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ORIGINAL ARTICLE

Riccardo Rosso · Epifanio G. Virga · Samo Kralj

Parallel transport and defects on nematic shells

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Abstract Nematic shells are thin films of nematic liquid crystal deposited on the boundary of colloidal particles, where liquid crystal molecules may freely glide, while remaining tangent to the surface substrate. The surface nematic order is described here by an appropriate tensor field $\mathbf{Q}$, which vanishes wherever a defect occurs in the molecular order. We show how the classical concept of parallel transport on a manifold introduced by Levi-Civita can be adapted to this setting to define the topological charge m of a defect. We arrive at a simple formula to compute m from a generic representation of $\mathbf{Q}$. In a number of separate appendices, we revisit in a unified language several, apparently disparate applications of Levi-Civita's parallel transport.

Keywords Parallel transport · Nematic shells · Nematic defects · Topological charge · Order tensor

1 Introduction

Defects in ordered media arise in configurations that cannot relax to the uniform state by *local* modifications of the order parameter [17,18,27]. The topological charge of a defect is its fingerprint, as it characterizes the qualitative behavior of the order parameter in the vicinity of the defect. In systems where the order parameter is a vector on a plane, as in a two-dimensional ferromagnet, the topological charge is the *index* of a vector field, introduced by Poincaré in his study of equilibria for ordinary differential equations [22]. If a vector field is constrained to lie everywhere tangent to an orientable surface $\mathscr{S}$ embedded in the three-dimensional Euclidean space, the definition of its topological charge can conveniently be related to another geometric notion, that of *parallel transport* along curves on $\mathscr{S}$.

Since its appearance in [16], the concept of parallel transport of a vector along a curve lying on a manifold has proved to have ubiquitous applications to physics, starting from general relativity. Parallel transport is connected to a concept of *covariant* derivative, which measures the variation of vector and tensor fields defined along a curve on a manifold, as seen, so to say, from the manifold itself. Thus, for a vector defined on a surface, the ordinary derivative is deprived of the component along the unit normal to the manifold to yield the covariant derivative. As perhaps first pointed out by Persico [20], Levi-Civita's parallel transport can also be given a telling *kinematic* interpretation in terms of a rigid *convex* surface $\mathscr{S}$ rolling without sliding nor

Dedicated to Ingo Müller on the occasion of his 75th birthday.

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R. Rosso · E. G. Virga (✉)
Dipartimento di Matematica, Università di Pavia,
via Ferrata 1, 27100 Pavia, Italy
E-mail: virga@imati.cnr.it

S. Kralj
Department of Physics, Faculty of Natural Sciences and Mathematics,
University of Maribor, Koroška cesta 160, 2000 Maribor, Slovenia

pivoting on a fixed plane $\mathscr{P}$: the ruled surface generated by all instantaneous axes of rotation of the moving surface can be developed in $\mathscr{P}$, upon which directions parallel transported along the curve $\mathscr{C}$ traced on $\mathscr{S}$ by the point of contact between $\mathscr{S}$ and $\mathscr{P}$ appear to be parallel in the usual sense. Pictorially, the parallel transport along a curve $\mathscr{C}$ on $\mathscr{S}$ is thus described in [20] by imagining that parallel directions on $\mathscr{P}$ impress their marks on $\mathscr{S}$ while this rolls without sliding nor pivoting along $\mathscr{C}$ on $\mathscr{P}$. Essentially the same image is evoked by Pfister [21] in his definition of the Levi-Civita motion of a pair of smooth rigid surfaces in mutual contact at one point, varying in time. Such a kinematic interpretation also purports the one adopted in this paper, which, as better explained below, regards a vector $\boldsymbol{e}$ tangent to a surface $\mathscr{S}$ as parallel transported along a curve $\mathscr{C}$ on $\mathscr{S}$ whenever $\boldsymbol{e}$ is rigidly conveyed in a frame that performs the *least* motion capable of accompanying the variations of the local unit normal $\boldsymbol{\nu}$ to $\mathscr{S}$ at all times.[1]

In the method employed in this paper to define the topological charge for a vector field tangent to a surface, parallel transport plays a central role: we call this method *intrinsic* as it requires a minimum knowledge of differential geometry, the main tools being re-obtained here, including an independent proof of the Gauss-Bonnet formula.

The rest of the paper is organized as follows. In Sect. 2, we introduce the naive definition of topological charge for a vector field on a plane. In Sect. 3, we elaborate on a kinematic interpretation of the parallel transport of a vector tangent to a surface $\mathscr{S}$ along a curve $\mathscr{C}$ on $\mathscr{S}$, which forms the basis for our method. Section 4 is devoted to the extension of our definition of topological charge to defects on nematic liquid crystal shells, which are described here as in [15] through a purely biaxial[2] order tensor field $\mathbf{Q}$ defined on a surface $\mathscr{S}$. In Sect. 5, we apply to spherical nematic shells our method to detect the topological charge of a defect directly from the knowledge of the order tensor field $\mathbf{Q}$ around it. The outcomes of the paper are summarized in Sect. 6.

Other methods can also be used to compute the topological charge of a tangent vector field: in Appendix A, we recall the notion of index and formulate in terms of it a definition of topological charge equivalent to the one based on parallel transport that we privilege here. The paper is closed by a number of other technical appendices, where we illuminate the connection between Levi-Civita's parallel transport along a curve on a surface and other, apparently unrelated concepts.

2 Topological charge in the plane

When the physical space is a plane $\mathscr{P}$ and the order parameter is a unit vector field $\boldsymbol{n}$ on it, defined everywhere away from possibly a finite number of isolated points, the topological charge of $\boldsymbol{n}$ at any point in $\mathscr{P}$ can be computed through a classical formula [14]. Here, we rephrase it in a form that may more easily justify its extension to two-dimensional manifolds. Following Trebin [27], we parameterize $\boldsymbol{n}$ as

$$\boldsymbol{n} = n_1 \boldsymbol{e}_1 + n_2 \boldsymbol{e}_2, \tag{1}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are orthogonal, unit vectors fixed in $\mathscr{P}$, n_1 and n_2 are scalar fields smooth everywhere, but a finite number of points, which obey the pointwise constraint

$$n_1^2 + n_2^2 \equiv 1.$$

We now prove that if $\boldsymbol{n}$ is singular at an isolated point p_0, then the *topological charge* of $\boldsymbol{n}$ associated with the point defect at p_0 is defined by

$$m := \frac{1}{2\pi} \oint_{\mathscr{C}} \boldsymbol{a} \cdot \boldsymbol{t} \, \mathrm{d}s, \tag{2}$$

where

$$\boldsymbol{a} := n_1 \nabla n_2 - n_2 \nabla n_1$$

[1] For a curve $\mathscr{C}$ in the three-dimensional Euclidean space, not necessarily lying on a surface, a similar point of view was adopted long ago by Bishop [3] who introduced a movable frame including the unit tangent vector $\boldsymbol{t}$, but which differs from Frénet-Serret's in that it is constrained not to rotate about $\boldsymbol{t}$. Such a frame, which according to Bishop can also be said to be parallel transported along $\mathscr{C}$, is the same as the frame involved in the Fermi-Walker *transport*, which Dandoloff [10] has related to the *geometric phase*, also called the Berry phase [1,26]. Alternative choices for the movable frame along a curve in three-dimensional space have also been discussed for Kirchhoff's theory of elastic rods [11, §2].

[2] A purely biaxial tensor is a non-zero, symmetric tensor such that both its trace and its determinant vanish.

and $\mathscr{C}$ is a regular closed circuit with unit tangent vector $\boldsymbol{t}$ that surrounds the point p_0. In (2), s denotes the arc-length parameter along $\mathscr{C}$. Letting n_1 and n_2 be represented on $\mathscr{C}$ as

$$n_1(s) := \cos\vartheta(s) \quad \text{and} \quad n_2 := \sin\vartheta(s),$$

where ϑ is the angle that $\boldsymbol{n}$ makes with $\boldsymbol{e}_1$, an easy computation shows that

$$\oint_{\mathscr{C}} \boldsymbol{a}\cdot\boldsymbol{t}\,\mathrm{d}s = \oint_{\mathscr{C}} \vartheta'\mathrm{d}s,$$

where a prime stands for differentiation with respect to s, and so m in (2) is clearly an integer that expresses the number of times $\boldsymbol{n}(p(s))$ winds about $\boldsymbol{e}_1$ as $p(s)$ travels along $\mathscr{C}$ once. The integer m is either positive or negative, depending on whether $\boldsymbol{n}$ winds coherently with $\boldsymbol{t}$ or not. Since m is a continuous mapping of $\mathscr{C}$ and can only attain integer values, it remains constant on all curves $\mathscr{C}$ that encircle the defect p_0 without crossing it.

Equivalently, we may define the differential form

$$\omega_{\boldsymbol{n},\boldsymbol{e}_1} := n_1\mathrm{d}n_2 - n_2\mathrm{d}n_1$$

and rephrase (2) as

$$m = \frac{1}{2\pi}\oint_{\mathscr{C}} \omega_{\boldsymbol{n},\boldsymbol{e}_1} =: \iota(\boldsymbol{n}, p_0) \in \mathbb{Z},$$

which is also known as the *index* of the vector field $\boldsymbol{n}$ at the point p_0 encircled by $\mathscr{C}$. This point of view is further pursued in Appendix A.

Yet another way of phrasing (2), which we find suggestive of a possible extension to a curved surface, is obtained by defining

$$\boldsymbol{n}_\perp := \boldsymbol{e}_3\times\boldsymbol{n} = n_1\boldsymbol{e}_2 - n_2\boldsymbol{e}_1, \tag{3}$$

where $\boldsymbol{e}_3 := \boldsymbol{e}_1\times\boldsymbol{e}_2$ is the unit *normal* to the $(\boldsymbol{e}_1,\boldsymbol{e}_2)$-plane that orients it. By (3), we can rewrite (2) in an intrinsic form, independent of the parametrization (1),

$$m = \frac{1}{2\pi}\oint_{\mathscr{C}} \boldsymbol{n}'\cdot\boldsymbol{n}_\perp\mathrm{d}s, \tag{4}$$

for which we shall find below the natural extension to a unit vector field $\boldsymbol{n}$ everywhere tangent to a surface $\mathscr{S}$ in space. In Sect. 3, apart from adding to (4) a term reflecting the curvature of $\mathscr{S}$, we shall replace $\boldsymbol{n}_\perp$ in (3) with $\boldsymbol{n}_\perp := \boldsymbol{\nu}\times\boldsymbol{n}$, where $\boldsymbol{\nu}$ is the outer unit normal to $\mathscr{S}$.

3 Parallel transport and topological charge on a surface

Let a smooth, closed, orientable surface $\mathscr{S}$ be given in the three-dimensional Euclidean space $\mathscr{E}$ and let $\boldsymbol{\nu}$ be the outer unit normal to $\mathscr{S}$. Consider a closed, simple curve $\mathscr{C}$ over $\mathscr{S}$ that can be continuously contracted in $\mathscr{S}$ to a point p_0 and denote by $\mathscr{S}_\mathscr{C}$ the subset of $\mathscr{S}$ enclosed by $\mathscr{C}$ whence, conventionally, the orientation chosen for $\mathscr{C}$ is seen to be *anti-clockwise* around $\boldsymbol{\nu}$. Thus, by changing the orientation of $\mathscr{C}$, we make $\mathscr{S}_\mathscr{C}$ represent either one or the other of the two subsets of $\mathscr{S}$ bounded by $\mathscr{C}$. As done above for a planar curve, we still denote by s the arc-length parameter along $\mathscr{C}$.

Given an orthonormal frame $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{\nu})$ defined on $\mathscr{C}$ such that $\boldsymbol{e}_1\times\boldsymbol{e}_2 = \boldsymbol{\nu}$, we imagine to follow its evolution along $\mathscr{C}$ as if it were a movable frame gliding in time over $\mathscr{S}$: ideally, this amounts to imagine that the origin of this frame traces $\mathscr{C}$ at unit speed. Since this is a rigid motion in space, building upon Persico's kinematic analogy [20], we can write

$$\boldsymbol{e}_1' = \boldsymbol{\Omega}\times\boldsymbol{e}_1, \qquad \boldsymbol{e}_2' = \boldsymbol{\Omega}\times\boldsymbol{e}_2, \qquad \boldsymbol{\nu}' = \boldsymbol{\Omega}\times\boldsymbol{\nu}, \tag{5}$$

where a prime denotes differentiation with respect to s and $\boldsymbol{\Omega}$ is the *spin* vector associated with the frame's evolution. Since $\boldsymbol{\nu}$ is unambiguously defined on $\mathscr{S}$, $\boldsymbol{\nu}'$ is uniquely determined by $\mathscr{C}$ and so is consequently

the component of $\boldsymbol{\Omega}$ orthogonal to $\boldsymbol{v}$. The component of $\boldsymbol{\Omega}$ parallel to $\boldsymbol{v}$, which does not affect $\boldsymbol{v}'$, represents the spin of the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{v})$ about $\boldsymbol{v}$. We say that the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{v})$ is *parallel transported* along $\mathscr{C}$ whenever

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\parallel} \quad \text{with} \quad \boldsymbol{\Omega}_{\parallel} \cdot \boldsymbol{v} \equiv 0. \tag{6}$$

By (6), $\boldsymbol{\Omega}_{\parallel}$ can easily be characterized as the *least* spin for which $(5)_3$ is valid. Since $\boldsymbol{v}$ is a unit vector field, $(6)_2$ also determines $\boldsymbol{\Omega}_{\parallel}$ explicitly from $(5)_3$ as

$$\boldsymbol{\Omega}_{\parallel} = \boldsymbol{v} \times \boldsymbol{v}'. \tag{7}$$

More generally, any vector $\boldsymbol{u}$ tangent to $\mathscr{S}$ is said to be parallel transported along $\mathscr{C}$ whenever it is rigidly conveyed by a parallel transported frame, so that

$$\boldsymbol{u}' = \boldsymbol{\Omega}_{\parallel} \times \boldsymbol{u}. \tag{8}$$

A simple, but important consequence of (8) is that

$$\boldsymbol{u}' \parallel \boldsymbol{v} \tag{9}$$

for all parallel transported vectors $\boldsymbol{u}$. Moreover, since by (8) $\boldsymbol{u}' \cdot \boldsymbol{u} \equiv 0$, the parallel transport of a vector does not affect its length.

