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Original article

On criticism of the nature of objectivity in classical continuum physics

A.I. Murdoch

Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, Scotland, UK (e-mail: aim@maths.strath.ac.uk)

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Abstract. Murdoch (J. Elasticity 60, 233–242, 2000) showed that restrictions imposed upon response functions by material frame-indifference are the consequences of five distinct aspects of observer agreement (that is, of 'objectivity') and involve only proper orthogonal tensors. Accordingly it is unnecessary to invoke the 'principle of invariance under superposed rigid motions' (in the sense of 'one observer, two motions'), which imposes a restriction upon nature. Liu (Continuum Mech. Thermodyn. 16, 177–183, 2003, and Continuum Mech. Thermodyn. 17, 125–133, 2005) has challenged, misinterpreted and misrepresented the content of both Murdoch's work and this work. Here all criticisms of Liu are answered, his 'counter-examples' are used to amplify the tenets of Murdoch's work, and a key modelling issue in the controversy is indicated. Further, the response function restrictions for a given observer, derived on the basis of considering other observers, are shown to be independent of possible differences in the scales of mass, length, and time employed by other observers.

Key words: material frame-indifference, objectivity
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1 Introduction

In classical macroscopic physics material behaviour is assumed in principle to be independent of its observation. However, since information concerning such behaviour is only available by observation/measurement, an observer-based formulation of this principle is required before its implications can be examined. To this end it is customary to postulate a 'principle of material frame-indifference' which according to Truesdell and Noll [4, Sect. 14] asserts that

'*the response of a material is the same for all observers.*' MFI

The manner in which this principle imposes restrictions upon response functions has been the subject of lengthy controversy (see, for example, [5–9]). There are essentially two viewpoints, one of which can be based solely upon observer agreement ('objectivity'), and the other involves aspects of observer agreement together with invariance of material response with respect to superposed rigid body motions ('*isrbm*'). The significant difference between these viewpoints is that the latter imposes an a priori restriction upon nature while the former does not. In particular, the latter viewpoint excludes possible response that is sensitive to rotation of material relative to an (hence any) inertial frame. Such sensitivity has been given theoretical support in the context of gas dynamics [10], and has been invoked to describe superfluid helium behaviour [11], and turbulence [12]. Further, '*isrbm*' seems to be inconsistent with models of magneto-elastic phenomena [13]. Accordingly the status of *isrbm* as a general principle for macroscopic behaviour is questionable. The apparent growing acceptance of *isrbm* as a principle which stands or falls with that of material frame-indifference led Murdoch [1] to argue that considerations of objectivity alone suffice to obtain standard restrictions upon response functions. In [1] five distinct aspects O.1-5 were listed, of which O.4 and O.5 together constitute a statement of MFI based upon observer consensus. Allowing the selection of distinct response functions by different observers, it was shown by example how corresponding functions for any pair of observers must be related, and how this results in restrictions upon any given response function which involve only *proper* orthogonal tensors. From this perspective constitutive dependence upon local material spin with respect to inertial frames *is* admissible[1].

Liu [2, 3] is severely critical of [1] and this note. Specifically, he claims to have exposed errors in logical reasoning, disagrees on how any one observer is able to consider the motions of others, regards O.4 as 'impractical', 'innocuous', 'superfluous' and 'vaguely stated', and holds that the arguments employed run 'against the rational spirit deeply cherished in modern continuum mechanics'. In studying these claims the reader should be aware that the term 'objective considerations' is interpreted differently by Liu and Murdoch. (The letter codified these from the outset in [1] as O.1–5.) To see how such misunderstanding colours Liu's criticism, consider the preamble in Sect. 4 of [2] in which he stated that Murdoch claimed in [1] to have obtained 'standard restrictions' on the basis of (Euclidean) objectivity alone, 'without invoking the form-invariance of the principle of MFI'. Murdoch made no such claim: his arguments were based on (Euclidean) objectivity (which derives from O.1–3) *together with* O.4–5. Thus MFI *was* invoked by Murdoch, albeit with a different interpretation (namely O.4–5) from that of Liu. Accordingly, far from exposing a logical flaw, Liu did not appreciate the basic assumptions upon which the arguments of [1] were based, despite these being explicitly invoked. In the same vein, Liu[2] takes issue with [1] over a perceived failure to recognise 'the mathematical implication of *the*[3] condition of objectivity'. Since the arguments of [1] specifically, and intentionally, omit the condition to which he refers (such 'condition' rules out the possibility of material response which is sensitive to the spin of a body relative to an (any) inertial frame, and thus imposes a restriction upon nature), it is hardly surprising that its implication is not explored in [1]. Liu's 'implication' is that the response function(s) for any observer O^* is (are) uniquely determined by the response function(s) chosen[4] by a given observer, O say. This may come as a surprise to O^*, who may not be in communication with O. Liu's interpretation of objectivity is thus consistently authoritarian in mandating restrictions both upon nature and observers.

It is the purpose of this work to draw attention to the different interpretations of MFI involved in the controversy, to discuss the differing viewpoints on how relative motions of observers enter the arguments, to use the 'counter-examples' of [2] to amplify the tenets of [1], and to indicate a key modelling issue in the controversy. Additional remarks are made on the invariance of the results of [1] to possible selection of different scales of mass, length, and time by different observers.

Before addressing Liu's criticisms in detail it is instructive to examine several interpretations of the above-cited statement of MFI:

I.1 *The response* **function(s)** *of a material is (are) the same for all observers.*

I.1′ (Liu [2], Sect. 3.2) *The constitutive function of an objective quantity must be independent of the frame.*

I.2 (Murdoch [1], Sect. 2.2, O.4, 5) *Observers are able to agree upon the nature and responses of any given ideal material, no matter what be their relative motions*.

Since observers may not be in communication when formulating constitutive relations for an ideal material which are to model the macroscopic behaviour of a specific material of interest, and may not employ the same physical units, interpretation I.1/I.1′ is *a priori* questionable. Liu codifies I.1′ as '*form-invariance*' which, together with the objective nature of displacements, (Cauchy) stresses etc., is equivalent to *isrbm* (see [9]). Accordingly form-invariance is at odds with the material behaviour addressed in [10–13]. Given that any form of words employed to delineate frame-indifference is open to misinterpretation, the essence of the concept is best elicited by considering

[1] This is the viewpoint of Noll, who has pointed out to Müller (see [10]) that such dependence does not violate material frame-indifference. [Personal communication, December 1983]

[2] See [3], Abstract.

[3] My italics.

[4] The possibility of choice can arise in, for example, the selection of a reference configuration for an elastic body and/or scales of mass, length, and time.

particular examples. While this was undertaken in [1] in respect of viscous fluids, elastica, and simple materials with memory, the 'counter-examples' of [2] are excellent in elucidating how I.2 works in practice.

2 Liu's criticisms

2.1 Preamble

Any scientist who wishes to model aspects of the behaviour of a particular material system must formulate constitutive relations which suffice to describe the behaviour of interest. In selecting appropriate response functions to be employed in these relations such scientist/observer should take account of how any other observer would describe the same behaviour, for only by so doing can meaningful communication between observers be envisaged in regard to this behaviour, and a body of science established. Such account requires assumptions concerning other observers, specifically what agreement would be possible were communication to be established. In [1] observers were assumed to be able to agree upon:

O.1. mathematical ideas and results, and relevant physical concepts,
O.2. inertial frames,
O.3. time lapses between events, and distances between simultaneous events,

and, no matter what be the relative motion of the observers,

O.4. the nature of any given ideal material, and
O.5. all possible responses of an ideal material.

As a consequence of O.3. there is, at any instant t, an isometry $\boldsymbol{\alpha}_t$ between space $\mathcal{E}$ perceived by an observer O and space $\mathcal{E}^*$ as regarded by another observer O^*. This isometry involves an orthogonal tensor $\mathbf{Q}_t$ which maps $\mathcal{V}$ onto $\mathcal{V}^*$, where $\mathcal{V}(\mathcal{V}^*)$ denotes the three-dimensional vector space used by $O(O^*)$ to describe vectorial quantities. Specifically, if $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ then

$$\mathbf{y}^* - \mathbf{x}^* = \mathbf{Q}_t(\mathbf{y} - \mathbf{x}), \tag{2.1}$$

where $\mathbf{y}^* := \boldsymbol{\alpha}_t(\mathbf{y})$ and $\mathbf{x}^* := \boldsymbol{\alpha}_t(\mathbf{x})$.

