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Original article

Hyperbolicity in Extended Thermodynamics of Fermi and Bose gases

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Abstract. We consider the balance system of Extended Thermodynamics with 13 Moments in the case of Fermi and Bose gases, for processes not far from equilibrium. In this case, the hyperbolicity of the differential system holds only in a neighborhood of the equilibrium state. The main aim of the paper is to evaluate the hyperbolicity region of the differential system. The knowledge of this region in the state variables is mandatory to check the admissibility of the solutions and the corresponding boundary and Cauchy data in the limit of the approximation considered. The results are obtained through numerical evaluations of the Fermi and Bose integral functions $I_n^{\pm}(\alpha)$ that appear in the characteristic polynomial. Particular attention is devoted to the completely degenerate case when Fermi gas reaches the $0K$ and when the Bose gas is in proximity of the transition temperature T_c. In these limiting cases, the hyperbolicity requirement is lost according to previous results. In the last section we make use of the Maxwellian iteration in order to evaluate the heat conductivity and the viscosity for the degenerate Fermi and Bose gas.

Key words: Extended Thermodynamics, hyperbolic systems, degenerate gases

1 Introduction

The entropy principle plays a fundamental role in Extended Thermodynamics (ET) [1]. In fact it provides a powerful constraint for selecting the physical constitutive equations in the case of classical solutions, and it becomes a fruitful selection rule for admissible weak solutions [2,3]. Furthermore, if the principle is combined with the stability requirement of the concavity of the entropy density, it allows one to rewrite the field equations in the form of a symmetric hyperbolic system through the introduction of the privileged main field components [4,5]. In the case of a moments system associated with the Boltzmann Transport Equation (BTE), the principle permits the closure of the system and the corresponding distribution function coincides with the one obtained by the Maximum Entropy Principle (MEP) [6–11].

The Entropy Principle (EP) is exploited in the full non-linear case without any assumptions about the non-equilibrium processes. If we consider processes not far from an equilibrium state, an approximate distribution function is usually derived through a formal expansion in the neighborhood of the local equilibrium and so, we obtain Extended Thermodynamics theories of M moments and degree α (ET_M^α theories). The Grad approach correspond to $M = 13$ and $\alpha = 1$.

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The non-linear closure suffers from some analytical problems that were first discovered by Junk and coworkers [12], in particular concerning the domain $\mathcal{D}$ of invertibility between the field and the main field (Lagrange multipliers) and the integrability of the moments (see also [13]). Instead, ET_M^α does not have these kinds of problems. In fact, thanks to the equilibrium distribution function that dominates any polynomial, all the expressions for the moments are integrable. Therefore, from now on, we will forget the original non-linear problem that justifies in a formal sense the expansion, and we will consider only the ET_{13}^1 and ET_{13}^2 theories in the case of degenerate Fermi and Bose gas. We have to recall that also in the case of non-degenerate gas for such ET_M^α theories the general statements of Extended Thermodynamics hold only in an approximate sense. Typically the concavity of the entropy density and the hyperbolicity are now valid only in a neighborhood of an equilibrium state (*hyperbolicity domain*).

The knowledge of the hyperbolicity region is mandatory for checking the admissibility of the solutions and the corresponding boundary and Cauchy data in the limit of the approximation considered. For non degenerate gases the problem was already investigated by Müller and Ruggeri in the case of 13-moments [1] and by Brini in the case of 14-moment [14].

Moreover, as it was pointed out by Ruggeri and Seccia [15] the full degenerate gas presents several problems concerning the non existence of hyperbolicity for the differential system.

The aim of this paper is to improve on the previous result with a great attention to the Fermi and Bose integral functions $I_n^\pm(\alpha)$ that permits one to evaluate, by an analysis of the characteristic polynomial, the hyperbolicity region for the degenerate gas. We verify that the hyperbolicity is lost in the complete degenerate case of Fermi gas corresponding to the temperature of 0 K and in the case of the change of phase for a completely degenerate Bose gas, in equilibrium conditions, near the critical temperature T_c.

2 The BTE and moment's method

Let us consider the Boltzmann transport equation for the distribution function $f(\boldsymbol{c},\boldsymbol{r},t)$ being $\boldsymbol{c}$ and $\boldsymbol{r}$, respectively the velocity and the position of single particle, at the time t, in the phase space. It follows that

$$\frac{\partial f}{\partial t} + f_i\,\frac{\partial f}{\partial c_i} + c_k\,\frac{\partial f}{\partial x_k} = Q(f) \tag{1}$$

where f_i is the resulting of the specific external forces while the right member describes the collisional processes. To pass from the kinetic level of the BTE to the Hydrodynamic (HD) level of the balance equations in the general framework of the moment theory [1], the following generalized kinetic fields must be considered:

$$\psi_A(\boldsymbol{c}) = \{m,\ m\,c_{i_1},\ m\,c_{i_1}\,c_{i_2},\ \ldots,\ m\,c_{i_1}\,c_{i_2}\cdots c_{i_s}\,,\ldots\} \tag{2}$$

where m is the single particle mass and $s = 1,2,\ldots N$ with arbitrary values for the integer N. By multiplying eq.(1) by $\psi_A(\boldsymbol{c})$ and integrating in velocity space we obtain the following set of (HD) moment equations [1]

$$\frac{\partial F_A}{\partial t} + \frac{\partial F_{Ak}}{\partial x_k} = S_A + P_A \quad , \qquad \text{with} \quad A = 1,\ldots \mathrm{M} \tag{3}$$

where M is the number of moments used, and F_A, F_{Ak}, S_A P_A indicate, respectively, the densities, the fluxes, the external field productions, and the collisional productions defined as:

$$F_A = \int \psi_A(\boldsymbol{c})\, f(\boldsymbol{c},\boldsymbol{r},t)\, d\boldsymbol{c} \qquad , \quad F_{Ak} = \int \psi_A(\boldsymbol{c})\, c_k\, f(\boldsymbol{c},\boldsymbol{r},t)\, d\boldsymbol{c} \tag{4}$$

$$S_A = f_i \int \frac{\partial \psi_A(\boldsymbol{c})}{\partial c_i}\, f(\boldsymbol{c},\boldsymbol{r},t)\, d\boldsymbol{c} \quad , \quad P_A = \int \psi_A(\boldsymbol{c})\, Q(f)\, d\boldsymbol{c}\,. \tag{5}$$

With this procedure, we obtain a system of partial differential equations of finite order where the flux of each equation of set (3) becomes the density of the successive equation. The structure of this system of equations shows that there are some unknown functions $H_A = \{\overline{F}_{Ak},\ P_A\}$ that must be determined in terms of the variables F_A ($\overline{F}_{Ak}$ represents the flux that are not present in the list of the densities F_A). In the ET theory the closure is obtained requiring that the truncated system satisfies an entropy law [1]. This procedure was proved equivalent to the MEP [6,7].

In the case of the first 13 moments we have:

$$\begin{aligned}
&\psi_A = \{m,\ m\,c_i,\ m\,c_i\,c_j,\ m\,c^2 c_i\} \\
&F_A = \{F,\ F_i,\ F_{ij},\ F_{ill}\} \\
&F_{Ak} = \{F_k,\ F_{ik},\ F_{ijk},\ F_{illk}\} \\
&S_A = \{0, F f_i,\ 2\,F_{(i} f_{j)},\ 3\,F_{(il} f_{l)}\} \\
&P_A = \{0,\ 0,\ P_{\langle ij \rangle},\ P_{ill}\}
\end{aligned} \tag{6}$$

where P, P_i, P_{ll} are null for the conservation of total particle number, momentum and energy, respectively and $H_A = \{F_{\langle ijk \rangle},\ F_{illk},\ P_{\langle ij \rangle},\ P_{ill}\}$.

