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ORIGINAL ARTICLE

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Global nonlinear stability for a triply diffusive convection in a porous layer

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Abstract A triply convective-diffusive fluid mixture saturating a porous horizontal layer in the Darcy–Oberbeck–Boussinesq scheme is studied. The nonlinear stability analysis of the conduction solution is performed when the layer is heated from below and salted from above by one salt and below by another salt. Denoting by P_i, $(i = 1, 2)$, the salts Prandtl numbers, it is shown that in the cases $\{P_1 = 1;\ P_2 = 1;\ P_1 = P_2\}$ do not exist subcritical instabilities and the thermal Rayleigh critical number of global stability in a simple closed form is given. The methodology used and the results obtained appear to be new in the existing literature and useful for the applications.

Keywords Multi-component fluid mixtures · Porous media · Convection · Stability

1 Introduction

Materials with very small voids (pores), distributed everywhere and interconnected, are called *porous media*. Many materials (sandstones, skin, bones, metallic foams,…) are porous media. The pores are generally occupied by a fluid. The convective-diffusive motions of the fluid in the interconnected pores can describe several phenomena. In particular, there are numerous applications in geophysical situations (like salt movement underground), in contaminant transport and underground water flow in ice melting {cfr. [1–3] and the references therein}.

The research concerned with the fluid motions in the porous media—very active in the past—is still very active in the nowadays, also because artificial porous materials (like fiber materials used in insulating purposes or metallic foams in heat transfer devices) occur everywhere and influence all of our lives. A porous medium is schematized via a body (generally rigid and called *skeleton*) having interconnected pores everywhere. In the present paper, we are concerned with convective-diffusive phenomena in a porous horizontal layer.

Generally, the fluid invading the pores is a mixture since are dissolved in chemical species (“salts”) and the layer is embedded in a temperature field.

Although the subject of double-diffusive convection is still a very active research area {cfr, for instance, [1–20,28] and the references therein}, the same subject with more than two components—although more difficult—in the past as nowadays has also attracted the attention of many authors {cfr [21–25]}. This is because the multicomponent diffusive convection presents a picture of behaviors increasing together with the number of components.

Dedicated to Professor Ingo Müller for his 75th birthday.

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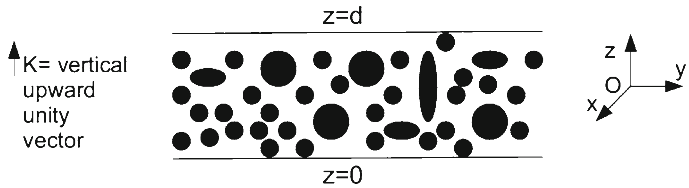

As concerns the *nonlinear stability* of the conduction solution in multicomponent (triply, quadruply,...) diffusive convection in a porous layer, as far as we know, the stability has been investigated only in the (symmetric) cases of a layer heated from below and salted by all salt fields either from above (the most stabilizing case) or from below (the most destabilizing case).

The present paper is devoted to the *nonlinear stability* of the conduction solution in the case of a triply convection-diffusion in a porous layer heated from below and salted from above by one salt ("salt 2") and from below by the other ("salt 1"). This case, not considered before (in particular neither in [23] nor in [28]), appears of notable interest since heat and "salt 2 " are destabilizing while "salt 1" is in competition and acts as a stabilizing agent. Our aim is precisely to show that, in this case, exists relevant salts for which

(i) *do not exist subcritical instability regions*;
(ii) *the thermal critical Rayleigh number R_c of nonlinear stability in the L^2-norm can be given in a simple closed form;*
(iii) *$R < R_c$ implies global nonlinear stability in the L^2-norm, that is, nonlinear stability with respect to any initial data.*

Denoting by R_i and P_i, $(i = 1, 2)$, respectively, the Rayleigh and Prandtl number of "salt i," we show that in the cases $\{P_1 = 1;\ P_2 = 1;\ P_1 = P_2 = P\}$, do not exist subcritical instability regions and the global nonlinear stability conditions are given by $R^2 < R_c^2$ with

$$R_c^2 = \min\left[R_1^2 - \frac{R_2^2}{P_2} + 4\pi^2\left(1 + \frac{1}{P_2}\right),\ R_1^2 - R_2^2 + 4\pi^2\right],\quad \text{when } P_1 = 1, \tag{1.1}$$

$$R_c^2 = \min\left[\frac{R_1^2}{P_1} - R_2^2 + 4\pi^2\left(1 + \frac{1}{P_1}\right),\ R_1^2 - R_2^2 + 4\pi^2\right],\quad \text{when } P_2 = 1, \tag{1.2}$$

$$R_c^2 = \min\left[\frac{1}{P}(R_1^2 - R_2^2) + 4\pi^2\left(1 + \frac{1}{P}\right),\ R_1^2 - R_2^2 + 4\pi^2\right],\quad \text{when } P_1 = P_2 = P. \tag{1.3}$$

Section 2 is devoted to some preliminaries concerned with the problem at stake. In Sect. 3, the main boundary value problem of the model equations is studied. Preliminaries to the stability of the conduction solution are given in Sect. 4. Section 5 is concerned with the cases $\{P_1 = 1;\ P_2 = 1;\ P_1 = P_2\}$. Via the introduction of auxiliary systems of P.D.Es, the absence of subcritical instabilities and the global stability condition $R^2 < R_c^2$, with R_c^2 given, respectively, by (1.2), (1.3), are obtained. The paper ends (Sect. 6) with some final remarks.