Agreement upon the notions of force (O.1. and O.2.), normals and areas associated with surfaces (O.1.), and tractions and stresses (O.1. and O.2.) leads to standard relations. In particular

$$\mathbf{T}^*(\mathbf{x}^*, t) = \mathbf{Q}_t\, \mathbf{T}(\mathbf{x}, t)\mathbf{Q}_t^T, \tag{2.2}$$

where $\mathbf{T}(\mathbf{x}, t)(\mathbf{T}^*(\mathbf{x}^*, t))$ denotes the value of the Cauchy stress tensor at point $\mathbf{x}(\mathbf{x}^*)$ at instant t for $O(O^*)$.

Remark 1. Concerning (2.2) there is no disagreement between [1] and [2], although (following Noll [14]) a conceptual distinction is made in [1] between $\mathcal{V}$ and $\mathcal{V}^*$. It should be noted that this distinction means $\det \mathbf{Q}_t$ *is undefined*: see [1], Remark 2.

2.2 Criticism 1: the manner in which a given observer considers other observers

The key criticism concerns relative motions of O and O^*. Specifically Liu asserts (in [2], Sect. 4.2 after (4.6))

> L.1. '*...by definition, any two observers (i.e., two frames of reference) are related by a (one and only one) relative motion of time-dependent rigid transformation...*'

A related criticism in ([2], last paragraph of Sect. 1)

> L.2. '*...the question of whether the Euclidean transformation*[5] *should involve general or only proper orthogonal tensors has always been regarded as a question of choice and should only be decided by experiments. This assertion was also disputed in Murdoch* [1] *in the claim that only proper orthogonal transformations had been proved to be allowed.*'

To appreciate the arguments of [1] it is necessary to consider how an observer O, faced with devising constitutive relations, can consider how any other observer O^* would regard the behaviour he/she is describing, in any conceivable motion of O^* with respect to O. As a consequence of O.4. and O.5., whatever relation(s) O selects

[5] That is, relation (2.1) herein.

which model the behaviour in question, he/she can appreciate how O^* could specify relations governing the same behaviour (O.4.), and contemplate agreement about all possible behaviour of interest with O^*, no matter what be their relative motions at any given instant (O.5). As a consequence O is able to deduce restrictions upon his/her response functions which guarantee satisfaction of O.4. and O.5. In the foregoing considerations no contact with other observers is involved. Indeed, observer O^*, and his/her motions are *hypothetical*. Accordingly it is possible to contemplate any number of relative motions of O^* with respect to O. Liu (see third paragraph in Sect. 2 of [3]) regards such considerations as metaphysical, and 'beyond our comprehension[6]'despite the rôle played by 'thought experiments' throughout science. Indeed, Liu's relation (2) in [3], giving the stress as a function of the history of the motion of a body, could similarly be regarded as metaphysical since the body can only undergo one motion in its 'lifetime'. In [1] restrictions on response functions were obtained for viscous fluids, elastica, simple materials, and materials whose response functions depended upon objective field values. These restrictions involve only proper orthogonal tensors $\mathbf{Q}$ on space $\mathcal{V}$ as a consequence of any given observer being able to *distinguish* between right- and left-handed screws (an aspect of O.1.). If not, such an observer would be unable to detect the difference between optically-active sugar solutions which rotate plane polarised light in opposing senses. For such a solution O^* must thus be expected to be aware of the corresponding sense of rotation at any instant t, no matter what be the relative motion of O and O^* (O.4.). If O envisages two such motions with $\mathbf{Q}_t$ (see (2.1)) values of $\mathbf{Q}_1$ and $\mathbf{Q}_2$, say, this awareness implies that $\mathbf{Q}_2^T\mathbf{Q}_1$ be proper orthogonal (this is proved in [1], Remark 3). It is combinations of form $\mathbf{Q}_2^T\mathbf{Q}_1$ that enter into restrictions upon response functions in [1]. An even simpler, but essentially-equivalent, example would be that of a carpenter screwing a right-handed or left-handed screw into a piece of wood. The very fact that this activity is understood by readers of this article involves an implicit assumption that any single observer would appreciate the nature of the screw no matter how he/she moves relative to the carpenter. Of course, this observer can actually undergo only one such motion. However, surely there is no disputing that were this observer to be undertaking *any* other motion relative to the carpenter (within sight of the screw!) then his/her conclusion about the nature of the screw would be unchanged? Notice that this conclusion is quite independent of notions of material symmetry (see [3], Remark 2).

While the foregoing disposes of L.1., statement L.2. indicates that Liu did not apparently notice the difference between the orthogonal transformation (from $\mathcal{V}$ into $\mathcal{V}^*$) $\mathbf{Q}_t$ in (2.1) and tensors of form $\mathbf{Q} = \mathbf{Q}_2^T\,\mathbf{Q}_1$ discussed above. Specifically (see [1], Remark 2 and Remark 1 herein), the conceptual distinction between $\mathcal{V}$ and $\mathcal{V}^*$ means det $\mathbf{Q}_t$ is not defined (and so no claim of $\mathbf{Q}_t$ being 'proper orthogonal' was advanced in [1]), while the tensors $\mathbf{Q}$ which appeared in the standard restrictions upon response functions derived in [1] were orthogonal transformations from $\mathcal{V}$ into itself, and were proved to be proper orthogonal on the basis of the foregoing argument concerning screws.

Remark 2. Distinguishing between $\mathcal{V}$ and $\mathcal{V}^*$ is helpful in making precise arguments. If the distinction is *not* made, and observers choose Cartesian co-ordinate systems, then O and O^* may select systems with the same orientation (that is, both right-handed or both left-handed) in which case $\det \mathbf{Q}_t = +1$ in (2.1). If O and O^* select systems with *different* orientations, then $\det \mathbf{Q}_t = -1$ in (2.1). Notice, however, that combinations $\mathbf{Q}_2^T\mathbf{Q}_1$ of such tensors yield $\det(\mathbf{Q}_2^T\mathbf{Q}_1) = +1$. Accordingly L.2. remains an invalid criticism.

2.3 Criticism 2: The manner in which observers agree upon the nature of a given material

In his introduction Liu [2] states

L.3: '*Essentially, the principle of material frame-indifference* **merely**[7] *states that* **the**[7] *constitutive function of an "objective" quantity should be independent of observers.*'

Just how and why two observers, not necessarily in contact, would or could select the same response function for such a quantity is not clear[8]. Imposition of L.3. is rendered unnecessary by the reasoning in [1] which invokes

[6] See [3], Sect. 2, before Remark 1.

[7] My emphasis.

[8] Indeed, since different observers may employ different scales of mass, length, and time, it is difficult to conceive how this might be possible.

O.5, together with O.4 and the objective nature of the quantity delivered by the function in question, to obtain the restriction upon any response function utilised by O.

We now explore the mathematical formulation of L.3. Confining attention to the Cauchy stress tensor in a purely mechanical context, Liu considers constitutive relations of the form[9]

$$\mathbf{T}(X,t) = \boldsymbol{\mathcal{F}}_{\phi}(\boldsymbol{\chi}^t, X, t). \tag{2.3}$$

Here $\mathbf{T}(X,t)$ denotes the stress at instant 't' for an observer O/frame ϕ and material point X, and $\boldsymbol{\mathcal{F}}_{\phi}$ is *the* response function. Frame-indifference is taken to imply satisfaction of (7) of [3] with

$$\boldsymbol{\mathcal{F}}_{\phi^*} = \boldsymbol{\mathcal{F}}_{\phi}. \tag{2.4}$$

