ \mathsf{tr}\left(\mathbf{F}_1^{-1}\mathsf{F}_1 - \mathbf{F}_1^{-1}\mathsf{F}_2[\mathbf{F}_2^{-1}\mathbf{F}_1, \mathbf{F}_2^{-1}\mathbf{F}_1]\right) = \mathbf{0} \end{cases}. \tag{72}$$

On letting:

$$\begin{aligned}(\mathbf{H}, \mathsf{K}) &= (\mathbf{F}_2, \mathsf{F}_2)^{-1} \circ (\mathbf{F}_1, \mathsf{F}_1)\\ &= \left(\mathbf{F}_2^{-1}, -\mathbf{F}_2^{-1}\mathsf{F}_2\left[\mathbf{F}_2^{-1}, \mathbf{F}_2^{-1}\right]\right) \circ (\mathbf{F}_1, \mathsf{F}_1)\\ &= \left(\mathbf{F}_2^{-1}\mathbf{F}_1, \mathbf{F}_2^{-1}\mathsf{F}_1 - \mathbf{F}_2^{-1}\mathsf{F}_2\left[\mathbf{F}_2^{-1}\mathbf{F}_1, \mathbf{F}_2^{-1}\mathbf{F}_1\right]\right),\end{aligned} \tag{73}$$

so that

$$\begin{aligned}\det\mathbf{H} &= \det\left(\mathbf{F}_2^{-1}\mathbf{F}_1\right)\\ \mathsf{tr}(\mathbf{H}^{-1}\mathsf{K}) &= \mathsf{tr}\left(\mathbf{F}_1^{-1}\mathsf{F}_1 - \mathbf{F}_1^{-1}\mathsf{F}_2\left[\mathbf{F}_2^{-1}\mathbf{F}_1, \mathbf{F}_2^{-1}\mathbf{F}_1\right]\right),\end{aligned} \tag{74}$$

we see that, as a consequence of (72), (73), (74) and definition (28),

$$\begin{cases} \rho_1 = \rho_2 \\ \operatorname{grad}\rho_1 = \operatorname{grad}\rho_2 \end{cases} \Leftrightarrow \begin{cases} \det\mathbf{H} = 1 \\ \mathsf{tr}(\mathbf{H}^{-1}\mathsf{K}) = \mathbf{0} \end{cases} \Leftrightarrow (\mathbf{H}, \mathsf{K}) \in \mathscr{F}.$$

In words, two pointwise deformations $(\mathbf{F}, \mathsf{F})$ and $(\mathbf{F}, \mathsf{F}) \circ (\mathbf{H}, \mathsf{K})$, with $(\mathbf{H}, \mathsf{K}) \in \mathscr{F}$, induce one and the same $(\rho, \operatorname{grad}\rho)$ pair. Thus, if (70) holds, then (69) must hold too. Conversely, if condition (69) holds, and hence, ψ has constant value over all pointwise deformations inducing a fixed $(\rho, \operatorname{grad}\rho)$ pair, then a function $\widetilde{\psi}$ satisfying (70) can be defined. □

As shown in [18], in the general case when $\mathbf{a} := \nabla\rho_0 \neq \mathbf{0}$, it can be proved that the set

$$\mathscr{F}_{\mathbf{a}} := \{(\mathbf{H}, \mathsf{K}) \in \mathscr{F} :\ \det\mathbf{H} = 1,\ \mathsf{tr}(\mathbf{H}^{-1}\mathsf{K}) = (\mathbf{I} - \mathbf{H}^T)\mathbf{a}\}$$

is a group, and that Proposition 6 holds true with $\mathscr{F}$ replaced by $\mathscr{F}_{\mathbf{a}}$.

8.2 Representation of the Cauchy-type stress mappings

In view of the foregoing discussion, let the constitutive dependence of $\mathbf{T}$ and T on the deformation be through ρ and $\operatorname{grad}\rho$:

$$\mathbf{T} = \widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho), \quad \mathsf{T} = \widehat{\mathsf{T}}(\rho, \operatorname{grad}\rho).$$

Invariance under observer changes places restrictions that we now derive on the form of both mappings $\widehat{\mathbf{T}}$ and $\widehat{\mathsf{T}}$.