2 Preliminaries

Let $Oxyz$ be a Cartesian frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards).

We assume that the fluid has dissolved in two different chemical components (or "salts") S_α $(\alpha = 1, 2)$, having the concentration C_α $(\alpha = 1, 2)$, respectively, and assume that the equation of state is given by

$$\rho = \rho_0\left[1 - \alpha(T - T_0) + A_1(C_1 - \hat{C}_1) + A_2(C_2 - \hat{C}_2)\right],$$

where ρ_0, T_0, $\hat{C}_\alpha$ $(\alpha = 1, 2)$ are reference values of density, temperature, and salt concentration, respectively, and the constants α, A_α denote, respectively, the thermal and solute S_α expansion coefficient, respectively $(\alpha = 1, 2)$. Combining the Darcy's Law

$$\nabla p = -\frac{\mu}{K}\mathbf{v} + \rho\mathbf{g},$$

together with the equations of conservation of temperature and solute in the Boussinesq approximation {cfr [1] and [29]}, the equations governing the isochoric motions can be written as

$$\begin{cases} \nabla p = -\dfrac{\mu}{K}\mathbf{v} - \mathbf{g}\rho_0[1-\alpha(T-T_0)+A_1(C_1-\hat{C}_1)+A_2(C_2-\hat{C}_2)], \\ \nabla\cdot\mathbf{v}=0, \\ T_t+\mathbf{v}\cdot\nabla T = k\Delta T, \\ C_{1t}+\mathbf{v}\cdot\nabla C_1 = k_1\Delta C_1, \\ C_{2t}+\mathbf{v}\cdot\nabla C_2 = k_2\Delta C_2, \end{cases} \tag{2.1}$$

where p, pressure field; μ, dynamic viscosity; K, porosity; $\mathbf{v}$, velocity; $\mathbf{g}$, gravity; k, thermal diffusivity; K_α, diffusivity of the solute S_α.

To (2.1) we append the boundary conditions

$$\begin{cases} T(0)=T_1,\ \ T(d)=T_2, \\ C_\alpha(0)=C_{\alpha_l},\ \ C_\alpha(d)=C_{\alpha_u}\ \ \alpha=1,2, \\ \mathbf{v}\cdot\mathbf{k}=0,\ \text{ at } z=0,d, \end{cases} \tag{2.2}$$

with $T_1, T_2, C_{\alpha_l}, C_{\alpha_u}$ $(\alpha = 1,2)$, positive constants. The boundary value problem (2.1), (2.2) admits the conduction solution $(\tilde{\mathbf{v}}, \tilde{p}, \tilde{T}, \tilde{C}_\alpha)$ given by [23]

$$\begin{cases} \tilde{\mathbf{v}}=0,\ \ \tilde{T}=T_1-\beta z,\ \ \beta=\dfrac{T_1-T_2}{d}, \\ \tilde{C}_\alpha = C_{\alpha_l}-\dfrac{z(\delta C_\alpha)}{d},\ \ C_{\alpha_l}-C_{\alpha_u}=\delta C_\alpha, \\ \tilde{p}=p_0+\rho_0 g z^2\left[-\dfrac{\alpha\beta}{2}+A_1\dfrac{(\delta C_1)}{2d}+A_2\dfrac{(\delta C_2)}{2d}\right]+ \\ -\rho_0 g z^2\left[1-\alpha(T_1-T_0)+A_1(C_{1l}-\hat{C}_1)+A_2(C_{2l}-\hat{C}_2)\right], \end{cases} \tag{2.3}$$

where p_0 is a constant. Setting

$$\mathbf{v}=\tilde{\mathbf{v}}+\mathbf{u},\quad p=\tilde{p}+\Pi,\quad T=\tilde{T}+\theta,\quad C_\alpha=\tilde{C}_\alpha+\Phi_\alpha, \tag{2.4}$$

and introducing the scalings

$$\begin{cases} t=t^*\dfrac{d^2}{k},\quad \mathbf{u}=\mathbf{u}^*\dfrac{k}{d},\quad \Pi=\Pi^*\dfrac{\mu k}{K},\quad \mathbf{x}=\mathbf{x}^*d,\ \ \theta=\theta^*T^\sharp, \\ \Phi_\alpha=(\Phi_\alpha)^*\varphi_\alpha,\ \ T^\sharp=\left(\dfrac{\mu k|\delta T|}{\alpha\rho_0 g K d}\right)^{\frac{1}{2}},\quad \varphi_\alpha=\left(\dfrac{\mu k P_\alpha|\delta C_\alpha|}{A_\alpha\rho_0 g K d}\right)^{\frac{1}{2}}, \\ R=\left(\dfrac{\alpha\rho_0 g K d|\delta T|}{\mu k}\right)^{\frac{1}{2}},\quad R_\alpha=\left(\dfrac{A_\alpha\rho_0 g K d P_\alpha|\delta C_\alpha|}{\mu k}\right)^{\frac{1}{2}}, \\ \delta T=T_1-T_2,\quad H=\mathrm{sgn}(\delta T),\quad H_\alpha=\mathrm{sgn}(\delta C_\alpha),\quad P_\alpha=\dfrac{k}{k_\alpha}, \end{cases} \tag{2.5}$$

the dimensionless equations governing the perturbation $\{\mathbf{u}^*, \Pi^*, \theta^*, (\Phi_\alpha)^*\}$, omitting the stars, in the case $H = H_1 = 1,\ H_2 = -1$, are