There are two aspects of the foregoing that differ significantly from the viewpoint of [1]:
(a) Relation (2.4) does not make sense if 'space' $\mathcal{E}$ as perceived by observer O is distinguished from that, $\mathcal{E}^*$ say, as apparent to observer O^* (that is, if Noll's neo-classical space-time is adopted). This is because the first argument of $\boldsymbol{\mathcal{F}}_{\phi}(\boldsymbol{\mathcal{F}}_{\phi^*})$ is associated with regions in $\mathcal{E}$ ($\mathcal{E}^*$), and hence the domains of $\boldsymbol{\mathcal{F}}_{\phi}$ and $\boldsymbol{\mathcal{F}}_{\phi^*}$ differ. In Remark 3 of [3] Liu argues that his viewpoint 'can easily be adapted to ...essentially equivalent one in neo-classical space-time'. This is difficult to envisage in view of the foregoing. On the other hand the arguments of [1], involving different relative motions of observer pairs, can be used to *derive* Liu's material objectivity relation ((3.5) of [2]) and hence *prove* that responses of materials of form (2.3) satisfy *isrbm*. That is, *Liu's result follows from* O.4 *and* O.5 *without assumption* (2.4).
(b) While very general, (2.3) cannot be directly compared with established constitutive relations which relate field values amenable to measurement/observation. In particular one deals with the stress $\mathbf{T}(\mathbf{x},t)$ as a function of location $\mathbf{x}$ in $\mathcal{E}$, not a function of material point X. The key to identifying material points and locations, in the case of simple materials, is for an observer to select a reference configuration, $\boldsymbol{\kappa}$ say, which establishes a bijection between $\mathcal{B}$ (the set of material points which constitute the 'body') and $\mathcal{E}$. Relation (2.3) can then be expressed as

$$\mathbf{T}(\mathbf{x},t) = \boldsymbol{\mathcal{F}}_{\phi,\boldsymbol{\kappa}}(\boldsymbol{\chi}^t_{\boldsymbol{\kappa}}, \hat{\mathbf{x}}, t), \tag{2.5}$$

where

$$\hat{\mathbf{x}} = \boldsymbol{\kappa}(X) \quad \text{and} \quad \mathbf{x} = \boldsymbol{\chi}(X,t) =: \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\hat{\mathbf{x}},t). \tag{2.6}$$

The response function $\boldsymbol{\mathcal{F}}_{\phi,\boldsymbol{\kappa}}$ clearly depends upon the choice of $\boldsymbol{\kappa}$. For Murdoch selection of reference configuration is a matter of choice for any given observer (see [1], Remark 6), while Liu requires a common configuration (only possible since he identifies $\mathcal{E}^*$ with $\mathcal{E}$) in order to be able to utilise (2.4) in (7) of [3].

The difference in viewpoints is particularly clear in the case of elastica. For Murdoch an elastic body is one in which, for any observer O, the stress is a function of the deformation gradient with respect to a reference configuration, $\boldsymbol{\kappa}$ say, *of his/her choice*. Thus in standard notation,

$$\mathbf{T} = \hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}). \tag{2.7}$$

Another observer O^* will, *as a consequence of* O.4, describe the behaviour as

$$\mathbf{T}^* = \overline{\mathbf{T}}^*_{\boldsymbol{\mu}^*}(\mathbf{F}^*), \tag{2.8}$$

where $\boldsymbol{\mu}^*$ is his/her choice of reference configuration, and $\mathbf{F}^*$ is the corresponding deformation gradient. Given that O and O^* may not be in communication, there is no reason to suppose that $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}^*$ are related in any other way than by a differentiable bijection[10] (unknown to O and O^* if they are not in communication)

$$\boldsymbol{\lambda} := \boldsymbol{\mu}^* \circ \boldsymbol{\kappa}^{-1}. \tag{2.9}$$

From (2.2), (2.7) and (2.8)

$$\overline{\mathbf{T}}^*_{\boldsymbol{\mu}^*}(\mathbf{F}^*) = \mathbf{Q}_t\, \hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F})\mathbf{Q}_t^T, \tag{2.10}$$

[9] See (3.1) of [2] and (2) of [3]. These relations differ: the latter involves the explicit dependence of $\boldsymbol{\mathcal{F}}_{\phi}$ upon t, as here in (2.3), while the former does not. This is presumably because the first argument relates to a one-parameter family of regions 'occupied' by the body in space as perceived by ϕ over all times in the 'past', without explicit indication of the 'current' instant, 't': that is, (3.1) of [2] has been amended to acknowledge this fact.

[10] Selection of a reference configuration $\boldsymbol{\kappa}$ by O identifies a fixed region $B_{\boldsymbol{\kappa}}$ in $\mathcal{E}$: $B_{\boldsymbol{\kappa}}$ is the region 'occupied' by the body in this configuration. To each geometrical point in $B_{\boldsymbol{\kappa}}$ corresponds a unique 'material point' of the body. Likewise selection of $\boldsymbol{\mu}^*$ by O^* involves a fixed region $B^*_{\boldsymbol{\mu}^*}$ in $\mathcal{E}^*$. Map $\boldsymbol{\lambda}$ identifies the locations of material points in the two reference configurations and is time-independent.

where (see [1], (3.37))

$$\mathbf{F}^* = \mathbf{Q}_t\,\mathbf{F}\mathbf{H}^{-1} \tag{2.11}$$

with

$$\mathbf{H} := \nabla\boldsymbol{\lambda}. \tag{2.12}$$

Thus

$$\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}) = \mathbf{Q}_t^T\,\overline{\mathbf{T}}^*_{\boldsymbol{\mu}^*}(\mathbf{Q}_t\,\mathbf{F}\mathbf{H}^{-1})\mathbf{Q}_t. \tag{2.13}$$

By O.5. this relation holds for all invertible $\mathbf{F}$ and all relative motions of O^* and O. Consider two conceivable relative motions in which at instant t $\mathbf{Q}_t$ takes the values $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and the deformation gradient for O is $\mathbf{F}_0$. Then from (2.13)

$$\begin{aligned}\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}_0) &= \mathbf{Q}_1^T\,\overline{\mathbf{T}}^*_{\boldsymbol{\mu}^*}(\mathbf{Q}_1\,\mathbf{F}_0\,\mathbf{H}^{-1})\mathbf{Q}_1\\ &= \mathbf{Q}_1^T\,\overline{\mathbf{T}}^*_{\boldsymbol{\mu}^*}(\mathbf{Q}_2(\mathbf{Q}_2^T\,\mathbf{Q}_1\,\mathbf{F}_0)\mathbf{H}^{-1})\mathbf{Q}_1\\ &= \mathbf{Q}_1^T\{\mathbf{Q}_2\,\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{Q}_2^T\,\mathbf{Q}_1\,\mathbf{F}_0)\mathbf{Q}_2^T\}\mathbf{Q}_1.\end{aligned} \tag{2.14}$$

The last step follows from (2.13) with $\mathbf{F} := \mathbf{Q}_2^T\,\mathbf{Q}_1\,\mathbf{F}_0$. Writing

$$\mathbf{Q} := \mathbf{Q}_2^T\,\mathbf{Q}_1, \tag{2.15}$$

(2.14) yields

$$\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{Q}\mathbf{F}_0) = \mathbf{Q}\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}_0)\mathbf{Q}^T. \tag{2.16}$$

The arbitrary nature both of $\mathbf{F}_0$ and possible relative motions of O^* and O imply that (2.16) should hold for all deformation gradients $\mathbf{F}_0$ and all *proper* orthogonal tensors $\mathbf{Q}$ (see Sect. 2.2).

Remark 3. The standard restriction (2.16) *has been derived without any restriction upon the choice of reference configuration by* O^*. The foregoing should not be confused with formally similar computations which correspond to a change of reference configuration by O. If $\boldsymbol{\mu}$ is another reference configuration chosen by O then the corresponding response function $\hat{\mathbf{T}}_{\boldsymbol{\mu}}$ must satisfy, for all deformation gradients $\mathbf{F}_{\boldsymbol{\mu}}$,

$$\hat{\mathbf{T}}_{\boldsymbol{\mu}}(\mathbf{F}_{\boldsymbol{\mu}}) = \mathbf{T} = \hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}_{\boldsymbol{\kappa}}). \tag{2.17}$$

Here the deformation gradients $\mathbf{F}_{\boldsymbol{\kappa}}$ and $\mathbf{F}_{\boldsymbol{\mu}}$ with respect to $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$, respectively, are related by

$$\mathbf{F}_{\boldsymbol{\mu}} = \mathbf{F}_{\boldsymbol{\kappa}}\mathbf{K}, \;\text{ where }\; \mathbf{K} := \nabla(\boldsymbol{\kappa}\circ\boldsymbol{\mu}^{-1}). \tag{2.18}$$

