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REVIEW ARTICLE

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The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool

In memory of Paul Germain (1920–2009)

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Abstract In a period of a few decades, the formulation known as the principle of virtual power (PVP) has gained a prominent place among the most efficient tools in the thermomechanics of continua. Strongly marked by a "continental" (French-Italian) influence, it has successfully incorporated the basic invariances of modern continuum mechanics while capturing the spirit of twentieth-century analysis (generalized functions or distributions) in which it became synonymous of weak formulation. It proved to provide the surest and safest way to formulate complex theories of continua (so-called "generalized continuum mechanics", theory of coupled fields, etc) and approximate or generalized theories of structural members and the associated natural boundary conditions while preparing the way for the full thermomechanical formulation, providing the best setting for the proof of various mathematical theorems, and paving the way for modern numerical methods. The present contribution, illustrated by many examples of varying complexity, emphasizes the role of Paul Germain (1920–2009) in this formulation. The author, himself an active contributor and a never tired propagandist of the method, has participated in these developments during four decades and presents here his witness but critical viewpoint, highlighting the difficult points and also the esthetically pleasing ones where necessary.

Keywords Variational formulation · Virtual power · Mechanics of deformable solids · Thermomechanics · Microstructure · Discontinuities · Mixtures · Structures

1 By way of a long introduction

Some thirty years ago, I published a long paper devoted to the application of the principle of virtual power to continuum physics [7], the paragon of variational formulation, or "weak" formulation. Following two brilliant essays by Paul Germain [8,9] and a series of more specialized papers by me and my co-workers, that paper had for main purpose to set forth in one place a kind of *doxa* or a set of canons or rules (some would say recipes) for exploiting the principle of virtual power (PVP) as an efficient and safe conceptual tool to construct the governing equations of a complex theory of continuous media. The ambition, partly fulfilled by the success met with graduate students and a somewhat influential diffusion, was to provide in a few lines, but illustrated by an already large number of examples, prescriptions that could be followed quite easily if not strictly. At the time, following Lagrange, we naturally paid our tax to the historical side, recalling in large traits some historical

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This contribution is not intended for providing a history of the principle of virtual power in pre-d'Alembertian times or of its development by applied mathematicians, mathematical physicists and mechanical engineers in the nineteenth century. For these, we refer the reader to historical and critical reviews such as those given in more general works by Mach [1], Dugas [2], Timoshenko [3], Budo [4], Szabo [5] and Crowe [6].

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steps that, not always safely founded in precisely identified texts, are altogether plausible. Thus, we reminded the possible role of the ancient Greek Aristotle (with pseudo-Aristotle writings), of the medieval Jordanus of Nemore, of the late sixteenth century Simon Stevin, and closer to our thinking, of John Bernoulli, Leonard Euler, Samuel Koenig and, inevitably, Jean Le Rond d'Alembert (whose name we took for our Institute in 2007; It happens that the author descends scientifically directly from d'Alembert according to "Mathematical genealogy"). We acknowledged then the essential role played by the *principle of virtual work* (also called the principle of *virtual velocities*) in so-called industrial dynamics, where some irreversibilities (e.g., friction) are at work. The name of Lazare Carnot, of French Revolution's fame under the name of the "organizer of victory" (practically against the rest of Europe), was associated with this trend in the late eighteenth century.

The fact that engineering mechanics soon recognized this efficient power in formulating the expression of looked for forces—and this even in the United Kingdom—is to be contrasted with the enduring controversy between British mechanics (this means Newton's and followers) and "continental" scientists (this meaning essentially the French, German and Italian "mechanicians"). Still the emphasis put by the latter was on the solution of problems and not, as we construe here, on the conceptual framework rebuilding continuum physics on sound bases acknowledging the ordered role of geometry, kinematics and then kinetics appearing in the happy union of partners, forces and velocities, in that unique notion of expended power. This, as I recollect, was the original viewpoint of Paul Germain. For a long time, the scientific career of this scientist was that of an analyst solving difficult problems in theoretical fluid mechanics (transonic flows, flows around delta wings, structure of shock waves, magnetohydrodynamics, perturbation methods, etc). It is through teaching that he was inevitably caught in this typically French type of presentation of the principles of mechanics, especially with two arguments that escaped his standard fluid-mechanicist pre-occupations: general energy theorems in elasticity and the theory of structural members and the wealth of new modellings (e.g., so-called generalized continua with a multiplicity of varied forms of kinematics) appearing in the 1960s in continuum mechanics. He and others recognized at once that some logical order was to be brought among these generalizations, if only for the sake of comparison. I immediately joined this band wagon realizing then that this was also the way to write down with some economical thought the basic equations of continuum theories dealing not only with mechanics but also with what are now called *coupled fields*. We shall return to this point in the main body of the text.

The above noted radical opposition between the Newtonian school and the continental one was to have, in our opinion, a very negative consequence even in modern times. This is duly illustrated by the repeated criticism made by Clifford A. Truesdell and Richard A. Toupin in numerous footnotes of their opus magnum [10] of ("continental") variational methods and the like. To reject in the darkness variational methods of which the PVP partakes was an error in the appraisal made on the importance of various approaches in science, because ignoring the PVP and its direct inheritors practically is to negate all progress made in mathematical physics and engineering in the nineteenth and twentieth century, i.e., in succession, the analytical mechanics of Lagrange and Hamilton, its role in optics, in existence and uniqueness theorems, then in general field theories and correlative applications of group theory (e.g., Noether's theorem), in the creation of wave mechanics, and more recently, the remarkable works of Eshelby on defects and mechanics of inhomogeneous media, and the essential role of the PVP in the conception of modern numerical means such as the finite-element method, itself an engineering implementation of the mathematical *weak formulations* and the theory of generalized functions (distributions in Laurent Schwartz's vocabulary). This lack of perspective from undoubtedly great scientists is remarkable in itself and should prevent us of being prejudiced and wearing blinkers. In France, the publication of Paul Germain's book [11] was to bring a new standard in the teaching of continuum mechanics in both universities and engineering schools. Furthermore, the wealth of continuum models proposed in the 1960s–1970s led him to a more research-like approach in the two noted papers [8,9] and some sequels [12,13]. Then, practically, all French mechanicians followed suit with the success we know. This was also implemented by many researchers in Italy (e.g., A. di Carlo, G. Capriz, F. dell'Isola, S. Quiligotti, G. Sciarra, G. Del Piero, etc) and also elsewhere (e.g., M. Epstein and M. Segev in Galgary), although often in a more abstract form.

[Now, this raises the natural but politically incorrect question whether there exist "national" styles in science? The question is generally improper as modern science is precisely identified as a field of knowledge where there is a consensus at least in the (momentarily) admitted paradigm and in results. But there may exist easily identified "national" styles of teaching the same matter. This obviously may happen under dictatorships when a style may be imposed from the top in a way not to contradict the dictator's views. But this may also occur through the influence of remarkable individuals. In particular, this may happen in medium-sized countries with a strong tendency (inherited from the past) at centralization or the obvious supremacy of some institutions. This was the case of Newton's and Cambridge's enduring influence in British mechanics and mathematics during the 18th century with an unfortunate choice in differential notation. The "continental" (especially French) influence of the mathematical education in the early nineteenth century was to remedy this by favoring the emergence of a flourishing British mathematical physics (William Whewell, Lord Kelvin, James C. Maxwell, etc). In Poland, the reorganization of Polish applied mathematics and mechanics by two scientists of great talent and persuasive power, Waclaw Olszak and Witold Nowacki, allied to a favorable centralization (under the Polish Academy of Sciences) was to yield the creation and success of a true Polish school in both style and focused interests (e.g., study of anelastic behavior, microstructured solids, etc) in the post-WWII period. In Italy, it was the indelible print left by Ricci and Levi-Civita on rational mechanics in the early twentieth century. In France, the publication of Paul Germain's book [11] was to bring a new standard in the teaching of continuum mechanics in both universities and engineering schools. Only the course at the Polytechnique School,—e.g., Jean Mandel's one—could compare and compete with it, but with a much more restricted audience and a less talented pedagogy. Professors decidedly settled in French provincial universities rarely published textbooks (Pierre Duhem in Bordeaux was an exception). The royal road in academic careers was to finally obtain a chair in Paris (this was still true in the 1950s-60s), and then to publish a book as the "national" holly book in the field. This is what happened to Germain's book—nurtured during Germain's years in Lille and Poitiers—in addition to its remarkable qualities. When Paul Germain reoriented his thoughts toward the principles of mechanics, he naturally engaged more deeply in the line of the principle of virtual power. As said above, the wealth of continuum models proposed in the 1960s brought him to a more research-like approach in the two noted papers [8,9] and some sequels [12,13]. Then, practically, all French mechanicians followed suit with the success we know. This was the source of a true "national style" in approaching continuum mechanics (see later).