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t + \mathbf{u} \cdot \nabla\theta = R\mathbf{u} \cdot \mathbf{k} + \Delta\theta, \\ P_1(\Phi_{1t} + \mathbf{u} \cdot \nabla\Phi_1) = R_1\mathbf{u} \cdot \mathbf{k} + \Delta\Phi_1, \\ P_2(\Phi_{2t} + \mathbf{u} \cdot \nabla\Phi_2) = -R_2\mathbf{u} \cdot \mathbf{k} + \Delta\Phi_2, \end{cases} \tag{2.6}$$

under the boundary conditions (free boundary conditions)

$$(\mathbf{u} \cdot \mathbf{i})_z = (\mathbf{u} \cdot \mathbf{j})_z = \mathbf{u} \cdot \mathbf{k} = \theta = \Phi_1 = \Phi_2 = 0 \text{ on } z = 0, 1. \tag{2.7}$$

In (2.5), (2.6), R and R_α are the thermal and salt Rayleigh numbers, while P_α are the salt Prandtl numbers.

We assume, as normally is done in stability problems in layers [1,2,5], that

- (i) the perturbations $(u, v, \omega, \theta, \Phi_1, \Phi_2)$ are periodic in the x and y directions, respectively, of periods $2\pi/a_x,\ 2\pi/a_y$;
- (ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;
- (iii) $\mathbf{u},\ \Phi_1,\ \Phi_2,\ \theta$ belong to $W^{2,2}(\Omega)$ and are such that all their first derivatives and second spatial derivatives can be expanded in a Fourier series uniformly convergent in Ω.

3 A boundary value problem

This section is devoted to the boundary value problem

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (\alpha\psi - \beta\psi_1 - \gamma\psi_2)\mathbf{k}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega, \\ \omega = \psi = \psi_1 = \psi_2 = 0, & \text{on } z = 0, 1, \end{cases} \tag{3.1}$$

with α, β, γ real constants. Equation (3.1), already present in (2.6) with $\{\alpha = R,\ \beta = R_1,\ \gamma = R_2,\ \psi = \theta,\ \psi_i = \Phi_i,\ i = 1, 2\}$, will be encountered in the sequel with different values for α, β, γ.

Let $L_2^*(\Omega)$ be the set of the functions Φ such that

- (i) $\Phi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \Phi(\mathbf{x}, t) \in \mathbb{R},\ \ \Phi \in W^{2,2}(\Omega),\ \ \forall t \in \mathbb{R}^+$; Φ is periodic in the x and y directions of periods $\dfrac{2\pi}{a_x}, \dfrac{2\pi}{a_y}$ respectively and $[\Phi]_{z=0} = [\Phi]_{z=1} = 0$;
- (iii) all the first derivatives and the second spatial derivatives of Φ can be expanded in a Fourier series absolutely uniformly convergent in $\Omega, \forall t \in \mathbb{R}^+$.

Since the sequence $\{\sin n\pi z\}$, $(n = 1, 2, \ldots)$, is a complete orthogonal system for $L_2[(0, 1)]$—by virtue of periodicity—it turns out that, $\forall \Phi \in L_2^*(\Omega)$, it exists a sequence $\{\Phi_n(x, y, t)\}$ such that

$$\begin{cases} \Phi = \sum_1^\infty \Phi_n(x, y, t) \sin n\pi z, & \dfrac{\partial\Phi}{\partial t} = \sum_1^\infty \dfrac{\partial\Phi_n}{\partial t} \sin n\pi z, \\ \Delta_1\Phi = -a^2\Phi, & \Delta\Phi = -\sum_1^\infty \xi_n\Phi_n \sin n\pi z, \end{cases} \tag{3.2}$$

$$\begin{cases} \xi_n = a^2 + n^2\pi^2, \quad a^2 = a_x^2 + a_y^2, \\ \Delta = \Delta_1 + \dfrac{\partial^2}{\partial z^2}, \quad \Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}, \end{cases} \tag{3.3}$$

the series appearing in (3.1) being absolutely uniformly convergent in Ω.

Lemma 1 *Let* $(\mathbf{u}, \psi, \psi_1, \psi_2)$*—with* $\omega, \psi, \psi_1, \psi_2 \in L_2^*(\Omega)$*—be solution of the b.v.p.*(3.1)*. Then*

(i) $(\omega, \psi, \psi_1, \psi_2)$ *is solution of the b.v.p.*

$$\begin{cases} \Delta\omega = \Delta_1(\alpha\psi - \beta\psi_1 - \gamma\psi_2) & \text{in } \Omega, \\ \omega = \psi = \psi_1 = \psi_2 = 0, & z = 0, 1, \end{cases} \tag{3.4}$$

(ii) *the first two components* u, v *of* $\mathbf{u}$ *are given by*

$$\begin{cases} u = \sum_{n=1}^{\infty} u_n(x, y, t) \frac{\mathrm{d}}{\mathrm{d}z}(\sin n\pi z), & u_n = \frac{1}{a^2} \frac{\partial \omega_n}{\partial x}, \\ v = \sum_{n=1}^{\infty} v_n(x, y, t) \frac{\mathrm{d}}{\mathrm{d}z}(\sin n\pi z), & v_n = \frac{1}{a^2} \frac{\partial \omega_n}{\partial y}, \end{cases} \tag{3.5}$$