Accordingly

$$\hat{\mathbf{T}}_{\boldsymbol{\mu}}(\mathbf{F}) = \hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F}\mathbf{K}^{-1}) \tag{2.19}$$

for all deformation gradients $\mathbf{F}$ computed with respect to $\boldsymbol{\mu}$, and hence the response function $\hat{\mathbf{T}}_{\boldsymbol{\mu}}$ is uniquely determined by $\hat{\mathbf{T}}_{\boldsymbol{\kappa}}$. The different natures of $\mathbf{H}$ (see (2.12)), an invertible linear map from $\mathcal{V}$ onto $\mathcal{V}^*$, and $\mathbf{K}$ (see $(2.18)_2$), an invertible linear map from $\mathcal{V}$ onto $\mathcal{V}$, has been rendered precise by distinguishing between 'space' as perceived by different observers in the manner of Noll [14][11]. It is relation (2.19) which is used to categorise the symmetry group of the material for O with respect to configuration $\boldsymbol{\kappa}$. In [1] (preceding (3.35)) it was remarked that in selecting a reference configuration it is natural for an observer to choose a configuration assumed by the body at some given instant. Of course, two different observers not in communication cannot be expected to choose the same instant: in such case $\mathbf{H}$ in the foregoing is unrestricted. If the *same* instant were to be selected then $\mathbf{H}$ would be an orthogonal map from $\mathcal{V}$ onto $\mathcal{V}^*$. The latter case was considered in Murdoch [15] for simplicity, and cited incorrectly in Sect. 4.3 of Liu [3] as evidence that $\mathbf{H}$ *must* be orthogonal. The general situation addressed in [1] was first presented as Remark 3 in Murdoch [16].

[11] Having made such distinction it becomes impossible to adopt L.3. in this context, since the domains of $\hat{\mathbf{T}}_{\boldsymbol{\kappa}}$ and $\overline{\mathbf{T}}_{\boldsymbol{\mu}^*}$ differ, being invertible linear transformations on $\mathcal{V}$ and $\mathcal{V}^*$, respectively.

3 Liu's 'counter-examples'

3.1 Preamble

By 'counter-example' Liu means a candidate model of material behaviour consistent with O.1–5 but not 'isrbm'/'form-invariance'. It is not clear just why Liu feels the need to add to the list of *established* and *physically-motivated* models in [10–13] which share this characteristic, unless it is to emphasise that requirements O.1–5 are less restrictive than those of Euclidean objectivity together with 'isrbm'/'form-invariance'. Of course, satisfaction of O.1–5 by a candidate model is no guarantee that the model describes any actual behaviour, but only that objective considerations do not preclude such possibility[12]. *However, it requires but a single example of actual behaviour consistent with* O.1–5 *but not with 'isrbm' to expose the failure of 'isrbm' as a 'principle'.*

3.2 'Counter-example' 1 (Liu [2], Sect. 4.1)

$$\mathbf{T} = \lambda(\mathbf{e}\,.\,\mathbf{D}\mathbf{e})\mathbf{1} + \mu\mathbf{D} =: \tilde{\mathbf{T}}(\mathbf{D}, \mathbf{e}). \tag{3.1}$$

Here, for a particular observer, O say, $\mathbf{T}$ is the stress tensor, $\mathbf{D}$ is the symmetric part of the velocity gradient, and $\mathbf{e}$ is a unit vector fixed in the reference frame adopted by O. The necessity of knowing *both* $\mathbf{D}$ *and* $\mathbf{e}$ in obtaining $\mathbf{T}$ is here emphasised by writing the combination as $\tilde{\mathbf{T}}(\mathbf{D}, \mathbf{e})$ in contrast to Liu who in [2] regards this expression to be a function $\hat{\mathbf{T}}(\mathbf{D})$ of $\mathbf{D}$ alone, presumably since by hypothesis $\mathbf{e}$ is fixed for observer O. *However, also by hypothesis, Liu states* $\mathbf{e}$ *is objective*, so that at instant t another observer O^* will regard this vector as[13] $\mathbf{e}^*(t) := \mathbf{Q}_t\,\mathbf{e}$. The stress response function is thus a tensor-valued function of an objective tensor *field value* $\mathbf{D}$ and an objective vector $\mathbf{e}$. The implications for $\tilde{\mathbf{T}}$ of O.4 and O.5 follow in the manner of Sect. 3.2 of [1] concerning constitutive dependence upon objective field values[14]: specifically, it follows that any such response function $\tilde{\mathbf{T}}(\mathbf{D}, \mathbf{e})$ must satisfy

$$\tilde{\mathbf{T}}(\mathbf{Q}\mathbf{D}\mathbf{Q}^T, \mathbf{Q}\mathbf{e}) = \mathbf{Q}\tilde{\mathbf{T}}(\mathbf{D}, \mathbf{e})\mathbf{Q}^T \tag{3.2}$$

for all proper orthogonal tensors on $\mathcal{V}$. It is a simple matter to verify that $\tilde{\mathbf{T}}(\mathbf{D}, \mathbf{e})$ delivered by (3.1) does indeed satisfy (3.2). Accordingly the material in question is consistent with all objective considerations O.1–5. *Of course, this is a matter quite distinct from whether such a material exists*. The particular peculiarity of this candidate model is that it selects a distinguished class of observers, namely O together with any other observer O' who may undergo arbitrary translatory motions, but not rotate, relative to O. Then for such an O' the relevant vector

$$\mathbf{e}' = \mathbf{Q}_0\,\mathbf{e}, \tag{3.3}$$

where $\mathbf{Q}_0$ is a fixed orthogonal map from $\mathcal{V}$ into $\mathcal{V}'$ and hence $\mathbf{e}'$ is a fixed unit vector for O'. For any other observer O^* the relevant vector $\mathbf{e}^*$ varies with time.

Remark 4. It is natural to ponder the physical plausibility of model (3.1) since it is not ruled out by O.1–5. Given that the only distinguished frames in classical mechanics are inertial, one is led to consider O above as an inertial observer. The model could reflect a possible oriented, spatially uniform and temporally constant (for inertial observers), microstructure reminiscent of nematic liquid crystals. Were such a material to exist it would be useful in compass design. Of course in practice one is faced with devising a model which represents *actual* behaviour, and employs O.1–5 to refine candidate models to guarantee observer consensus and thus ensure candidacy for physical legitimacy. Said differently, any proposal to adopt (3.1) in practice would be motivated by a physical interpretation of $\mathbf{e}$.

Remark 5. From (2.2) and (3.2) one easily obtains

$$\mathbf{T}^* = \lambda(\mathbf{e}^*.\,\mathbf{D}^*\mathbf{e}^*)\mathbf{1}^* + \mu\mathbf{D}^*, \tag{3.4}$$

[12] The same can be said if a model satisfies Euclidean objectivity plus *isrbm*. For example, $\mathbf{T} = \mu\mathbf{D}$, where μ is constant, corresponds to a material in which the stress vanishes in the absence of motion, no matter what be the density.

[13] The time-dependence of $\mathbf{e}^*$ *forces* O^* to regard $\mathbf{T}^*$ to be a function both of $\mathbf{D}^*$ *and* $\mathbf{e}^*$ (see (3.4) and (3.5)): this is particularly evident if O^* writes (3.4) in standard Cartesian co-ordinate format for which the basis vectors define fixed directions in $\mathcal{E}^*$.