There is no guarantee that the parallel transport of a vector $\boldsymbol{u}$ along $\mathscr{C}$ brings it back to the same vector after a complete turn around $\mathscr{C}$. Actually, the angle *mismatch* $\Delta\vartheta_{\mathscr{C}}$ between any tangent vector $\boldsymbol{u}$ and its image under parallel transport along a complete turn around $\mathscr{C}$ is a good measure of the distortion induced by the surface $\mathscr{S}$. A classical result says that

$$\Delta\vartheta_{\mathscr{C}} = \int_{\mathscr{S}_{\mathscr{C}}} K \, \mathrm{d}a, \tag{10}$$

where K is the Gaussian curvature of $\mathscr{S}$ (see, for example, [24, p. 193]). In Appendix D, we shall give a non-standard proof of (10), which shall be then extended in Appendix E to prove a similar formula for the *vector* mismatch induced by parallel transport, whence a new relation between parallel transport on $\mathscr{S}$ and Gaussian curvature of $\mathscr{S}$ will emerge. Though such a relation is not directly involved in our development, we reckon that it merits being derived and recorded here. Equation (10) makes it clear that the total rotation of parallel transported vectors along a closed curve $\mathscr{C}$ on $\mathscr{S}$ is *controlled* solely by the geometry of the surface. This makes parallel transported bases the natural frames through which the topological charge of a vector field on $\mathscr{S}$ can be defined.

Let $\boldsymbol{n}$ be a unit vector field, tangent to $\mathscr{S}$ and defined within the region $\mathscr{S}_{\mathscr{C}}$ surrounded by $\mathscr{C}$, and let $\boldsymbol{e}_0$ be a unit tangent vector to $\mathscr{S}$, defined at a point $q \in \mathscr{C}$. Suppose that $\boldsymbol{n}$ is regular in $\mathscr{S}_{\mathscr{C}}$, apart from a point p_0 where it fails to be defined. Since $\boldsymbol{n}$ is properly defined and continuous on $\mathscr{C}$, we can assume that the angle $\Delta\varphi_{\mathscr{C}}$ by which it rotates about $\boldsymbol{e}_0$ along a complete turn around $\mathscr{C}$ is given by

$$\Delta\varphi_{\mathscr{C}} = 2\pi m_{\mathscr{C}} \quad \text{with} \quad m_{\mathscr{C}} \in \mathbb{Z}.$$

We say that $m_{\mathscr{C}}$, which is positive or negative depending on whether the complete turn of $\boldsymbol{n}$ is coherent or not with the orientation of $\mathscr{C}$, is the *charge* of the field $\boldsymbol{n}$ along $\mathscr{C}$. By the way it is defined, $m_{\mathscr{C}}$ appears to depend on $\mathscr{C}$. However, since it is continuous under deformation of $\mathscr{C}$ on $\mathscr{S}$, it must be constant, as such is any continuous mapping that can only attain isolated values. Thus, $m_{\mathscr{C}}$ is invariant under deformations of $\mathscr{C}$, and we shall simply denote it by m: it is a topological invariant associated with the defect of $\boldsymbol{n}$ at p_0 and shall be called the *topological charge* of this defect.

Let $\boldsymbol{e}$ be a unit tangent vector field parallel transported along $\mathscr{C}$ and let $\boldsymbol{n}$ be represented as

$$\boldsymbol{n} = \cos\alpha \, \boldsymbol{e} + \sin\alpha \, \boldsymbol{e}_{\perp}, \tag{11}$$

where $\boldsymbol{e}_{\perp} := \boldsymbol{v} \times \boldsymbol{e}$. We may also write

$$2\pi m = \Delta\varphi_{\mathscr{C}} = \Delta\alpha_{\mathscr{C}} + \Delta\vartheta_{\mathscr{C}}, \tag{12}$$

where $\Delta\vartheta_{\mathscr{C}}$ is the angle mismatch of $\boldsymbol{e}$ and

$$\Delta\alpha_{\mathscr{C}} = \oint_{\mathscr{C}} \alpha' \mathrm{d}s. \tag{13}$$

This latter angle can be given a more intrinsic form by computing the inner product $\boldsymbol{n}' \cdot \boldsymbol{n}_\perp$, where $\boldsymbol{n}_\perp := \boldsymbol{\nu} \times \boldsymbol{n}$. It readily follows from (11) that

$$\boldsymbol{n}' = -\alpha' \sin\alpha\, \boldsymbol{e} + \cos\alpha\, \boldsymbol{e}' + \alpha' \cos\alpha\, \boldsymbol{e}_\perp + \sin\alpha\, \boldsymbol{e}'_\perp,$$

whence, by (9), since both $\boldsymbol{e}$ and $\boldsymbol{e}_\perp$ are parallel transported along $\mathscr{C}$, we arrive at

$$\boldsymbol{n}' \cdot \boldsymbol{n}_\perp = \alpha', \tag{14}$$

once use is also made of the equation

$$\boldsymbol{n}_\perp = \cos\alpha\, \boldsymbol{e}_\perp - \sin\alpha\, \boldsymbol{e},$$

which follows from the definition of $\boldsymbol{n}_\perp$ and the fact that $\boldsymbol{\nu} \times \boldsymbol{e}_\perp = -\boldsymbol{e}$. By combining (14), (13), (12), and (10), we then conclude that

$$m = \frac{1}{2\pi} \oint_{\mathscr{C}} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s + \frac{1}{2\pi} \int_{\mathscr{S}_{\mathscr{C}}} K \mathrm{d}a. \tag{15}$$

We note that the integrals on the right-hand side of (15) need not be separately integers: only their sum needs to be so. Actually, by (12), we may distinguish two contributions in $\Delta\varphi_{\mathscr{C}}$, the former—$\Delta\vartheta_{\mathscr{C}}$—only depending on the geometry of $\mathscr{S}$, the latter—$\Delta\alpha_{\mathscr{C}}$—accounting for the rotation of $\boldsymbol{n}$ relative to a parallel transported basis. It is instructive to compare (4) and (15): the latter, which reduces to the former in the plane, reveals the role of the Gaussian curvature of the underlying surface $\mathscr{S}$ in computing the topological charge m.

Equation (15) for m is somehow unpractical from the computational point of view as it requires evaluation of both a line and a surface integral. Below we shall derive from (15) a formula for m that involves only a line integral. Here, we apply (15) to establish the *additivity* of topological charge. Imagine that in the surface patch $\mathscr{S}_{\mathscr{C}}$ delimited by $\mathscr{C}$, there are N point defects, located at the points p_i, $i = 1, \ldots, N$. We may say that (15) delivers the *total* topological charge m enclosed by $\mathscr{C}$. We now surround the point p_i by a closed, regular contour $\mathscr{C}_i$ and let $\mathscr{S}_i$ be the surface patch containing p_i and bounded by $\mathscr{C}_i$. According to (15), the topological charge m_i of the i-th defect is

$$m_i = \frac{1}{2\pi} \oint_{\mathscr{C}_i} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s + \frac{1}{2\pi} \int_{\mathscr{S}_i} K \mathrm{d}a. \tag{16}$$

Consider the surface patch $\mathscr{S}^*_{\mathscr{C}} := \mathscr{S}_{\mathscr{C}} \setminus \bigcup_{i=1}^N \mathscr{S}_i$ with boundary $\partial\mathscr{S}^*_{\mathscr{C}} = \mathscr{C} \bigcup_{i=1}^N (-\mathscr{C}_i)$, where $-\mathscr{C}_i$ is $\mathscr{C}_i$ with opposite orientation. To reobtain a simply connected region, we also insert cuts $\mathscr{C}^+_{ij}$ and $\mathscr{C}^-_{ij}$ connecting the holes and endowed with opposite orientations (see Fig. 1). Another pair of oppositely oriented contours, $\mathscr{C}^+_{01}$ and $\mathscr{C}^-_{01}$, is required to connect the outer boundary $\mathscr{C}$ with the inner boundary $\mathscr{C}_1$.

Since the field $\boldsymbol{n}$ is *regular* inside $\mathscr{S}^*_{\mathscr{C}}$ we have that

$$0 = \frac{1}{2\pi} \oint_{\partial\mathscr{S}^*_{\mathscr{C}}} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s + \frac{1}{2\pi} \int_{\mathscr{S}^*_{\mathscr{C}}} K \mathrm{d}a, \tag{17}$$

as the integrals along $\mathscr{C}^\pm_{ij}$ cancel in pairs. By using the additivity of integrals, (17) becomes

$$0 = \frac{1}{2\pi} \oint_{\mathscr{C}} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s - \sum_{i=1}^N \oint_{\mathscr{C}_i} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s + \frac{1}{2\pi} \int_{\mathscr{S}_{\mathscr{C}}} K \mathrm{d}a - \sum_{i=1}^N \frac{1}{2\pi} \int_{\mathscr{S}_i} K \mathrm{d}a$$

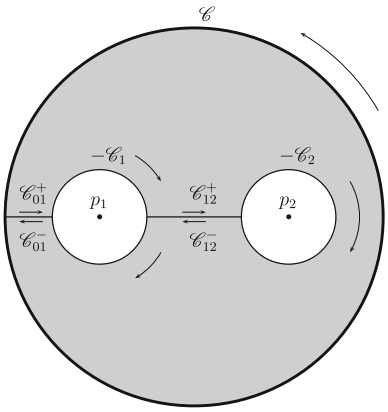

Fig. 1 Sketch of a simply connected domain employed to prove additivity of the topological charge. The *gray* region is the singularity-free domain $\mathscr{S}^*_{\mathscr{C}}$. Singular points of the vector field $\boldsymbol{n}$ are located at p_1 and p_2

whence, by use of (15) and (16), we obtain the desired result

$$m = \sum_{i=1}^{N} m_i. \tag{18}$$

By applying (15) to the unit tangent $\boldsymbol{t}$ to $\mathscr{C}$, under the assumption that $\mathscr{C}$ is everywhere smooth so that $\boldsymbol{t}$ is a continuous function of s, since then $\Delta\varphi_{\mathscr{C}} = 2\pi$, we find that

$$2\pi = \oint_{\mathscr{C}} \boldsymbol{t}' \cdot \boldsymbol{t}_\perp \mathrm{d}s + \int_{\mathscr{S}_{\mathscr{C}}} K \mathrm{d}a, \tag{19}$$

where $\boldsymbol{t}_\perp := \boldsymbol{v} \times \boldsymbol{t}$. As shown in Appendix B, $\boldsymbol{t}_\perp$ is opposite to the outer conormal vector $\boldsymbol{\nu}_{\mathscr{C}}$ to $\mathscr{C}$, and so, by $(49)_1$ $\boldsymbol{t}' \cdot \boldsymbol{t}_\perp$ equals the geodesic curvature κ_g of $\mathscr{C}$. Thus, (19) reduces to the Gauss-Bonnet formula for a smooth curve $\mathscr{C}$ (see, for example, [24, p. 195]):

$$2\pi = \oint_{\mathscr{C}} \kappa_g \mathrm{d}s + \int_{\mathscr{S}_{\mathscr{C}}} K \mathrm{d}a. \tag{20}$$

For a *piecewise* smooth contour $\mathscr{C} = \bigcup_{i=1}^{n} \mathscr{C}_i$, where all n curves $\mathscr{C}_i$ are regular, the Gauss-Bonnet formula acquires a form different from (20). With the aid of (14) applied to the piecewise continuous mapping $s \mapsto \boldsymbol{t}(s)$, we now see that

$$\Delta\alpha_{\mathscr{C}} = \sum_{i=1}^{n} \left(\int_{\mathscr{C}_i} \kappa_g \mathrm{d}s + \Delta\alpha_i \right),$$

where $\Delta\alpha_i$ represents the jump of the angle α at the i-the vertex of $\mathscr{C}$ where $\boldsymbol{t}$ is discontinuous. Letting $\beta_i := \pi - \Delta\alpha_i$ be the *internal* angle at the i-th vertex of $\mathscr{C}$, we arrive at

$$(2-n)\pi + \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \int_{\mathscr{C}_i} \kappa_g \mathrm{d}s + \int_{\mathscr{S}_{\mathscr{C}}} K \mathrm{d}a, \tag{21}$$

which is the Gauss-Bonnet formula for a piecewise smooth curve $\mathscr{C}$.