If we define the mean velocity v_i of the gas and the peculiar velocity $C_i = c_i - v_i$,, it is possible to decompose, by Galilean invariance, both the fields F_A and the functions H_A in convective and non-convective parts [16, 1]. Thus by introducing the new set of kinetic fields $\psi_A(\boldsymbol{C})$ and the new set of variables $m_A = \{\rho,\ m_{ll} = 3\,p,\ m_{\langle ij \rangle},\ m_{ill} = 2\,q_i\}$ with ρ the *mass density*, p the *pressure*, $m_{\langle ij \rangle}$ the *stress deviator* and q_i the *heat flux*, we obtain

$$\begin{aligned}
&F = \rho \quad , \quad F_i = \rho\, v_i \quad , \quad F_{ll} = 3\,p + \rho\, v^2\ , \\
&F_{\langle ij \rangle} = m_{\langle ij \rangle} + \rho\, v_{\langle i}\, v_{j \rangle} \quad , \quad F_{ill} = 2\,q_i + 2\,m_{\langle il \rangle}\, v_l + 5\,p\, v_i + \rho\, v^2\, v_i\ ,
\end{aligned} \tag{7}$$

Analogously in the case of unknown functions we can write, for the fluxes

$$\begin{aligned}
&F_{\langle ijk \rangle} = m_{\langle ijk \rangle} + 3\,m_{(\langle ij \rangle}\, v_{k)} - \frac{2}{5} v_r\, \left(m_{\langle ri \rangle}\, \delta_{jk} + m_{\langle rk \rangle}\, \delta_{ij} + m_{\langle rj \rangle}\, \delta_{ki}\right) + \rho\, v_{\langle i}\, v_j\, v_{k \rangle}\ , \\
&F_{illk} = m_{illk} + 2\,v_r\, m_{\langle rik \rangle} + 7\,p\, v_i\, v_k + \frac{28}{5}\, q_{(i}\, v_{k)} + \frac{4}{5}\,(q_r\, v_r)\,\delta_{ik} + v^2\, m_{\langle ik \rangle} + p\, v^2\, \delta_{ik} + \\
&\qquad 2\,v_r\,(m_{\langle ri \rangle}\, v_k + m_{\langle rk \rangle}\, v_i) + \rho\, v^2\, v_i\, v_k\ ,
\end{aligned} \tag{8}$$

and for the collisional productions

$$P_{\langle ij \rangle} = R_{\langle ij \rangle}\ , \quad P_{ill} = R_{ill} + 2\,R_{\langle il \rangle}\, v_l\ , \tag{9}$$

Using relations (7–9) it is possible to rewrite the set of balance equations (3) in terms of the new variables $\{\rho,\ v_i,\ p,\ m_{\langle ij \rangle},\ q_i\}$ (see eqs. (3.1) in [1]) with the new set of constitutive functions $G_A = \{m_{\langle ijk \rangle},\ m_{llik},\ R_{\langle ij \rangle},\ R_{ill}\}$, being

$$m_{<ijk>} = \int m\, C_{<i}\, C_j\, C_{k>}\, f\, d\boldsymbol{C}\ , \quad m_{llik} = \int m\, C^2\, C_i\, C_k\, f\, d\boldsymbol{C} \tag{10}$$

$$R_{\langle ij \rangle} = \int m\, C_{\langle i}\, C_{j \rangle}\, Q(f)\, d\boldsymbol{C} \quad , \quad R_{ill} = \int m\, C^2\, C_i\ Q(f)\, d\,\boldsymbol{C} \tag{11}$$

where, in order to determine the productions R_A, it is necessary to take into consideration specific physical problems, because the production terms will depend on the nature of the collisions.

By the entropy principle requirement all unknown constitutive functions can be determined systematically together with the analytic expression for the non-equilibrium distribution function. In this way we obtain a closed system of balance equations for the densities F_A, and each solution of this set will be named a *thermodynamic process* for the system [1].

3 Closure

If we introduce the entropy density of the system

$$h = -k_B \int \left[f\, ln\left(\frac{1}{y} f\right) \pm y\, \left(1 \mp \frac{1}{y} f\right)\, ln\left(1 \mp \frac{1}{y} f\right)\right]\, d\boldsymbol{C} \tag{12}$$

where the upper and lower signs correspond to fermions and bosons, respectively, k_B is Boltzmann's constant and $y = (2s + 1)\,(m/h)^3$ with $s\hbar$ particle spin. The entropy principle or the maximum entropy principle [1, 6–11] implies the non-equilibrium distribution function

$$f = \frac{y}{\exp \widehat{\Sigma} \pm 1} \quad \text{with} \quad \widehat{\Sigma} = \sum_{A=1}^{13}\ \psi_A(\boldsymbol{C})\, \widehat{\lambda}_A \quad . \tag{13}$$

We shall expand the carriers distribution function around an appropriate local equilibrium state defined by a *local equilibrium function* depending on the time and space by means of temperature $T(\boldsymbol{r}, t)$ and mass density $\rho(\boldsymbol{r}, t)$.

3.1 General properties of the variables of local equilibrium

One can write the quantity $\widehat{\Sigma}$ defined by $(13)_2$ explicitly and decompose it into a *local equilibrium part* Σ_0

$$\Sigma_0 = \alpha + \beta \frac{1}{2} m C^2 \tag{14}$$

and a non equilibrium contribution Σ

$$\Sigma = \lambda + \lambda_{ll} mC^2 + \lambda_i C_i + \lambda_{<ij>} m C_{<i}C_{j>} + \lambda_{ill} mC^2 C_i \tag{15}$$

where $\alpha = \widehat{\lambda}|_E$ and $\beta = 2\,\widehat{\lambda}_{ll}|_E$ denote the local equilibrium values, whereas the remaining *Lagrange multipliers* will vanish in local equilibrium. Let λ_A denote the non equilibrium parts of $\widehat{\lambda}_A$. For a Fermi and a Bose gas we have

$$f|_E = \frac{y}{\exp\left(\alpha + \frac{1}{2}\beta m C^2\right) \pm 1} = \frac{y}{\exp\left[(m/2\, C^2 - \mu)/(k_B\, T)\right] \pm 1} \tag{16}$$

being $\alpha = -\mu/(k_B T)$ and $\beta = 1/(k_B T)$, with μ the gas chemical potential.