In spite of great names, both Germany and the United States did not have to suffer fundamentally from this type of behavior. This is the result of the absence of central intellectual and academic power in Germany in the nineteenth century although the masterly writings of such great figures as Hermann von Helmholtz and Ludwig Boltzmann in the nineteenth century and August Föppl and George Hamel in the twentieth century may have played a role equivalent to Germain's book. The USA with their size and multiplicity of intellectual centers was a priori exempt from such a fact although the influence of such journals as the *Journal of Rational Mechanics and Analysis* and the *Archive for Rational Mechanics and Analysis*, together with the Encyclopedic articles of Clifford A. Truesdell, Richard A. Toupin and Walter Noll in the *Handbuch der Physik*, while participating vividly in a tremendous revival of continuum mechanics, introduced an inclination toward the creation of a unique superior authority in the field. Fortunately, the simultaneous creation of many scientific journals in the 1960s (especially the "International Journals" created by Pergamon Press) was to counterbalance this emerging "intellectual terrorism" with its accompanying spirit of chapel, with excommunication and the like for the nonacknowledged believers or the identified blasphemists.]

2 The essential statement of the PVP

The already noticed Britain-Continent controversy started with a disagreement about the primary role played by either the force or the displacement (see Chapter I in my book [14]). In point mechanics, the geometry of the object under study, the *point*, is reduced to nothing although Newton himself can define mass only by introducing first a density (something rarely noted). It is then natural to emphasize the role of force. But for many people of the period—including Leibniz—that notion of force is mysterious (think of gravity) and somewhat metaphysical. Several eighteenth-century scientists wanted to purge mechanics of this metaphysical notion. This is one of the purposes of d'Alembert's vision, attributing thus the primary role to displacement (or velocity), moreover arguing that this is what is primarily observed and measured. Forces are just coefficients to be found in factor of these "displacements" in the (then) recently introduced notion of work or "power." Geometry enters fully the picture. This becomes even truer in continua where one must describe not only the traditional kinematics of a "point" but also the deformation of two neighboring points. Indeed, deformation is introduced as opposed to *rigidity*. The latter is geometrically defined by the celebrated Killing's theorem concerning isometries over the material body (this is a condition imposed on the gradient of a displacement). Typically, (here, we use Cartesian tensorial notation),

$$\xi_{i,j} + \xi_{j,i} = 0. \tag{2.1}$$

This integrates over 3D space to yield the notion of rigid displacements (rigid translations and rigid rotations, the latter usually represented by means of orthogonal transformations of E^3 onto itself). A body in a state of true deformation does not satisfy (2.1). In our modern view forces, whether standard vectors or more complicated objects from the tensorial viewpoint are introduced as factors of displacements, velocities or generalized such quantities. But it was soon remarked that if one performs in thought the operation of *rigidifying* a deformable body, then the corresponding generalized forces must not expend any power. This is the fundamental property to be taken into account in a mechanical system made of interconnected rigid bodies or in a deformable body. This requirement is translated in modern terms in the following:

Hypothesis H1: Internal forces should not expend any power in a rigidified deformation field.

Accordingly, they must be introduced as factors of kinematic quantities that are said to be *objective* (form invariant under superimposition of a rigid body motion—we do (and wish) not enter here the never-ending debate about the precise formulation of "objectivity"). In more mathematical terms, this yields:

Hypothesis H1′: The virtual power of internal forces of a mechanical system must be written as a continuous linear form on a set of objective virtual velocities.

This brings us in close contact with the theory of distributions. It suffices to add the notion of *separating duality* and obviously to specify the relevant set of virtual velocities. This is at our disposal. It brings to the foreground something that Euler and Cauchy, although the true creators of continuum mechanics, could not envisage, the notion of *gradient theory* of various orders.

Example first-order gradient theory of deformable bodies

The basic kinematic field is the physical velocity $\mathbf{v}(\mathbf{X}, t)$, where t is Newtonian's time and $\mathbf{X}$ stands for the selected space parametrization (three material coordinates). This notion is valid for both a point in point mechanics and a "point" in continuum mechanics. But in the latter case, we need to consider at least the first gradient of this $\mathbf{v}$, i.e., $\nabla\mathbf{v}$, in order to include the notion of deformation. Standard continuum mechanics (as conceived by Euler, Cauchy and their immediate followers) is satisfied with this. Thus, symbolically, we have to consider the extended set

$$V = \{\mathbf{v}, \nabla\mathbf{v}\}. \tag{2.2}$$

The second argument within braces decomposes into a symmetric part and a skewsymmetric part, respectively, the rate of strain $\mathbf{D}$ and the rate of rotation or vorticity $\boldsymbol{\Omega}$. Both the latter and the velocity $\mathbf{v}$ are not objective (cf. any book on continuum mechanics). From (2.2), there remains the set of symmetric tensors $\mathbf{D}$ such that (T = transpose)

$$V_{\text{obj}} = V/C = \left\{\mathbf{D} = \mathbf{D}^T\right\}, \tag{2.3}$$

where V_{obj} is the quotient space of V by the equivalence relation represented by the set C—a so-called *distributor*—of rigid body motions. Accordingly, the power expended by internal forces for a starting first-order gradient velocity of the type (2.2) is written, for a 3D regular continuous material body of volume B and using a Cartesian tensorial notation, as

$$P^*_{\text{int}}(V) = -\int_B \sigma_{ji} D^*_{ij} \mathrm{d}B, \tag{2.4}$$

where a star denotes a virtual field or the result of its application, and the symmetric tensor of components σ_{ji} is called the *intrinsic stress* tensor. Einstein's summation convention over repeated indices is enforced. For a rigidifying virtual motion, $\mathbf{v}^*$ satisfies the constraint (2.1) and both H1′ and H1 are verified. There is no such constraint imposed on the virtual power of inertial forces, prescribed body forces and surface forces acting in B or its supposedly regular boundary ∂B. The sign introduced in (2.4) is conventional (physicists dealing with a *pressure* tensor rather than a stress tensor would not put a minus sign in 2.4). For a theory based on (2.2), the virtual powers of inertial forces, body and surface forces are given by

$$P^*_{\text{inertia}}(B) = \int_B \mathbf{a}.\mathbf{v}^* \mathrm{d}B, \tag{2.5}$$

$$P^*_{\text{vol}}(B) = \int_B \rho \mathbf{f}.\mathbf{v}^* \mathrm{d}B, \tag{2.6}$$

$$P^*_{\text{surf}}(B) = \int_{\partial B} \mathbf{T}^d.\mathbf{v}^* \mathrm{d}S, \tag{2.7}$$

where $\mathbf{a} = \rho\dot{\mathbf{v}}$ is the "quantity of acceleration", $\mathbf{f}$ is the body force per unit mass and $\mathbf{T}^d$ is the surface traction. Then, the principle of virtual power (PVP) for the velocity field (2.2) is posited in the following generic form:

$$P^*_{\text{inertia}}(B) = P^*_{\text{int}}(B) + P^*_{\text{vol}}(B) + P^*_{\text{surf}}(B), \tag{2.8}$$

valid for any element of volume and surface (axiom of continuity of continuum mechanics) and any virtual field $\mathbf{v}^*$ (axiom of the PVP). In the present case, this yields

$$\rho\dot{\mathbf{v}} = \text{div}\sigma + \rho\mathbf{f}, \quad \text{at any } \mathbf{X} \in B; \tag{2.9}$$

$$\mathbf{n}.\sigma = \mathbf{T}^d, \quad \text{at } \mathbf{X} \in \partial B, \tag{2.10}$$

after application of the divergence theorem and localization.

The following is always true in this approach:

Remark 1 The *natural* boundary conditions associated with the field equation (2.9) always follow from the application of (2.8).

Remark 2 The symmetry of the stress tensor $\boldsymbol{\sigma} - \sigma = \sigma^T$—which can now be identified with the *Cauchy stress*, follows from the writing of (2.4) and the fact that (2.6) does not contain *here* any term linear in $\nabla\mathbf{v}^*$. But this is our choice as such a term is not ruled out in principle. In particular, such a term may exist in electromagnetic bodies that develop a nonvanishing ponderomotive couple (see my book [15]).

Remark 3 The application of (2.8) replaces the application of the two balance laws of linear momentum and moment of momentum, and the Cauchy "tetrahedron argument" traditionally exploited to introduce the Cauchy stress.

But the following recipe has been strictly followed:

Step 1: Select the kinematic space (2.2), i.e., select the gradient order of the theory. This, although not obvious to most witnesses, is a *constitutive assumption*. That is, how *precisely* we like to describe the velocity field of the continuous body at any material point and, by duality, the set of internal forces at play in B;
Step 2: Apply the restriction H1' or H1 here illustrated by (2.3) in order to write (2.4);
Step 3: Write (2.5) according to one's knowledge of physics (the acceleration term follows Galilei and Newton for classical mechanics). Here, no restriction of objectivity. This term will never be objective (on the contrary, if we may say so).
Step 4: Write (2.6) as a continuous linear form on V, but not necessarily the whole of V; Here, also no restriction of objectivity as the force contribution is external and given by some physical theory (e.g., gravitation, electromagnetic interactions).
Step 5: As is clear (2.7) can only be written as a continuous linear form on a set of velocities corresponding to the $(n-1)$-gradient order. The divergence theorem (or Stokes' theorem) is behind this restriction. This makes us say that Cauchy may not have needed any tetrahedron argument if he had known the divergence theorem due to George Green (cf. the result 2.10). However, Cauchy's tetrahedron argument was recently extended to the case including a bounding surface with discontinuous tangent plane and the concomitant appearance of edge contact forces along lines (dell'Isola and Seppecher [16], Noll and Virga [17], after an early work by Gurtin et al. [18]).