[14] Here one has a special case in which, for any given observer, at any given instant, there is a dependence on a *spatially-constant* objective vector field.

where

$$\mathbf{D}^* := \mathbf{Q}_t\,\mathbf{D}\mathbf{Q}_t^T, \;\; \mathbf{e}^* := \mathbf{Q}_t\,\mathbf{e}, \;\text{ and }\; \mathbf{1}^* := \mathbf{Q}_t\,\mathbf{Q}_t^T \tag{3.5}$$

denotes the identity on $\mathcal{V}^*$ (see [1], (2.3)). Hence $\mathbf{T}^*$ is a function of the symmetric part $\mathbf{D}^*$ of the velocity gradient, as computed by O^*, and $\mathbf{e}^*$. Notice that *formally*, from (3.1),

$$\tilde{\mathbf{T}}(\mathbf{D}^*, \mathbf{e}^*) = \lambda(\mathbf{e}^*.\,\mathbf{D}^*\mathbf{e}^*)\mathbf{1} + \mu\mathbf{D}^*. \tag{3.6}$$

However, $\tilde{\mathbf{T}}(\mathbf{D}^*, \mathbf{e}^*)$ does not make sense since we have consistently distinguished[15] $\mathcal{V}^*$ from $\mathcal{V}$: the domain of $\tilde{\mathbf{T}}$ in (3.1) is (Sym $\mathcal{V}$) $\times$ (Unit $\mathcal{V}$), where Sym $\mathcal{V}$ denotes the space of symmetric linear transformations on $\mathcal{V}$ and Unit $\mathcal{V}$ represents the set of unit vectors in $\mathcal{V}$, while $\mathbf{D}^* \in$ Sym $\mathcal{V}^*$ and $\mathbf{e}^* \in \mathcal{V}^*$. If such distinction is *not* made then (3.4) and (3.6) coincide and I.1 of Sect. 1 would be satisfied. Liu regards (3.1) and (3.4) as frame-dependent relations, and hence at variance with his version I.1$'$ (see Sect. 1). To try to appreciate this viewpoint one may regard (3.4) as a special case of

$$\mathbf{T}^* = \hat{\mathbf{T}}(\mathbf{D}^*, \mathbf{Q}_t; \mathbf{e}), \tag{3.7}$$

with (3.1) and the relation for O' with corresponding vector $\mathbf{e}'$ given by (3.3) yielding

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{D}, \mathbf{1}; \mathbf{e}) \quad\text{and}\quad \mathbf{T}' = \hat{\mathbf{T}}(\mathbf{D}', \mathbf{Q}_0; \mathbf{e}). \tag{3.8}$$

From this viewpoint the constitutive function common to all observers is $\hat{\mathbf{T}}(\,.\,,.\,;\mathbf{e})$ and the response depends upon the instantaneous orientation of the observer frame of reference with respect to a ‘fixed’ frame. Such a viewpoint is, however, misleading: the dependence upon $\mathbf{Q}_t$ and **e** in (3.7) is upon the composite variable $\mathbf{Q}_t\,\mathbf{e}$, as manifest in (3.6). One meets similar confusion in the literature when material response depends upon spin $\mathbf{W} + \mathbf{S}$ relative to an (any) inertial frame. (Here $\mathbf{W}$ denotes the skew part of the velocity gradient and $\mathbf{S}$ the spin of the observer frame with respect to any inertial frame.) If the dependence is written as a function of both $\mathbf{W}$ *and* $\mathbf{S}$ then the response appears to be frame-dependent. Of course, in combination $\mathbf{W} + \mathbf{S}$ this is *not* the case.

3.3 ‘Counter-example’ 2 (Liu [2], Example 2 of Sect. 5)

Liu considers a possible model of elastic behaviour of the form

$$\mathbf{T} = s_0\,\mathbf{1} + s_1\,\mathbf{B} + s_2\,\mathbf{Be}\otimes\mathbf{Be} =: \overline{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{B}, \mathbf{e}), \tag{3.9}$$

where s_0, s_1 and s_2 are constants and $\mathbf{B} := \mathbf{F}\mathbf{F}^T$, where **F** denotes the gradient of the deformation from a reference configuration and **e** is a *fixed* vector in the frame of the observer. As in the ‘counter-example’ of (3.2), **e** is said to be objective.

As in the foregoing subsection, dependence of $\mathbf{T}$ upon both an objective tensor-valued field $\mathbf{B}$ *and* vector **e** has been made explicit by writing $\overline{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{B}, \mathbf{e})$ rather than $\hat{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{F})$ or $\tilde{\mathbf{T}}_{\boldsymbol{\kappa}}(\mathbf{B})$ for the constitutive expression. The discussion of Sect. 3.2 may be repeated save only for replacing **D** by **B** and $\tilde{\mathbf{T}}$ by $\overline{\mathbf{T}}_{\boldsymbol{\kappa}}$. Accordingly the material satisfies the analogue of (3.2) but is frame-dependent in the manner of (3.7) and (3.8).

Remark 6. The candidate constitutive relations (3.1) and (3.9) are similar in that they could describe the behaviour of materials which have a one-dimensional microstructure invariant relative to inertial frames and modelled in such a frame by[16] **e**. Subjecting a body composed of such material to a superposed rigid body motion would alter the orientation of the body but not that of its microstructure. Accordingly it is not surprising that the change in stress does not correspond to a corresponding rotated version of the original stress: that is, that *isrbm* is violated. On the other hand, an observer who walks round the body sees both the body *and* its microstructure suffer an apparent rigid body motion, with consequent corresponding rotation of the stress field. This is the essence of the disagreement between the theses of [1] and [2], namely whether *isrbm* should be interpreted in the ‘one observer, two motions’ sense (Liu) or ‘one motion, two observers’ interpretation (Murdoch). In the former it is the body that is rotated relative to the rest of the material universe, while the latter corresponds to one observer putting himself/herself in the shoes of another observer walking round the body for whom *both* the body *and* the rest of the material universe appear to rotate together.

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[15] It should be noted that such distinction renders L.3 in Sect. 2.3 *a priori* meaningless in general and hence should worry anyone who wishes to emulate the rigour and clarity of Noll: see [14].

[16] Such behaviour is analogous to that observed in Helium II where the microstructure corresponds to a vortex structure.

3.4 Example 3 (Liu [2], Example 1 of Sect. 5)

Liu cites the example of a transversely-isotropic elastic material given by

$$\mathbf{T} = s_0\,\mathbf{1} + s_1\,\mathbf{F}\mathbf{F}^T + s_2\,\mathbf{F}\mathbf{n} \otimes \mathbf{F}\mathbf{n}. \tag{3.10}$$

Here **n** is a fixed vector in the reference configuration. It is not clear why this example is introduced, unless merely to contrast its superficial similarity with relation (3.9). It is, however, useful to see how (3.10) is considered from the viewpoint of [1].

The nature of the material modelled by (3.10), for the purposes of observer agreement O.4., is that

(i) it is an elastic material (that is, the stress is a function of the deformation gradient with respect to a choice of reference configuration),
(ii) there is a choice of reference configuration, $\boldsymbol{\kappa}$ say, with respect to which it is transversely-isotropic, and
(iii) the stress is the sum of three terms: the first a constant scalar multiple of the identity, the second a constant scalar multiple of $\mathbf{F}\mathbf{F}^T$ (where $\mathbf{F}$ denotes the deformation gradient with respect to the configuration $\boldsymbol{\kappa}$ of (ii)), and the third a constant scalar multiple of the tensor product of $\mathbf{F}\mathbf{n}$ with itself, where **n** is either unit vector parallel to the axis of symmetry in $\boldsymbol{\kappa}$.

Such agreement means that if observer O adopts relation (3.10) as his/her model then any other observer O^* will take the corresponding stress field to be

$$\mathbf{T}^* = s_0^*\,\mathbf{1}^* + s_1^*\,\mathbf{F}^*(\mathbf{F}^*)^T + s_2^*\,\mathbf{F}^*\,\mathbf{m}^* \otimes \mathbf{F}^*\mathbf{m}^*, \tag{3.11}$$

where $\mathbf{m}^*$ is a unit vector parallel to the axis of symmetry in a reference configuration $\boldsymbol{\mu}^*$ with respect to which the material is transversely isotropic. Here (see [1], Sect. 3.3)

$$\mathbf{F}^* = \mathbf{Q}_t\,\mathbf{F}\mathbf{H}^{-1}, \tag{3.12}$$

where **H** denotes the gradient of the bijective map $\boldsymbol{\lambda}$ which identifies material points in $\boldsymbol{\kappa}$ with those in $\boldsymbol{\mu}^*$. (Notice that **H** is an invertible linear map from $\mathcal{V}$ into $\mathcal{V}^*$.) We now use (3.12) and (2.2) to explore how $\boldsymbol{\mu}^*, \mathbf{m}^*, s_0^*, s_1^*$ and s_2^* must be related to $\boldsymbol{\kappa}, \mathbf{n}, s_0, s_1$ and s_2.