Firstly, we take up $\widehat{\mathbf{T}}$. While ρ is unaffected by an observer change, its spatial gradient transforms as follows:

$$\operatorname{grad}\rho \mapsto \mathbf{Q}\operatorname{grad}\rho.$$

Thus, we have that

$$\widehat{\mathbf{T}}(\rho, \mathbf{Q}\operatorname{grad}\rho) = \mathbf{Q}\widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho)\mathbf{Q}^T, \tag{75}$$

for all rotations $\mathbf{Q}$ (cf., in the "Appendix" to follow, the first of (85)). Let $\mathrm{Rot}(\mathbf{n})$ be the group of rotations about

$$\mathbf{n} := \frac{\operatorname{grad}\rho}{|\operatorname{grad}\rho|}.$$

Since

$$\mathbf{Q}\operatorname{grad}\rho = \operatorname{grad}\rho \quad \forall \mathbf{Q} \in \mathrm{Rot}(\mathbf{n}),$$

we deduce from (75) that

$$\widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho) = \mathbf{Q}\widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho)\mathbf{Q}^T \quad \forall \mathbf{Q} \in \mathrm{Rot}(\mathbf{n}).$$

Thus, well-known results form the spectral theory for second-order tensors imply that the span of $\mathbf{n}$ and $\mathbf{n}^\perp$, the plane orthogonal to $\mathbf{n}$, are proper spaces of $\mathbf{T} = \widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho)$:

$$\mathbf{T} = \widetilde{\alpha}(\rho, \operatorname{grad}\rho)(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}) + \widetilde{\beta}(\rho, \operatorname{grad}\rho)\mathbf{n}\otimes\mathbf{n},$$

or rather, equivalently because $\mathbf{n}$ and $\operatorname{grad}\rho$ are parallel,

$$\widehat{\mathbf{T}}(\rho, \operatorname{grad}\rho) = \widetilde{\alpha}(\rho, \operatorname{grad}\rho)\mathbf{I} + \check{\beta}(\rho, \operatorname{grad}\rho)\operatorname{grad}\rho\otimes\operatorname{grad}\rho. \tag{76}$$

But, when written for (76), condition (75) is satisfied if and only if

$$\widetilde{\alpha}(\rho, \mathbf{Q}\operatorname{grad}\rho) = \widehat{\alpha}(\rho, \operatorname{grad}\rho), \quad \check{\beta}(\rho, \mathbf{Q}\operatorname{grad}\rho) = \check{\beta}(\rho, \operatorname{grad}\rho),$$

for *all* rotations $\mathbf{Q}$. We conclude that

$$\mathbf{T} = \alpha\mathbf{I} + \beta\operatorname{grad}\rho\otimes\operatorname{grad}\rho \tag{77}$$

with

$$\alpha = \widehat{\alpha}(\rho, |\operatorname{grad}\rho|) \quad \beta = \widehat{\beta}(\rho, |\operatorname{grad}\rho|)$$

for some functions $\widehat{\alpha}$ and $\widehat{\beta}$. Notice that Korteweg stress (4) is the special case of (77) when all dependences on second derivatives of the density field are omitted.

As to $\widehat{\mathsf{T}}$, we begin to note that it must transform under observer change in such a way that the following condition is fulfilled for all rotations $\mathbf{Q}$:

$$\widehat{\mathsf{T}}(\rho, \mathbf{Q}\operatorname{grad}\rho) = \mathbf{Q}\widehat{\mathsf{T}}(\rho, \operatorname{grad}\rho)[\mathbf{Q}^T, \mathbf{Q}^T].$$

In particular, for $\mathbf{n}$ and $\mathrm{Rot}(\mathbf{n})$ just as before, we have that

$$\widehat{\mathsf{T}}(\rho, \operatorname{grad}\rho) = \mathbf{Q}\widehat{\mathsf{T}}(\rho, \operatorname{grad}\rho)[\mathbf{Q}^T, \mathbf{Q}^T], \quad \forall \mathbf{Q} \in \mathrm{Rot}(\mathbf{n}).$$

Thus, tensor $\mathsf{T} = \widehat{\mathsf{T}}(\rho, \operatorname{grad}\rho) \in \mathsf{Sym}^{(2,3)}$ is such that

$$\mathsf{T} = \mathbf{Q}\mathsf{T}[\mathbf{Q}^T, \mathbf{Q}^T] \quad \forall \mathbf{Q} \in \mathrm{Rot}(\mathbf{n}),$$

a condition which, in view of (12) and (13), is equivalent to

$$\mathsf{T}(\mathbf{Qa}, \mathbf{Qb}, \mathbf{Qc}) = \mathsf{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}) \quad \forall \mathbf{Q} \in \mathrm{Rot}(\mathbf{n}).$$

Any such *transversely hemitropic* tensor that, in addition, satisfies the symmetry condition $\mathsf{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathsf{T}(\mathbf{a}, \mathbf{c}, \mathbf{b})$ has the following representation:

$$\mathsf{T} = \lambda_1\mathsf{T}^1 + \mu(\mathsf{T}^2 + \mathsf{T}^3) + \lambda_4\mathsf{T}^4, \tag{78}$$

where

$$\mathsf{T}^1(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n})(\mathbf{c}\cdot\mathbf{n})$$
$$\mathsf{T}^2(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{n})$$
$$\mathsf{T}^3(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{n})$$
$$\mathsf{T}^4(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{n})$$

or, in components,

$$T^1_{ijk} = n_i n_j n_k, \quad T^2_{ijk} = I_{ij} n_k, \quad T^3_{ijk} = I_{ik} n_j, \quad T^4_{ijk} = I_{jk} n_i.$$