The remarkable fact is that the statement (2.8) potentially contains the formulation of all modern theories of complex and/or microstructured continua. Suffice it to follow the above given recipe when the space generalizing (2.2) is chosen. But the theory that has to be *thermomechanical* to include both reversible and irreversible effects and to be able to propose a complete description of the behavior would not be complete without the simultaneous statement of the first and second laws of thermodynamics for continua. These are written down as follows for the body B.

First law of thermodynamics:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(K(B) + E(B)\right) = P_{\text{ext}}(B, \partial B) + \dot{Q}(B, \partial B), \tag{2.11}$$

Second law of thermodynamics:

$$\frac{\mathrm{d}}{\mathrm{dt}}N(B) \geq \dot{\overline{N}}(B, \partial B), \tag{2.12}$$

wherein, classically,

$$K(B) = \int_B \frac{1}{2} \rho \mathbf{v}^2 \mathrm{d}B, \tag{2.13}$$

$$E(B) = \int_B \rho e \mathrm{d}B, \tag{2.14}$$

$$P_{\text{ext}}(B, \partial B) = P_{\text{vol}}(B) + P_{\text{surf}}(\partial B), \tag{2.15}$$

$$N(B) = \int_B \rho \eta \mathrm{d}B, \tag{2.16}$$

$$\dot{Q}(B, \partial B) = \int_B \rho h \mathrm{d}B - \int_{\partial B} q \mathrm{d}S, \tag{2.17}$$

$$\dot{\overline{N}}(B, \partial B) = \int_B \rho \overline{\eta} \mathrm{d}B - \int_{\partial B} \tilde{\eta} \mathrm{d}S. \tag{2.18}$$

Here, e is the internal energy density, η is the entropy density, h is the external heat source, q is the influx of heat at the boundary and in standard thermo*dynamics*, $q = \mathbf{n}.\mathbf{q}$. It is admitted that

$$\overline{\eta} \equiv h/\theta, \tilde{\eta} = q/\theta = \mathbf{n}.\mathbf{q}/\theta, \tag{2.19}$$

where θ is the thermodynamical temperature such that $\theta > 0$ and $\inf \theta = 0$.

The remarkable fact here stems from the expression (2.15), which corresponds to the last two contributions in (2.8) and also that in a well-constructed theory we have the following *identity* written for real fields (no asterisk):

$$P_{\text{inertia}}(B) = \frac{\mathrm{d}}{\mathrm{dt}} K(B). \tag{2.20}$$

Thanks to these, combination of (2.8) and (2.11) for real fields yields the so-called *theorem of internal energy* in the global form

$$\frac{\mathrm{d}}{\mathrm{dt}} E(B) + P_{\text{int}}(B) = \dot{Q}(B, \partial B). \tag{2.21}$$

The localization of this equation and its combination with the local form of (2.12) yield the now celebrated *Clausius–Duhem inequality* that imposes restrictions on the constitutive equations, for both reversible and irreversible processes. In transforming integrals and establishing the local forms, repeated use has been made of the conservation of mass written as the time constancy of the mass measure:

$$\frac{\mathrm{d}}{\mathrm{dt}} dm = 0, \quad dm = \rho \mathrm{d}B. \tag{2.22}$$

This concludes the introduction of the standard *first-order gradient* theory of continua as it is now sometimes presented in graduate lectures and books (especially in France).

3 Generalizations

Of course, the interest in the PVP would be limited if we stopped with the example of the previous section. The first main interest is to be found in the formulation of theories of special media or continua in which the velocity field (2.2) is specialized for some geometrical reasons, such as in slender bodies (plates, shells, rods). This will be examined last. First, we note that the very formulation (2.2) through (2.8) hints at different forms of generalization: (i) accounting for higher-order gradients than the first one in the extended set (2.2) and define the other powers than that of the "internal forces" in accordance; we have then to deal with so-called *gradient theories of continua*; (ii) describe the primary kinematics not by that of a point (as in 2.2) but that of a point equipped itself with a microstructure, itself rigid or deformable, so that internal degrees of freedom describing this additional structure are necessary; we then deal with *microstructured continua*; (iii) introducing additional physical fields on an equal footing with the basic kinematics present in (2.3); we then deal with *theories of coupled fields* in which the additional degrees of freedom (in the sense of field theory) are not of mechanical nature. The idea of gradient order can also be introduced for these new fields; (iv) the special cases of so-called *internal variables of state* must be singled out. We examine briefly these various cases and the difficulties they generate for each of them, in a row.

3.1 Higher-order gradient theories

A straightforward generalization of (2.2) consists in considering the set of velocities

$$V = \{\mathbf{v}, \nabla\mathbf{v}, \nabla\nabla\mathbf{v}, \ldots\}. \tag{3.1}$$

But this is just written for the sake of generality as we shall be **content** with the second gradient of $\mathbf{v}$, and it does not seem to be advisable to go further up (see below). Indeed, both $\nabla\mathbf{v}$ and $\nabla\nabla\mathbf{v}$ admit canonical decompositions in symmetric and skewsymmetric contributions, noting however the symmetry due to the two ∇ symbols so that, in principle, unless more kinematical constraints are applied, $\nabla\mathbf{v}$ and $\nabla\nabla\mathbf{v}$ admit at most 9 and 18 independent components. Germain [8] has presented a solution for the unambiguous representation of the various virtual powers to be involved in (2.8). We refer the reader to his work. But the following are essential remarks:

Remark 4 An objective set extracted from (3.1) is

$$V_{\text{obj}} = \left\{\mathbf{D} = (\nabla\mathbf{v})_S\, , \nabla\nabla\mathbf{v}\right\}, \tag{3.2}$$

having of course dimension $30 - 6 = 24$ according to the quotient operation by the distributor of rigid body motions. With this, one can write down a priori the corresponding P^*_{internal}. But in further transformations of the integrals, one has to account for a transformation not only to a smooth surface ∂B (standard divergence theorem) but also next of surface integrals to possible lines over that surface (Stokes'theorem), lines along which we may have a discontinuity of the tangent plane to ∂B. Thus, while dealing with the term issued from $\nabla\nabla\mathbf{v}$, we will have to account for the geometry of a surface not only at the first order—the local unit normal—but also at the second order (curvature describing how the unit normal changes direction along the nonflat surface). Tangential derivatives will be involved, and the necessity to consider edges (discontinuous tangent plane of the surface) and perhaps even apices may appear. This is far from Cauchy's original tetrahedron argument and the sole introduction of a stress tensor. The internal force dual to $\nabla\nabla\mathbf{v}$ is a third-order tensor (having some obvious symmetry) called the *hyperstress* $\mathbf{m}$. Such a quantity was introduced by Mindlin, Eshel and Tiersten [19,20] in a direct approach to the linear elasticity accounting for the gradient of strain (hence, a second-gradient theory in the present classification). The corresponding set of boundary conditions at a nonregular boundary was painstakingly devised. In the present formulation, complicated as they are, they follow from the PVP statement on an equal footing with the balance of momentum (see Germain [8]).

Remark 5 **On inertia**: The above construct has for only purpose to describe the local variation of the velocity field of an otherwise normal continuum with a greater accuracy. This is essential in the vicinity of the bounding surfaces (so-called boundary layers), in the neighborhood of singularities of the stress field and across transition layers with sharp field variations. Outside of these regions, the fields are much more regular and the usual first-order theory may be sufficient, from which we deduce the usefulness of the technique of matched asymptotic expansions. But, here, no additional field is considered (compare next section) as only the classical velocity is considered. Accordingly, the expression (2.5) is left unchanged. This also is essential as the identity (2.20) remains satisfied allowing us to pursue the general development just like in Sect. 2.

Remark 6 Of course, gradient theories abound in physics (see Maugin [21]) and may be qualified of *weakly nonlocal theories* [22]. They necessarily involve *characteristic lengths* if gradients of different orders are present simultaneously favoring the *dispersion* of waves. The effect of *capillarity* related to the curvature of surfaces (remember Laplace's law for bubble films) also emerges in a natural way. This naturally explains the early interest for such theories (gradient of density by Korteweg, elasticity in the pioneering work of Le Roux [23] and the more recent works of Casal, e.g., [24]). The latter author, in fact, considers a kind of PVP but he does not impose the objectivity argument to internal forces.

Remark 7 For higher-order gradient theories than the second in 3D physical space, we see that all new internal forces will have to satisfy *homogeneous* boundary conditions. That is why theories with such higher gradient order may be practically useless. In addition, one should not forget that they render all mathematical problems very stiff.