From (3.11), (2.2) and (3.10)

$$\begin{aligned} s_0^*\mathbf{1}^* + s_1^*\mathbf{F}^*(\mathbf{F}^*)^T + s_2^*\mathbf{F}^*\mathbf{m}^* \otimes \mathbf{F}^*\mathbf{m}^* &= \mathbf{T}^* = \mathbf{Q}_t\,\mathbf{T}\mathbf{Q}_t^T \\ &= \mathbf{Q}_t(s_0\mathbf{1} + s_1\mathbf{F}\mathbf{F}^T + s_2\mathbf{F}\mathbf{n} \otimes \mathbf{F}\mathbf{n})\mathbf{Q}_t^T. \end{aligned} \tag{3.13}$$

However, use of $(3.5)_3$ and (3.12) yields

$$\begin{aligned} &s_0^*\mathbf{1}^* + s_1^*\mathbf{F}^*(\mathbf{F}^*)^T + s_2^*\mathbf{F}^*\mathbf{m}^* \otimes \mathbf{F}^*\mathbf{m}^* \\ &= s_0^*\mathbf{Q}_t\,\mathbf{Q}_t^T + s_1^*\mathbf{Q}_t(\mathbf{F}\mathbf{H}^{-1})(\mathbf{F}\mathbf{H}^{-1})^T\mathbf{Q}_t^T + s_2^*\mathbf{Q}_t(\mathbf{F}\mathbf{H}^{-1}\mathbf{m}^* \otimes \mathbf{F}\mathbf{H}^{-1}\mathbf{m}^*)\mathbf{Q}_t^T. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), specifically equating the right-hand sides of these relations, pre-multiplying by $\mathbf{Q}_t^T$ and post-multiplying by $\mathbf{Q}_t$, and noting (see [1], $(2.3)_1$) $\mathbf{Q}_t^T\mathbf{Q}_t = \mathbf{1}$,

$$s_0\mathbf{1} + s_1\mathbf{F}\mathbf{F}^T + s_2\mathbf{F}\mathbf{n} \otimes \mathbf{F}\mathbf{n} = s_0^*\mathbf{1} + s_1^*(\mathbf{F}\mathbf{H}^{-1})(\mathbf{F}\mathbf{H}^{-1})^T + s_2^*\mathbf{F}\mathbf{H}^{-1}\mathbf{m}^* \otimes \mathbf{F}\mathbf{H}^{-1}\mathbf{m}^*. \tag{3.15}$$

Agreement on all possible responses (O.5) means (3.15) must hold for all invertible **F** with positive determinant. Setting first $\mathbf{F} = \mathbf{1}$ and then $\mathbf{F} = \alpha\mathbf{1}$ $(\alpha > 0, \alpha \neq 1)$ yields

$$s_0\mathbf{1} + s_1\mathbf{1} + s_2\mathbf{n} \otimes \mathbf{n} = s_0^*\mathbf{1} + s_1^*\mathbf{H}^{-1}\mathbf{H}^{-T} + s_2^*\mathbf{H}^{-1}\mathbf{m}^* \otimes \mathbf{H}^{-1}\mathbf{m}^* \tag{3.16}$$

and

$$s_0\mathbf{1} + \alpha^2\{s_1\mathbf{1} + s_2\mathbf{n} \otimes \mathbf{n}\} = s_0^*\mathbf{1} + \alpha^2\{s_1^*\mathbf{H}^{-1}\mathbf{H}^{-T} + s_2^*\mathbf{H}^{-1}\mathbf{m}^* \otimes \mathbf{H}^{-1}\mathbf{m}^*\}. \tag{3.17}$$

Thus $\alpha^2\times$ (3.16) – (3.17) gives

$$(\alpha^2 - 1)s_0 = (\alpha^2 - 1)s_0^*,$$

whence

$$s_0^* = s_0. \tag{3.18}$$

Accordingly (3.15) reduces to

$$s_1\mathbf{F}\mathbf{F}^T + s_2\mathbf{F}\mathbf{n} \otimes \mathbf{F}\mathbf{n} = s_1^*\mathbf{F}\mathbf{H}^{-1}\mathbf{H}^{-T}\mathbf{F}^T + s_2^*\mathbf{F}\mathbf{H}^{-1}\mathbf{m}^* \otimes \mathbf{F}\mathbf{H}^{-1}\mathbf{m}^*. \tag{3.19}$$

Pre-multiplication by $\mathbf{F}^{-1}$ and post-multiplication by $\mathbf{F}^{-T}$ yields

$$s_1\,\mathbf{1}+s_2\,\mathbf{n}\otimes\mathbf{n}=\mathbf{H}^{-1}(s_1^*\,\mathbf{1}^*+s_2^*\mathbf{m}^*\otimes\mathbf{m}^*)\mathbf{H}^{-T}. \tag{3.20}$$

Relations (3.18) and (3.20) are necessary and sufficient conditions for observer agreement. The time-independent tensor $\mathbf{H}^{-1}$ (which maps $\mathcal{V}^*$ into $\mathcal{V}$) admits a polar decomposition

$$\mathbf{H}^{-1}=\mathbf{VR} \tag{3.21}$$

where[17]

$$\mathbf{V}\in \mathrm{Sym}^+\mathcal{V} \quad\text{and}\quad \mathbf{R}\in \mathrm{Orth}(\mathcal{V}^*,\mathcal{V}). \tag{3.22}$$

Using (3.21), (3.20) becomes

$$s_1\,\mathbf{1}+s_2\,\mathbf{n}\otimes\mathbf{n}=s_1^*\mathbf{V}^2+s_2^*\mathbf{Vm}\otimes\mathbf{Vm}, \tag{3.23}$$

where

$$\mathbf{m}:=\pm\,\mathbf{Rm}^*. \tag{3.24}$$

Relation (3.20) holds provided (3.23) is satisfied, with $\mathbf{H}$ delivered via (3.21) *for any* $\mathbf{R}$. Clearly (3.23) is satisfied by $\mathbf{V}=\mathbf{1},\ \mathbf{m}=\mathbf{n},\ s_1^*=s_1$ and $s_2^*=s_2$. It follows that any isometric map $\boldsymbol{\lambda}$ of the reference configuration $\boldsymbol{\kappa}$ selected by O, into space $\mathcal{E}^*$ as perceived by any other observer O^* (which will accordingly have a gradient $\mathbf{R}^T$ with value in Orth $(\mathcal{V},\mathcal{V}^*)$), will serve as a reference configuration for O^* with (see also (3.18)) the same material constants as for O and (see (3.24)) with $\mathbf{m}^*=\pm\,\mathbf{R}^T\mathbf{n}$. Of course, this result is not surprising. However, other non-trivial possibilities exist. For brevity we merely list two other cases, but note that a complete and systematic characterisation of solution pairs $(\mathbf{V},\mathbf{m})$, with corresponding values of s_1^* and s_2^* delivered these results.

Case 1: $\mathbf{V}=\alpha\mathbf{1}\ (\alpha>0),\ \mathbf{m}=\pm\mathbf{n};\ s_1^*=s_1/\alpha^2$, and $s_2^*=s_2/\alpha^2$. This corresponds to map $\boldsymbol{\lambda}$ being the composition of a dilatation, with scaling factor α^{-1}, of the body in configuration $\boldsymbol{\kappa}$ with an arbitrary isometry having gradient $\mathbf{R}^T$. Of course (see (3.24)), $\mathbf{m}^*=\pm\,\mathbf{R}^T\mathbf{n}$.

Case 2: Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ denote any orthonormal basis with $\mathbf{e}_1=\mathbf{n}$, and suppose s_1 and (s_1+s_2) have the same sign (see Remark 7 below). Then for arbitrary positive λ_1,λ_3, we can have

$$\mathbf{V}=\lambda_1\mathbf{e}_1\otimes\mathbf{e}_1+\lambda_2\mathbf{e}_2\otimes\mathbf{e}_2+\lambda_3\mathbf{e}_3\otimes\mathbf{e}_3,$$

where $\lambda_2^2=s_1\lambda_1^2/(s_1+s_2),\ \mathbf{m}=\pm\,\mathbf{e}_3,\ s_1^*=(s_1+s_2)/\lambda_1^2$, and $s_2^*=s_1(\lambda_3^{-2}-\lambda_2^{-2})$. Here $\boldsymbol{\mu}^*$ is obtained from $\boldsymbol{\kappa}$ via three simple stretches (two of which are related) of magnitudes λ_1^{-1} in the $\mathbf{n}$ direction and $\lambda_2^{-1},\lambda_3^{-1}$ in mutually orthogonal directions transverse to the axis of symmetry defined by $\mathbf{n}$, followed by an arbitrary isometry having gradient $\mathbf{R}^T$. Here $\mathbf{m}^*=\pm\,\mathbf{R}^T\mathbf{m}=\pm\,\mathbf{R}^T\mathbf{e}_3$ defines the axis of symmetry in $\boldsymbol{\mu}^*$.