(iii) $\mathbf{u}$ *verifies* $(3.1)_2$ *with*

$$\begin{cases} \omega = \sum_{n=1}^{\infty} \omega_n(x, y, t) \sin n\pi z, \quad \psi = \sum_{n=1}^{\infty} \psi_n(x, y, t) \sin n\pi z, \\ \psi_i = \sum_{n=1}^{\infty} \psi_{in}(x, y, t) \sin(n\pi z), \quad i = 1, 2. \end{cases} \tag{3.6}$$

Proof In view of

$$\begin{cases} \mathbf{k} \cdot \{\nabla \times \nabla \times \mathbf{u}\} = -\Delta\omega, \\ \mathbf{k} \cdot \{\nabla \times \nabla \times \Phi\} = -\Delta_1 \Phi, \quad \Phi \in \{\psi, \psi_1, \psi_2\} \end{cases} \tag{3.7}$$

Equation $(3.1)_1$ implies $(3.4)_1$. Setting

$$\zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{k}, \tag{3.8}$$

in view of $(3.1)_2$, one obtains

$$\begin{cases} \Delta_1 u = -\frac{\partial^2 \omega}{\partial x \partial z} - \frac{\partial \zeta}{\partial y} \\ \Delta_1 v = -\frac{\partial^2 \omega}{\partial y \partial z} + \frac{\partial \zeta}{\partial y} \end{cases} \tag{3.9}$$

On the other hand, $(3.1)_1$ implies $\zeta = 0$, hence

$$\Delta_1 u = -\frac{\partial^2 \omega}{\partial y \partial z}, \quad \Delta_1 v = -\frac{\partial^2 \omega}{\partial y \partial z} \tag{3.10}$$

that is,

$$\begin{cases} \Delta_1 u = -\sum_1^{\infty} \frac{\partial \omega_n}{\partial x} \frac{\mathrm{d}}{\mathrm{d}z}(\sin n\pi z) \\ \Delta_1 v = -\sum_1^{\infty} \frac{\partial \omega_n}{\partial y} \frac{\mathrm{d}}{\mathrm{d}z}(\sin n\pi z) \end{cases} \tag{3.11}$$

By virtue of the periodicity in the x and y directions and (3.11), one obtains that $(3.4)_1$, $(3.4)_3$, and (3.11) together with

$$\Delta_1 u_n = -a^2 u_n, \quad \Delta_1 v_n = -a^2 v_n \tag{3.12}$$

imply $(3.4)_2$ and $(3.4)_4$. Finally in view of

$$\nabla \cdot \mathbf{u} = \sum_1^\infty \left(\frac{1}{a^2} \Delta_1 \omega_n + \omega_n \right) \frac{\mathrm{d}}{\mathrm{d}z} (\sin n\pi z) \tag{3.13}$$

Equation $(3.1)_2$ immediately follows.
Setting

$$\begin{cases} \tilde{\psi}_{in} = \psi_{in}(x, y, t) \sin n\pi z, \quad i = 1, 2, \\ \tilde{\omega}_n = \omega_n(x, y, t) \sin n\pi z, \quad \tilde{\psi}_n = \psi_n(x, y, z) \sin n\pi z, \end{cases} \tag{3.14}$$

the following theorem holds. □

Theorem 1 *Let* $\tilde{\omega}_n, \tilde{\psi}_n, \tilde{\psi}_{1n}, \tilde{\psi}_{2n} \in L_2^*(\Omega), \ \forall n \in \mathbb{N}^+$. *Then, a complete orthogonal system of solutions of the b.v.p.* (3.4) *is given by*

$$\begin{cases} \tilde{\omega}_n = \eta(\alpha \tilde{\psi}_n - \beta \tilde{\psi}_{1n} - \gamma \tilde{\psi}_{2n}), \qquad (n = 1, 2, \ldots), \\ \mathbf{u}_n = \dfrac{1}{a^2} \left(\dfrac{\partial^2 \tilde{\omega}_n}{\partial x \partial z} \mathbf{i} + \dfrac{\partial^2 \tilde{\omega}_n}{\partial y \partial z} \mathbf{j} \right) + \tilde{\omega}_n \mathbf{k}, \quad \eta_n = \dfrac{a^2}{\xi_n}. \end{cases} \tag{3.15}$$

Proof In view of (3.2), (3.3), it easily follows that (3.15) implies (3.4) $\forall n \in \mathbb{N}^+$. On the other hand, by virtue of

$$\int_0^1 \sin n\pi z \cdot \sin m\pi z \, \mathrm{d}z = 0 \qquad n \neq m, \quad n, m \in \mathbb{N}^+, \tag{3.16}$$

the system $\{\tilde{\omega}_n, \tilde{\psi}_n, \tilde{\psi}_{1n}, \tilde{\psi}_{2n}, (n = 1, 2, \ldots)$, with $\tilde{\omega}_n$ given by $(3.15)_1$ is a complete orthogonal system of solutions of (3.1) since $\{\sin n\pi z\}$ is a such system for $L^2(0, 1)$. □

Remark 1 By virtue of (3.15), the independent unknown fields are reduced only to ψ, ψ_1, ψ_2.