By utilizing the *distribution function* $f|_E$, it is possible to calculate the variables of local equilibrium $\{\rho, p\}$

$$\rho(\boldsymbol{r}, t) = \int f|_E\, d\boldsymbol{C} = \frac{T^{\frac{3}{2}}}{\gamma}\, I_2^{\pm}(\alpha) \quad , \tag{17}$$

$$p(\boldsymbol{r}, t) = \frac{1}{3} \int m C^2 f|_E\, d\boldsymbol{C} = T^{5/2} F(z), \tag{18}$$

where assuming $z = \rho T^{-3/2}$ we have

$$F(z) = \frac{2}{3} \frac{k_B}{m} \frac{1}{\gamma} I_4^{\pm}(\alpha) \quad , \qquad \gamma = \frac{\sqrt{\pi}}{4} \frac{1}{m(2s+1)} \left(\frac{2\pi\hbar^2}{m k_B}\right)^{\frac{3}{2}} \quad , \tag{19}$$

being, in general, $I_n^{\pm}(\alpha)$ the Fermi and the Bose integral functions

$$I_n^{\pm}(\alpha) = \int_0^{+\infty} \frac{x^n}{\exp(\alpha + x^2) \pm 1}\, dx. \tag{20}$$

The $I_n^{\pm}(\alpha)$ satisfy the differentiation property with the recurrence relation

$$\frac{d^r\, I_n^{\pm}(\alpha)}{d\,\alpha^r} = (-1)^r \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} - r\right)}\, I_{n-2r}^{\pm}(\alpha) \tag{21}$$

where, it should be noted that, the functions $I_n^{\pm}(\alpha)$ can be continued analytically for complex value of n [17, 18] and the recurrence relation (21) can be not restricted necessarily to positive values [17,18] of n. Being $\{\alpha,\, p,\, F(z)\}$ functions of $\{\rho,\, T\}$ it is easy to show, in terms of integral $I_n^{\pm}(\alpha)$, that

$$\frac{\partial\, \alpha}{\partial\, \rho} = -\frac{2}{\rho} \frac{I_2^{\pm}}{I_0^{\pm}} \qquad \frac{\partial\, \alpha}{\partial\, T} = \frac{3}{T} \frac{I_2^{\pm}}{I_0^{\pm}} \tag{22}$$

$$\frac{\partial\, p}{\partial\, \rho} = 2 \frac{k_B}{m} T \frac{I_2^{\pm}}{I_0^{\pm}} \qquad \frac{\partial\, p}{\partial\, T} = \frac{5}{2} \frac{p}{T} - 3 \frac{k_B}{m} \rho \frac{I_2^{\pm}}{I_0^{\pm}} \tag{23}$$

$$F'(z) = 2 \frac{k_B}{m} \frac{I_2^{\pm}}{I_0^{\pm}} \qquad F''(z) = 2 \frac{k_B}{m} \gamma \left\{ \frac{1}{I_0^{\pm}} + I_2^{\pm} \frac{I_{-2}^{\pm}}{\left(I_0^{\pm}\right)^3} \right\} \tag{24}$$

where the function $I_{-2}^{\pm}(\alpha)$ is define by the relation (21) with $r = 1$ and $n = 0$.

3.2 Non equilibrium distribution function

By expanding f up to second order around $f|_E$, with respect to the deviations from the local equilibrium state, and inserting this expansion into the moment expressions we obtain a non-linear system in the non-equilibrium variables λ_B

$$m_A - m_A\,|_E = \sum_{B=1}^{13} \lambda_B \int \psi_A(C) \left[\frac{\partial\, f}{\partial\, \lambda_B}\right]_E d\boldsymbol{C} \qquad (25)$$
$$+ \frac{1}{2} \sum_{D=1}^{13} \sum_{B=1}^{13} \lambda_D\, \lambda_B \int \psi_A(C) \left[\frac{\partial^2\, f}{\partial\lambda_D\, \partial\lambda_B}\right]_E d\boldsymbol{C}\ .$$

In order to express the quantities λ_B as functions of the *objective* variables $\{\rho,\, p,\, q_i,\, m_{<ij>}\}$, we shall use the representation for isotropic functions until quadratic terms in non equilibrium variables:

$$\begin{aligned}
&\lambda_i = \gamma_1\, q_i + \gamma_2\, m_{<ij>}\, q_j \\
&\lambda_{ill} = \alpha_1\, q_i + \alpha_2\, m_{<ij>}\, q_j \\
&\lambda_{<ij>} = \beta_1\, m_{<ij>} + \beta_2\, q_{<i}\, q_{j>} + \beta_3\, m_{<<ir>}\, m_{<rj>>} \\
&\lambda = \delta_1\, q_r\, q_r + \delta_2\, m_{<pq>}\, m_{<qp>} \\
&\lambda_{ll} = \sigma_1\, q_r\, q_r + \sigma_2\, m_{<pq>}\, m_{<qp>}
\end{aligned} \qquad (26)$$

where all the coefficients will be functions of $(\rho,\, T)$ to be determined.

By inserting the relations (26), into the system (25) and taking into account only the quadratic terms we obtain a non linear system of 11 equations for the coefficients $\{\delta_1,\, \delta_2,\, \sigma_1,\, \sigma_2,\, \alpha_1,\, \alpha_2,\, \gamma_1,\, \gamma_2,\, \beta_1,\, \beta_2,\, \beta_3\}$

3.3 Constitutive functions

Having obtained the *distribution function* as function of the fields $(\rho,\, p,\, m_{<ij>},\, q_i)$, we can determine the constitutive functions $(m_{<ijk>},\, m_{llpp},\, m_{ll<ij>})$ appearing in the fluxes of the balance equations. In this way, using (10), we obtain

$$\begin{aligned}
&m_{<ijk>} = A\, q_{<i}\, m_{<jk>>}, \\
&m_{ll<ik>} = A_1\, m_{<ik>} + A_2\, q_{<i}\, q_{k>} + A_3\, m_{<<ij>}\, m_{<jk>>}, \\
&m_{llpp} = 3\,[B_0 + B_1\, q_r\, q_r + B_2\, m_{<pq>}\, m_{<qp>}],
\end{aligned} \qquad (27)$$

where the coefficients $\{A,\, A_1,\, A_2,\, A_3,\, B_0,\, B_1,\, B_2\}$ are expressed through the relations

$$A = \left\{\frac{6}{35}\left[-\frac{70}{3}\,\frac{I_4^\pm}{I_2^\pm} + \frac{126}{5}\,\frac{I_6^\pm}{I_4^\pm}\right]\left[-\frac{10}{3}\,\frac{I_4^\pm}{I_2^\pm} + \frac{14}{5}\,\frac{I_6^\pm}{I_4^\pm}\right]^{-1}\right\}\frac{1}{p} \qquad (28)$$

$$A_1 = \left\{\frac{14}{5}\,\frac{I_6^\pm}{I_4^\pm}\right\}\frac{k_B}{m}\,T, \qquad (29)$$

$$A_2 = \left\{-\frac{4}{25}\left[84\,\frac{I_6^\pm}{I_2^\pm} + \frac{1764}{25}\left(\frac{I_6^\pm}{I_4^\pm}\right)^2 - \frac{700}{9}\left(\frac{I_4^\pm}{I_2^\pm}\right)^2 - \frac{396}{5}\,\frac{I_8^\pm}{I_4^\pm}\right]\right.$$
$$\left.\cdot\left[-\frac{10}{3}\,\frac{I_4^\pm}{I_2^\pm} + \frac{14}{5}\,\frac{I_6^\pm}{I_4^\pm}\right]^{-2}\right\}\frac{1}{p},$$

$$A_3 = \left\{\frac{4}{5}\,\frac{I_6^\pm}{I_4^\pm}\right\}\frac{k_B}{m}\,\frac{T}{p},$$

$$B_0 = \left\{ 2\, \frac{I_6^{\pm}}{I_4^{\pm}} \right\} \, \frac{k_B}{m} \, T\, p \,, \tag{30}$$