Remark 7. In configuration $\boldsymbol{\kappa}$ the material described by (3.10) has residual stress $(s_0+s_1)\mathbf{1}+s_2\,\mathbf{n}\otimes\mathbf{n}$. A deformation with gradient $\mathbf{F}=\alpha\mathbf{n}\otimes\mathbf{n}+\beta(\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3)$, here $\mathbf{e}_2,\mathbf{e}_3$ are as in Case 2 – yields a zero stress for choice $\alpha=(-s_0/(s_1+s_2))^{1/2}$ and $\beta=(-s_0/s_1)^{1/2}$. That is, it is possible to find a stress-free configuration provided s_1 and s_1+s_2 have the same sign *and* s_0 and s_1 have opposite signs.

Remark 8. In Sect. 4.3 of [3] Liu criticises the foregoing analysis since it apparently embodies 'a denial of the assumption of free choice of reference configuration'. This criticism is misconceived. While any choice of reference configuration serves to describe elastic behaviour, the existence of any specific material symmetry will delineate a distinguished class of such configurations in which this symmetry is evident.[18] The transverse isotropy of material (3.10) was supposed to have been recognised by both O and O^* in our analysis, together with its specific form (see (i)–(iii)). As a consequence it was possible for O to delineate all possible reference configurations that could be selected by O^* so as to agree on the symmetry and on the form of (3.10). A simpler example is provided by Liu in footnote 6 of [3], namely

$$\mathbf{T}=s_1\,\mathbf{B}\quad\text{for}\quad O\quad\text{and}\quad \mathbf{T}^*=s_1^*\,\mathbf{B}^*\quad\text{for}\quad O^*. \tag{3.25}$$

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[17] $\mathrm{Sym}^+\mathcal{V}$ denotes the set of symmetric positive-definite tensors on $\mathcal{V}$ and $\mathrm{Orth}(\mathcal{V}^*,\mathcal{V})$ the set of orthogonal tensors from $\mathcal{V}^*$ onto $\mathcal{V}$.

[18] Selection of such configurations in practice can be difficult: see, for example, Jones [17] in which the problem of identifying principal material directions for orthotropic materials is discussed on pages 41 and 55. Notice that the very practice of selection runs counter to Liu's insistence on a common reference configuration.

Here O and O^* have recognised the existence of configurations with respect to which the material is isotropic, and each has chosen a reference configuration which reflects this symmetry.[19] From (2.2), (3.12) and (3.25) we have

$$\begin{aligned} s_1^*(\mathbf{QFH}^{-1})(\mathbf{QFH}^{-1})^T &= s_1^*\,\mathbf{F}^*(\mathbf{F}^*)^T = s_1^*\,\mathbf{B}^* \\ &= \mathbf{T}^* = \mathbf{QTQ}^T = s_1\,\mathbf{QBQ}^T = s_1\,\mathbf{QFF}^T\,\mathbf{Q}^T. \end{aligned} \tag{3.26}$$

Consequently

$$s_1^*\,\mathbf{H}^{-1}\mathbf{H}^{-T} = s_1\,\mathbf{1}. \tag{3.27}$$

It follows that $\mathbf{H} = \alpha\mathbf{R}$ for *any* orthogonal tensor $\mathbf{R}$ from $\mathcal{V}$ into $\mathcal{V}^*$ and *any* $\alpha \in \mathbb{R}^+$, in which case $s_1^* = \alpha^2\,s_1$. Accordingly the recognition of isotropy by O and O^* in their selection of reference configurations means these configurations must be related by dilatations and/or rotations.

Remark 9. In Sect. 5 of [3] Liu incorrectly claims that O.1-5 do not suffice to exclude material response of the form

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{x}, t). \tag{3.28}$$

That is, for any observer O the stress at a given point $\mathbf{x} \in \mathcal{E}$ and *time*[20] t is an explicit function[21] of both $\mathbf{x}$ and t. O.4 requires that for any other observer O^* the stress is given by

$$\mathbf{T}^*(\mathbf{x}^*, t^*) = \check{\mathbf{T}}^*\,(\mathbf{x}^*, t^*). \tag{3.29}$$

If $\mathbf{x}_0 \in \mathcal{E}$ is a fixed point and in (2.1) we set $\mathbf{y} = \mathbf{x}_0$ and $\mathbf{c}^*(t) := \alpha_t(\mathbf{x}_0)$ then

$$\mathbf{x}^* = \mathbf{c}^*(t) + \mathbf{Q}_t(\mathbf{x} - \mathbf{x}_0). \tag{3.30}$$

Hence from (2.2) and (3.28)–(3.30),

$$\check{\mathbf{T}}^*(\mathbf{c}^*(t) + \mathbf{Q}_t(\mathbf{x} - \mathbf{x}_0), at + b) = \mathbf{Q}_t\,\hat{\mathbf{T}}(\mathbf{x}, t)\mathbf{Q}_t^T. \tag{3.31}$$

From O.5 this is to hold for all possible relative motions of O^* and O. In particular consider O and O^* at relative rest. That is, $\mathbf{c}^*(t) = \mathbf{c}^*$ (any fixed point in $\mathcal{E}^*$) and $\mathbf{Q}_t = \mathbf{Q}_0$ (any orthogonal transformation from $\mathcal{V}$ into $\mathcal{V}^*$). Thus (3.31) becomes

$$\check{\mathbf{T}}^*(\mathbf{c}^* + \mathbf{Q}_0(\mathbf{x} - \mathbf{x}_0), at + b) = \mathbf{Q}_0\,\hat{\mathbf{T}}(\mathbf{x}, t)\mathbf{Q}_0^T. \tag{3.32}$$

The right-hand side of (3.32) does not change as different choices of $\mathbf{c}^*$ are made. Thus $\check{\mathbf{T}}^*$ does not depend upon its first argument. Similarly, different choices of time origin by O^*, corresponding to arbitrary choice of b, do not change the right-hand side of (3.32) and hence $\check{\mathbf{T}}^*$ does not depend upon its second argument. Thus $\check{\mathbf{T}}^*$ is constant. Now (3.32) implies $\hat{\mathbf{T}}$ must be independent of its arguments, so that from (3.28) and (3.29) both $\mathbf{T}$ and $\mathbf{T}^*$ are constant linear transformations, $\mathbf{T}_0$ and $\mathbf{T}_0^*$ say. However, (2.2) requires that for general relative motions

$$\mathbf{T}_0^* = \mathbf{Q}_t\,\mathbf{T}_0\,\mathbf{Q}_t^T \tag{3.33}$$

so that $\mathbf{T}_0^*$ *does* depend upon t. This contradiction implies that the initial hypothesis must be false. That is, material response of form (3.28) is ruled out (by O.4 and O.5). The *reductio ad absurdum* nature of the foregoing argument seems not to have been recognised by Liu who confuses the hypothesis (3.28) in this logical argument with the assumption of arbitrary relative motions contained in O.4, 5.

[19] Only via such selection can dependence upon $\mathbf{F}$ be reduced to one upon $\mathbf{B}$. Notice that the matter is not entirely trivial since the material has no 'natural' (that is, stress-free) configuration.

[20] On selecting a time scale and origin observer O can associate a time $t \in \mathbb{R}$ with any given instant. From O.3 (specifically agreement on time *lapses*) the corresponding time for O^* is $t^* = at + b$. Here $a, b \in \mathbb{R} : a > 0$ is a specific constant associated with (if $a \neq 1$) a different choice of time *units* and b is *arbitrary*, corresponding to choice of time origin.

[21] In distinction to implicit dependence as, for example, a viscous fluid for which $\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\rho(\mathbf{x}, t(), \mathbf{D}(\mathbf{x}, t))$: see [1], $(3.17)_1$.

4 A key modelling issue

As indicated in (a) of Sect. 2.3, Liu's general form (2.3) of the constitutive relation for Cauchy stress can be shown (by the reasoning of [1], which invokes O.4 and O.5 and accordingly different hypothetical relative motions of observer pairs) to *imply isrbm* for the response of such a material. This is to be contrasted with Liu's *postulation* of 'form invariance' (2.4), which is equivalent to *isrbm* modulo the generally-accepted relation (2.2). At this point the reader may wonder what the fuss is all about, since (2.3) is very general, and *isrbm is* satisfied. *The problem stems from how* O.4 *is to be regarded when* (2.3) *holds for an* **inertial** *observer* $O_{\rm in}$/frame $\phi_{\rm in}$. There are two ways to proceed.