Both (20) and (21) can be used to express the topological charge of m of a unit vector field $\boldsymbol{n}$ through only line integrals. For a smooth curve $\mathscr{C}$, it easily follows from (15) and (20) that

$$m = \frac{1}{2\pi} \oint_{\mathscr{C}} \left(\boldsymbol{n}' \cdot \boldsymbol{n}_\perp - \kappa_g \right) \mathrm{d}s + 1. \tag{22}$$

In particular, if $\mathscr{C}$ is a closed, smooth geodesic of $\mathscr{S}$, κ_g vanishes identically in (22). For a planar curve, also (22) reduces to (4), as the geodesic curvature κ_g then coincides with the total curvature κ, whose integral along $\mathscr{C}$ equals 2π. For a non-planar curve, since the integral of κ_g fails in general to be an integer multiple of 2π, the integral of $\boldsymbol{n}' \cdot \boldsymbol{n}_\perp$ also fails to be so.[3]

A similar formula can be obtained for a piecewise smooth curve $\mathscr{C}$ with the aid of (21):

$$m = \frac{1}{2\pi} \sum_{i=1}^{n} \oint_{\mathscr{C}_i} \left(\boldsymbol{n}' \cdot \boldsymbol{n}_\perp - \kappa_g \right) \mathrm{d}s + 1 - \frac{n}{2} + \frac{1}{2\pi} \sum_{i=1}^{n} \beta_i . \tag{23}$$

Reasoning as in [24, p. 237] to prove that the integral of the Gaussian curvature over the whole surface $\mathscr{S}$ (the *curvatura integra*) is a topological invariant of $\mathscr{S}$, we may think of triangulating $\mathscr{S}$ with sufficiently small geodesic triangles, each enclosing at most one defect of the field $\boldsymbol{n}$. Thus, adapting (23) to the triangle $\mathscr{C}_i$ enclosing the i-th defect, we have that

$$m_i = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{\mathscr{C}_{ij}} \boldsymbol{n}' \cdot \boldsymbol{n}_\perp \mathrm{d}s - \frac{1}{2} + \frac{1}{2\pi} \sum_{j=1}^{3} \beta_{ij},$$

where $\mathscr{C}_{ij}$ is the j-th arc of $\mathscr{C}_i$ and β_{ij} is the j-th internal angle of $\mathscr{C}_i$. Summing over all triangles on $\mathscr{S}$, over those that include one of the N defects of $\boldsymbol{n}$ as well as over those that include none, since all curves $\mathscr{C}_{ij}$ are traversed twice in opposite senses, by (18), we may define the *total* topological charge $m(\mathscr{S})$ of $\boldsymbol{n}$ on $\mathscr{S}$ as

$$m(\mathscr{S}) := \sum_{i=1}^{N} m_i = -\frac{1}{2} N + \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{j=1}^{3} \beta_{ij}. \tag{24}$$

By a classical argument (see, for example, [24, p. 238]), the right-hand side of (24) can easily be identified with the Euler characteristics $\chi(\mathscr{S})$, the topological invariant of $\mathscr{S}$ defined in terms of the number F of faces, the number E of edges, and the number V of vertices of any tessellation of $\mathscr{S}$ as

$$\chi(\mathscr{S}) := F - E + V.$$

Since

$$\chi(\mathscr{S}) = 2(1 - g(\mathscr{S})), \tag{25}$$

where $g(\mathscr{S})$ is the *genus* of $\mathscr{S}$, which by an appropriate definition of connectivity[4] is recognized to be the number of "handles" possessed by the two-sided surface $\mathscr{S}$, we finally write (24) as

$$m(\mathscr{S}) = 2(1 - g(\mathscr{S})), \tag{26}$$

which is the assertion of a theorem of Poincaré [22]. In particular, for a sphere and for all closed surfaces $\mathscr{S}$ that can be obtained by smoothly deforming a sphere, $g(\mathscr{S}) = 0$ and $m(\mathscr{S}) = 2$, while for a torus and for all surfaces with its shape, $g(\mathscr{S}) = 1$ and $m(\mathscr{S}) = 0$. Besides its elegance, the merit of (26) is establishing that the total topological charge for any unit vector field $\boldsymbol{n}$ tangent to $\mathscr{S}$ is a topological invariant of $\mathscr{S}$. Moreover, (26) requires that for $g(\mathscr{S}) \neq 1$ any unit tangent field on $\mathscr{S}$ must be singular.

Though equation (22) has been instrumental to relate the total topological charge of a unit tangent vector field $\boldsymbol{n}$ on a surface $\mathscr{S}$ to a topological invariant of $\mathscr{S}$, it is in general inconvenient for the explicit computation of the topological charge m of a single defect, as it also requires computing the geodesic curvature κ_g of $\mathscr{C}$. Another formula for m can be obtained from (15), which only involves a line integral. To this end, we consider any unit vector field $\boldsymbol{n}^*$ tangent to $\mathscr{S}$ and defined everywhere on $\mathscr{S}_\mathscr{C}$, possibly away from a finite number of points. Let m^* be the topological charge of $\boldsymbol{n}^*$ in $\mathscr{S}_\mathscr{C}$, which by (15) can be written as

$$m^* = \frac{1}{2\pi} \oint_{\mathscr{C}} \boldsymbol{n}^{*\prime} \cdot \boldsymbol{n}^*_\perp \mathrm{d}s + \frac{1}{2\pi} \int_{\mathscr{S}_\mathscr{C}} K \mathrm{d}a. \tag{27}$$

[3] Again, as in (15), though the left-hand side of (22) is necessarily an integer, it does not result from the sum of integers.

[4] See, for example, [12, §§ 44,45], for a classical and suggestive definition of this notion and a derivation of (25).

It follows from (15) and (27) that

$$m - m^* = \frac{1}{2\pi} \oint_{\mathscr{C}} \left(\boldsymbol{n}' \cdot \boldsymbol{n}_\perp - \boldsymbol{n}^{*\prime} \cdot \boldsymbol{n}^*_\perp \right) \mathrm{d}s. \tag{28}$$

Here, $\boldsymbol{n}^*$ acts as a *comparison field*, which can be chosen freely on $\mathscr{S}$. In particular, if it is chosen so as to be free from defects in $\mathscr{S}_{\mathscr{C}}$, (28) takes the easier form[5]

$$m = \frac{1}{2\pi} \oint_{\mathscr{C}} \left(\boldsymbol{n}' \cdot \boldsymbol{n}_\perp - \boldsymbol{n}^{*\prime} \cdot \boldsymbol{n}^*_\perp \right) \mathrm{d}s. \tag{29}$$

In the following section, we shall build upon both (28) and (29) to compute the topological charge of defects when the unit vector field $\boldsymbol{n}$ on the surface $\mathscr{S}$ is replaced by a tensor field $\mathbf{Q}$, which better describes the ordering of a two-dimensional molecular assembly lying upon a surface $\mathscr{S}$, which is also called a *nematic shell*.

4 Topological charge in the order tensor representation

Nematic shells, which are the very motivation of this study, are thin films of nematic liquid crystals deposited on the boundary of colloidal particles whose linear size is typically below the range of microns. We imagine that liquid crystal molecules as elongated ribbons able to glide freely on the colloids' surface and induced to lie flat on it by the intermolecular forces exerted by the colloids' surface substrate. Simplistic as this representation may appear, it conveys the main features of the microscopic picture that underlies our model for a nematic shell.

Elaborating on this, we showed in [15] that a two-dimensional order tensor $\mathbf{Q}$ can be defined on a surface $\mathscr{S}$ as

$$\mathbf{Q} := \left\langle \boldsymbol{\ell} \otimes \boldsymbol{\ell} - \frac{1}{2}\mathbf{P}(\boldsymbol{\nu}) \right\rangle, \tag{30}$$

where the brackets $\langle \cdots \rangle$ denote an *ensemble average*, $\boldsymbol{\ell}$ is a unit vector representing the molecular long axis, and

$$\mathbf{P}(\boldsymbol{\nu}) := \mathbf{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu} \tag{31}$$

is the projection onto the local tangent plane to $\mathscr{S}$. The postulation leading to (30) is that $\boldsymbol{\ell}$ is subject to a *degenerate* tangential anchoring on $\mathscr{S}$, which amounts to require that

$$\boldsymbol{\ell} \cdot \boldsymbol{\nu} \equiv 0. \tag{32}$$

As also shown in [15], it follows from (32) that $\boldsymbol{\nu}$ is an eigenvector of $\mathbf{Q}$ with zero eigenvalue, and so $\det \mathbf{Q} = 0$. Since $\mathbf{Q}$ is by construction symmetric and traceless, it can be uniquely represented as

$$\mathbf{Q} = \lambda(\boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{n}_\perp \otimes \boldsymbol{n}_\perp), \tag{33}$$

where λ, which is positive by definition, is the *scalar* order parameter. In (33), $\boldsymbol{n}$ denotes the eigenvector with positive eigenvalue, while $\boldsymbol{n}_\perp := \boldsymbol{\nu} \times \boldsymbol{n}$ is the eigenvector with negative eigenvalue. Moreover, as shown in [15], λ is bound to range in the interval $[0, \frac{1}{2}]$. For $\lambda = \frac{1}{2}$, $\mathbf{Q}$ represents the limiting state in which all molecules are oriented along $\boldsymbol{n}$, whereas for $\lambda = 0$, $\mathbf{Q}$ vanishes, representing a disordered molecular state with no preferred orientation, a state that our theory regards as a *defect* in orientation, where the unit tangent vector field $\boldsymbol{n}$ fails to be defined.

In a local basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit tangent vector fields on $\mathscr{S}$ such that $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{\nu}$, the tensor $\mathbf{Q}$ can be represented as follows

$$\mathbf{Q} = q_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + q_2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1), \tag{34}$$

[5] Other ways to compute the topological charge m for vectors fields defined on a surface $\mathscr{S}$ have been introduced in the literature, which appear even simpler as they ignore the underlying differential structure of $\mathscr{S}$. For instance, on studying the continuum Heisenberg model to describe ferro- or antiferromagnets distributed on a sphere [19] or on a torus [9], the spins are parameterized in terms of their spherical polar coordinates, both viewed as functions of the Gaussian parameters that describe the surface. For some particular spin fields studied there, their apparently simpler formulas for m coincide indeed with ours.

where q_1 and q_2 are scalar fields on $\mathscr{S}$. It is apparent from (34) that $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is a basis of eigenvectors for $\mathbf{Q}$, if and only if $q_2 \equiv 0$, in which case q_1 coincides with the scalar order parameter λ. In general, computing the square norm $\mathbf{Q} \cdot \mathbf{Q}$ of $\mathbf{Q}$ using both (33) and (34), we easily conclude that

$$\lambda = \sqrt{q_1^2 + q_2^2}. \tag{35}$$

Let η be the angle defined by

$$\boldsymbol{n} = \cos\eta\, \boldsymbol{e}_1 + \sin\eta\, \boldsymbol{e}_2, \tag{36a}$$

so that

$$\boldsymbol{n}_\perp = -\sin\eta\, \boldsymbol{e}_1 + \cos\eta\, \boldsymbol{e}_2. \tag{36b}$$

By inserting this representation into (33) and comparing the result with (34), with the aid of (35), we readily obtain that

$$\cos 2\eta = \frac{q_1}{\sqrt{q_1^2 + q_2^2}} \quad \text{and} \quad \sin 2\eta = \frac{q_2}{\sqrt{q_1^2 + q_2^2}}. \tag{37}$$

Since $\mathbf{Q}$ vanishes wherever $\boldsymbol{n}$ is undefined, though a point $p_0 \in \mathscr{S}$ where this is the case is by no means a singular point of $\mathbf{Q}$, we call it a *defect* of $\mathbf{Q}$, as there $\boldsymbol{n}$, the eigenvector of $\mathbf{Q}$ with positive eigenvalue, is singular (as so is also its orthogonal companion $\boldsymbol{n}_\perp$). To a point p_0 on $\mathscr{S}$ where the order tensor field $\mathbf{Q}$ has an isolated zero, we attribute the same topological charge as to the unit tangent vector field $\boldsymbol{n}$. However, since $\boldsymbol{n}$ is now defined through $\mathbf{Q}$ in (30), it should not be regarded as having an orientation, though it can always be given one along any curve $\mathscr{C}$ on $\mathscr{S}$ enclosing p_0 so that (36) still make sense with η defined to within a multiple integer of π. Thus, the field $\boldsymbol{n}$ becomes akin to a field of *line elements* on $\mathscr{S}$, for whose first study in connection with the properties of closed surfaces in the large Stoker [24, p. 244] refers to H. Hopf's lecture notes [13]. The net effect of defining $\boldsymbol{n}$ through (33) is that its topological charge m is no longer an integer, but half an integer, $2m \in \mathbb{Z}$.

To compute m, we find it convenient employing (28) above with $\boldsymbol{n}^* = \boldsymbol{e}_1$. Since $\boldsymbol{e}_2 = \boldsymbol{\nu} \times \boldsymbol{e}_1$, an easy computation based on (36a) shows that

$$\boldsymbol{n}' \cdot \boldsymbol{n}_\perp - \boldsymbol{e}_1' \cdot \boldsymbol{e}_2 = \eta',$$

and then (37) implies that

$$\eta' = \frac{1}{2} \frac{q_1 q_2' - q_2 q_1'}{q_1^2 + q_2^2},$$

so that (28) can finally be expressed in terms of the components q_1 and q_2 of $\mathbf{Q}$ in the generic basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ as

$$m - m^* = \frac{1}{4\pi} \oint_{\mathscr{C}} \frac{q_1 q_2' - q_2 q_1'}{q_1^2 + q_2^2} \mathrm{d}s, \tag{38}$$

where m^* is the topological charge of the reference field $\boldsymbol{e}_1$, which vanishes wherever $\boldsymbol{e}_1$ has no singularity on $\mathscr{S}_\mathscr{C}$.