3.2 Mechanically structured media

Generalized continuum theories are often introduced, not through the notion of higher gradients, but by considering that each material point **X** in the continuum, usually subjected only to displacement, is also endowed with *internal degrees of freedom* that, in a sense, equip this point with a *structure*. If this structure can only rotate rigidly, we say that the medium is *micropolar* or a *Cosserat continuum* after the celebrated work of the Cosserat brothers of 1909 [25]; if this structure can deform affinely, the medium is called a *micromorphic continuum* or an *Eringen–Mindlin continuum* (our coinage). A special case corresponding to an elongation only is referred to as a *microstretch continuum*. This easily visualized classification is due to Eringen (see [26]) who has been instrumental in developing many aspects of these theories. The Cosserats themselves exploit not the PVP but a kind of variational principle (the so-called "Euclidean action") and present one of the first application of group theory in pre-Noetherian times.

We call $\chi\,(\mathbf{X}, t)$ the geometric object representing the relevant microstructure at material point **X** and Newtonian time t. This is in strict similarity with the classical motion $\mathbf{x}\,(\mathbf{X}, t)$ except that the object may be a scalar, a vector, a second-order tensor, etc. In the illustrative case of micropolar-Cosserat continua, it is an orthogonal tensor satisfying the usual conditions

$\chi^T = \chi^{-1}, det\chi = +1$ (SO(3) group). The time derivative of χ satisfies thus

$$\dot{\chi} = \omega.\chi, \omega = -\omega^T = \dot{\chi}\chi^T. \tag{3.3}$$

For a first-order gradient theory with respect to both the classical motion and the internal degree of freedom, the set generalizing (2.2) is

$$V = \{\mathbf{v}, \mathbf{D}, \Omega, \omega, \nabla\omega\}. \tag{3.4}$$

The quotient space (2.3) is enlarged to

$$V_{\text{obj}} = \{\mathbf{D}, \Omega - \omega, \nabla\omega\}. \tag{3.5}$$

Thus, (2.4) is replaced by

$$P^*_{\text{int}}\,(B) = -\int_B \left(\overline{\sigma}_{ij} D^*_{ji} + \tilde{\sigma}_{ij}\left(\Omega^*_{ji} - \omega^*_{ji}\right) + \mu_{kji}\omega^*_{ij,k}\right)\mathrm{d}B, \tag{3.6}$$

with

$$\overline{\sigma}_{ij} = \overline{\sigma}_{ji}, \tilde{\sigma}_{ij} = -\tilde{\sigma}_{ji}, \mu_{kji} = -\mu_{kij}, \tag{3.7}$$

that are, respectively, an intrinsic (symmetric) stress (six independent components at most), an intrinsic (skew-symmetric) stress (three independent components at most), and a so-called *couple-stress tensor* (nine independent components at most). The latter is a new concept here although it resembles a hyperstress. There is no problem in writing consistent generalizations of (2.6) and (2.7):

$$P^*_{\text{vol}}\,(B) = \int_B \left(\rho\mathbf{f}.\mathbf{v}^* + \rho c_{ij}\omega^*_{ji}\right)\mathrm{d}B,\; P^*_{\text{surf}}\,(B) = \int_{\partial B}\left(\mathbf{T}^d.\mathbf{v}^* + M^d_{ij}\omega^*_{ji}\right)\mathrm{d}S, \tag{3.8}$$

where $c_{ij} = -c_{ji}$ and $M^d_{ij} = -M^d_{ji}$ are applied couples in the bulk and at the surface of B.

Remark 8 **Microinertia**

Eringen [26] introduced the notion of microinertia associated with the internal rotation $\boldsymbol{\omega}$. In parallel with the usual mass conservation (2.22), there is thus a "conservation of microinertia" $\boldsymbol{J}$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{J}\,dm = 0, \quad \mathbf{S} = \left\{S_i = J_{ij}\overline{\omega}_j\right\}, \tag{3.9}$$

where $\mathbf{S}$ is an *intrinsic spin* per unit mass and $\overline{\omega}_j = \frac{1}{2}\varepsilon_{jkl}\omega_{lk}$.

Accordingly, the generalization of (2.5) reads

$$P^*_{\text{inertia}}(B) = \int_B \rho\left(\dot{\mathbf{v}}.\mathbf{v}^* + \dot{S}_i\overline{\omega}_i^*\right)\mathrm{d}B, \tag{3.10}$$

and the essential property (2.20) is checked for this generalized motion.

Application of the PVP (2.8) then yields the following local dynamical equations and *natural* boundary conditions:

$$\rho\dot{\mathbf{v}} = \operatorname{div}\sigma + \rho\mathbf{f}, \ \rho\dot{\overline{S}}_{ij} = \mu_{kij,k} + \sigma_{[ij]} + \rho c_{ij} \quad \text{at } \mathbf{X} \in B, \tag{3.11}$$

$$\mathbf{n}.\sigma = \mathbf{T}^d, \ \mathbf{n}.\mu = \mathbf{M}^d \text{ at } \mathbf{X} \in \partial B, \tag{3.12}$$

with a total (nonsymmetric) Cauchy stress given by

$$\sigma_{ij} = \overline{\sigma}_{ij} + \tilde{\sigma}_{ij}, \sigma_{(ij)} \equiv \overline{\sigma}_{ij}, \sigma_{[ij]} \equiv \tilde{\sigma}_{ij}. \tag{3.13}$$

Since all quantities appearing in $(3.11)_2$ and $(3.12)_2$ are skewsymmetric, the dual of this equation can be taken (by applying the alternation symbol ε_{pij}) yielding a vectorial equation of an axial nature. These equations or the couple (3.11)–(3.12) are the basic field equations of the mechanics of 3D micropolar or Cosserat continua. The construction of the model can be pursued just like in Sect. 2 since the identity (2.20) holds good. We refer to the original works for these developments and many applications (e.g., in Eringen [27] and [28]) for deformable solids and fluids, respectively). The most notable feature of this modeling obviously is the nonsymmetry of the Cauchy stress. The rarity of the occurrence of the external couples such as c_{ij} and $\mathbf{M}^d$ is to be emphasized, the classical example provided by such quantities (in the bulk) being of electromagnetic origin, cf. [15]. In case of small strains and small angular excursions of the internal structure, displacement and a vectorial angle facilitate the formulation, for which more than often the micropolar inertial tensor is taken as isotropic and thus characterized by a unique scalar.

Remark 9 **Constrained Cosserat media**

Sometimes, a constraint such as the equality of macro and microrotations is imposed, making the central term in (3.6)—with a skewsymmetric stress tensor—to disappear. But, simultaneously, since macrorotation is none other than the skew part of the velocity gradient, we have then

$$\omega_{ij,k} = v_{[i,j],k} = \frac{1}{2}\left(v_{i,jk} - v_{j,ik}\right). \tag{3.14}$$

so that the theory in principle transforms to a special case of second-gradient theory of the classical motion called the *indeterminate couple-stress theory*. This theory is ill conceived because it makes the stress to be not only constitutive but to depend on inertia, the latter taking a strange form indeed (see the discussion in Eringen [19], pp. 698–701)

Remark 10 **Micromorphic media**

In this case where the structure of a material point X is akin to that of a little deformable body, the approach using the present PVP approach in more or less the present style was established by Germain [9,12]. The number of internal degrees of freedom is obviously larger by six compared to the micropolar case. Accordingly, there must exist an additional internal force with the same number of independent components (and noted s_{kl} in Eringen [27] Eq. (2.2.25)): This is a symmetric second-order tensor that contributes to an equation of the type of $(3.11)_2$ that is no longer a pure skewsymmetric equation. Trying to define micromorphic theories of higher order, Germain [9,12] had noted a difficulty with the definition of microinertia since only an identity such as (2.20) allows one to proceed without further ado with thermomechanics and real velocity fields.

3.3 Physically structured media

Electro-magneto–mechanical interactions in continua generally manifest themselves through nonzero ponderomotive forces and couples. The presence of the latter hints at the existence of a *nonsymmetric* Cauchy stress (compare foregoing paragraphs). This is indeed one of the arguments that kindled the interest for Cosserat continua in the 1960s–1970s. But there is better than that because there exist electric and magnetic states of matter, the so-called *ferroïc* states, such as ferroelectricity, ferromagnetism, antiferromagnetism, etc, where a true *nonmechanical microstructure* exists even in the absence of applied (electric or magnetic) fields. The author has been particularly active in this field. It was his idea to apply the above-sketched out PVP formulation to these cases and to show that this was in fact the best method for their unambiguous continuum formulation. Indeed, in a ferroelectric body exhibiting a density of permanent electric dipoles, it is normal to consider as a primitive quantity the electric dipole density—or electric polarization $\mathbf{p}$—per unit mass, since this is a material quantity. The density of magnetic dipoles—or magnetization $\boldsymbol{\mu}$—plays an identical role in ferromagnetism.