4 Preliminaries to nonlinear stability

Lemma 2 *The eigenvalues of the matrix*

$$L = \begin{pmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \tag{4.1}$$

with real entries α_{ij}, *have negative real part if and only if*

$$\alpha_{11} < 0, \quad \mathcal{I} = \alpha_{22} + \alpha_{33} < 0, \quad A = \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} > 0. \tag{4.2}$$

Proof Since the invariants $\mathtt{I}_1, \mathtt{I}_2, \mathtt{I}_3$ of (4.1) and $\mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3$ are given by

$$\mathtt{I}_1 = \alpha_{11} + \mathtt{I}, \quad \mathtt{I}_2 = \alpha_{11}\mathtt{I} + A, \quad \mathtt{I}_3 = \alpha_{11}A, \quad \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 = \alpha_{11}\mathtt{I}\left(\mathtt{I} + \frac{\alpha_{11}^2 + A}{\alpha_{11}} \right), \tag{4.3}$$

it immediately follows that (4.2) implies the Routh–Hurwitz stability condition [26]

$$\mathtt{I}_1 < 0, \quad \mathtt{I}_3 < 0, \quad \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 < 0. \tag{4.4}$$

Vice versa, let (4.4) hold. Since one easily verifies that α_{11} is a real root of (4.1), by virtue of (4.4), it follows that $\alpha_{11} < 0$. Then $(4.3)_3$ and $(4.4)_2$ imply $A > 0$. It remains to obtain $\mathtt{I} < 0$. But (4.4) implies

$$(-\alpha_{11})\mathtt{I}\left(\mathtt{I} + \frac{\alpha_{11}^2 + A}{\alpha_{11}}\right) = (-\alpha_{11})\mathtt{I}^2 - (\alpha_{11}^2 + A)\mathtt{I} > 0,$$

and the roots of

$$(-\alpha_{11})\mathtt{I}^2 - (\alpha_{11}^2 + A)\mathtt{I} = 0,$$

are 0 and $\dfrac{\alpha_{11}^2 + A}{-\alpha_{11}} > 0$, hence $\mathtt{I} \notin \left[0, -\dfrac{\alpha_{11}^2 + A}{\alpha_{11}}\right]$. Since $-\dfrac{\alpha_{11}^2 + A}{\alpha_{11}} > -\alpha_{11}$, $(4.4)_1$ does not allow, in view of $(4.3)_1$, $\mathtt{I} > -\alpha_{11}$, hence $\mathtt{I} < 0$. □

Lemma 3 *The temporal derivative of*

$$W = \frac{1}{2}\left[X_1^2 + A(X_2^2 + X_3^2) + (\alpha_{22}X_3 - \alpha_{32}X_2)^2 + (\alpha_{23}X_3 - \alpha_{33}X_3)^2\right], \tag{4.5}$$

along the solutions of

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = L\mathbf{X} + \mathbf{F}, \tag{4.6}$$

with L given by (4.1), A by $(4.2)_3$ and $\mathbf{X} = (X_1, X_2, X_3)^T$, $\mathbf{F} = (F_1, F_2, F_3)^T$, is given by

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{2}\left[\alpha_{11}X_1^2 + \mathcal{I}A(X_2^2 + X_3^2)\right] + \Phi^*, \tag{4.7}$$

with $\mathtt{I}$ given by $(4.2)_2$ and

$$\begin{cases} \Phi^* = F_1X_1 + (A_1X_2 - A_3X_3)F_2 + (A_2X_3 - A_3X_2)F_3, \\ A_1 = A + \alpha_{32}^2 + \alpha_{33}^2, \quad A_2 = A + \alpha_{22}^2 + \alpha_{23}^2, \quad A_3 = \alpha_{22}\alpha_{32} + \alpha_{23}\alpha_{33}. \end{cases} \tag{4.8}$$

Proof A detailed proof can be found in [27–30]. □

Lemma 4 *Let $\mathbf{F} = \mathbf{F}(\mathbf{X})$ be a nonlinear function of $\mathbf{X}$ such that $\mathbf{F}(0) = 0$ and let $\Phi^* \leq 0$ and (4.2) hold. Then the zero solution of (4.6) is nonlinearly globally stable, and subcritical instabilities do not occur.*

Proof By virtue of $\Phi^* \leq 0$, (4.2) gives

$$\dot{W} \leq -\delta(X_1^2 + X_2^2 + X_3^2) \leq 0, \tag{4.9}$$

with

$$\delta = \min(|\alpha_{11}|, |\mathtt{I}A|). \tag{4.10}$$

Therefore, $\dot{W}$ is negative definite *for any initial data* when $\Phi^* \leq 0$ and W is positive definite when (4.2) holds. Since—by virtue of Lemma 2—(4.2) is equivalent to the Routh–Hurwitz conditions of linear stability, subcritical instabilities cannot occur. □

5 Absence of subcritical instabilities and global nonlinear L^2-stability in the case $P_1 = 1$

In the case $P_1 = 1$ (2.6) reduces to

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u} \cdot \nabla\theta, \\ \Phi_{1t} = R_1\omega + \Delta\Phi_1 - \mathbf{u} \cdot \nabla\Phi_1, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_2 - \mathbf{u} \cdot \nabla\Phi_2. \end{cases} \tag{5.1}$$

Setting

$$\varphi = R_1\theta - R\Phi_1, \tag{5.2}$$

it turns out that

$$\Phi_1 = \frac{1}{R}(R_1\theta - \varphi), \tag{5.3}$$

and (5.1) becomes

$$\begin{cases} \nabla \Pi = -\mathbf{u} + \left(\dfrac{R^2 - R_1^2}{R}\theta + \dfrac{R_1}{R}\varphi - R_2\Phi_2\right)\mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \varphi_t = \Delta\varphi - \mathbf{u} \cdot \nabla\varphi, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u} \cdot \nabla\theta, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_2 - \mathbf{u} \cdot \nabla\Phi_2. \end{cases} \tag{5.4}$$