In the following section, we shall employ (38) to compute the topological charge of the defects exhibited by a spherical nematic shell at equilibrium.

5 Topological charge of defects on a spherical nematic shell

A simple phenomenological model to describe stable equilibrium order textures on a nematic shell $\mathscr{S}$ was based in [15] on the following free-energy functional

$$\mathscr{F} := \int_{\mathscr{S}} \left\{ \frac{k}{2} |\nabla_s \mathbf{Q}|^2 + \frac{1}{2} (A - k_{24} K) \operatorname{tr} \mathbf{Q}^2 + \frac{C}{4} (\operatorname{tr} \mathbf{Q}^2)^2 \right\} \mathrm{d}a. \tag{39}$$

In (39), k and k_{24} are elastic constants, K is the Gaussian curvature of $\mathscr{S}$, ∇_s denotes the surface gradient and $A = A_0 t$, where A_0 is a material constant, like C, and t is the dimensionless, reduced temperature defined by

$$t := \frac{T - T_c}{T_c} \tag{40}$$

in terms of the absolute temperature T and its critical value T_c. This latter is defined as the value of T at which the condensation potential

$$f_p(\mathbf{Q}) := \frac{A}{2} \operatorname{tr} \mathbf{Q}^2 + \frac{C}{4} (\operatorname{tr} \mathbf{Q}^2)^2 \tag{41}$$

ceases to be minimized by $\mathbf{Q} = \mathbf{0}$, which represents the isotropic state. Moreover, in (39), elastic distortions are penalized by both $|\nabla_s \mathbf{Q}|^2$ and $K \operatorname{tr} \mathbf{Q}^2$, the latter affecting the temperature T_c at which the ordering transition from a homogeneous melt with $\lambda \equiv 0$ would take place for a developable surface (with $K \equiv 0$). Thus, the elastic terms that appear in (39) have two distinct origins: one results from order distortions and the other from geometric distortions.

The theory based on the energy functional $\mathscr{F}$ in (39) treats states with opposite order tensors, $\mathbf{Q}$ and $-\mathbf{Q}$, as equivalent. Such an identification stems from being $\operatorname{tr} \mathbf{Q}^3 = 0$ as a consequence of the definition (30), so that, at variance with its three-dimensional analog, the potential f_p in (41) misses a cubic term. By (33), reverting the sign of $\mathbf{Q}$ amounts to exchange $\boldsymbol{n}$ and $\boldsymbol{n}_\perp$, which for a purely uniaxial state would mean that a $\frac{\pi}{2}$ rotation of the nematic director on the plane tangent to $\boldsymbol{\nu}$ has no consequence on the energy density. Pictorially, such an energy degeneracy could be represented by imagining molecules as square ribbons. Though the theory based on (39) has already proven able to predict defect aggregations promoted by curvature, as will be recalled shortly below, it would be desirable to extend it so as to remove the energy degeneracy it suffers from. Intuitively, contributions to the energy density that couple $\mathbf{Q}$ to the curvature tensor $\nabla_s \boldsymbol{\nu}$ (see (53) in Appendix B below) emerge as a natural choice; among these, the easiest that distinguishes $\mathbf{Q}$ from $-\mathbf{Q}$ while being invariant under reversal of $\boldsymbol{\nu}$ (which means invariance under reversal of orientation of $\mathscr{S}$) is

$$k_s \mathbf{Q} \cdot (\nabla_s \boldsymbol{\nu})^2, \tag{42}$$

where k_s is a further elastic constant. By (33), (42) also reads

$$k_s \lambda [\boldsymbol{n} \cdot (\nabla_s \boldsymbol{\nu})^2 \boldsymbol{n} - \boldsymbol{n}_\perp \cdot (\nabla_s \boldsymbol{\nu})^2 \boldsymbol{n}_\perp],$$

which resembles closely the coupling between surface order and curvature outlined in [23] for a spin system. We shall study elsewhere the effects of (42) on the equilibrium order textures in nematic shells. Here, we shall be contented with considering the energy functional $\mathscr{F}$ in (39) as the formula (38) we wish to illustrate is clearly independent of the energy functional being employed.

It was proved in [15] that the elastic free-energy density in (39) is positive definite whenever

$$k > 0 \quad \text{and} \quad -k < k_{24} < k.$$

Moreover, since $\operatorname{tr} \mathbf{Q}^2 = 2\lambda^2$, one easily sees that for $t < 0$ the potential f_p in (41) attains its absolute minimum when λ equals the *condensation* value

$$\lambda_c := \sqrt{-\frac{A_0}{2C} t}, \tag{43}$$

which is required to lie in the admissible interval $[0, \frac{1}{2}]$. Similarly, a characteristic length, which measures the scale over which the order tensor $\mathbf{Q}$ varies on $\mathscr{S}$ at equilibrium, is given by

$$\xi_0 := \sqrt{\frac{k}{A_0}}, \tag{44}$$

which is also called the nematic *coherence* length.

In [15], the functional in (39) was minimized numerically for a family of axisymmetric ellipsoids parameterized by the angle $u \in [0, 2\pi]$ running along parallels and the angle $v \in [0, \pi]$ running along meridians.

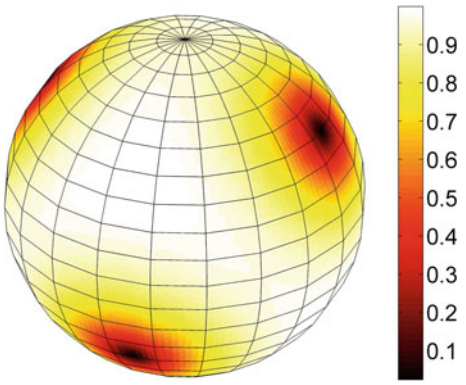

Fig. 2 Two-dimensional plot of the scalar order parameter λ on a sphere of radius R. The *shade code* on the *right* side of the figure represents the ratio λ/λ_c, where λ_c is defined by (43). This plot was obtained for $k_{24} = 0$, $R/\xi_0 = 20$, and $t = -0.03$, where ξ_0 and t are defined as in (44) and (40), respectively

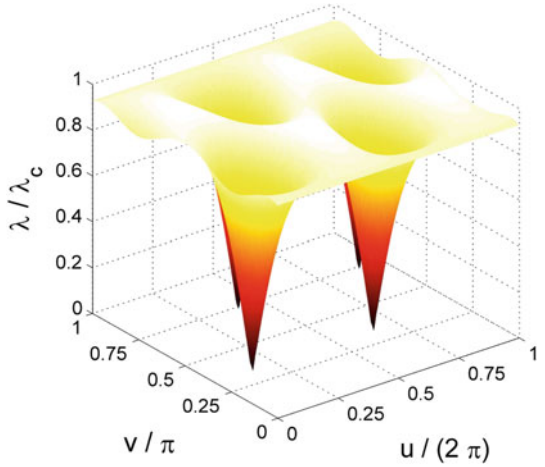

Fig. 3 Representation of the scalar order parameter λ shown in Fig. 2 as a *graph* on the (u, v) plane

It was proved that when the surface $\mathscr{S}$ is a sphere the equilibrium order texture with least free energy $\mathscr{F}$ exhibits four defects, each with topological charge $m = \frac{1}{2}$, so that the global topological constraint (26) is satisfied: these defects appeared symmetrically disposed on the sphere, at the vertices of an inscribed regular tetrahedron, thus confirming an earlier result reached analytically in [29] within a purely elastic model. It was further shown in [15] that upon gradually transforming the sphere into an oblate ellipsoid, the four defects with $m = \frac{1}{2}$ first migrate toward the equator and then dispose themselves symmetrically along it, at the vertices of a square, as the degree of oblation is further increased. Conversely, upon gradually transforming the sphere into a prolate ellipsoid, the four defects migrate in pairs toward the poles until, once a critical degree of prolation is reached, they coalesce into two defects with $m = 1$, each at one pole, thus showing that the number of defects at equilibrium on a nematic shell $\mathscr{S}$ can be affected by the local curvature of $\mathscr{S}$, while keeping the global topological constraint (26) valid.

Figure 2 illustrates the equilibrium order texture on a sphere with radius $R = 20\xi_0$ through a plot of the scalar order parameter λ scaled to the condensation value λ_c, which easily visualizes the occurrence of defects: the darker the shade, the closer is λ to zero. The same scalar order parameter λ is depicted in Fig. 3, but in the form of a graph over the (u, v) plane that parameterizes the sphere: here defects appear as deep depressions in the graph of λ. Finally, Fig. 4 shows the different paths we have considered to compute the topological charge m through (38). Here, the unit tangent vector fields $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in (34) were either along parallels or along meridians, thus exhibiting singularities only at the poles. Since none of the contours displayed in Fig. 4 encircles a pole, we set $m^* = 0$ in (38). Within numerical accuracy, the computations of m performed along these contours confirmed that $m = \frac{1}{2}$ for each defect.

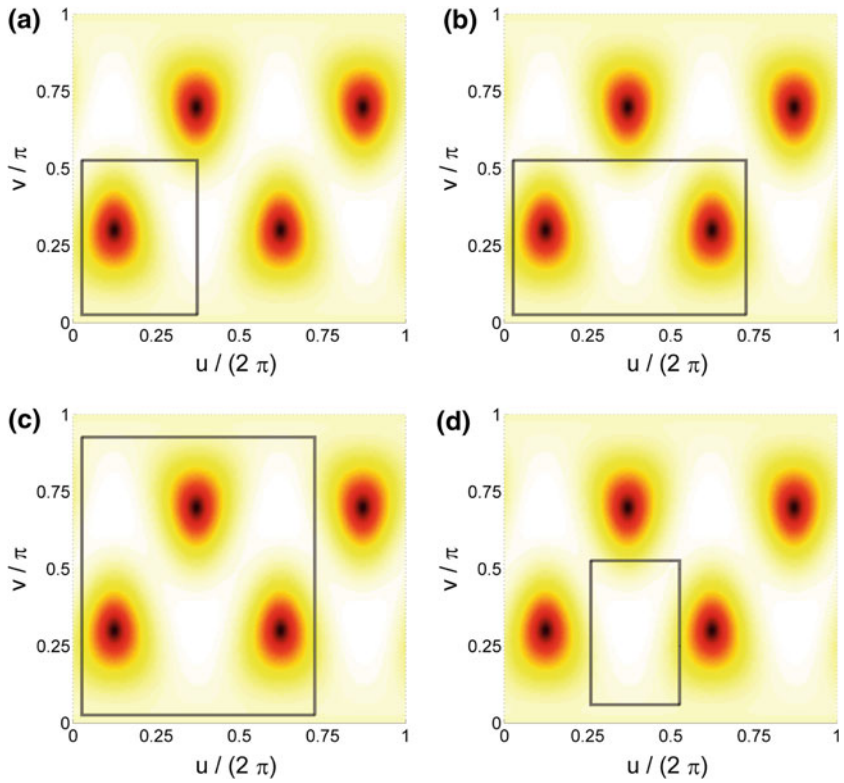

Fig. 4 Contours chosen on the (u, v) plane to compute, according to (38), the topological charge m of the defects shown in Figs. 2 and 3. The results obtained confirm the expected values to within numerical accuracy: **a** $m = \frac{1}{2}$, **b** $m = 1$, **c** $m = \frac{3}{2}$, **d** $m = 0$

6 Conclusions

We proposed to define through parallel transport the topological charge m of a point defect in a nematic liquid crystal film coating the surface of a colloidal particle. Definitions have been given that apply when a nematic texture is described by a director field $\boldsymbol{n}$ or by a tensor field $\mathbf{Q}$; only for the former, defects are singularities, as for the latter they are nodal points. We applied the formula obtained in the tensorial case, (38), to compute the topological charge of defects on an ellipsoidal colloid. Since the equilibrium tensorial texture is obtained by solving numerically non-linear partial differential equations, a defect is likely to be mistaken for a point where $\mathbf{Q}$ is *close* to vanish, but indeed it does not. Since the topological charge must be either an integer or half an integer, equation (38) can easily decide whether a true defect has indeed be caught. It is also conceivable that points where $\mathbf{Q}$ vanishes fail to be defects. In this case, equation (38) will also single out, among points eligible as defects, the true ones, for which m is different from zero.

In our treatment, we assumed that the director field $\boldsymbol{n}$ is everywhere tangent to the substrate $\mathscr{S}$. It might be of interest also to consider situations in which $\boldsymbol{n}$ may have a component out of $\mathscr{S}$. In this case, some defects for the projection $\boldsymbol{n}_{\parallel}$ of $\boldsymbol{n}$ onto $\mathscr{S}$ may not be defects for $\boldsymbol{n}$, if this fully escapes from the tangent plane to $\mathscr{S}$.

The method proposed here revisited some classical issues in global differential geometry of surfaces and made it possible to arrive in a self-consistent way to the Gauss-Bonnet formula, as well as to express intrinsically the *vector mismatch*, that is, the difference between a vector and its image under parallel transport along a closed curve of $\mathscr{S}$.