$$\begin{aligned} B_1 = \Bigg\{ \frac{2}{15} \, \Bigg[& -\frac{4000}{27} \, \left(\frac{I_4^{\pm}}{I_2^{\pm}} \right)^3 + \frac{2600}{3} \, \frac{\left(I_4^{\pm}\right)^2}{I_2^{\pm}\, I_0^{\pm}} - 1344 \, \frac{I_6^{\pm}}{I_0^{\pm}} \\ & + \frac{560}{3} \, \frac{I_6^{\pm}\, I_4^{\pm}}{\left(I_2^{\pm}\right)^2} + \frac{1176}{5} \, \frac{\left(I_6^{\pm}\right)^2}{I_4^{\pm}\, I_2^{\pm}} - 264 \, \frac{I_8^{\pm}}{I_2^{\pm}} + \frac{2376}{5} \, \frac{I_8^{\pm}\, I_2^{\pm}}{I_4^{\pm}\, I_0^{\pm}} \Bigg] \\ & \cdot \left[-\frac{10}{3} \, \frac{I_4^{\pm}}{I_2^{\pm}} + 6 \, \frac{I_2^{\pm}}{I_0^{\pm}} \right]^{-1} \, \left[-\frac{10}{3} \, \frac{I_4^{\pm}}{I_2^{\pm}} + \frac{14}{5} \, \frac{I_6^{\pm}}{I_4^{\pm}} \right]^{-2} \Bigg\} \, \frac{1}{p} \,, \\ B_2 = \Bigg\{ \frac{1}{6} \, \Bigg[& -20 \, \frac{I_4^{\pm}}{I_0^{\pm}} - \frac{28}{3} \, \frac{I_6^{\pm}}{I_2^{\pm}} + \frac{168}{5} \, \frac{I_2^{\pm}\, I_6^{\pm}}{I_0^{\pm}\, I_4^{\pm}} \Bigg] \cdot \left[-\frac{10}{3} \, \frac{I_4^{\pm}}{I_2^{\pm}} + 6 \, \frac{I_2^{\pm}}{I_0^{\pm}} \right]^{-1} \Bigg\} \, \frac{k_B}{m} \, \frac{T}{p} \,. \end{aligned}$$

4 Analysis of the hyperbolicity zone

In the one-dimensional case, by defining with $F_A = \{F, F_1, F_{ll}, F_{<11>}, F_{1ll}\}$ the vector of the variables, with F_{Ax} the vector of the fluxes, the system of balance equation (3) can be rewritten as

$$\frac{\partial F_A}{\partial t} + \frac{\partial F_{Ax}}{\partial x} = S_A + P_A \,, \qquad \text{with} \quad A = 1, \dots 5$$

We have that $F_A = F_A(U_B)$ and $F_{Ax} = F_{Ax}(U_B)$ being $U_B = \{\rho, v, T, \sigma, q\}$ the vector of the independent variables with $v = v_1$, $q = q_1$, $\sigma = -m_{<11>}$.

In this way we obtain the system

$$\mathcal{A}^0_{AC} \, \frac{\partial U_C}{\partial t} + \mathcal{A}_{AC} \, \frac{\partial U_C}{\partial x} = S_A + P_A \,, \tag{31}$$

where

$$\mathcal{A}^0_{AC} = \frac{\partial F_A}{\partial U_C} \,, \quad \mathcal{A}_{AC} = \frac{\partial F_{Ax}}{\partial U_C} \,. \tag{32}$$

The system (31) is said to be hyperbolic if:

1) The determinant of $\mathcal{A}^{\mathbf{0}}$ is not zero, i.e.,

$$\det \mathcal{A}^0_{AB} = -6\rho \, \frac{\partial p}{\partial T} \neq 0 \,; \tag{33}$$

2) all the roots $\lambda^{(i)}$ (with $i = 1, 2, \dots$) of the characteristic polynomial of $J_{AC} = (\mathcal{A}^0_{AB})^{-1} \, \mathcal{A}_{BC}$ are real,
3) if $\lambda^{(i)}$ has multiplicity $r^{(i)}$ there exist in correspondence $r^{(i)}$ linearly independent eigenvectors.

If we study the characteristic polynomial of $\mathbf{J}$, we obtain

$$\lambda^5 + g_4 \, \lambda^4 + g_3 \, \lambda^3 + g_2 \, \lambda^2 + g_1 \, \lambda + g_0 = 0 \,, \tag{34}$$

where the g_i coefficients are function of the variables $\{\rho, v, T, \sigma, q\}$. In order to simplify our analysis, it is convenient to define a *suitable sound speed* c

$$c = \left[\frac{10}{9} \, \frac{I_4^{\pm}(\alpha)}{I_2^{\pm}(\alpha)} \, \frac{k_B}{m} \, T \right]^{1/2} \tag{35}$$

and to introduce the dimensionless quantities

$$\tilde{\lambda} = \frac{(\lambda - v)}{c}, \quad \widetilde{\sigma} = \frac{\sigma}{c^2 \, \rho}, \quad \widetilde{q} = \frac{q}{c^3 \, \rho}. \tag{36}$$

In this case the polynomial assumes the simplified form

$$\tilde{\lambda}^5 + \tilde{g}_4 \, \tilde{\lambda}^4 + \tilde{g}_3 \, \tilde{\lambda}^3 + \tilde{g}_2 \, \tilde{\lambda}^2 + \tilde{g}_1 \, \tilde{\lambda} + \tilde{g}_0 = 0 \,, \tag{37}$$

where the new coefficients $\tilde{g}_i$ depend only on the dimensionless variables $\{\widetilde{\sigma},\, \widetilde{q},\, \alpha\}$ and they have a form which depends of the order of the series expansion of constitutive functions.

Besides, introducing the density entropy h,, we suppose that the function $-h$ is a strictly convex function with respect to the components $\{\rho,\, p,\, \sigma,\, q\}$, and in particular in the equilibrium state, we have that the matrix of second derivatives of $-h_E$ with respect to $\{\rho, p\}$ must be positive definite [1,15]:

$$0 < F'(z) < \frac{5}{3}\,\frac{F(z)}{z}\ . \tag{38}$$

As well known the convexity condition implies that the differential system is symmetric hyperbolic and the local Cauchy problem is well posed [19,3]. The conditions (38) guarantees the hyperbolicity only in the equilibrium state and, therefore, is very important to check the region, in the neighborhood of equilibrium, in which the hyperbolicity remains valid.

In the work that follows, we determine the hyperbolicity domain through a numerical computation of the roots of the characteristic polynomial. The coefficients of this polynomial depend on the Fermi and Bose integrals, consequently only by means an efficient evaluation of these integrals the study of hyperbolicity will be treated with numerical precision. The new section is devoted to give a summary of the results on the evaluation and approximation of these integrals to describe the details of the calculation procedure and to discuss the calculational error.