Considering first the case of *ferroelectrics* and the fact that an order is exhibited for the electric dipoles—hence the introduction of $\nabla\mathbf{p}$ to measure the deviation from a strict parallel alignment typical of ferroelectric ordering (in the Landau-Ginzburg tradition)—, we propose then for a classical deformable medium to generalize (2.2) in the following manner:

$$V = \{\mathbf{v}, \mathbf{D}, \Omega, \dot{\mathbf{p}}, \nabla\dot{\mathbf{p}}\}. \tag{3.15}$$

Writing the generalization of the objective set (2.3) in order to write down a priori a continuous linear form on an objective set of virtual velocities consists in building a *minimal* (i.e., made of linearly independent members) objective basis form of (3.15). This is easily shown to be given by (cf. Proof in Maugin [7])

$$V_{\text{obj}} = \left\{\mathbf{D}, \hat{\mathbf{p}}, \hat{\Pi}\right\}, \tag{3.16}$$

wherein

$$\hat{\mathbf{p}} = D_J\mathbf{p} := \dot{\mathbf{p}} - \Omega.\mathbf{p},\ \hat{\Pi} \equiv (\nabla\dot{\mathbf{p}})^T - \Omega.(\nabla\mathbf{p})^T = \left(D_J(\nabla\mathbf{p})^T\right) + (\nabla\mathbf{p})^T.\mathbf{D}, \tag{3.17}$$

where D_J denotes the Jaumann derivative. Introducing co-factors, we can write the generalization of (2.4) as

$$P^*_{\text{int}}(B) = -\int_B \left(\overline{\sigma}_{ij}D^*_{ji} - \rho\mathbf{E}^L.\hat{\mathbf{p}}_* + \overline{E}^L_{ij}\hat{\Pi}^*_{ji}\right)\mathrm{d}B, \tag{3.18}$$

where $\overline{\sigma}_{ij} = \overline{\sigma}_{ji}$ is a symmetric intrinsic stress, $\mathbf{E}_L$ is a local electric field that accounts for the interactions between deformable matter and the distribution of electric dipoles and $\overline{E}^L_{ij}$ is a general second-order tensor representative of interactions between neighboring electric dipoles. Constitutive equations must be formulated for these three "internal forces." The theory is closed by writing a power of inertial forces containing a polarization inertia (in a continuum theory, a physical datum sometimes computed from a microscopic model), so that the generalization of (2.20) holds good, and introducing in the powers of external forces, the energy terms, forces and couples coming from the analysis of electromagnetic continua (see [15]). Anyhow, the result of the application of the PVP to this theory yields easily understandable equations. That is, quoting just the result:

$$\rho\dot{\mathbf{v}} = \operatorname{div}\sigma + \rho\mathbf{f} + \rho\mathbf{f}^{em} \text{ in } \mathbf{X} \in B \tag{3.19}$$

$$\rho d^e\ddot{\mathbf{p}} = \mathbf{E}^C + \mathbf{E}^L + \rho^{-1}\operatorname{div}\overline{\mathbf{E}}^L \text{ in } \mathbf{X} \in B, \tag{3.20}$$

along with the natural boundary conditions

$$\mathbf{n}.\sigma = \mathbf{T}^d + \mathbf{T}^{em},\ \mathbf{n}.\overline{\mathbf{E}}^L = \mathbf{E}^d, \tag{3.21}$$

where $\mathbf{f}^{em}$ and $\mathbf{T}^{em}$ have known expressions from electromagnetism (cf. [15]), $\mathbf{T}^d$ and $\mathbf{E}^d$ are data, $\mathbf{E}^C$ is the convected electric field (also called electromotive intensity) that appears in Maxwell's equations in moving matter, d^e is the polarization inertia (here taken as a scalar) and the *nonsymmetric* Cauchy stress is given by

$$\sigma_{ij} = \overline{\sigma}_{ij} + \rho E^L_{[j}p_{i]} - \overline{E}^L_{[j|k|}p_{i],k}. \tag{3.22}$$

The remarkable equation here is Eq. (3.20), which looks somewhat like a motion equation. It shows the essentially different ontological status of the electric fields $\mathbf{E}^C$ and $\mathbf{E}^L$, even in the case when typical ferroelectric effects are discarded, i.e., with $d^e = 0$ and $\overline{\mathbf{E}}^L$ set to zero, in which case (3.2) reduces to the balance equation among these two fields:

$$\mathbf{E}^C + \mathbf{E}^L = \mathbf{0}, \tag{3.23}$$

and $\mathbf{E} + \mathbf{E}^L = \mathbf{0}$ in the quasi-electrostatics of normal dielectrics. It is such a reduction which yields the usual fact that there exists a *direct* constitutive relationship between the Maxwellian field $\mathbf{E}$ and the electric polarization $\mathbf{P} = \rho\mathbf{p}$, what is not true in ferroelectrics according to (3.20). Of course, Maxwell's equations themselves are not deduced from the PVP while at least a couple of them would be deduced from a variational principle accounting for the second couple by introducing electromagnetic potentials.

Remark 11 **The case of ferromagnetism**

One can be tempted to consider the case of ferromagnetism by analogy with ferroelectricity. But the analogy is misleading because a magnetization is altogether different from an electric polarization. A magnetization or density of magnetic dipoles is an axial vector, which itself is proportional to a spin via the celebrated gyromagnetic ratio γ: $\mathbf{s} = \gamma^{-1}\mu$, where μ is the magnetization per unit mass. Furthermore, it is assumed in micromagnetism that in each magnetic domain, magnetization has reached saturation, i.e., $|\mu| = \mu_S$. Thus, the vector $\boldsymbol{\mu}$ can only be subjected to a *precession* at each material point $\mathbf{X}$, i.e., $\dot{\mu} = \omega \times \mu$. It follows from this that

$$\dot{\mathbf{s}}.\omega = \gamma^{-1}\dot{\mu}.\omega = 0. \tag{3.24}$$

A couple such as $\dot{\mathbf{s}}$ that expends no power in a **real** precessional velocity field is called a *d'Alembertian inertia couple*. Accordingly, it can be introduced in the dynamical setting of the theory only through a formulation considering nonzero **virtual** velocities. Therefore, in the present case, (2.2) is replaced by

$$V = \{\mathbf{v}, \mathbf{D}, \Omega, \dot{\mu} = \omega \times \mu, \nabla\dot{\mu}\}, \tag{3.25}$$

and thus,

$$V_{\text{obj}} = \left\{\mathbf{D}, \hat{\mathbf{m}}, \hat{M}_{ij}\right\}, \tag{3.26}$$

with (compare (3.17))

$$\hat{\mathbf{m}} = D_J\mu := \dot{\mu} - \Omega.\mu,\ \hat{M} \equiv (\nabla\dot{\mu})^T - \Omega.(\nabla\mu)^T = \left(D_J(\nabla\mu)^T\right) + (\nabla\mu)^T.\mathbf{D} \tag{3.27}$$

and

$$P^*_{\text{inertia}}(B) = \int_B \left(\dot{\mathbf{v}}.\mathbf{v}^* + \gamma^{-1}\dot{\mathbf{s}}.\omega^*\right)\rho \mathrm{d}B,\ (\dot{\mu})^* = \omega^* \times \mu. \tag{3.28}$$

Similarly to (3.18), we introduce the generalized "internal" forces $\mathbf{B}^L$ and $\overline{B}^L_{ij}$, so that (3.18) is replaced by

$$P^*_{\text{int}}(B) = -\int_B \left(\overline{\sigma}_{ij}D^*_{ji} - \rho\mathbf{B}^L.\hat{\mathbf{m}}^* + \overline{B}^L_{ij}\hat{M}^*_{ji}\right)\mathrm{d}B. \tag{3.29}$$

We just quote the final result of the application of the PVP to this case. Equations (3.19) through (3.20) are replaced by

$$\rho\dot{\mathbf{v}} = \text{div}\sigma + \rho\mathbf{f} + \rho\mathbf{f}^{em} \text{ in } \mathbf{X} \in B, \tag{3.30}$$

$$\dot{\mu} = \omega \times \mu, \text{ in } \mathbf{X} \in B, \tag{3.31}$$

and (*natural* boundary conditions)

$$\mathbf{n}.\sigma = \mathbf{T}^d + \mathbf{T}^{em},\ \left(\mathbf{n}.\overline{\mathbf{B}}^L\right) \times \mu = \mathbf{B}^d, \tag{3.32}$$

wherein

$$\sigma_{ij} = \overline{\sigma}_{ij} + \rho B^L_{[j}\mu_{i]} \overline{B}^L_{[j|k|}\mu_{i],k}, \tag{3.33}$$

$$\omega = -\gamma \mathbf{B}^{\text{eff}}, \mathbf{B}^{\text{eff}} := \mathbf{B} + \mathbf{B}^L + \rho^{-1}\text{div}\,\overline{\mathbf{B}}^L. \tag{3.34}$$