P.1. Disregard the inertial nature of $O_{\rm in}$, reason as above, and deduce satisfaction of *isrbm* by $\boldsymbol{\mathcal{F}}_{\phi_{\rm in}}$, or

P.2. note the special character of inertial frames (O.2) and accordingly (via O.4) take the constitutive relation for an arbitrary observer O^* to be

$$\mathbf{T}^*(X,t) = \boldsymbol{\mathcal{F}}_{\phi^*}((\boldsymbol{\chi}_{\rm in}^t)^*, X, t), \tag{4.1}$$

where $(\boldsymbol{\chi}_{\rm in}^t)^*$ is the motion with respect to an inertial frame chosen by O^*. Invocation of O.5 does *not* now imply that *isrbm* holds for $\boldsymbol{\mathcal{F}}_{\phi_{\rm in}}$ but only that this response function be invariant under Galilean transformations.

The foregoing is exemplified by the history of confusion associated with the dependence of stress upon spin (skew part of the velocity gradient) derived in the kinetic theory of gases and attributed to Burnett [18]. Failure to recognise that Burnett's equations were derived in an inertial frame led many authors (see, for example, [5], [19], [20]) to note incompatibility with *isrbm* and conclude that material frame-indifference (identified as *isrbm*) could not be a principle, but rather a useful approximation. Truesdell [21] argued that the form and approximative nature of the Burnett equations could not be regarded as constitutive relations in the sense of continuum mechanics. Müller [10] alone recognised that the spin-dependence was that upon spin relative to an (hence any) inertial frame, so demonstrating that *in this context procedure* P.2 *must be adopted*. The Burnett equations are Galilean invariant and hence consistent with O.4, 5. Müller's general form of these equations is also consistent with O.4, 5: see Remark 11 in [1].

An entirely analogous situation has arisen in the phenomenological description of Helium II in which the balance of linear momentum for the superfluid involves a diffusive force density $\mathbf{F}_{sn}$ which accounts for the differing motions of superfluid and normal fluid (see [11], p.189). Independently, Hall and Bekarevich & Khalatnikov derived an explicit relation for $\mathbf{F}_{sn}$ which involves superfluid spin (see (3.15)–(3.17) in [11]) and consequently violates *isrbm*. However, interpreting such spin as that with respect to an (any) inertial frame[22] results in a constitutive relation for $\mathbf{F}_{sn}$ in complete accord with O.4, 5.

Remark 10. The foregoing distinction between P.1 and P.2 highlights the crucial nature of O.4 in the controversy over the implications of material frame-indifference, in contrast to Liu's dismissive view of this assumption.

5 Scales of mass, length, and time

Two observers cannot in general be expected to select the same scales of mass, length, and time. This implies that the foregoing requires modification. Specifically, agreement between O and O^* upon distances between simultaneous events results in the modification of (2.1) to

$$\mathbf{y}^* - \mathbf{x}^* = \alpha \mathbf{Q}_t(\mathbf{y} - \mathbf{x}). \tag{5.1}$$

Here α is the dimensionless factor necessary to convert length units employed by O to those adopted by O^*. Of course, there will be a different scaling factor for each category of physical quantities which have the same mass M, length L, and time T, dimensions. In particular suppose β is the factor relevant to $ML^{-1}T^{-2}$ quantities (e.g. tractions). Thus if $\mathbf{t}(\mathbf{t}^*)$ denotes the traction field on a surface with unit normal field $\mathbf{n}$ $(\mathbf{n}^*)$ and the corresponding stress field is $\mathbf{T}$ $(\mathbf{T}^*)$ for observer O (O^*) then

$$\mathbf{t}^* = \beta \mathbf{Q}_t\, \mathbf{t} \quad \text{and} \quad \mathbf{n}^* = \mathbf{Q}_t\, \mathbf{n} \tag{5.2}$$

[22] To many physicists the appearance of spin in constitutive relations only makes sense with this interpretation. This may explain why many researchers have no problems with the cited relations (3.15)–(3.17), in contrast to continuum mechanicians who subscribe to *isrbm*: see [22].

at any instant 't'. (Of course, 'unit normal vectors' are dimensionless and no scaling factor is involved in $(5.2)_2$.) Since

$$\mathbf{t} = \mathbf{T}\mathbf{n} \quad \text{and} \quad \mathbf{t}^* = \mathbf{T}^*\mathbf{n}^* \tag{5.3}$$

it follows (since $\mathbf{n}$ is *any* unit vector) that

$$\mathbf{T}^* = \beta\mathbf{Q}_t\,\mathbf{T}\mathbf{Q}_t^T. \tag{5.4}$$

Revisiting the arguments of [1], and taking due account of the foregoing (and appropriate scaling factors when calculating spatial and temporal derivatives of relations between quantities associated with O and O^*), it can be shown that the restrictions imposed upon the response function(s) relevant to O involve only proper orthogonal tensors $\mathbf{Q}$ on $\mathcal{V}$. Essentially this is because in all cases $\mathbf{Q}$ is of the form $(\alpha\mathbf{Q}_2)^{-1}(\alpha\mathbf{Q}_1)$, with $\mathbf{Q}_1$ and $\mathbf{Q}_2$ orthogonal maps from $\mathcal{V}$ into $\mathcal{V}^*$: that is, $\mathbf{Q} = \mathbf{Q}_2^T\,\mathbf{Q}_1$ as in [1]. Thus the standard restrictions upon response functions for any given observer O remain unchanged as a consequence of the arguments advanced in [1], no matter what choices of units of mass, length, and time are made by other observers.

6 Concluding remarks

It is not always clear just what viewpoint is adopted by those who subscribe to the *isrbm* interpretation of material frame-indifference[23] in respect of the spin (relative to inertial frames) that appears to play a rôle in determining the behaviour of gases [10], superfluid helium [11], turbulence [12] and the lack of 'form-invariance' associated with modelling magneto-elastic phenomena [13]. An extreme viewpoint is to deny any possibility of spin-dependence, and hence call into question any theories associated therewith.[24] Another is to regard this version of material frame-indifference as an approximation which is very good for most materials. While clearly any statement of the principle cannot be exact in an ultimate sense (since all behaviour is modified by the process of observation, codified in its quantum-mechanical description) the viewpoint here advanced has addressed and (modulo acceptance of O.1–5) resolved issues which have caused confusion. Specifically, the viewpoint here advanced

(i) is systematic in delineating what is entailed in reaching observer consensus and provides a rationale for obtaining restrictions upon *any* candidate response function(s)[25],
(ii) does not require different observers to employ the same response functions[26] in modelling any specific behaviour nor to adopt the same scales of mass, length, and time,
(iii) leads to standard restrictions on response functions in terms only of proper orthogonal tensors,
(iv) in delivering the standard restrictions on response functions obtained in [1] for elastic and viscous fluids (and simple materials in general), has shown that the responses of these materials satisfy *isrbm* as a *consequence* of O.1-5 rather than via an *a priori* assumption of *isrbm* as a 'principle' (indeed the arguments deliver *isrbm* for *all* materials considered by Liu), and,
(v) for materials with a constitutive dependence upon motion history relative to inertial frames (for example, spin relative to such frames), requires only that the response functions be Galilean invariant.

[23] Interestingly, Bertram and Svendsen [23] did not regard *isrbm* to be a general principle for all materials. However, they did not address restrictions upon response functions which do not satisfy *isrbm*.

[24] Sceptics should consult [11], Sect. 3(e) for theories which describe vortex motion in Helium II, particularly the interpretation of *isrbm* in the last paragraph. That 'spin' therein should more properly be that with respect to inertial frames was given support in [22] (see Eq. (1.3)).

[25] Liu criticises the lack of a mathematical formulation of O.4, 5. Given the impossibility of foreseeing all possible future models it is difficult to see how such might be drafted. Liu's own formalism, while general, does not include any microstructural features as required to describe liquid crystalline behaviour nor address constitutive relations of form (4.1).

[26] A variety of response functions were explicitly given in Sect. 3.4 which model, for different observers, the material exemplified by Liu in (3.10). Accordingly this is a counter-example to I.1′.

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