5 Analytical and numerical approximations for the Fermi and Bose integrals

The past years have seen numerous published studies and several tabulations for various $I_n^{\pm}(\alpha)$. Same authors did not tabulate the $I_n^{\pm}(\alpha)$ functions but rather the family of integrals

$$\mathcal{F}_j^{\pm}(\alpha) = \frac{1}{\Gamma(j+1)} \int_0^{\infty} \frac{\varepsilon^j}{\exp(\varepsilon+\alpha) \pm 1}\, d\varepsilon\,. \tag{39}$$

that are relate to the functions (20) by the relation

$$I_n^{\pm}(\alpha) = \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\, \mathcal{F}^{\pm}_{(n-1)/2}(\alpha)\,. \tag{40}$$

Several investigations have been made into the mathematical properties of expressions (39) or (40) and of their relationships to other mathematical functions, and in particular Dingle [20] has been discussed in detail, for the Fermi integrals, the advantages and disadvantages of these two forms.

The evaluation of these integrals can be classified in different way: 1) series expansions [17,18,20–26], 2) tabulations [17,27,28] 3) polynomial or rational approximations [29–34], 4) numerical integration schemes [27, 35–37].

The expansions of the Bose integral functions in power of α are desirable to evaluate them behavior for small positive α (in proximity of the transition temperature T_c). Robinson [18] explore a number of analytic aspects of the functions $\mathcal{F}_j^-$ and obtain a expansion in terms of power series whose functional behavior for small α is translucent and which are particularly well suited to numerical computation when $\alpha < 1$. Gautschi [36] consider a recent Gauss-type quadrature formulae based on rational functions to evaluate generalized Bose integrals to high accuracy.

Analogously, several Fermi integrals can become involved in semiconductor physics [28] and astrophysics [35]. The application range of the Fermi integrals functions may be shown in the treatment of transport effects, e.g., electrical and thermal conductivities, thermoelectric effects, magnetotransport [28] and their accurate evaluation is needed in the modelling of optoelectronic devices [38].

Sommerfeld [21] and Nordheim [22] examined the Fermi integrals for strongly degenerate conditions and were concerned with asymptotic mathematical forms that would be suitable for the Fermi gas of a normal metal.

These contributed to the basis both for a paper by McDougal and Stoner [17] containing an analysis and tabulation of same Fermi integral functions $\mathcal{F}_j^+$ and for results obtained by Rhodes [23] when j is a small positive integer.

Tabulations by Beer et al. [27] included reprinting the McDougal and Stoner entries and then extension of the tabulation order for half-integer values of j, using an Euler-Maclaurin numerical integration, while for arbitrary values of j several asymptotic expansions have been proposed by Dingle [20].

Extensive computations of $\mathcal{F}_j^+$ for different orders j is expressed in diverse forms in the literature. In particular Blakemore [28] provided a fairly comprehensive table of the more important $\mathcal{F}_j^+$ both for half-integer and integer j values. In the last years, the solution of same physical problems (for example in quantum structures [34]) has required an efficient evaluation of the Fermi-Dirac integrals, consequently in recent papers [24–26,34] several methods for computing $\mathcal{F}_j^+$ have been proposed with very high relative accuracy (up to $10^{-10} - 10^{-13}$).

The existence of these tables has not curbed attempts at providing analytic approximations [30–33] of usable accuracy especially for I_2^+ and I_4^+ (i.e. $\mathcal{F}_{1/2}^+$ and $\mathcal{F}_{3/2}^+$) which are related to the carrier density and to the energy density of Fermi gas. Thus Bednarczyk [30] and Aymerich-Humet et al. [31] used an analytic expression to fit for all α values (including the transition region, when the Fermi level is close to the band edge) the integral $\mathcal{F}_{1/2}^+$ and $\mathcal{F}_{3/2}^+$ with a relative error below 0.7%. This approach has been generalized in [32,33] where the authors have proposed different approximate expressions, which satisfy to an high accuracy, to fit the integrals $\mathcal{F}_j^+$ for different orders j and $-\infty < \alpha < \infty$.

In this section we considered the several analytical investigations that have been made in the literature and using an extensive numerical computations (obtained with the Numerical Algorithms Group (NAG) integration routine) we show the fractional error $E_n^\pm(\alpha)$ associated with the use of these analytic approximations, being

$$E_n^\pm(\alpha) = \frac{\triangle I_n^\pm(\alpha)}{I_n^\pm(\alpha)} \,. \tag{41}$$

The existence of analytic approximations, together with the relation (21), offers a valuable tool in obtaining one order from another for the integrals $I_n^\pm(\alpha)$ and them derivatives. Nevertheless we remark that if we use the (21) to calculate the Fermi and the Bose integrals we obtain from the fractional error (41) the relation

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{I_{n-2}^\pm}{I_n^\pm} \left[E_n^\pm(\alpha) - E_{n-2}^\pm(\alpha)\right] = \frac{d\,E_n^\pm(\alpha)}{d\,\alpha}. \tag{42}$$

The eq.(42) shows that the extremal positions of the fractional error $E_n^\pm(\alpha)$ are intersection points of the two curves $E_{n-2}^\pm(\alpha)$ and $E_n^\pm(\alpha)$. Therefore, the presence of a absolute extreme, to internal of interval $\mathcal{H}$, for the function $E_n^\pm(\alpha)$ implies that

$$\max_{\alpha\in\mathcal{H}} \mid E_{n-2}^\pm(\alpha) \mid > \max_{\alpha\in\mathcal{H}} \mid E_n^\pm(\alpha) \mid, \tag{43}$$

i.e., by using the relation (21), the maximum relative error in the calculation of $I_{n-2}^\pm(\alpha)$ is greater than the corresponding maximum relative error in the calculation of $I_n^\pm(\alpha)$.

5.1 Statistics of a weakly degenerate gas

For a weakly degenerate gas, it may readily be shown that [17,39], when $n > -1$ and $\alpha > 1$ the following series can be employed

$$I_n^\pm(\alpha) = \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\,\exp(-\alpha)\,\left\{\sum_{r=0}^{+\infty} \frac{(\mp 1)^r}{(r+1)^{\frac{n+1}{2}}}\,\exp(-r\alpha)\right\}. \tag{44}$$

We shall consider, as first approximation, the classical gas obtained for $r = 0$ in (44)

$$I_n^\pm(\alpha) \approx \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\,\exp(-\alpha)\,. \tag{45}$$

In this case from the relations (17),(19) we have to the zero order

$$\exp(-\alpha) \approx \exp(-\alpha_0) = \frac{4}{\sqrt{\pi}}\,\gamma\,z \quad , \qquad F(z) = \frac{k_B}{m}\,z\,. \tag{46}$$

As second approximation, we consider for the integral functions $I_n^\pm(\alpha)$ only the first two terms of the expansion (44). In this way we have

$$I_n^\pm(\alpha) \approx \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\,\exp(-\alpha)\,\left[1 \mp \frac{1}{2^{(n+1)/2}}\,\exp(-\alpha)\right]. \tag{47}$$