In view of the parallelism between grad ρ and $\mathbf{n}$, we rewrite (78) as

$$\mathsf{T} = \lambda_1 \,\mathrm{grad}\,\rho \otimes \mathrm{grad}\,\rho \otimes \mathrm{grad}\,\rho + \mu \sum_{l=1}^{3} \big(\mathbf{e}_l \otimes (\mathbf{e}_l \otimes \mathrm{grad}\,\rho + \mathrm{grad}\,\rho \otimes \mathbf{e}_l)\big) + \lambda_4 \,\mathrm{grad}\,\rho \otimes \mathbf{I}, \tag{79}$$

or, in components,

$$T_{ijk} = \lambda_1 \rho_{,i}\rho_{,j}\rho_{,k} + \mu(I_{ij}\rho_{,k} + I_{ik}\rho_{,j}) + \lambda_4 I_{jk}\rho_{,i}. \tag{80}$$

The scalar-valued coefficient functions in (79) all depend in general on ρ and grad ρ; by an argument quite similar to the one used just above for $\mathbf{T}$, it can be shown that their dependence on grad ρ is through $|\,\mathrm{grad}\,\rho|$.

It is worthwhile to compare (80) with (66), a representation result derived by postulating invariance of the stress power under time-dependent material-symmetry transformations: in spite of the disparate nature of the two physical requirements and of the differences in their algebraic realization, the latter representation is a special case of the former. We plan to return on this issue in a forthcoming paper, where we characterize solidity for second-gradient materials.

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9 Appendix: Consequences of observer changes

Let a body in motion be observed in its current placement by two synchronous observers $\mathcal{O}$ and $\mathcal{O}^+$ undergoing a relative motion parameterized by a point-valued mapping $t \mapsto o^+(t)$ and a rotation-valued mapping $t \mapsto \mathbf{Q}(t)$:

$$\mathcal{E} \ni y \mapsto y^+(t) = o^+(t) + \mathbf{Q}(t)(y - o) \in \mathcal{E}.$$

An application of the chain rule yields the following relation between the deformation gradients and their time rates, as measured by the two observers:

$$\mathbf{F}^+ = \mathbf{QF},^{8} \quad \mathsf{F}^+ = \mathbf{Q}\mathsf{F}, \tag{81}$$

whence

$$\dot{\mathbf{F}}^+ = \dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}, \quad \dot{\mathsf{F}}^+ = \dot{\mathbf{Q}}\mathsf{F} + \mathbf{Q}\dot{\mathsf{F}}. \tag{82}$$

On the other hand, $(6)_2$ implies that

$$\dot{\mathbf{F}}^+ = (\mathrm{grad}\,\mathbf{v})^+\mathbf{F}^+. \tag{83}$$

[8] Thus, in particular,

$$(\det \mathbf{F})^+ = \det \mathbf{F} \quad \text{and hence} \quad dV_{y^+} = dV_y.$$

Combining (81)$_1$, (82)$_1$, and (83), we have:

$$\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}} = (\operatorname{grad}\mathbf{v})^+\mathbf{Q}\mathbf{F},$$

whence

$$(\operatorname{grad}\mathbf{v})^+ = \mathbf{Q}(\operatorname{grad}\mathbf{v})\mathbf{Q}^T + \mathbf{W}, \quad \mathbf{W} = \dot{\mathbf{Q}}\mathbf{Q}^T \in \mathrm{Skw}, \tag{84}$$

where $\mathbf{W}$, a skew-symmetric tensor, is the spin of $\mathscr{O}^+$ relative to $\mathscr{O}$. Moreover, further applications of the chain rule yield:

$$(\operatorname{grad}^2\mathbf{v})^+ = \mathbf{Q}\operatorname{grad}\mathbf{v}[\mathbf{Q}^T, \mathbf{Q}^T].$$

Customarily, *the stress power is required to be generically invariant under observer changes*. In the first-gradient case, when formulated in terms of current density and Cauchy stress, this requirement takes the form:

$$\mathbf{T}^+ \cdot (\operatorname{grad}\mathbf{v})^+ = \mathbf{T} \cdot \operatorname{grad}\mathbf{v}$$

for all motions, whatever the change in observer; with the use of (84), it yields:

$$\mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \quad \text{and} \quad \mathbf{T} \in \mathrm{Sym}$$

(it is perhaps worth-noting that the second of these conditions, an expected result, is usually and unduly derived from balance of angular momentum).