By virtue of Lemma 1 and Theorem 1 for $(\alpha = \dfrac{R^2 - R_1^2}{R}, \beta = -\dfrac{R_1}{R}, \gamma = R_2, \psi = \theta,\ \psi_1 = \varphi,\ \psi_2 = \Phi_2)$, it follows that

$$\tilde{\omega}_n = \eta_n\left(\frac{R^2 - R_1^2}{R}\tilde{\theta}_n + \frac{R_1}{R}\tilde{\varphi}_n - R_2\tilde{\Phi}_{2n}\right), \quad n \in \mathbb{N}, \tag{5.5}$$

with $\tilde{\theta}_n,\ \tilde{\Phi}_{2n}$ given by (3.14) and—analogously—$\tilde{\varphi}_n \in L_2^*(\Omega)$ given by

$$\tilde{\varphi}_n = \varphi_n(x, y, t) \sin n\pi z. \tag{5.6}$$

Setting

$$\begin{cases} \bar{a}_{1n} = -\xi_n, \quad \bar{a}_{2n} = \bar{a}_{3n} = 0, \\ \bar{b}_{1n} = R_1\eta_n, \quad \bar{b}_{2n} = (R^2 - R_1^2)\eta_n - \xi_n, \quad \bar{b}_{3n} = -RR_2\eta_n, \\ \bar{c}_{1n} = -\dfrac{R_1R_2}{RP_2}\eta_n \quad \bar{c}_{2n} = -\dfrac{R_2(R^2 - R_1^2)}{RP_2}\eta_n \quad \bar{c}_{3n} = \dfrac{R_2^2\eta_n - \xi_n}{P_2}, \end{cases} \tag{5.7}$$

one obtains

$$\begin{cases} \varphi_t = \sum_1^\infty \bar{a}_{1n}\varphi_n - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = \sum_1^\infty (\bar{b}_{1n}\varphi_n + \bar{b}_{2n}\theta_n + \bar{b}_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{2t} = \sum_1^\infty (\bar{c}_{1n}\varphi_n + \bar{c}_{2n}\theta_n + \bar{c}_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\Phi_2, \end{cases} \tag{5.8}$$

under the boundary conditions (2.7), and $\mathbf{u} = \sum_{n=1}^\infty \mathbf{u}_n$ with $\mathbf{u}_n$ given by (3.15).
Since

$$\varphi = \lim_{m\to\infty} S_m^{(\varphi)}, \quad \theta = \lim_{m\to\infty} S_m^{(\theta)}, \quad \Phi_2 = \lim_{m\to\infty} S_m^{(\Phi_2)}, \quad \mathbf{U}_m = \sum_{n=1}^\infty \mathbf{u}_n, \tag{5.9}$$

with

$$S_m^{(\varphi)} = \sum_{n=1}^m \varphi_n, \quad S_m^{(\theta)} = \sum_{n=1}^m \theta_n, \quad S_m^{(\Phi_2)} = \sum_{n=1}^m \Phi_{2n}, \tag{5.10}$$

the nonexistence of subcritical instabilities and the global stability is guaranteed by showing that the asymptotic stability of the null solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} S_m^{(\varphi)} = \sum_{n=1}^m \bar{a}_{1n}\varphi_n - \mathbf{U}_m\cdot\nabla S_m^{(\varphi)}, \\ \frac{\mathrm{d}}{\mathrm{d}t} \S_m^{(\theta)} = \sum_{n=1}^m (\bar{b}_{1n}\varphi_n + \bar{b}_{2n}\theta_n + \bar{b}_{3n}\Phi_{2n}) - \mathbf{U}_m\cdot\nabla S_m^{(\theta)}, \\ \frac{\mathrm{d}}{\mathrm{d}t} S_m^{(\Phi_2)} = \sum_{n=1}^m (\bar{c}_{1n}\varphi_n + \bar{c}_{2n}\theta_n + \bar{c}_{3n}\Phi_{2n}) - \mathbf{U}_m\cdot\nabla S_m^{(\Phi_2)}, \end{cases} \tag{5.11}$$

under the initial boundary conditions

$$\begin{aligned} &(\varphi_n)_{t=0} = \varphi_n^{(0)}, \quad (\theta_n)_{t=0} = \theta_n^{(0)}, \quad (\Phi_{2n})_{t=0} = \Phi_{2n}^{(0)}, \\ &\varphi_n = \theta_n = \Phi_{2n} = 0, \quad z = 0, 1, \end{aligned} \tag{5.12}$$

is guaranteed *for any initial data* $(\varphi_n^{(0)}, \theta_n^{(0)}, \Phi_{2n}^{(0)})$ and for any $m \in \mathbb{N}$ if and only if the Routh–Hurwitz conditions

$$\bar{\mathrm{I}}_{1n} < 0, \quad \bar{\mathrm{I}}_{3n} < 0, \quad \bar{\mathrm{I}}_{1n}\bar{\mathrm{I}}_{2n} - \bar{\mathrm{I}}_{3n} < 0, \tag{5.13}$$

with $\bar{\mathrm{I}}_{1n}, \bar{\mathrm{I}}_{2n}, \bar{\mathrm{I}}_{3n}$ invariants of the matrix

$$L_n = \begin{pmatrix} \bar{a}_{1n} & 0 & 0 \\ \bar{b}_{1n} & \bar{b}_{2n} & \bar{b}_{3n} \\ \bar{c}_{1n} & \bar{c}_{2n} & \bar{c}_{3n} \end{pmatrix}, \tag{5.14}$$

are verified for any $n \in \mathbb{N}$.