Appendix A: Curvature and connection

To illustrate the intimate relationship between curvature and connection on manifolds, we follow the elementary approach contained in Thorpe's introductory textbook [25, Chap. 21]. To this end, we observe that the skew-symmetric tensor field $\mathbf{N}$ defined on the surface $\mathscr{S}$ by

$$\mathbf{N}\boldsymbol{u} := \boldsymbol{\nu} \times \boldsymbol{u}, \tag{45}$$

for all vectors $\boldsymbol{u}$ tangent to $\mathscr{S}$, induces a *complex structure* on the local tangent plane, since $\mathbf{N}^2 = -\mathbf{P}(\boldsymbol{\nu})$, where $\mathbf{P}(\boldsymbol{\nu})$ is the projector defined in (31), which here plays the role of a two-dimensional identity. Let now $\boldsymbol{X}$ be any smooth unit tangent vector field defined on an open set $U \subset \mathscr{S}$ containing the point p. The *connection* form ω at p is the differential form defined by

$$\omega(\boldsymbol{u}) := (D_{\boldsymbol{u}}\boldsymbol{X}) \cdot \mathbf{N}\boldsymbol{X}(p),$$

where $D_{\boldsymbol{u}}\boldsymbol{X}$ is the *covariant* derivative of $\boldsymbol{X}$ along $\boldsymbol{u}$ at p. Given a vector $\boldsymbol{u}$ tangent to $\mathscr{S}$ at a point p and a curve $\gamma(t) \subset \mathscr{S}$ such that $\gamma(0) = p$, the covariant derivative $D_{\boldsymbol{u}}\boldsymbol{X}$ of a vector field $\boldsymbol{X}$ tangent to $\mathscr{S}$ with respect to $\boldsymbol{u}$ is defined as the tangent vector to $\mathscr{S}$ at p such that

$$D_{\boldsymbol{u}}\boldsymbol{X} := \mathbf{P}(\boldsymbol{\nu})\nabla_{\boldsymbol{u}}\boldsymbol{X},$$

where $\nabla_{\boldsymbol{u}}\boldsymbol{X} := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}(\gamma(t))\big|_{t=0}$. This definition actually does not depend on the choice of the curve γ on $\mathscr{S}$, provided it passes through p at $t = 0$ and obeys $\dot{\gamma}(0) = \boldsymbol{u}$.

Since $\boldsymbol{X} \cdot \boldsymbol{X} \equiv 1$, the covariant derivative of $\boldsymbol{X}$ is directed along $\mathbf{N}\boldsymbol{X}$ and we see that

$$D_{\boldsymbol{u}}\boldsymbol{X} = \omega(\boldsymbol{u})\mathbf{N}\boldsymbol{X}(p) \qquad \text{and} \qquad D_{\boldsymbol{u}}\mathbf{N}\boldsymbol{X} = -\omega(\boldsymbol{u})\boldsymbol{X}(p).$$

The relevance to our setting of the differential form ω is revealed by the following Lemmas, which we quote from [25, pp. 191–193]. In what follows, $\boldsymbol{X}$, U, and ω have the meaning just introduced.

Lemma 1 *If $\gamma : [a, b] \to U$ is a parameterized curve in U and $\mathbf{Z}$ is a parallel unit vector field along γ, then $\omega(\dot{\gamma})$ is equal to the opposite of the rotation rate of $\mathbf{Z}$, relative to $\boldsymbol{X}$, along γ and $\int_\gamma \omega := \int_a^b \omega(\dot{\gamma})$ is the negative of the total angle of rotation of $\mathbf{Z}$ relative to $\boldsymbol{X}$ along γ.*

Lemma 2 *If now γ is a unit speed curve in U, the angle of rotation of $\dot{\gamma}$ with respect to $\boldsymbol{X}$ is $\int_a^b \kappa_g - \int_\gamma \omega$.*

Lemma 3 *If K is the Gaussian curvature of $\mathscr{S}$ and ζ is the volume two-form[6] on $\mathscr{S}$ then*

$$\mathrm{d}\omega = -K\zeta$$

where d *stands for the exterior derivative of the one-form ω.*[7]

By use of these Lemmas, and also resorting to Stokes theorem for differential forms,

$$\int_U \mathrm{d}\omega = \int_{\partial U} \omega,$$

we conclude that

$$\int_U K\zeta$$

is equal to the total angle of rotation of the parallel field $\mathbf{Z}$, relative to $\boldsymbol{X}$, along the closed contour ∂U.

We are now in a position to define the *index* of a unit tangent vector field $\boldsymbol{X}$ having an isolated singularity at p with respect to a *regular* unit vector field $\boldsymbol{Y}$ defined in U. If we set

$$f := \boldsymbol{X} \cdot \boldsymbol{Y} \qquad \text{and} \qquad g := \boldsymbol{X} \cdot \mathbf{N}\boldsymbol{Y}$$

and introduce the one-form

$$\omega_{\boldsymbol{XY}} := f\,\mathrm{d}g - g\,\mathrm{d}f,$$

[6] We recall that the mapping $(\boldsymbol{e}_1, \boldsymbol{e}_2) \mapsto \zeta(\boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{e}_1 \times \boldsymbol{e}_2 \cdot \boldsymbol{\nu}$ is a volume two-form.

[7] See [25, p. 182] for the definition of the exterior derivative of a one-form.

then $\mathrm{d}\omega_{\boldsymbol{XY}} = 0$ and $\int_\gamma \omega_{\boldsymbol{XY}}$ coincides with the total angle of rotation of $\boldsymbol{X}$ relative to $\boldsymbol{Y}$ along any parameterized curve γ (see [25, pp. 201–204]). This results makes it possible to define the *index* $\iota(\boldsymbol{X}, p)$ of $\boldsymbol{X}$ at the isolated singularity p by setting

$$\iota(\boldsymbol{X}, p) := \frac{1}{2\pi} \int_\gamma \omega_{\boldsymbol{XY}}. \tag{46}$$

Actually, it is possible to prove (see [25, pp. 202–204]) that $\iota(\boldsymbol{X}, p)$ is an integer independent of both $\boldsymbol{Y}$ and the simple, closed curve γ embracing p. The curve γ can be obtained as the image of a circle lying in the tangent plane $\mathscr{T}_p$ to $\mathscr{S}$ at p under the exponential mapping, which is a local diffeomorphism between a neighborhood of p in $\mathscr{T}_p$ and a neighborhood of p on $\mathscr{S}$. The vector field $\boldsymbol{Y}$ can be taken as the image of a *constant* vector field in $\mathscr{T}_p$ but, in any case, it plays an ancillary role.

The reader will easily appreciate that, despite the different formalism employed here, the definition of index in (46) parallels closely the definition of topological charge given in (28).

Appendix B: Surface calculus

We collect here a few basic results concerning differential calculus on a smooth orientable surface $\mathscr{S}$, which shall be repeatedly used in the following. Let $\mathscr{C}$ be a smooth, closed curve on $\mathscr{S}$. We denote by $\boldsymbol{\nu}$ the unit normal that orients $\mathscr{S}$, conventionally chosen to be the *outer* unit normal to $\mathscr{S}$. Correspondingly, we define along $\mathscr{C}$ the *conormal* $\boldsymbol{\nu}_{\mathscr{C}}$ to be the unit vector orthogonal to $\mathscr{C}$ on $\mathscr{S}$, oriented away from $\mathscr{S}_{\mathscr{C}}$, the portion of $\mathscr{S}$ surrounded by $\mathscr{C}$. We call the orthonormal frame $(\boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{\nu}_{\mathscr{C}})$, where

$$\boldsymbol{t} := \boldsymbol{\nu} \times \boldsymbol{\nu}_{\mathscr{C}}, \tag{47}$$

the *Darboux frame*. Seen from $\mathscr{S}_{\mathscr{C}}$, the orientation induced on $\mathscr{C}$ by $\boldsymbol{t}$ defined as in (47) is *anti-clockwise* around $\boldsymbol{\nu}$. This definition of the Darboux frame departs from the traditional one in the orientation of $\boldsymbol{t}$ (see, e.g., [8, p. 261]). Such a slightly heretic choice is made both to maintain here for $\mathscr{C}$ the same conventional orientation adopted in the text above and to express both the surface-divergence theorem in (52) and the surface Stokes theorem in (64) below in their traditional forms.[8]

The evolution along $\mathscr{C}$ of the frame $(\boldsymbol{t}, \boldsymbol{\nu}, \boldsymbol{\nu}_{\mathscr{C}})$ is governed by the Darboux equations,

$$\boldsymbol{t}' = -\kappa_g \boldsymbol{\nu}_{\mathscr{C}} + \kappa_n \boldsymbol{\nu}, \tag{48a}$$

$$\boldsymbol{\nu}_{\mathscr{C}}' = \kappa_g \boldsymbol{t} + \tau_g \boldsymbol{\nu}, \tag{48b}$$

$$\boldsymbol{\nu}' = -\kappa_n \boldsymbol{t} - \tau_g \boldsymbol{\nu}_{\mathscr{C}}, \tag{48c}$$

where a prime denotes differentiation with respect to the arc-length s oriented like $\boldsymbol{t}$, and

$$\kappa_g := -\boldsymbol{t}' \cdot \boldsymbol{\nu}_{\mathscr{C}}, \quad \kappa_n := \boldsymbol{t}' \cdot \boldsymbol{\nu}, \quad \tau_g := -\boldsymbol{\nu}' \cdot \boldsymbol{\nu}_{\mathscr{C}} \tag{49}$$

are the *geodesic curvature*, the *normal curvature*, and the *geodesic torsion* of $\mathscr{C}$, respectively.[9]

Let a vector-valued mapping $\boldsymbol{u}$ be defined on $\mathscr{S}$ and let $\widehat{\boldsymbol{u}}$ denote a smooth extension of $\boldsymbol{u}$ in a three-dimensional neighborhood of $\mathscr{S}$ that agrees with $\boldsymbol{u}$ on $\mathscr{S}$. Suppose that $\widehat{\boldsymbol{u}}$ is differentiable at all points of $\mathscr{S}$ and denote by $\nabla\widehat{\boldsymbol{u}}$ its (three-dimensional) gradient. We define the *surface gradient* of $\boldsymbol{u}$ on $\mathscr{S}$ as

$$\nabla_{\mathrm{s}}\boldsymbol{u} := (\nabla\widehat{\boldsymbol{u}})\mathbf{P}(\boldsymbol{\nu}), \tag{50}$$

where $\mathbf{P}(\boldsymbol{\nu})$ is the orthogonal projector on $\mathscr{S}$ as in (31). It is easily seen that $\nabla_{\mathrm{s}}\boldsymbol{u}$ is independent of the extension $\widehat{\boldsymbol{u}}$ of $\boldsymbol{u}$ in (50). Relative to a *fixed* Cartesian frame $(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$, the components of $\nabla_{\mathrm{s}}\boldsymbol{u}$ are denoted by $u_{i;j}$, where a semicolon denotes surface differentiation, and (50) becomes

$$u_{i;j} = u_{i,h} P_{hj}, \tag{51}$$

where a colon denotes the usual differentiation in three space, $P_{hj} := \delta_{hj} - \nu_h \nu_j$ are the Cartesian components of $\mathbf{P}(\boldsymbol{\nu})$, and summation over repeated indices is understood.

[8] Choosing the traditional orientation of the Darboux frame would indeed imply an annoying minus sign in both (52) and (64).

[9] In particular, the geodesic curvature κ_g is defined by $(49)_1$ in accordance with equation (7.21) of [24, p. 173].

The *surface divergence* of $\boldsymbol{u}$ is defined as

$$\operatorname{div}_{\mathrm{s}} \boldsymbol{u} := \operatorname{tr} \nabla_{\mathrm{s}} \boldsymbol{u},$$

where tr denotes the *trace* of a tensor. In component form, by (51), $\operatorname{div}_{\mathrm{s}} \boldsymbol{u} = u_{i;i} = u_{i,i} - u_{i,j} \nu_i \nu_j$. A classical result of differential calculus on surfaces is the *surface-divergence theorem* (see, e.g., [28, p. 84]), which states that

$$\int_{\mathscr{S}_{\mathscr{C}}} \operatorname{div}_{\mathrm{s}} \boldsymbol{u} \mathrm{d}a = \int_{\mathscr{S}_{\mathscr{C}}} 2H \boldsymbol{u} \cdot \boldsymbol{\nu} \mathrm{d}a + \oint_{\mathscr{C}} \boldsymbol{u} \cdot \boldsymbol{\nu}_{\mathscr{C}} \mathrm{d}s \tag{52}$$

where a stands for the area measure on $\mathscr{S}$, while H is the mean curvature of $\mathscr{S}$. If $\mathscr{S}$ is a surface of class C^2, that is, if it can locally be regarded as the graph of a C^2-mapping, $\nabla_{\mathrm{s}} \boldsymbol{\nu}$ is a symmetric tensor that can be represented in its principal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ as

$$\nabla_{\mathrm{s}} \boldsymbol{\nu} = \sigma_1 \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \sigma_2 \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \tag{53}$$

where σ_1 and σ_2 are the *principal curvatures* of $\mathscr{S}$ and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are the corresponding *principal directions* of curvature. The mean curvature H and the Gaussian curvature K are related to the principal curvatures by

$$H := \frac{1}{2}(\sigma_1 + \sigma_2), \qquad K := \sigma_1 \sigma_2. \tag{54}$$

In particular, it follows from (53) and $(54)_1$ that $\operatorname{div}_{\mathrm{s}} \boldsymbol{\nu} = 2H$.