In these equations, $\mathbf{f}^{em}$ and $\mathbf{T}^{em}$ are the reduced form of the ponderomotive force and its surface contribution in quasi-magnetostatics (sufficient for this problem), $\mathbf{B}^L$ is a local magnetic induction representative of the interaction between the magnetic dipole distribution and the crystal lattice and $\overline{\mathbf{B}}^L$ is a general second-order tensor representative of the interactions between neighboring magnetic spins, the so-called *exchange* interactions in micromagnetism. More than often the source $\mathbf{B}^d$ in $(3.32)_2$ vanishes so that the latter reduces to the boundary conditions of the Robin type

$$\mathbf{n}.\overline{\mathbf{B}}^L + \lambda\mu = 0, \tag{3.36}$$

where λ is a proportionality coefficient varying between zero and infinity ($\lambda = 0$: Dirichlet condition; $\lambda \to \infty$: vanishing spin at the boundary). The above recalled construction of the theory of deformable ferromagnets was first given by Maugin [29]—in oral presentations in 1973. Note that the internal forces considered may be conservative or dissipative or contain both types of contributions. The theory is completed by the thermodynamical statements such as in Sect. 2, noting that (2.20) holds good with a kinetic energy reduced to that of the usual displacement. Only the present type of construction allows a straightforward generalization to the case of antiferromagnets (two continua of co-existing opposed magnetic spins) and ferrimagnets (the local magnetization is the sum of a series of magnetic spin collections according to L.Néel's model of ferrimagnetism; see Maugin [30]). In case of paramagnetism or soft ferromagetism, Eqs. (3.31) and (3.34) reduce to a "balance " equation between Maxwellian magnetic induction and local magnetic induction :

$$\mathbf{B} + \mathbf{B}^L = \mathbf{0}, \tag{3.37}$$

so that the latter disappears completely from the formulation in the absence of magnetic relaxation and hysteresis.

Examples from this section were used in symposia presentations by Paul Germain in the 1970s–1980s (e.g., [13]) in order to popularize the method of virtual power for sufficiently complex cases.

3.4 Other theories: dielectrics with electric quadrupoles, liquid crystals,…

Other theories involving coupled fields (i.e., mechanical effects coupled to other physical effects) have been formulated on the basis of the present scheme of the PVP. In particular, we note the cases of *elastic dielectrics* accounting not only for electric dipoles but also for *electric quadrupoles* (the thermodynamical dual of an electric quadrupole is a gradient of electric field, so that the theory has some points in common with a theory involving polarization gradients—see above) and also the theory of *liquid crystals* in which the additional internal degree of freedom is represented by a vector field (a so-called "director" $\mathbf{n}(\mathbf{X}, t)$) whose gradients need to be introduced. The inertia associated with $\mathbf{n}$ is not a very well-defined physical quantity and is often discarded as relaxation effects related to $\mathbf{n}$ are more relevant—a matter of time scales. These two cases are documented in the review [7].

But it may happen that the physical field entered as an additional internal degree of freedom has a spatial range of interaction that may resonate with the material microstructure. This means that the gradient order for the deformation field and that for this new field may be different. This is the case in some refined theories of micromagnetism, especially in the neighborhood of limiting surfaces, i.e, in boundary layers. Indeed, consider the natural spin boundary condition $(3.32)_2$. The applied field in the right-hand side is a couple. But the classical mechanical behavior chosen in Remark 11 is a first-order gradient that does not allow for a counter mechanical response to that surface couple. Following an idea of Harry F. Tiersten, Collet and the author [31] have formulated a theory that is first-order gradient in the magnetic spin or magnetization and second-order gradient for deformation. Of course, the coupled boundary conditions take a formidable form, and the resulting problems have to be envisaged only in the immediate vicinity of boundaries and solved asymptotically.

4 More exotic cases: discontinuous fields, mixtures, etc

4.1 Discontinuity surfaces

In many problems (propagation of shock waves, phase-transition fronts, etc), one has to deal with fields that are not sufficiently continuous to allow for all manipulations done previously assuming that all derivatives of any order existed. These fields may now suffer finite discontinuities at the crossing of fixed or moving interfaces. For fixed interfaces, there is no problem as the formulation is applied on both sides of the fixed interface. For discontinuity surfaces moving through the material, one has to account for the motion of the interface and the equations generated by this motion, whether virtual or real. The solution to this in the PVP formulation was given by Daher and Maugin [32] considering a virtual motion of the discontinuity surface along with a virtual motion of material points. Basically, these authors distinguish between the two regions separated by the discontinuity surface Σ, and virtual velocity fields have to be introduced for material points on both sides and the surface itself. If that surface in addition to moving relatively to the material particles in the bulk is itself endowed with its own material particles and is equipped with density and inertia and some deformation properties—although of mathematically zero thickness—, then an additional velocity field must be introduced for this one; In all, with an obvious notation, we have velocities $\mathbf{v}^{\pm}$, $\boldsymbol{\nu}$ and $\hat{\mathbf{v}}$; $\mathbf{v}^{\pm}$ of the bulk material points is the uniform limit in approaching Σ from its "plus" and "minus" sides, the unit normal $\mathbf{N}$ to Σbeing oriented from the minus to the plus side. We denote by $[A] = A^{+} - A^{-}$ the finite jump of A across Σ; $\boldsymbol{\nu}$ is the absolute velocity of Σ through the material body, and $\hat{\mathbf{v}}$is the velocity of a material point that belongs to Σ. It is tangential to Σ. All quantities pertaining to $\Sigma$are also noted with a superimpose caret.

First, we consider *the case where Σ carries no material properties*. Then, in writing down the virtual power expended by internal forces and inertial forces, we must distinguish between terms in the bulk (on each side of Σ) and terms at Σ. The first terms are standard (e.g., for a first-gradient theory). For the terms at Σ, Daher and the author were led to considering the following relative (objective) velocity set at Σ:

$$V_{\text{rel}}(\Sigma) = \left\{\mathbf{v}^{+} - \nu, \mathbf{v}^{-} - \nu\right\}. \tag{4.1}$$

Then the virtual powers of internal forces and inertial forces due to the presence of Σ are given by (cf. Daher and Maugin [32])

$$P^{*}_{\text{int}}(\Sigma) = -\int_{\Sigma} \left[\tau . \left(\mathbf{v}^{*} - \nu^{*}\right)\right] \mathrm{d}\Sigma, \tag{4.2}$$

and

$$P^{*}_{\text{inertia}}(\Sigma) = \int_{\Sigma} m\, [\mathbf{v}].\nu^{*} \mathrm{d}\Sigma, \tag{4.3}$$

where $m = \rho\,(\mathbf{v} - \nu)\,.\mathbf{N}$ is the mass flux density across Σ, and $\boldsymbol{\tau}$ is an *"internal" traction* at Σ. This, in fact, as shown by these authors, is equal to the normal component of the Cauchy stress at Σ. The field equation generated by the PVP at Σ first reads as

$$m\,[\mathbf{v}] = [\tau], \tag{4.4}$$

which is shown to be none other than the classical jump equation for linear momentum at a moving discontinuity surface Σ:

$$\mathbf{N}.\,[\rho\,(\mathbf{v} - \nu) \otimes \mathbf{v} - \sigma] = \mathbf{0}. \tag{4.5}$$

The rest of the formulation is straightforward (see [32]).

The *case when Σ carries its own material properties* is much less trivial. We refer to the original work [32] for details and quote only some of the final results with a view to emphasizing the difficult points. With superimposed carets denoting all quantities pertaining to the material particles that belong to Σ, it is shown that (4.4) is replaced by

$$\hat{\rho}\frac{\hat{\mathrm{d}}\hat{\mathbf{v}}}{\mathrm{dt}} + \left[m\left(\mathbf{v} - \hat{\mathbf{v}}\right)\right] = [\tau] + \left(\hat{\nabla} + 2\Omega\mathbf{N}\right).\hat{\sigma} + \hat{\mathbf{f}}, \tag{4.6}$$

in which we easily identify the new terms. Here, $\hat{\nabla}$ is a surface nabla, Ω is the scalar curvature of Σ, $\hat{\sigma}$is a surface stress and $\hat{\mathbf{f}}$ in principle is a surface external force. The final form taken by (4.6) reads (compare to (4.5):

$$\hat{\rho}\frac{\hat{\mathrm{d}}\hat{\mathbf{v}}}{\mathrm{dt}}+\mathbf{N}.\left[\rho\left(\mathbf{v}-\nu\right)\otimes\left(\mathbf{v}-\hat{\mathbf{v}}\right)-\sigma\right]=\hat{\nabla}.\hat{\sigma}+\hat{\mathbf{f}}. \tag{4.7}$$

It is the next step in the construct that causes some difficulty. Indeed, one would like to have an identity such as (2.20) to have an easy combination of the PVP written for real velocities and the first law of thermodynamics. The latter of course reads:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(E+K\right)+\frac{\hat{\mathrm{d}}}{\mathrm{dt}}\left(\hat{E}+\hat{K}\right)=P_{\mathrm{external}}\left(B-\Sigma,\Sigma,\partial B-\Sigma\right)+\dot{Q}\left(B-\Sigma,\Sigma,\partial B-\Sigma\right). \tag{4.8}$$

The combination of the PVP and (4.8) is shown to yield the global *internal* energy theorem in the form:

$$\frac{\mathrm{d}}{\mathrm{dt}}E+\frac{\hat{\mathrm{d}}}{\mathrm{dt}}\hat{E}+P_{\mathrm{int}}\left(B^{+},B^{-},\Sigma\right)+\dot{K}_{\mathrm{excess}}\left(\Sigma\right)=\dot{Q}, \tag{4.9}$$

where the last quantity introduced is the *excess of kinetic energy* given by

$$\dot{K}_{\mathrm{excess}}\left(\Sigma\right)=\int_{\Sigma}\left[\frac{1}{2}m\left(\mathbf{v}-\hat{\mathbf{v}}\right)^{2}\right]\mathrm{d}\Sigma. \tag{4.10}$$

The above theory, complicated as it is, was nonetheless exploited by Daher and Maugin to study first general electromagnetic continuous bodies including propagating discontinuity surfaces [33] and then deformable semiconductors including interfaces [34] where various recombinations occur between different constituents (charged particles, holes). To our knowledge, this is so far the most advanced and complex application ever made of the present formulation of the PVP.