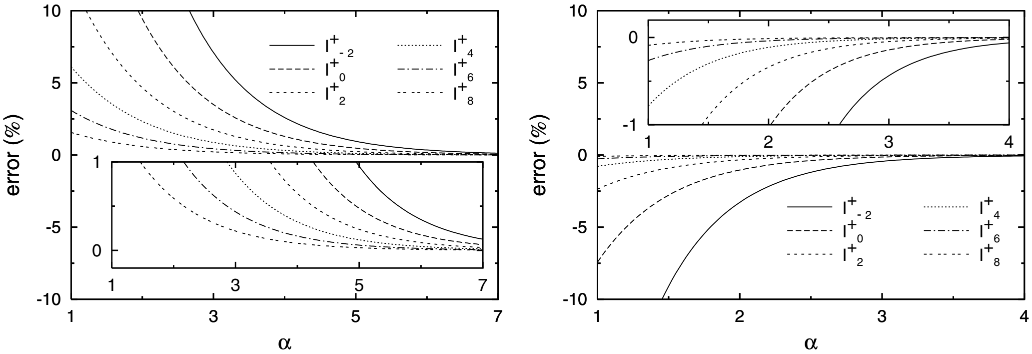

Fig. 1. The fractional percentage error involved in representation of Fermi Integral I_n^+, when $\alpha \geq 1$, using (for $n = 0, 2, 4, 6, 8$) both the expression (45) (on the left) for a classical gas and terms of correction (47) (on the right) for a weakly degenerate gas, while the function $I_{-2}^+(\alpha)$ is define by the relation (21) with $r = 1$ and $n = 0$

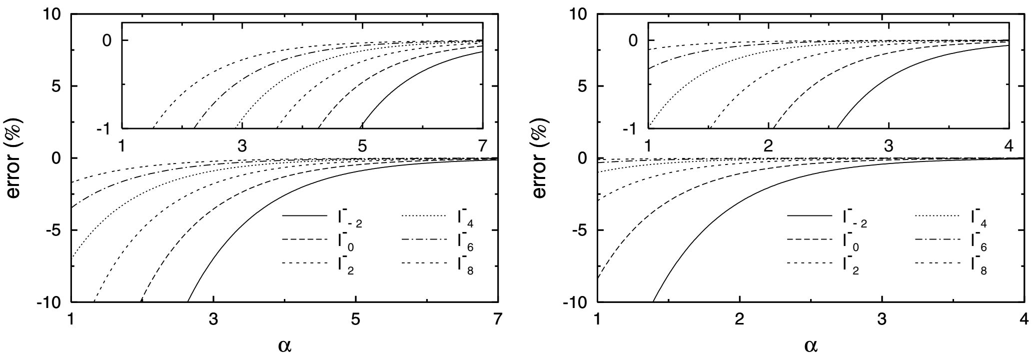

Fig. 2. The fractional percentage error involved in representation of Bose integral I_n^-, when $\alpha \geq 1$, using (for $n = 0, 2, 4, 6, 8$) both the expression (45) (on the left) for a classical gas and terms of correction (47) (on the right) for a weakly degenerate gas, while the function $I_{-2}^-(\alpha)$ is define by the relation (21) with $r = 1$ and $n = 0$

Analogously, by using the $(46)_1$ we obtain the second correct approximation [39,40], in the variable $\exp(-\alpha_0)$ for the quantities $\{\exp(-\alpha),\ F^{\pm}(z)\}$.

$$\exp(-\alpha) \approx \frac{4}{\sqrt{\pi}}\,\gamma\, z \left[1 \pm \sqrt{\frac{2}{\pi}}\,\gamma\, z\right] \quad , \quad F^{\pm}(z) \approx \frac{k_B}{m}\, z \left[1 \pm \frac{\gamma}{\sqrt{2\pi}}\, z\right] . \tag{48}$$

The interesting difference between these results and that for a classical gas is at once apparent. The measure of the degeneracy is the value of $\exp(-\alpha)$ and if $\alpha \gg 1$ the approximation in $(48)_1$ is a very good one in this case. As a consequence of these results, a weakly degenerate Bose gas is more compressible than an classical ideal gas, while a weakly degenerate Fermi gas is less compressible that an classical ideal gas. The relative percentage error $E_n^+(\alpha) \times 100$ of the expressions (45) and (47), in relation to the value of Fermi and Bose integrals, is reported in Figs. 1 and 2, respectively.

These figures show that the expansion is not very strongly convergent for small positive α and small n, with a maximum percentage error below $\pm 10\%$ when $\alpha \geq 2.5$ and below $\pm 1\%$ when $\alpha \geq 5$ in the classic approximation. The error can be reduced by using the first additional term ((47)) with an accuracy of less than $\pm 10\%$ for $\alpha \geq 1.45$ and below $\pm 1\%$ when $\alpha \geq 2.6$, while to obtain highly precise calculations, for more small α values and different orders n, many terms of expansion (44) are required (see for example [24,25]).

For the Fermi–Dirac integrals the convergence of expansion (44) can be greatly improved by the application of sequence transformation techniques [41], and the transformed series will be very efficient using a small number of terms (within a relative error $\varepsilon_r \leq 10^{-9}$) also in the interval $\alpha \in]0, 1]$ and for small n [24,25].

At last we note that, for the Fermi integrals, a high accuracy both for $\alpha \geq 0$ and a wide range of n can be obtained by using a simple polynomial approximation, which is very closely related to short forms of the classic series expansion (44). Thus Van Halen and Pulfrey [33] have calculated the functions $\mathcal{F}_j^+$ within a relative error under 10^{-5} by a polynomial expression for $\alpha \geq 0$. Coefficients of the proposed approximation requires the use of standard curve-fitting techniques and are given only for $-1/2 \leq j \leq 7/2$ (i.e. $0 \leq n \leq 8$ for I_n^+).

5.2 Fermi Statistics of a strongly degenerate gas

For strong degeneracy in the Fermi–Dirac statistics an asymptotic expansion may be obtained through a method due to Sommerfeld [21]. It follows that

$$I_n^+(\alpha) = \frac{1}{n+1} \left\{ (-\alpha)^{\frac{n+1}{2}} + \sum_{m=1}^{+\infty} 2 \left[1 - \frac{1}{2^{2m-1}} \right] \xi(2m) \, \frac{d^{2m}(-\alpha)^{\frac{n+1}{2}}}{d(-\alpha)^{2m}} \right\} \tag{49}$$

where $\xi(x)$ denote the Riemann-function, $n > -1$ and $-\alpha \gg 1$.

Deficiencies of expansion (49) were demonstrated by Rhodes [23] for small odd values of n. This analysis was extended by Dingle with a more general asymptotic expansion [20] that, when n is even, includes Sommerfeld's result (49) as a special case.

For $T \to 0$K we have the *Completely Degenerate Fermi gas* (CDF) and the energy approaches a non-vanishing value at the absolute zero. In this case we have:

$$I_n^+(\alpha) \approx \frac{1}{n+1} (-\alpha)^{\frac{n+1}{2}} \quad , \qquad -\alpha \, k_B \, T = \mu_0 = \frac{5}{2} \, m \, \nu_F \, \rho^{2/3} \, , \tag{50}$$

$$F(z) = \nu_F \, z^{5/3} \quad , \qquad \nu_F = \frac{2}{5} \, \frac{k_B}{m} \, (3\gamma)^{2/3} = \frac{\hbar^2}{5 \, m^{\frac{8}{3}}} \left[\frac{6\pi^2}{(2s+1)} \right]^{\frac{2}{3}} . \tag{51}$$

We note that some coefficients ($\{A, A_2, B_1, B_2\}$) associated to terms of second order of constitutive relations (27) diverge. In this way only to first order, we get no singular expressions in the constitutive functions. Analogously, some singularities are present (also to first order of expansion) in the coefficients of the Lagrange multipliers $\{\gamma_1, \, \gamma_2, \, \alpha_1, \, \alpha_2, \, \beta_2, \, \delta_1, \, \delta_2, \, \sigma_1, \, \sigma_2\}$.