A result similar to (52) also holds for a tensor field $\mathbf{L}$. To establish it, we let $\boldsymbol{a}$ be a constant, but otherwise arbitrary, vector defined on $\mathscr{S}_{\mathscr{C}}$, and we apply (52) to $\boldsymbol{u} = \mathbf{L}^{\mathsf{T}} \boldsymbol{a}$, so as to obtain

$$\int_{\mathscr{S}_{\mathscr{C}}} \operatorname{div}_{\mathrm{s}}(\mathbf{L}^{\mathsf{T}} \boldsymbol{a}) \mathrm{d}a = \int_{\mathscr{S}_{\mathscr{C}}} 2H (\mathbf{L}^{\mathsf{T}} \boldsymbol{a}) \cdot \boldsymbol{\nu} \mathrm{d}a + \oint_{\mathscr{C}} (\mathbf{L}^{\mathsf{T}} \boldsymbol{a}) \cdot \boldsymbol{\nu}_{\mathscr{C}} \mathrm{d}s = \boldsymbol{a} \cdot \int_{\mathscr{S}_{\mathscr{C}}} 2H \mathbf{L} \boldsymbol{\nu} \mathrm{d}a + \boldsymbol{a} \cdot \oint_{\mathscr{C}} \mathbf{L} \boldsymbol{\nu}_{\mathscr{C}} \mathrm{d}s.$$

Defining $\operatorname{div}_{\mathrm{s}} \mathbf{L}$ as the vector such that $\operatorname{div}_{\mathrm{s}}(\mathbf{L}^{\mathsf{T}} \boldsymbol{a}) = \boldsymbol{a} \cdot \operatorname{div}_{\mathrm{s}} \mathbf{L}$, for all constant vectors $\boldsymbol{a}$, we conclude that

$$\int_{\mathscr{S}_{\mathscr{C}}} \operatorname{div}_{\mathrm{s}} \mathbf{L} \mathrm{d}a = \int_{\mathscr{S}_{\mathscr{C}}} 2H \mathbf{L} \boldsymbol{\nu} \mathrm{d}a + \oint_{\mathscr{C}} \mathbf{L} \boldsymbol{\nu}_{\mathscr{C}} \mathrm{d}s. \tag{55}$$

The Cartesian components of the tensor $\mathbf{N}$ defined on $\mathscr{S}$ by (45) are given by

$$N_{ij} = \varepsilon_{ikj} \nu_k, \tag{56}$$

where ε_{ijk} is the Levi-Civita alternator, skew-symmetric under the exchange of any pair of indices. Since $\nabla_{\mathrm{s}} \boldsymbol{\nu}$ is symmetric, it readily follows from (56) that

$$N_{ij;j} = \varepsilon_{ihj} \nu_{h;j} = 0.$$

Hence

$$\operatorname{div}_{\mathrm{s}} \mathbf{N} = \mathbf{0}. \tag{57}$$

Another surface differential operator which we shall use below is the *surface curl* of a vector field defined on $\mathscr{S}$; it is obtained by the following construction, which mimics the usual definition in three space dimensions. Let $\operatorname{skw}(\nabla_{\mathrm{s}} \boldsymbol{u})$ be the skew-symmetric part of $\nabla_{\mathrm{s}} \boldsymbol{u}$:

$$\operatorname{skw}(\nabla_{\mathrm{s}} \boldsymbol{u}) := \frac{1}{2}[\nabla_{\mathrm{s}} \boldsymbol{u} - (\nabla_{\mathrm{s}} \boldsymbol{u})^{\mathsf{T}}].$$

The surface curl of $\boldsymbol{u}$, which we denote as $\mathrm{curl}_\mathrm{s}\,\boldsymbol{u}$, is defined[10] as twice the axial vector associated with $\mathrm{skw}(\nabla_\mathrm{s}\boldsymbol{u})$. In terms of the Cartesian components $u_{i;j}$ of $\nabla_\mathrm{s}\boldsymbol{u}$, those of $\mathrm{curl}_\mathrm{s}\,\boldsymbol{u}$ are given by

$$(\mathrm{curl}_\mathrm{s}\,\boldsymbol{u})_i = \varepsilon_{ikj}u_{j;k} = -\varepsilon_{ijk}u_{j;k}. \tag{58}$$

Since $\nabla_\mathrm{s}\boldsymbol{v}$ is a symmetric tensor, we conclude that

$$\mathrm{curl}_\mathrm{s}\,\boldsymbol{v} = \boldsymbol{0}. \tag{59}$$

A surface curl can also be defined for a second-rank tensor field on $\mathscr{S}$. Let $\mathbf{L}$ be such a field, and let $\boldsymbol{a}$ be a *constant* vector field on $\mathscr{S}$. Then, by applying (58) to $\boldsymbol{u} = \mathbf{L}^\mathsf{T}\boldsymbol{a}$, we obtain that

$$[\mathrm{curl}_\mathrm{s}(\mathbf{L}^\mathsf{T}\boldsymbol{a})]_i = -\varepsilon_{ijk}(L^\mathsf{T}_{jh}a_h)_{;k} = -\varepsilon_{ijk}L_{hj;k}a_h = (\mathrm{curl}_\mathrm{s}\,\mathbf{L})^\mathsf{T}_{ih}a_h, \tag{60}$$

where we have set

$$(\mathrm{curl}_\mathrm{s}\,\mathbf{L})_{ij} := -\varepsilon_{jhk}L_{ih;k} = \varepsilon_{jkh}L_{ih;k},$$

so that in absolute notation (60) also reads as

$$\mathrm{curl}_\mathrm{s}(\mathbf{L}^\mathsf{T}\boldsymbol{a}) =: (\mathrm{curl}_\mathrm{s}\,\mathbf{L})^\mathsf{T}\boldsymbol{a}, \tag{61}$$

which, being valid for all vectors $\boldsymbol{a}$, defines $\mathrm{curl}_\mathrm{s}\,\mathbf{L}$ as a tensor field on $\mathscr{S}$.

The analog to the Stokes theorem for surfaces can be obtained from the surface-divergence theorem as follows. We first compute

$$\int_{\mathscr{S}_\mathscr{C}} \mathrm{curl}_\mathrm{s}\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} \varepsilon_{ijk}u_{j;k}v_i\,\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} [(\varepsilon_{ijk}u_jv_i)_{;k} - \varepsilon_{ijk}u_jv_{i;k}]\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} (\varepsilon_{ijk}v_iu_j)_{;k}\mathrm{d}a,$$

where use has also been made of $\varepsilon_{ijk}u_jv_{i;k} = \mathrm{curl}_\mathrm{s}\,\boldsymbol{v}\cdot\boldsymbol{u} = 0$, which follows from (59). Finally, reverting to intrinsic notation and recalling (56), we conclude that

$$\int_{\mathscr{S}_\mathscr{C}} \mathrm{curl}_\mathrm{s}\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} \mathrm{div}_\mathrm{s}(\mathbf{N}\boldsymbol{u})\mathrm{d}a \tag{62}$$

and, since $\mathbf{N}\boldsymbol{u}\cdot\boldsymbol{v} = 0$, by (52), the right-hand side of the latter equation reduces to

$$\int_{\mathscr{S}_\mathscr{C}} \mathrm{div}_\mathrm{s}(\mathbf{N}\boldsymbol{u})\mathrm{d}a = \oint_\mathscr{C} \mathbf{N}\boldsymbol{u}\cdot\boldsymbol{v}_\mathscr{C}\mathrm{d}s = -\oint_\mathscr{C} \boldsymbol{u}\cdot\mathbf{N}\boldsymbol{v}_\mathscr{C}\mathrm{d}s = -\oint_\mathscr{C} \boldsymbol{u}\cdot\boldsymbol{t}\mathrm{d}s, \tag{63}$$

where we invoked the skew-symmetry of $\mathbf{N}$ and the orientation of the Darboux frame prescribed by (47). By combining (62) and (63), we arrive at the desired result:

$$\int_{\mathscr{S}_\mathscr{C}} \mathrm{curl}_\mathrm{s}\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}a = \oint_\mathscr{C} \boldsymbol{u}\cdot\boldsymbol{t}\mathrm{d}s. \tag{64}$$

Equation (64) can also be extended to a tensor field $\mathbf{L}$ on $\mathscr{S}$. To this end, we consider the following integral

$$\int_{\mathscr{S}_\mathscr{C}} [(\mathrm{curl}_\mathrm{s}\,\mathbf{L})\boldsymbol{v}]_i\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} \varepsilon_{jhk}L_{ih;k}v_j\mathrm{d}a = -\int_{\mathscr{S}_\mathscr{C}} \varepsilon_{kjh}v_jL_{ih;k}\mathrm{d}a,$$

[10] The surface differential operators div_s and curl_s were introduced by C. Burali-Forti and P. Burgatti. Precisely, Burali-Forti defined the surface gradient for a *scalar* function on $\mathscr{S}$ in [4], whereas surface differential operators acting on surface vector fields were introduced by Burgatti in [5] (see also [6]). Some of Burgatti's results simplify formulas obtained by E. Beltrami in his work on complex analysis on a surface. These results were reobtained independently by Weatherburn (see [30, Chap. I]). The reader interested in the achievements of the Italian school of differential geometry originated by Levi-Civita, and expressed in vectorial notation, should consult [7].

which, by (56), can be recast as

$$\int_{\mathscr{S}_{\mathscr{C}}} [(\mathrm{curl_s}\,\mathbf{L})\boldsymbol{\nu}]_i \mathrm{d}a = -\int_{\mathscr{S}_{\mathscr{C}}} [(N_{kh}L_{ih})_{;k} - N_{kh;k}L_{ih}]\mathrm{d}a = -\int_{\mathscr{S}_{\mathscr{C}}} [-\mathrm{div_s}(\mathbf{LN}) + \mathbf{L}(\mathrm{div_s}\,\mathbf{N})]_i \mathrm{d}a,$$

whence, by use of (55) and (57), and since $\mathbf{N}\boldsymbol{\nu} = \mathbf{0}$, we arrive at

$$\int_{\mathscr{S}_{\mathscr{C}}} (\mathrm{curl_s}\,\mathbf{L})\boldsymbol{\nu}\mathrm{d}a = \oint_{\mathscr{C}} \mathbf{LN}\boldsymbol{\nu}_{\mathscr{C}}\mathrm{d}s = \oint_{\mathscr{C}} \mathbf{L}(\boldsymbol{\nu}\times\boldsymbol{\nu}_{\mathscr{C}})\mathrm{d}s = \oint_{\mathscr{C}} \mathbf{L}\boldsymbol{t}\mathrm{d}s, \tag{65}$$

which is the desired extension of (64).

Appendix C: Parallel transport in the Darboux frame

In this appendix, we look at the parallel transport of a vector along a curve $\mathscr{C}$ on the surface $\mathscr{S}$ within the Darboux frame introduced in Appendix B.

Let a frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ be given along $\mathscr{C}$ such that $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{\nu}$. In general, an angle $\phi(s)$ can be introduced along $\mathscr{C}$ so that

$$\boldsymbol{e}_1 = \cos\phi\boldsymbol{t} + \sin\phi\boldsymbol{\nu}_{\mathscr{C}}, \tag{66a}$$

$$\boldsymbol{e}_2 = -\sin\phi\boldsymbol{t} + \cos\phi\boldsymbol{\nu}_{\mathscr{C}}. \tag{66b}$$

For $\phi \equiv 0$, the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ clearly reduces to the Darboux frame. For a generic $\phi(s)$, the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ evolves along $\mathscr{C}$ according to (5), with a spin vector $\boldsymbol{\Omega}$ whose components in the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ we now proceed to compute. By use of (48), we readily obtain from (66) that

$$\boldsymbol{e}_1' = (\phi' - \kappa_g)\boldsymbol{e}_2 + (\kappa_n\cos\phi + \tau_g\sin\phi)\boldsymbol{\nu}, \tag{67a}$$

$$\boldsymbol{e}_2' = -(\phi' - \kappa_g)\boldsymbol{e}_1 - (\kappa_n\sin\phi - \tau_g\cos\phi)\boldsymbol{\nu}, \tag{67b}$$

$$\boldsymbol{\nu}' = -(\kappa_n\cos\phi - \tau_g\sin\phi)\boldsymbol{e}_1 + (\kappa_n\sin\phi - \tau_g\cos\phi)\boldsymbol{e}_2. \tag{67c}$$

Equations (67) can be written in the compact form (5) by setting

$$\boldsymbol{\Omega} = -(\kappa_n\sin\phi - \tau_g\sin\phi)\boldsymbol{e}_1 - (\kappa_n\cos\phi + \tau_g\sin\phi)\boldsymbol{e}_2 + (\phi' - \kappa_g)\boldsymbol{\nu}. \tag{68}$$

It then follows from (6) that the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$ is parallel transported along $\mathscr{C}$, if and only if

$$\phi' \equiv \kappa_g, \tag{69}$$

in which case, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\parallel}$. A further consequence of (68) is that the modulus of $\boldsymbol{\Omega}_{\parallel}$ depends only on the differential geometric characteristics of $\mathscr{C}$ on $\mathscr{S}$, as

$$\Omega_{\parallel}^2 = \kappa_n^2 + \tau_g^2.$$

It is finally instructive to use (69) to reobtain the Gauss-Bonnet formula in (20). Since the angle α that $\boldsymbol{t}$ makes with a regular unit vector field $\boldsymbol{w}_1$ in $\mathscr{S}_{\mathscr{C}}$ can be written as $\alpha = \phi + \vartheta$, where ϑ is the angle made by $\boldsymbol{e}_1$ with $\boldsymbol{w}_1$, the corresponding total mismatch angles upon a complete turn around $\mathscr{C}$ are related through

$$\Delta\alpha_{\mathscr{C}} = \Delta\phi_{\mathscr{C}} + \Delta\vartheta_{\mathscr{C}}, \tag{70}$$

where, by (69),

$$\Delta\phi_{\mathscr{C}} = \oint_{\mathscr{C}} \phi'\mathrm{d}s = \oint_{\mathscr{C}} \kappa_g\mathrm{d}s. \tag{71}$$

Since $\Delta\alpha_{\mathscr{C}} = 2\pi$, by inserting (10) and (71) into (70), we readily arrive at (20).