Proposition 7 *The second-gradient stress power is invariant under observer changes, in the sense that*

$$\mathbf{T}^+ \cdot (\operatorname{grad}\mathbf{v})^+ + \mathsf{T}^+ \cdot (\operatorname{grad}^2\mathbf{v})^+ = \mathbf{T} \cdot \operatorname{grad}\mathbf{v} + \mathsf{T} \cdot \operatorname{grad}^2\mathbf{v},$$

if and only if

$$\mathbf{T}^+ = \mathbf{Q}\mathbf{T}\mathbf{Q}^T, \quad \mathsf{T}^+ = \mathbf{Q}\mathsf{T}[\mathbf{Q}^T, \mathbf{Q}^T], \quad \mathbf{T} \in \mathrm{Sym}. \tag{85}$$

Proof Combination of condition (85) with (81)$_2$ and (82)$_2$ yields:

$$\mathbf{T}^+ \cdot \mathbf{Q}(\operatorname{grad}\mathbf{v})\mathbf{Q}^T + \mathbf{T}^+ \cdot \mathbf{W} + \mathsf{T}^+ \cdot \mathbf{Q}\operatorname{grad}^2\mathbf{v}[\mathbf{Q}^T, \mathbf{Q}^T] = \mathbf{T} \cdot \operatorname{grad}\mathbf{v} + \mathsf{T} \cdot \operatorname{grad}^2\mathbf{v},$$

whence the desired result follows, because each of $\operatorname{grad}\mathbf{v}$, $\mathbf{W}$ and $\operatorname{grad}^2\mathbf{v}$ can be chosen arbitrarily and independently of the stress pairs $(\mathbf{T}^+, \mathsf{T}^+)$, $(\mathbf{T}, \mathsf{T})$. □

The first two of conditions (85) tell us that both Cauchy-type stress measures transform under observer changes by orthogonal conjugation, just as their work-conjugate velocity gradients do.
The transformation laws of Piola-type stresses $(\mathbf{S}, \mathsf{S})$ can be derived from (85) with the help of (38). We find it interesting, however, to reach the desired results directly, from the expression of the referential stress-power density given in (37)$_2$, whose invariance under observer changes reads:

$$\mathbf{S}^+ \cdot \dot{\mathbf{F}}^+ + \mathsf{S}^+ \cdot \dot{\mathsf{F}}^+ = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}}. \tag{86}$$

Proposition 8 *The second-gradient stress power satisfies satisfies (86) for all motions and all observer changes if and only if*

$$\mathbf{S}^+ = \mathbf{Q}\mathbf{S}, \quad \mathsf{S}^+ = \mathbf{Q}\mathsf{S}, \quad \mathbf{S}\mathbf{F}^T + \mathsf{S} : \overrightarrow{\mathsf{F}} \in \mathrm{Sym} \tag{87}$$

Proof A more explicit way to write condition (86) is:

$$\mathbf{S}^+ \cdot [\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}] + \mathsf{S}^+ \cdot [\dot{\mathbf{Q}}\mathsf{F} + \mathbf{Q}\dot{\mathsf{F}}] = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}}. \tag{88}$$

Pick a kinematical process in which, at the current time, $(\mathbf{F}, \mathsf{F})$ are arbitrarily chosen while the time derivatives $(\dot{\mathbf{F}}, \dot{\mathsf{F}})$ are both zero. Then, (88) reduces to

$$\mathbf{S}^+ \cdot \dot{\mathbf{Q}}\mathbf{F} + \mathsf{S}^+ \cdot \dot{\mathbf{Q}}\mathsf{F} = 0, \tag{89}$$

or rather, equivalently by setting $\dot{\mathbf{Q}} = \mathbf{Q}\mathbf{W}$ with $\mathbf{W} \in \mathrm{Skw}$ arbitrary and recalling (15),

$$\mathbf{Q}^T\mathbf{S}^+\mathbf{F}^T \cdot \mathbf{W} + (\mathbf{Q}^T\mathsf{S}^+ : \overrightarrow{\mathsf{F}}) \cdot \mathbf{W} = 0.$$

Thus, a necessary and sufficient condition for (89) to be satisfied is:

$$\mathbf{Q}^T\mathbf{S}^+\mathbf{F}^T + \mathbf{Q}^T\mathsf{S}^+ : \overrightarrow{\mathsf{F}} \in \mathrm{Sym}. \tag{90}$$

Consider now what is left of equality (88), that is:

$$\mathbf{S}^+ \cdot \mathbf{Q}\dot{\mathbf{F}} + \mathsf{S}^+ \cdot \mathbf{Q}\dot{\mathsf{F}} = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathsf{S} \cdot \dot{\mathsf{F}}.$$

Since, for given $(\mathbf{F}, \mathsf{F})$, each of $\dot{\mathbf{F}}$ and $\dot{\mathsf{F}}$ can be chosen independently, we have the first two of (87). Finally, by substitution of this result in (90), $(87)_3$ is established. □

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