In the case of a *Strongly Degenerate Fermi gases* (SDF), with $T > 0$ K, we consider the asymptotic expansion of integrals $I_n^+(\alpha)$ for even values of n. For the same applications it will be sufficient to confine our attention to the first terms in the summation in the bracket in (49). As a first approximation we obtain

$$I_n^+(\alpha) \approx \frac{1}{n+1} (-\alpha)^{\frac{n+1}{2}} \left[1 + \frac{\pi^2}{24} \, (n^2 - 1) \, (-\alpha)^{-2} \right] , \tag{52}$$

if we assume, for the term in the bracket, that $(-\alpha)^{-2} \approx (k_B \, T/\mu_0)^2 \ll 1$ we write, to the first order of approximation in the variable $(k_B \, T/\mu_0)^2$, the correction term to the completely degenerate Fermi gas [39]

$$\mu \approx \mu_0 \left[1 - \frac{\pi^2}{12} \left(\frac{k_B \, T}{\mu_0} \right)^2 \right] , \quad F(z) \approx \nu_F \, z^{\frac{5}{3}} \left[1 + \frac{5}{12} \left(\frac{2 \, \pi k_B}{5 \, m \, \nu_F} \right)^2 z^{-\frac{4}{3}} \right] . \tag{53}$$

The fractional percentage errors (for $n = -2, 0, 2, 4, 6, 8$) obtained using different terms of the asymptotic expansion (49) is plotted in Fig. 3.

For same functions of Fermi, the error oscillates in sign as a function of α, and it shows extremal positions for $\alpha < 0$. It is possible to verify that the differential property (21) is satisfied for different order of the truncated series, and according to (42) the extremal positions of the fractional error $E_n^+(\alpha)$ are intersection points of the two curves $E_{n-2}^+(\alpha)$ and $E_n^+(\alpha)$ (see on the left of Fig. 3 the curves obtained for $n = -2, \, 0, \, 2, \, 4$ and analogously on the right the curves for $n = 4, \, 6, \, 8$). In particular by the curves on the left, the relative errors associated with the use of eqn. (52) lies below 10% when $\alpha \leq -5$ and never more that 1% for $\alpha \leq -10$. The maximum error can be reduced by using another additional term of the expansion (49) and the curves on the right indicates an excursions never more that 10% for $\alpha \leq -3.5$ and of less than 1% for $\alpha \leq -5$.

Several authors have proposed different approximations to cover a wide range of $\alpha < 0$ for various orders of the Fermi–Dirac integral. One of the most accurate approximations for calculating the functions $\mathcal{F}_j^+$ also for small negative α values have been developed by Goano [24,25]. The resulting expansion involve the Kummer's confluent hypergeometric functions of first and second kind $M(a, b, z)$ and $U(a, b, z)$. Thus by using the relation

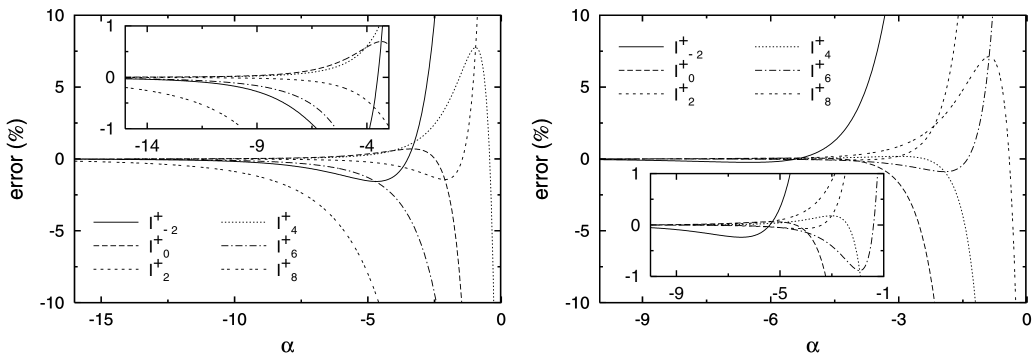

Fig. 3. The fractional percentage error involved in representation of the Fermi Integral I_n^+, when $\alpha < 0$, using (for $n = 0, 2, 4, 6, 8$) both the first (see 52) two terms (on the left) and the first tree terms (on the right) of the asymptotic expansion (49) for a strongly degenerate gas, while the function $I^+_{-2}(\alpha)$ is define by the relation (21) with $r = 1$ and $n = 0$

(40) we obtain for the integrals I_n^+

$$I_n^+(\alpha) = \frac{1}{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+3}{2})}\,(-\alpha)^{\frac{n+1}{2}}\left\{1 + \sum_{m=1}^{+\infty}(-1)^{m-1}\left[\left(\frac{n+1}{2}\right)\right.\right.$$
$$\left.\left.\times U\left(1, \frac{n+3}{2}, -m\alpha\right) - M\left(1, \frac{n+3}{2}, m\alpha\right)\right]\right\} \tag{54}$$

that converges also for small $\alpha < 0$ values. It should be noted that an efficient implementation of (54) requires the application of transforms [41] to accelerate the summation of the alternating series. In this way using a small number of terms, we obtain an algorithm for approximating, with arbitrary accuracy (see Table III in [25] where $\varepsilon_r \leq 10^{-9}$), the Fermi–Dirac integrals also in a region of the domain where the asymptotic expansion (49) not is applicable.

Van Halen and Pulfrey [33] have calculated, within a relative error $\varepsilon_r < 10^{-5}$, different polynomial expressions to fit the Fermi–Dirac integrals $\mathcal{F}_j^+$, for different ranges of $\alpha < 0$ and several j values. Other authors as Cody and Thacher [29] and recently Macleod [26], Trellakis et al. [34] have used the Chebyshev approximations to calculate, with better accuracy (up to $\varepsilon_r < 10^{-13}$), the functions $\mathcal{F}_j^+$. Unfortunately, it has been obtained only for same values of j and different expressions are needed for different intervals of α.

All the previous approximations introduced to describe the Fermi–Dirac integrals, with very high accuracy, both for weak degeneracy and strong degeneracy involve two, or more, intervals of α with different expressions for each interval. Nevertheless, this accuracy is unnecessary in most physical problems and, consequently, a single analytical expression which covers the whole range of α, for any real value of $n > -1$, is more useful despite its lower accuracy. In this sense Aymerich et al. [32] presented the following analytical approximation valid for all values of α and any real value of $n > -1$:

$$I_n^+(\alpha) = \frac{1}{2}\left\{\frac{(n+1)\,2^{\frac{n-1}{2}}}{\left[b - \alpha + (|\alpha + b|^c + a^c)^{1/c}\right]^{\frac{n+1}{2}}} + \frac{\exp(\alpha)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}^{-1} \tag{55}$$

being

$$a = \left[1 + \frac{15}{4}\left(\frac{n+1}{2}\right) + \frac{1}{40}\left(\frac{n+1}{2}\right)^2\right]^{1/2}, \quad b = 1.8 + 0.61\left(\frac{n-1}{2}\right), \quad c = 2 + (2-\sqrt{2})\,2^{\frac{1-n}{2}},$$

where for $0 \leq n \leq 10$ the maximum relative error is near to 1% while the errors increase with n for higher n values [32]. The proposed approximation (55), together with the relation (21), offers a straightforward analytic advantage to calculate $I_n^+(\alpha)$ and the derivatives.