Appendix D: Angle mismatch induced by parallel transport

Here, we prove equation (10), which in the text above was instrumental to establish the expression in (15) for the topological charge m of a unit tangent vector field $\boldsymbol{n}$.

Let $\boldsymbol{e}$ be a unit tangent vector to $\mathscr{S}$ that is parallel transported along a curve $\mathscr{C}$ in $\mathscr{S}$ and let $(\boldsymbol{u}, \boldsymbol{u}_\perp)$ be any orthonormal frame defined along $\mathscr{C}$, with $\boldsymbol{u}$ and $\boldsymbol{u}_\perp$ both tangent to $\mathscr{S}$ and such that $\boldsymbol{u}_\perp := \boldsymbol{\nu} \times \boldsymbol{u}$. Hence, we can set

$$\boldsymbol{e} = \cos\vartheta\,\boldsymbol{u} + \sin\vartheta\,\boldsymbol{u}_\perp$$

and then compute

$$\boldsymbol{e}' = -\vartheta' \sin\vartheta\,\boldsymbol{u} + \cos\vartheta\,\boldsymbol{u}' + \vartheta' \cos\vartheta\,\boldsymbol{u}_\perp + \sin\vartheta\,\boldsymbol{u}'_\perp.$$

Since $\boldsymbol{e}$ is parallel transported along $\mathscr{C}$, by (9), $\boldsymbol{e}' \cdot \boldsymbol{u} = \boldsymbol{e}' \cdot \boldsymbol{u}_\perp = 0$. Moreover, it follows from $\boldsymbol{u} \cdot \boldsymbol{u}_\perp = 0$ that $\boldsymbol{u}' \cdot \boldsymbol{u}_\perp = -\boldsymbol{u}'_\perp \cdot \boldsymbol{u}$, and since both $\boldsymbol{u}$ and $\boldsymbol{u}_\perp$ are unit vectors, $\boldsymbol{u}' \cdot \boldsymbol{u} = \boldsymbol{u}'_\perp \cdot \boldsymbol{u}_\perp = 0$. Thus, we easily conclude that $\boldsymbol{e}'$ is parallel transported along $\mathscr{C}$, if and only if

$$\vartheta' + \boldsymbol{u}' \cdot \boldsymbol{u}_\perp = \vartheta' - \boldsymbol{u}'_\perp \cdot \boldsymbol{u} = 0.$$

If we now imagine both $\boldsymbol{u}$ and $\boldsymbol{u}_\perp$ defined as smooth fields on $\mathscr{S}_\mathscr{C}$, we can write $\boldsymbol{u}'_\perp = (\nabla_{\mathrm{s}}\boldsymbol{u}_\perp)\boldsymbol{t}$, where $\boldsymbol{t}$ is the unit tangent to $\mathscr{C}$, and we also have that

$$\vartheta' = \boldsymbol{u} \cdot \boldsymbol{u}'_\perp = \boldsymbol{u} \cdot (\nabla_{\mathrm{s}}\boldsymbol{u}_\perp)\boldsymbol{t} = (\nabla_{\mathrm{s}}\boldsymbol{u}_\perp)^{\mathsf{T}}\boldsymbol{u} \cdot \boldsymbol{t} = \boldsymbol{w} \cdot \boldsymbol{t}, \tag{72}$$

where we have set $\boldsymbol{w} := (\nabla_{\mathrm{s}}\boldsymbol{u}_\perp)^{\mathsf{T}}\boldsymbol{u}$. If we suppose that $\mathscr{C}$ is closed, by (64), it follows from (72) that

$$\Delta\vartheta_\mathscr{C} := \oint_\mathscr{C} \vartheta' \mathrm{d}s = \int_{\mathscr{S}_\mathscr{C}} \mathrm{curl}_{\mathrm{s}}\,\boldsymbol{w} \cdot \boldsymbol{\nu} \mathrm{d}a. \tag{73}$$

To expand the surface integral in (73), we revert to Cartesian components, so that

$$(\mathrm{curl}_{\mathrm{s}}\,\boldsymbol{w})_\ell = \varepsilon_{\ell m j} w_{j;m}$$

and since $\boldsymbol{u}_\perp = \boldsymbol{\nu} \times \boldsymbol{u}$, we obtain that

$$w_j = \varepsilon_{ihk} u_i (\nu_h u_{k;j} + u_k \nu_{h;j}),$$

whence it follows that

$$(\mathrm{curl}_{\mathrm{s}}\,\boldsymbol{w})_\ell = \varepsilon_{\ell m j}\varepsilon_{ihk}[u_{i;m}(\nu_h u_{k;j} + \nu_{h;j}u_k) + u_i(\nu_{h;m}u_{k;j} + \nu_h u_{k;jm} + \nu_{h;jm}u_k + \nu_{h;j}u_{k;m})].$$

To simplify this latter expression, we note that $\varepsilon_{ihk}\nu_{h;j}[u_{i;m}u_k + u_{k;m}u_i] = 0$ since the bracketed term is symmetric in the exchange of indices $i \leftrightarrow k$, whereas ε_{ihk} is skew-symmetric. Moreover, for the same reason, $\varepsilon_{ihk}u_i u_k \nu_{h;jm} = 0$, and so

$$(\mathrm{curl}_{\mathrm{s}}\,\boldsymbol{w})_\ell = \varepsilon_{\ell m j}\varepsilon_{ihk}[u_{i;m}\nu_h u_{k;j} + u_i(\nu_{h;m}u_{k;j} + \nu_h u_{k;jm})]. \tag{74}$$

A further simplification is possible, but it requires a closer inspection of (74). The product $\varepsilon_{\ell m j}\varepsilon_{ihk}u_i\nu_h u_{k;jm}$ does not vanish because, as also recalled in [15], second-order surface derivatives fail in general to be symmetric in the differentiation indices. By using (51) and the symmetry of the tensor $\nabla_{\mathrm{s}}\boldsymbol{\nu}$, one can prove that

$$u_{k;jm} - u_{k;mj} = u_{k,q}(\nu_{q;j}\nu_m - \nu_{q;m}\nu_j),$$

which, since $\mathbf{P}\nabla_{\mathrm{s}}\boldsymbol{\nu} = \nabla_{\mathrm{s}}\boldsymbol{\nu}$, also becomes

$$u_{k;jm} - u_{k;mj} = u_{k,q}P_{qr}(\nu_{r;j}\nu_m - \nu_{r;m}\nu_j) = u_{k;r}(\nu_{r;j}\nu_m - \nu_{r;m}\nu_j).$$

Hence, also by use of (56), we obtain that

$$\begin{aligned}\varepsilon_{\ell m j}(u_{k;jm} - u_{k;mj}) &= 2\varepsilon_{\ell m j}u_{k;jm} = u_{k;q}[\varepsilon_{\ell m j}(\nu_{q;j}\nu_m - \nu_{q;m}\nu_j)] \\ &= u_{k;q}(N_{\ell j}\nu_{q;j} + N_{\ell m}\nu_{q;m}) = 2u_{k;q}N_{\ell j}\nu_{q;j},\end{aligned}$$

which in turn implies that

$$\varepsilon_{\ell mj} u_{k;jm} = u_{k;q} N_{\ell j} \nu_{q;j}, \tag{75}$$

an identity valid for any surface field $\boldsymbol{u}$, be it tangent or not. By applying (75) and also remarking that $\nu_\ell N_{\ell j} = 0$, we finally obtain that

$$(\mathrm{curl}_\mathrm{s}\, \boldsymbol{w}) \cdot \boldsymbol{\nu} = \nu_\ell \varepsilon_{\ell mj} \varepsilon_{ihk} [u_{i;m} \nu_h u_{k;j} + u_i \nu_{h;m} u_{k;j}].$$

By recalling again (56) and setting

$$\mathbf{U}\boldsymbol{a} := \boldsymbol{u} \times \boldsymbol{a}, \quad \forall\, \boldsymbol{a}, \quad \text{that is,} \quad U_{hk} := \varepsilon_{hik} u_i,$$

we finally arrive at

$$(\mathrm{curl}_\mathrm{s}\, \boldsymbol{w}) \cdot \boldsymbol{\nu} = \mathbf{N}(\nabla_\mathrm{s} \boldsymbol{u})\mathbf{N} \cdot \nabla_\mathrm{s} \boldsymbol{u} - \mathbf{U}(\nabla_\mathrm{s} \boldsymbol{\nu})\mathbf{N} \cdot \nabla_\mathrm{s} \boldsymbol{u}, \tag{76}$$

where the inner product $\mathbf{A} \cdot \mathbf{B}$ of the tensors $\mathbf{A}$ and $\mathbf{B}$ is defined in such a way that in Cartesian components $\mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ij}$.[11] The symmetric, second-rank tensor $\nabla_\mathrm{s} \boldsymbol{\nu}$ in (53) can also be referred to the local frame $(\boldsymbol{u}, \boldsymbol{u}_\perp, \boldsymbol{\nu})$ through the equation

$$\nabla_\mathrm{s} \boldsymbol{\nu} = \nu_{11} \boldsymbol{u} \otimes \boldsymbol{u} + \nu_{22} \boldsymbol{u}_\perp \otimes \boldsymbol{u}_\perp + \nu_{12} [\boldsymbol{u} \otimes \boldsymbol{u}_\perp + \boldsymbol{u}_\perp \otimes \boldsymbol{u}].$$

A similar expression can also be obtained for $\nabla_\mathrm{s} \boldsymbol{u}$. Since $\boldsymbol{u} \cdot \boldsymbol{\nu} \equiv 0$, we have that $(\nabla_\mathrm{s} \boldsymbol{u})^\mathsf{T} \boldsymbol{\nu} = -(\nabla_\mathrm{s} \boldsymbol{\nu}) \boldsymbol{u}$. Moreover, since $\boldsymbol{u} \cdot \boldsymbol{u} \equiv 1$, $(\nabla_\mathrm{s} \boldsymbol{u})^\mathsf{T} \boldsymbol{u} = \mathbf{0}$, and finally, by definition, $(\nabla_\mathrm{s} \boldsymbol{u}) \boldsymbol{\nu} = \mathbf{0}$. Putting all these requirements together, we can write

$$\nabla_\mathrm{s} \boldsymbol{u} = -\boldsymbol{\nu} \otimes (\nu_{11} \boldsymbol{u} + \nu_{12} \boldsymbol{u}_\perp) + \boldsymbol{u}_\perp \otimes (u_{21} \boldsymbol{u} + u_{22} \boldsymbol{u}_\perp),$$

from which we see that

$$\mathbf{N}(\nabla_\mathrm{s} \boldsymbol{u})\mathbf{N} \cdot \nabla_\mathrm{s} \boldsymbol{u} = 0.$$

Similarly,

$$\mathbf{U}(\nabla_\mathrm{s} \boldsymbol{\nu})\mathbf{N} \cdot \nabla_\mathrm{s} \boldsymbol{u} = \nu_{12}^2 - \nu_{11} \nu_{22} = -K,$$

where K is the Gaussian curvature of $\mathscr{S}$, which, being a scalar invariant of $\nabla_\mathrm{s} \boldsymbol{\nu}$, does not depend upon the choice of the local basis used to expand $\nabla_\mathrm{s} \boldsymbol{\nu}$. As a result, by (76), we can recast (73) precisely as (10).

Appendix E: Vector mismatch induced by parallel transport

Equation (13) is a classical result, for which in Appendix D we gave a different proof, based on the differential calculus on surfaces recalled in Appendix B. The angle $\Delta\vartheta_\mathscr{C}$ expressed by (13) measures the distortion induced in space by the surface $\mathscr{S}$. A similar, but independent measure of distortion can be introduced as follows.