We call *auxiliary system of order n* of (5.11) and denote it by $(\mathrm{AS})_n$, the system

$$\begin{cases} \dfrac{\partial \varphi_n}{\partial t} = \bar{a}_{1n}\varphi_n - \mathbf{U}_m \cdot \nabla\varphi_n, \\ \dfrac{\partial \theta_n}{\partial t} = \bar{b}_{1n}\varphi_n + \bar{b}_{2n}\theta_n + \bar{b}_{3n}\Phi_{2n} - \mathbf{U}_m \cdot \nabla\theta_n, \\ \dfrac{\partial \Phi_{2n}}{\partial t} = \bar{c}_{1n}\varphi_n + \bar{c}_{2n}\theta_n + \bar{c}_{3n}\Phi_{2n} - \mathbf{U}_m \cdot \nabla\Phi_{2n}, \end{cases} \tag{5.15}$$

under the initial boundary conditions (5.12), (5.13) with

$$\mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n, \quad \mathbf{u}_n = \frac{1}{a^2}\left(\frac{\partial^2 \omega_n}{\partial x \partial z}\mathbf{i} + \frac{\partial^2 \omega_n}{\partial y \partial z}\mathbf{j} + \omega_n \mathbf{k}\right), \tag{5.16}$$

and remark that (5.11) is immediately obtained by adding with respect to n, from $n = 1$ to $n = m$, each equation of the $(\mathrm{AS})_n$ (5.15). Therefore, the conditions guaranteeing the nonlinear stability of the null solution of $(\mathrm{AS})_n$, $\forall n \in \mathbb{N}$ and *for any initial data*, guarantee the stability of the null solution of (5.11), $\forall m \in \mathbb{N}$ and for any initial data. Further, the nonexistence of subcritical instabilities is guaranteed if the stability conditions are equivalent to (5.13) for any n. Setting

$$\mathbf{X}_n = (\varphi_n, \theta_n, \Phi_{2n})^T, \quad \mathbf{F}_n = -\left(\mathbf{U}_m \cdot \nabla\varphi_n,\, \mathbf{U}_m \cdot \nabla\theta_n,\, \mathbf{U}_m \cdot \nabla\Phi_{2n}\right)^T, \tag{5.17}$$

Equation (5.15) can be written

$$\frac{\partial}{\partial t}\mathbf{X}_n = L_n \mathbf{X}_n + \mathbf{F}_n, \tag{5.18}$$

and is of the type (4.6). Therefore, introducing the functional (analogous to (4.5))

$$W_n = \frac{1}{2}\int_\Omega \left[\varphi_n^2 + \mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2) + (\bar{b}_{2n}\Phi_{2n} - \bar{c}_{2n}\theta_n)^2 + (\bar{b}_{3n}\Phi_{2n} - \bar{c}_{3n}\theta_n)^2\right] \mathrm{d}\Omega, \tag{5.19}$$

with

$$\mathcal{A}_n = \bar{b}_{2n}\bar{c}_{3n} - \bar{c}_{2n}\bar{b}_{3n} = \frac{\eta_n \xi_n}{P_2}\left(R_1^2 - R_2^2 + \frac{\xi_n^2}{a^2} - R^2\right), \tag{5.20}$$

its temporal derivative along the solutions of (5.15) is given by

$$\dot{W}_n = \frac{1}{2}\int_\Omega \left[-\xi_n^2\varphi_n^2 + \mathcal{I}_n\mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2)\right] \mathrm{d}\Omega + \Phi_n^*, \tag{5.21}$$

with

$$\mathcal{I}_n = \bar{b}_{2n} + \bar{c}_{3n} = \eta_n\left[R^2 - R_1^2 + \frac{R_2^2}{P_2} - \left(1 + \frac{1}{P_2}\right)\frac{\xi_n^2}{a^2}\right], \tag{5.22}$$

and

$$\begin{cases} \Phi_n^* = \langle F_{1n}, X_n\rangle + \langle A_{1n}X_{2n} - A_{3n}X_{3n}, F_{2n}\rangle + \langle A_{2n}X_{3n} - A_{3n}X_{2n}, F_{3n}\rangle, \\ A_{1n} = \mathcal{A}_n + \bar{c}_{2n}^2 + \bar{c}_{3n}^2, \;\; A_{2n} = \mathcal{A}_n + \bar{b}_{2n}^2 + \bar{b}_{3n}^2, \;\; A_{3n} = \bar{b}_{2n}\bar{c}_{2n} + \bar{b}_{3n}\bar{c}_{3n}, \\ \langle\cdot,\cdot\rangle \text{ scalar product of } L^2(\Omega). \end{cases} \tag{5.23}$$