4.2 Mixtures of continuous media

The formulation of a continuum theory of mixtures has always posed fundamental questions (see Truesdell [35]) not only with respect to thermodynamical notions such as temperature (is it defined for each constituent or for the whole?) but also as concerns kinetic quantities such as the kinetic energy. Of course, temperature and kinetic energy (of agitation) are related notions in the kinetic theory of gases so that these companion difficulties should not come as a surprise. In turn, because of the looked for identity (2.20), these difficulties are carried to the formulation of the power of inertial forces in the PVP formulation. Early attempts by the author in the 1970s failed. However, returning to this problem some twenty-five years later with the help of Sara Quiligotti and Franceso dell'Isola provided a more satisfactory result [36]. Remember that in this type of continuum theory all constituents are supposed to be simultaneously present, but in various evolving proportions, at each material point. Let subscript α labels all quantities pertaining to constituent α. We will specialize to two constituents for this reminder.

For a first-order gradient theory the basic set of velocities will be

$$V=\{\mathbf{v}_{\alpha},\nabla\mathbf{v}_{\alpha}\}. \tag{4.11}$$

The corresponding elementary power of internal forces then reads

$$p_{\mathrm{int}}=-\sum_{\alpha}\left(\pi_{\alpha}.\mathbf{v}_{\alpha}+\overline{\sigma}_{\alpha}:(\nabla\mathbf{v}_{\alpha})^{T}\right), \tag{4.12}$$

where the π_{α} is a vector of exchange of momentum, and the $\overline{\sigma}_{\alpha}$ is a partial (or peculiar) intrinsic stress tensor. The principle of "rigidification" (equivalent to the requirement of objectivity for generalized internal forces) requires that

$$\sum_{\alpha}\pi_{\alpha}=\mathbf{0},\ \sum_{\alpha}\overline{\sigma}_{\alpha}\text{ is symmetric}. \tag{4.13}$$

The important point, however, is the formulation of the kinetic energy. We set

$$\hat{\mathbf{v}}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, \tag{4.14}$$

where the mixture velocity $\mathbf{v}$ is such that

$$\mathbf{v} := \sum_\alpha \xi_\alpha \mathbf{v}_\alpha,\, \xi_\alpha := \rho_\alpha/\rho, \tag{4.15}$$

where ξ_α is the mass fraction associated with the α constituent. Of course,

$$\sum_\alpha \xi_\alpha = 1. \tag{4.16}$$

The kinetic energy of the mixture as a whole has a classical expression. By calculations accounting for above given expressions, we obtain (mixture of two constituents)

$$K = \frac{1}{2}\rho\mathbf{v}^2 = \sum_\alpha \frac{1}{2}\rho_\alpha(\mathbf{v}_\alpha)^2 + K_{\text{excess}},\, K_{\text{excess}} = \frac{1}{2}\rho\left(\hat{\mathbf{v}}_1.\hat{\mathbf{v}}_2\right). \tag{4.17}$$

It is thanks to this result that we can state the following important result [27]: The time derivative of the kinetic energy associated with any smooth region B_t of the current configuration enveloped by a migrating surface, which follows the motion of the mixture as a whole, equals the integral of the power expended by inertial forces on the mean velocity field $\mathbf{v}$. That is,

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{B_\tau}\frac{1}{2}\rho\mathbf{v}^2\mathrm{d}B_\tau\right\}_{\tau=t} = \int_{B_t}\rho\mathbf{a}.\mathbf{v}\mathrm{d}B_t. \tag{4.18}$$

The (classical) acceleration $\mathbf{a} := d\mathbf{v}/\mathrm{d}t$ is *not* the mean of peculiar accelerations $\mathbf{a}_\alpha := d_\alpha\mathbf{v}_\alpha/dt$. As a matter of fact, the following can be proved:

$$\mathbf{a} = \sum_\alpha(\xi_\alpha\mathbf{a}_\alpha) - \rho^{-1}\mathrm{div}\,\tilde{\sigma}, \tag{4.19}$$

where the symmetric tensor $\tilde{\sigma}$ is the *apparent stress* due to diffusive motion given by

$$\tilde{\sigma} = \sum_\alpha \rho_\alpha\hat{\mathbf{v}}_\alpha \otimes \hat{\mathbf{v}}_\alpha. \tag{4.20}$$

Then, the principle of virtual power

$$P^*_{\text{inertia}}(B_t) = P^*_{\text{int}}(B_t) + P^*_{\text{vol}}(B_t) + P^*_{\text{surf}}(\partial B_t), \tag{4.21}$$

is built with the following contributions:

$$P^*_{\text{inertia}}(B_t) = \int_\alpha\left(\sum_\alpha \rho_\alpha\mathbf{a}.\mathbf{v}^*_\alpha\right)\mathrm{d}B_t, \tag{4.22}$$

$$P^*_{\text{int}}(B_t) = -\int_{B_t}\sum_\alpha\left(\pi_\alpha.\mathbf{v}^*_\alpha + \overline{\sigma}_\alpha : \left(\nabla\mathbf{v}^*_\alpha\right)\right)\mathrm{d}B_t, \tag{4.23}$$

$$P^*_{\text{vol}}(B_t) = \int_{B_t}\left(\sum_\alpha \rho_\alpha\mathbf{f}.\mathbf{v}^*_\alpha\right)\mathrm{d}B_t,\, \rho\mathbf{f} = \sum_\alpha \rho_\alpha\mathbf{f}, \tag{4.24}$$

and

$$P^*_{\text{surf}}(\partial B_t) = \int_{\partial B_t}\left(\sum_\alpha \xi_\alpha\mathbf{T}^d.\mathbf{v}^*_\alpha\right)\mathrm{d}S_t. \tag{4.25}$$

By localization and for any virtual field, we obtain the following local peculiar equations of linear momentum and associated natural boundary condition ($\sigma_\alpha = \overline{\sigma}_\alpha$):

$$\rho_\alpha \mathbf{a} = \text{div}\,\sigma_\alpha + \rho_\alpha \mathbf{f} - \pi_\alpha, \tag{4.26}$$

and

$$\mathbf{n}.\sigma_\alpha = \xi_\alpha \mathbf{T}^d. \tag{4.27}$$

Summing over α, it is checked that the whole mixture has a standard equation of linear momentum and associated boundary condition:

$$\rho \mathbf{a} = \text{div}\,\sigma + \rho \mathbf{f}, \mathbf{n}.\sigma = \mathbf{T}^d, \sigma = \sum_\alpha \sigma_\alpha. \tag{4.28}$$

The splitting of the boundary condition is a discussed matter.

For completeness, we note that if we set

$$\tilde{\sigma}_\alpha := \sigma_\alpha + \xi_\alpha \tilde{\sigma}, \tag{4.29}$$

we can rewrite the Cauchy stress of the mixture as

$$\sigma = \sum_\alpha \left(\tilde{\sigma}_\alpha - \rho_\alpha \hat{\mathbf{v}}_\alpha \otimes \hat{\mathbf{v}}_\alpha\right), \tag{4.30}$$

perhaps a more familiar formula (cf. [35]).

Note that conclusions drawn here might be consistently extended to higher-order gradient theories, provided that meaningful physical interpretation of further emerging boundary conditions can be taken for granted.

4.3 Dissipative media with internal variables of state

According to the accepted definition of internal variables of state that follows the vision of Kestin [37]—see also the book of Maugin [38]—such variables are introduced to describe phenomenologically complex underlying dissipative processes. Their "essence" is to be purely dissipative, and while they can be identified experimentally and measured to some extent, they cannot be controlled directly by means of local body or surface forces. They will show up only in the Clausius–Duhem inequality when one has specified the functional dependence of the free energy density. Their dual forces are given by Gibbs relations (definition of partial derivatives of the free energy). Accordingly, they do not modify before hand the general statements (PVP and laws of thermodynamics); in particular, they cannot appear in the expression of the virtual power of both internal and external forces. Some authors (e.g., Frémond and Nédjar [39]) do introduce them in these expressions, but this, in our opinion, is principally wrong because of the just-recalled definition. The same authors claimed that they also introduce gradients of these variables (which are no longer "internal") in the various powers for the first time. This is obviously wrong if we think of the cases of ferroelectrics and ferromagnets introduced some thirty-eight years ago (and reviewed in the long paper of 1980 [7]). In their case, the variable introduced is a scalar (damage parameter indeed usually considered as an internal variable of state) so that the algebra is much more simple than in the case where we had to write Eqs. (3.18) or (3.29). A discussion about the discrepancy and agreement with the theory of these false "internal variables" and true internal degrees of freedom is given in a recent work (Maugin [40]).