Consider a closed contour $\mathscr{C}$ on $\mathscr{S}$ and a point p_0 in the interior of the surface $\mathscr{S}_\mathscr{C}$ delimited by $\mathscr{C}$. We conceive $\mathscr{C}$ as a member of a family F of regular, closed curves shrinking to p_0: one can imagine these curves as geodesic circles, though this will not be required in the following. Let $\mathscr{C}_0$ be another curve, starting at p_0 and crossing each member of the family F once. Let $\boldsymbol{e}_0$ be a unit tangent vector defined along $\mathscr{C}_0$. We parallel transport $\boldsymbol{e}_0$ along the curves of F, that is, for every point p_0^* on $\mathscr{C}_0$, letting $\mathscr{C}_0^*$ be the unique member of F passing through p_0^*, we extend $\boldsymbol{e}_0$ along $\mathscr{C}_0^*$ by applying the law (8) of parallel transport. By repeating this construction for all points in $\mathscr{C}_0$, we obtain a vector field $\boldsymbol{e}$ defined on $\mathscr{S}_\mathscr{C}$, *apart* from the curve $\mathscr{C}_0$, as there is no guarantee that a vector parallel transported along a closed curve returns upon itself upon completion of the circuit. We denote by $\Delta\boldsymbol{e}_\mathscr{C}$ the vector on the tangent plane to $\mathscr{S}$ that fills this gap for the curve $\mathscr{C}$; we call it the *vector mismatch* induced by parallel transport. Here we shall learn how to compute $\Delta\boldsymbol{e}_\mathscr{C}$.

[11] It is easily seen that for any triple of second-rank tensors, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, $\mathbf{A} \cdot \mathbf{BC} = \mathbf{B}^\mathsf{T}\mathbf{A} \cdot \mathbf{C} = \mathbf{A}\mathbf{C}^\mathsf{T} \cdot \mathbf{B}$.

Equation (8) makes it possible to define the *surface gradient* of $\boldsymbol{e}$ everywhere in $\mathscr{S}_{\mathscr{C}}$, but along $\mathscr{C}_0$. Assuming that a curve in the family F passes through the point $p \in \mathscr{S}_{\mathscr{C}} \setminus \mathscr{C}_0$ with tangent unit vector $\boldsymbol{t}$, we can write (8) for the evolution of $\boldsymbol{e}$ as

$$\boldsymbol{e}' = \boldsymbol{\Omega}_{\|} \times \boldsymbol{e} =: (\nabla_{\mathrm{s}} \boldsymbol{e})\boldsymbol{t}, \tag{77}$$

where $\boldsymbol{\Omega}_{\|}$ depends implicitly on $\boldsymbol{t}$. Equation (38) properly defines $\nabla_{\mathrm{s}}\boldsymbol{e}$ only if the spin $\boldsymbol{\Omega}_{\|}$ depends *linearly* on $\boldsymbol{t}$, which in (77) can be regarded as an independent variable ranging in the tangent plane $\mathscr{T}_p$ to $\mathscr{S}$ at p. The tensor $\mathbf{A}$ such that $\boldsymbol{\Omega}_{\|} = \mathbf{A}\boldsymbol{t}$ can be determined through the equation

$$\boldsymbol{v}' = (\nabla_{\mathrm{s}} \boldsymbol{v})\boldsymbol{t} = \boldsymbol{\Omega}_{\|} \times \boldsymbol{v} = \mathbf{A}\boldsymbol{t} \times \boldsymbol{v}, \quad \forall \boldsymbol{t} \in \mathscr{T}_p, \tag{78}$$

which results from applying (8) for $\boldsymbol{u} = \boldsymbol{v}$. Since $\boldsymbol{\Omega}_{\|} \cdot \boldsymbol{v} = 0$, $\mathbf{A}\boldsymbol{t} \in \mathscr{T}_p$ for all $\boldsymbol{t} \in \mathscr{T}_p$, and so $\mathbf{A}$ maps linearly $\mathscr{T}_p$ onto itself. Equation (78) is of the form

$$\boldsymbol{x} \times \boldsymbol{a} = \boldsymbol{b},$$

and so it is solved by

$$\boldsymbol{x} = \frac{\boldsymbol{b} \times \boldsymbol{a}}{|\boldsymbol{a}|^2} + \lambda \boldsymbol{a},$$

where λ is an undetermined scalar. By making the appropriate identifications, we obtain that

$$\mathbf{A}\boldsymbol{t} = \boldsymbol{v} \times (\nabla_{\mathrm{s}} \boldsymbol{v})\boldsymbol{t} + \lambda \boldsymbol{v},$$

which satisfies $\mathbf{A}\boldsymbol{t} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{t} \in \mathscr{T}_p$, only if $\lambda = 0$. Thus, we find that

$$\mathbf{A} = \mathbf{N}\nabla_{\mathrm{s}} \boldsymbol{v}. \tag{79}$$

where the skew-symmetric tensor $\mathbf{N}$ is defined as in (45) above. By calling $\mathbf{E}$ the skew-symmetric tensor associated with the unit vector field $\boldsymbol{e}$ through the equation

$$\mathbf{E}\boldsymbol{a} = \boldsymbol{e} \times \boldsymbol{a}, \quad \forall \boldsymbol{a},$$

we rewrite (77) as

$$(\nabla_{\mathrm{s}} \boldsymbol{e})\boldsymbol{t} = -\mathbf{E}\mathbf{A}\boldsymbol{t}, \quad \forall \boldsymbol{t} \in \mathscr{T}_p,$$

and so we conclude that

$$\nabla_{\mathrm{s}} \boldsymbol{e} = -\mathbf{E}\mathbf{A} = -\mathbf{E}\mathbf{N}\nabla_{\mathrm{s}} \boldsymbol{v}, \tag{80}$$

which can also be interpreted as a linear differential equation for the unit vector field $\boldsymbol{e}$ induced on $\mathscr{S}_{\mathscr{C}} \setminus \mathscr{C}_0$ by parallel transport of $\boldsymbol{e}_0$, arbitrarily defined on $\mathscr{C}_0$.

We shall now see how (80) leads us to a formula for $\Delta \boldsymbol{e}_{\mathscr{C}}$. By (65), we have that

$$\Delta \boldsymbol{e}_{\mathscr{C}} = \oint_{\mathscr{C}} \boldsymbol{e}' \mathrm{d}s = \oint_{\mathscr{C}} (\nabla_{\mathrm{s}} \boldsymbol{e})\boldsymbol{t} \mathrm{d}s = \int_{\mathscr{S}_{\mathscr{C}}} (\mathrm{curl}_{\mathrm{s}}\, \nabla_{\mathrm{s}} \boldsymbol{e})\boldsymbol{v} \mathrm{d}a, \tag{81}$$

where the discontinuity of $(\mathrm{curl}_{\mathrm{s}}\, \nabla_{\mathrm{s}} \boldsymbol{e})$ along $\mathscr{C}_0$, being concentrated on a set of zero surface measure, does not affect the value of the surface integral. In components, (80) reads as

$$e_{i;j} = -\varepsilon_{ihk} e_h A_{kj},$$

whence it follows that

$$(\mathrm{curl}_{\mathrm{s}}\, \nabla_{\mathrm{s}} \boldsymbol{e})_{\ell m} = \varepsilon_{mhk}\varepsilon_{\ell pq}(e_p A_{qh})_{;k} = \varepsilon_{mhk}\varepsilon_{\ell pq}[e_{p;k} A_{qh} + e_p A_{qh;k}] = \Gamma_{\ell m} + \Delta_{\ell m}, \tag{82}$$

where, for simplicity, we have set

$$\Gamma_{\ell m} := \varepsilon_{mhk}\varepsilon_{\ell pq} e_{p;k} A_{qh} \quad \text{and} \quad \Delta_{\ell m} := \varepsilon_{mhk}\varepsilon_{\ell pq} e_p A_{qh;k}.$$

If we further expand $e_{p;k} = -\varepsilon_{prt} e_r A_{tk}$ and employ the identity

$$\varepsilon_{pq\ell}\varepsilon_{prt} = (\delta_{qr}\delta_{\ell t} - \delta_{qt}\delta_{\ell r}), \tag{83}$$

by lengthly, but easy computations, we obtain that

$$\Gamma_{\ell m} = -\varepsilon_{mhk}(e_r A_{\ell k} A_{rh} - e_\ell A_{qk} A_{qh}) = -\varepsilon_{mhk} e_r A_{\ell k} A_{rh}, \tag{84}$$

since $A_{qk}A_{qh}$ is symmetric in the exchange of indices h and k. By (79), (84) finally becomes

$$\Gamma_{\ell m} = -\varepsilon_{mhk} e_r \varepsilon_{\ell pq}\varepsilon_{rst}\nu_p \nu_{q;k}\nu_s \nu_{t;h}.$$

To simplify $\Delta_{\ell m}$, we make use of

$$A_{qh;k} = \varepsilon_{qrt}(\nu_r \nu_{t;h})_{;k},$$

which follows from (79), thus getting

$$\Delta_{\ell m} = \varepsilon_{mhk}\varepsilon_{\ell pq}\varepsilon_{qrt} e_p(\nu_{r;k}\nu_{t;h} + \nu_r \nu_{t;hk}).$$

By using (83) and recalling that $\boldsymbol{\nu}\cdot\boldsymbol{e} = \nu_p e_p = 0$, we obtain

$$\Delta_{\ell m} = \varepsilon_{mhk} e_p(\nu_{\ell;k}\nu_{p;h} + \nu_\ell \nu_{p;hk} - \nu_{p;k}\nu_{\ell;h}),$$

which, by exchanging dummy indices h and k in the last term, is further reduced to

$$\Delta_{\ell m} = \varepsilon_{mhk} e_p \nu_\ell \nu_{p;hk} + 2\varepsilon_{mhk} e_p \nu_{\ell;k}\nu_{p;h}.$$

By applying (75) for $\boldsymbol{u} = \boldsymbol{\nu}$ and exploiting both the symmetry of $\nabla_{\mathrm{s}}\boldsymbol{\nu}$ and the skew-symmetry of $\mathbf{N}$, we easily see that

$$\varepsilon_{mhk}\nu_{p;hk} = N_{hm}\nu_{h;r}\nu_{p;r},$$

and so, we can recast (82) as

$$(\mathrm{curl}_{\mathrm{s}}\,\nabla_{\mathrm{s}}\boldsymbol{e})_{\ell m} = -\varepsilon_{mhk}N_{\ell q}N_{rt}e_r\nu_p\nu_{q;k}\nu_{t;h} + 2\varepsilon_{mhk}e_p\nu_{\ell;k}\nu_{p;h} + N_{hm}e_p\nu_\ell\nu_{h;r}\nu_{p;r},$$

whence, by repeated use of (56), the symmetry of $\nabla_{\mathrm{s}}\boldsymbol{\nu}$, and the skew-symmetry of $\mathbf{N}$, in intrinsic notation, we obtain

$$(\mathrm{curl}_{\mathrm{s}}\,\nabla_{\mathrm{s}}\boldsymbol{e})\boldsymbol{\nu} = \mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\mathbf{N}\boldsymbol{e} + 2(\nabla_{\mathrm{s}}\boldsymbol{\nu})\mathbf{N}(\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{e},$$

which, by (79), finally becomes

$$(\mathrm{curl}_{\mathrm{s}}\,\nabla_{\mathrm{s}}\boldsymbol{e})\boldsymbol{\nu} = [\mathbf{A}^2\mathbf{N} + 2(\nabla_{\mathrm{s}}\boldsymbol{\nu})\mathbf{A}]\boldsymbol{e}. \tag{85}$$

Letting $\nabla_{\mathrm{s}}\boldsymbol{\nu}$ be represented as in (53), with $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{\nu}$, from (79) we arrive at

$$\mathbf{A} = \sigma_1 \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 - \sigma_2 \boldsymbol{e}_1 \otimes \boldsymbol{e}_2,$$

from which we easily obtain

$$(\nabla_{\mathrm{s}}\boldsymbol{\nu})\mathbf{A} = \sigma_1\sigma_2(\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_1 \otimes \boldsymbol{e}_2) = K\mathbf{N} \tag{86}$$

and

$$\mathbf{A}^2 = -\sigma_1\sigma_2(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) = -K\mathbf{P}(\boldsymbol{\nu}). \tag{87}$$

Making use of both (86) and (87) in (85), we conclude that

$$(\mathrm{curl}_{\mathrm{s}}\,\nabla_{\mathrm{s}}\boldsymbol{e})\boldsymbol{\nu} = K\mathbf{N}\boldsymbol{e},$$

thus arriving from (81) to the desired formula for $\Delta\boldsymbol{e}_{\mathscr{C}}$,

$$\Delta\boldsymbol{e}_{\mathscr{C}} = \int_{\mathscr{S}_{\mathscr{C}}} K\boldsymbol{e}_\perp \mathrm{d}a, \tag{88}$$

where $\boldsymbol{e}_\perp = \mathbf{N}\boldsymbol{e}$.

Equations (88) becomes perhaps more stringent in a special case. Let $\mathscr{C}$ be a *closed* geodesic.[12] Since $\kappa_g = 0$ along $\mathscr{C}$, by (48a) and (9), we conclude that the unit tangent $\boldsymbol{t}$ to $\mathscr{C}$ is parallel transported along $\mathscr{C}$. Since $\boldsymbol{t}$ is continuous along $\mathscr{C}$, $\Delta \boldsymbol{t}_{\mathscr{C}} = \mathbf{0}$, and so (88) implies that

$$\int_{\mathscr{S}_{\mathscr{C}}} K \mathbf{N} \boldsymbol{t} \mathrm{d}a = \mathbf{0},$$

where now $\boldsymbol{t}$ is defined on $\mathscr{S}_{\mathscr{C}}$ as the unit tangent vector to the curves of the family F nested within $\mathscr{C}$. It would be interesting to check this result against the surfaces such as the Tannery pears or the Zoll surfaces that only possess closed geodesics [2].

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[12] Geodesics on a surface are characterized as curves, not necessarily closed, having zero geodesic curvature.