Since $\mathbf{U}_m \cdot \nabla f_n$, with $f_n \in \{\varphi_n, \theta_n, \Phi_{2n}\}$, is given by

$$\mathbf{U}_m \cdot \nabla f_n = \sum_{p=1}^{m} \mathbf{u}_p \cdot \nabla f_n = \sum_{p=1}^{m} \left[\frac{p\pi}{a^2} \left(\frac{\partial \tilde{\omega}_p}{\partial x} \frac{\partial \tilde{f}_n}{\partial x} + \frac{\partial \tilde{\omega}_p}{\partial y} \frac{\partial \tilde{f}_n}{\partial y} \right) \cos(p\pi z) \sin(n\pi z) + n\pi \tilde{\omega}_p \tilde{f}_n \sin(p\pi z) \cos(n\pi z) \right], \tag{5.24}$$

and $\langle f_n^*, \mathbf{U}_m \cdot \nabla f_n \rangle$ with $f_n^* \in \{\varphi_n, \theta_n, \Phi_{2n}\}$ is given by $\langle \tilde{f}_n^* \sin(n\pi z), \mathbf{U}_m \cdot \nabla f_n \rangle$, by virtue of

$$\int_0^1 \sin(q\pi z) \cos(p\pi z) \sin(n\pi z)\, \mathrm{d}z = 0 \quad \text{for } p + q \neq n, \tag{5.25}$$

one easily obtains that $< f_n^*, \mathbf{U}_m \cdot \nabla f_n >= 0$. Since any scalar product appearing in $(5.23)_1$ is given by a finite numbers of scalar products of the type $< f_n^*, \mathbf{U}_m \cdot \nabla f_n >$, it follows that $\{\Phi_n^* = 0, \forall n \in \mathbb{N}\}$ and (5.21) reduces to

$$\dot{W}_n \leq \frac{1}{2} \int_\Omega \left[-\xi_n \varphi_n^2 + \mathcal{I}_n \mathcal{A}_n (\theta_n^2 + \Phi_{2n}^2) \right] \mathrm{d}\Omega. \tag{5.26}$$

Theorem 2 *Let* (1.1) *hold. Then the zero solution of the auxiliary system of order n* (5.15) *is globally asymptotically stable* $\forall n \in \mathbb{N}$ *and do not exist subcritical instabilities.*

Proof In fact, since

$$\begin{cases} \bar{a}_{1n} = -\xi_n = -(n^2\pi^2 + a^2) < -\pi^2, \\ \inf\limits_{(a^2,n) \in \mathbb{R}^+ \times \mathbb{N}} \dfrac{\xi_n^2}{a^2} = 4\pi^2, \quad \eta_n^2 \xi_n > 0, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}, \end{cases} \tag{5.27}$$

it follows that (1.1) guarantees

$$\mathcal{A}_n > 0, \quad \mathcal{I}_n < 0, \quad \forall n \in \mathbb{N}. \tag{5.28}$$

Hence W_n is positive definite and—in view of (5.26), (5.27), (5.28)—$\dot{W}_n$ is negative definite $\forall n \in \mathbb{N}$, *for any initial data*. Finally subcritical instabilities, since (5.28) together with $-\xi_n < 0$ are equivalent to the Routh–Hurwitz conditions for the matrix L_n, cannot exist. □

Remark 2 Setting

$$W = \sum_{n=1}^{m} W_n, \tag{5.29}$$

Equation (1.1) guarantee that W is positive definite and its temporal derivative along the solutions of (5.11) is negative definite.

Remark 3 In the case $P_2 = 1$, setting

$$\varphi = R_2\theta + R_1\Phi_2, \tag{5.30}$$

Equation (2.6) reduces to

$$\begin{cases} \nabla\Pi = -\mathbf{u} + \left(\dfrac{RR_1 + R_2^2}{R_1}\theta - \dfrac{R_2}{R_1}\varphi - R_1\Phi_1\right)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \varphi_t = \Delta\varphi - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ P_1(\Phi_{1t} + \mathbf{u}\cdot\nabla\Phi_1) = R_1\omega + \Delta\Phi_1. \end{cases} \tag{5.31}$$

Analogously, in the case $P_1 = P_2 = P$, setting

$$\varphi = R_2\Phi_1 + R_1\Phi_2, \tag{5.32}$$

Equation (2.6) reduces to

$$\begin{cases} \nabla\Pi = -\mathbf{u} + \left(R\theta - \dfrac{R_1}{R_2}\varphi + \dfrac{R_1^2 - R_2^2}{R_2}\Phi_2\right)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \varphi_t = \Delta\varphi - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ P(\Phi_{2t} + \mathbf{u}\cdot\nabla\Phi_2) = -R_2\omega + \Delta\Phi_2. \end{cases} \tag{5.33}$$

Either (5.31) or (5.33) is of the type (5.1). In fact both contain the equation

$$\varphi_t = \Delta\varphi - \mathbf{u}\cdot\nabla\varphi, \tag{5.34}$$

which guarantees that

$$\langle\varphi,\varphi\rangle = \langle\varphi(0),\varphi(0)\rangle e^{-\alpha t}, \quad \alpha = \text{const.} > 0. \tag{5.35}$$

Then, following step by step, the previous methodology, the absence of subcritical instabilities and the conditions of global asymptotic stability $R^2 < R_c^2$ with R_c^2 given by (1.2), (1.3), can be obtained.

6 Final remarks

(i) *The paper is concerned with the stability of the conduction solution in a triply convective fluid mixture saturating a porous horizontal layer when the layer is heated from below and salted from above by one salt and from below by another*;

(ii) *denoting by P_1 and P_2 the salts Prandtl numbers, either the case $P_1 = 1$ or $P_2 = 1$ or $P_1 = P_2$, are studied*;

(iii) *the absence of subcritical instability is shown, and the critical Rayleigh number of nonlinear global stability—in closed form—is obtained*;

(iv) *a new methodology aimed to reduce the triply diffusive convection to the double-diffusive convection has been applied;*

(v) *as far as we know, the results obtained appear to be new in the existing literature and useful for the applications.*

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