5 Approximate kinematic fields: refined theories of structural members

Along a different line from the previous sections, and quite easier to apply than some techniques such as the asymptotic zoom technique of Gold'enweizer [41], Ciarlet and Destuynder [42,43], the PVP is also an efficient and safe way to construct theories of specific structural members such as rods, plates and shells at different degrees of approximation. To achieve this, rather than adding new generalized kinematic fields (hence, enlarging the set of relevant virtual velocities), remaining in a purely mechanical frame we can specialize at will the basic field (2.2). This matter was dealt with in detail by Germain himself [44]. Since this is not our field of expertise, we simply remind the reader of some elementary facts and provide a few examples. For instance, in order to formulate the so-called *natural theory of plates* contained between the planes $x_3 = -h_1$ and $x_3 = +h_2$, the 3D field of virtual velocities of the set (2.1) can be reasonably described by the expressions (Greek indices take values 1 and 2 only for in-plane components)

$$v_\alpha^* = u_\alpha^* (x_1, x_2) + x_3 l_\alpha^* (x_1, x_2), \, v_3^* = w^* (x_1, x_2). \tag{5.1}$$

The spin associated with l^* is given by $\omega^* = \mathbf{e}_3 \times l^*$. The motion described by (5.1) is "*five dimensional*." It is, therefore, equivalent to the motion of a micropolar-Cosserat continuum (cf. Sect. 3.2) with the kinematic constraint that the spin is always contained in the plane$x_3 = 0$. The remaining components of (2.3) are given by

$$D_{\alpha\beta}^* = d_{\alpha\beta}^* + x_3 K_{\alpha\beta}^*, 2D_{\alpha 3}^* = \beta_\alpha^*, D_{33}^* = 0, \tag{5.2}$$

where we have defined

$$d_{\alpha\beta}^* = u_{(\alpha,\beta)}^*, K_{\alpha\beta}^* = l_{(\alpha,\beta)}^*, \beta_\alpha^* = l_\alpha^* + w_{,\alpha}^*. \tag{5.3}$$

We emphasize the constraint of *plane strains* represented by the third of (5.2). As to (5.3), it provides the objective set of velocities $V_{obj}$of the *natural theory of plates*. A rigidifying virtual motion corresponds to the simultaneous vanishing of the three quantities defined in that equation. Accordingly, the consistent virtual power of *internal forces* of the present theory is written as a continuous linear form on that set, i.e.,

$$P_{\rm int}^*(S) = -\int\limits_S \left(N_{\alpha\beta} d_{\alpha\beta}^* + \Pi_{\alpha\beta} K_{\alpha\beta}^* + Q_\alpha \beta_\alpha^* \right) \mathrm{d}S, \tag{5.4}$$

where S denotes the surface of the plate, and the three cofactors introduced are, respectively, the in-plane stress tensor (membrane forces; normal and in-plane shear), the tensor of bending torsion (bending and twisting moments) and the normal shear force. The theory (at least in quasi-statics) is completed by the writing down of the power of body forces over S and of the power of boundary forces along the contour ∂S. We do not elaborate further as the writing of (5.4) was indeed the crucial point.

5.0.1 Particular and more general cases

Equation $(5.2)_3$ was already an example of mathematical constraint. Another example is provided by the so-called *Love–Kirchhoff theory of plates* in which one sets $\beta_\alpha^* = 0$. With this, any normal section to the plate remains normal to the deformed material surface S. Eliminating l_α^* between the third and second of (5.3) results in the reduced set

$$d_{\alpha\beta}^* = u_{(\alpha,\beta)}^*, K_{\alpha\beta}^* = -w_{,\alpha\beta}^*, \tag{5.5}$$

while (5.4) simplifies to

$$P_{int}^*(S) = -\int\limits_S \left(N_{\alpha\beta} d_{\alpha\beta}^* - M_{\alpha\beta} w_{,\alpha\beta}^* \right) \mathrm{d}S. \tag{5.6}$$

The Q_β^* present in (5.4) becomes a Lagrange multiplier accounting for the newly introduced constraint. But it does not appear in the direct approach (5.6). The case exemplified by (5.5)–(5.6) may also be viewed as an obvious *second-gradient theory*. A drawback of this is that one may have to introduce finite normal forces at corners (if any) along the contour ∂S. But the resulting theory is poorer than the "natural" one. On the contrary, one may envisage a richer (from both kinematical and force viewpoints) than the natural theory based on (5.1) by starting with a more complicated representation of the basic 3D field v_i^*. This yields improved theories of plates. An example of such sophistication is given by Touratier [45] where (5.1) is replaced by

$$v_\alpha^* = u_\alpha^* (x_1, x_2) - x_3 w_{,\alpha}^* (x_1, x_2) + f(x_3) \gamma_\alpha^{0*}, v_3^* = w^* (x_1, x_2), \tag{5.7}$$

where the new 2D vector field γ_α^0 accounts for *shear deformation.* The function $f(x_3)$ can be chosen so as to satisfy exactly boundary conditions on the top and bottom surfaces of the plate (e.g., a sinusoidal function such as $f = (h/\pi)sin(\pi x_3/h)$ if h is the thickness of the plate). More generalizations would involve bending of normal components and stretch-thickness effects.

In case of *shells* where, unfortunately, the whole paraphernalia of the differential geometry of nonflat 2D objects embedded in the 3D physical space needs to be considered, approaches considering expansions and representations of the basic velocity field (2.2) are to be considered up to the chosen degree of approximation (cf. Germain [44]). It is difficult to imagine before hand the richness of the ensuing descriptions. What is clear, however, is that this method of the PVP seems again to be the surest and safest way to formulate the basic equations, *and* associated natural boundary conditions, in these sophisticated theories. From the previously quoted examples, it also provides the best framework to place in evidence the relations between the various theories and to offer a real rational classification of the many formulated theories of structural members. One may think of rods but also of wave guides with different cross-sections (cf. Touratier and Maugin [46]).

6 Conclusion

There is a long way between the original idea of getting rid of the (supposedly) metaphysical Newtonian notion of force by continental mechanicians in the eighteenth century and the beautiful tool, in full resonance with modern mathematical concepts such as those of test functions, distributions, weak formulation, variational inequalities, etc, dealt with in the previous sections. This tool has taken a rigorous form that is easily grasped by students and engineers, at least by those familiar with such techniques as the method of finite elements in computations. It has demonstrated its efficiency as a co-operative tool in the construction of complex theories of continua, that is, those that go further than the usual Euler–Cauchy standard modeling. The large selection of examples provided in the previous sections is witness of this efficiency and richness. We can parody a statement of Jean Salençon—succeeding Paul Germain in his teaching at Ecole Polytechnique ([47], vol. 1, pp. 233)—that the "systematic way that was followed in these many examples, has something that is both seducing and securing. Its implementation needs as starting point ideas issued from observation and physics under various aspects. The PVP thus acquires the status of a shape- and mind-forming instrument. Its clarity helps a better understanding of the constructed models, which in turn allows for their exploitation in depth."

As we saw repeatedly, on the one hand, the modern concept of "objectivity"—indeed a rather simple application of group theory—, is essential in this modern view. It plays a fundamental role in distinguishing these quantities for which one later has to formulate a constitutive equation (what distinguishes one material from another one in its response to external loads in an otherwise identical setting). The existing algebraic structure was underlined by Nayroles [48], and we exploited it in our paper of 1980 [7]. The other critical point, emphasized in each modeling but increasing in intensity with the complication of the model, remains the writing down of the virtual power of inertial forces. This is what truly transforms the principle of virtual work into d'Alembert's principle, *per se*. In spite of this difficulty that seems to be overcome, we are far from the cloudy and almost incomprehensible technical writing of d'Alembert in his famous opus of 1743 devoted to dynamics ([49], but the introduction of the book is a pearl in eighteenth-century French according to the own wife of the present writer). As a matter of fact, d'Alembert was more proud of his literary works (in the general introduction and the many articles, he wrote in the famous Encyclopedia and also late literary works) than of his scientific ones. That is a strange final word for someone who invented partial differential equations and gave us the first wave solution, but he was not exempt of a "paradox" and of "dreams" (cf. d'Alembert's paradox in fluid mechanics and the play "The dream of d'Alembert" by Denis Diderot, his fellow co-editor of the Encyclopedia). For the true aficionados of the PVP, we mention as a final point the generalization to 4D relativistic continuum mechanics by the author [50]. For an authorized scientific biography of Paul Germain see [51,52].

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