The curves, on the left, of Fig. 4 indicates the fractional percentage error associated to the approximation (55) for values of $n = 0, 2, 4, 6, 8, 10$, while on the right we report the fractional percentage error for the integrals $I_0^+(\alpha)$ and $I^+_{-2}(\alpha)$.

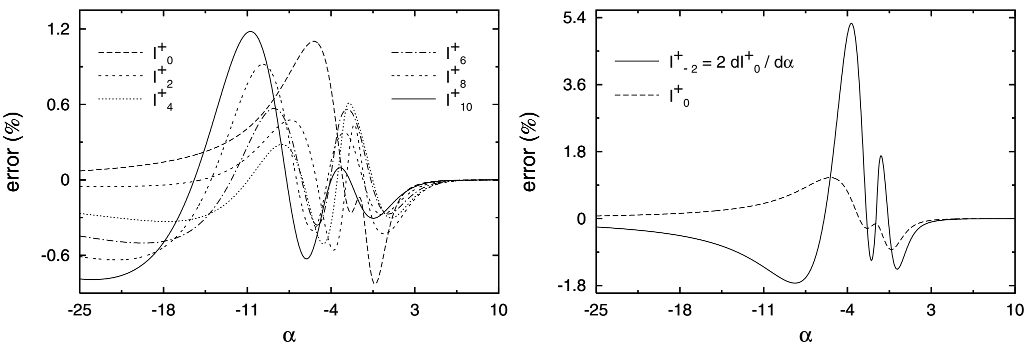

Fig. 4. The fractional percentage error involved in representation of Fermi Integral I_n^+, with $\alpha \in [-25, 10]$, using the analytic approximation (55) proposed by Aymerich et al. [32] for $n = 0, 2, 4, 6, 8, 10$, while the function $I_{-2}^-(\alpha)$ is defined by the relation (21) with $r = 1$ and $n = 0$

We note that the maximum relative error of approximation (55) is below 1.2% for $n = 0, 8, 10$ and below 0.7% for $n = 2, 4, 6$.

Nevertheless, being the function $I_{-2}^+(\alpha) = 2\, dI_0^+/d\alpha$, in accordance with the relations (42) and (43) (with $n = 0$), we obtain that: i) the extremal positions of the fractional error $E_0^+(\alpha)$ are intersection points of the two curves $E_{-2}^+(\alpha)$ and $E_0^+(\alpha)$, ii) the fractional error $E_0^+(\alpha)$ has a absolute maximum near to $\alpha \approx -5.5$,. Consequently the maximum relative error obtained in the calculation of $I_{-2}^+(\alpha)$, in the interval $[-25, 10]$, is greater (but below 5.4%) of the corresponding maximum relative error of $I_0^+(\alpha)$.

5.3 Bose Statistics of a strongly degenerate gas

For strong degeneracy in the Fermi statistics $\exp(\alpha) \ll 1$, such a situation in the Bose case would lead to negative values of density for a considerable range of values of energy. In particular for the Bose gas the parameter α, assumes no negative values. The α value becomes zero at a critical temperature T_c which, for a given density ρ, is define by relation

$$T_c = \left[\frac{\gamma}{I_2^-(0)}\, \rho \right]^{2/3} = \frac{2\, \pi\, \hbar^2}{k_B\, m^{5/3}} \left[\frac{\sqrt{\pi}}{4\, I_2^-(0)\, (2s+1)} \right]^{2/3} \rho^{2/3}. \tag{56}$$

For $T < T_c$ the α value remains equal to zero but (17) is no longer satisfied if $T < T_c$. Mathematically this is shown by the fact that, when $\alpha = 0$, $f\, |_E$ has a singularity [40] for $C = 0$ which corresponds to a fraction of particle in the state of zero microscopic energy. Consequently for $T < T_c$ a certain kind of *condensation* remove from the higher energy states a certain number of particles and transferring them to the lowest energy state, where they make no contribution to the pressure of the gas. At the limit $T = 0$ K the particles are all in their lowest state both with energy and momentum zero.

Therefore for $T < T_c$, we have a *phase transition* for the system that will be considered as a mixture of two thermodynamic phases, the *normal phase* and the *condensate phase* (Bose–Einstein condensation).

We remark that at $T = T_c$ all the thermodynamic quantities (17), (18), (22), (23), the functions $F(z)$, $F'(z)$ and second derivatives of h_E with respect to $\{\rho, p\}$ are continuous. One can show however, that the *Bose condensation* is a phase transition for the system in which the second derivative $F''(z)$ and consequently the second derivatives of h_E with respect to $\{\rho, T\}$ have a discontinuity at $T = T_c$ (see Appendix). We consider separately two different cases:

Bose statistics for $T \geq T_c$

To analyze the properties of a degenerate Bose gas with $T \geq T_c$ (i.e., $\alpha \geq 0$) it is convenient to rewrite the relations (17–19), in the form

$$\rho = \frac{3}{2}\, \frac{m}{k_B}\, \nu_B\, T^{3/2}\, \frac{I_2^-(\alpha)}{I_4^-(0)}\,, \quad p = \nu_B\, T^{5/2}\, \frac{I_4^-(\alpha)}{I_4^-(0)}\,, \quad F(z) = \nu_B\, \frac{I_4^-(\alpha)}{I_4^-(0)}\,, \tag{57}$$

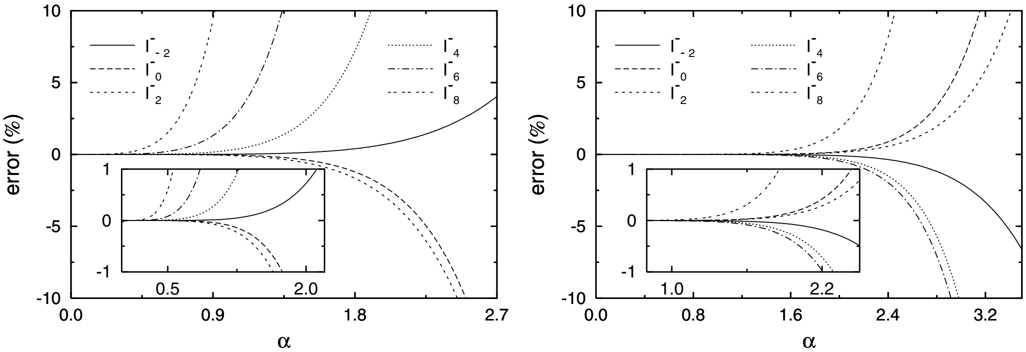

Fig. 5. The fractional percentage error involved in representation of Bose integral I_n^-, when $\alpha > 0$ ($T > T_c$), using (for $n = -2, 0, 2, 4, 6, 8$) both the first five terms (on the left) and the first seven terms (on the right) of the expansion (58) for a Bose degenerate gas

where $\nu_B = (2\,k_B\,I_4^-(0))/(3\,\gamma\,m)$.

The integral functions $I_n^-(\alpha)$, for strongly degenerate cases, have been analyzed and tabulated in the literature. In particular one can express [18] the Bose Functions in terms of power series of α with the expansion

$$I_n^-(\alpha) = \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\left\{\Gamma\left(\frac{1-n}{2}\right)\,\alpha^{\frac{n-1}{2}} + \sum_{r=0}^{+\infty}\frac{(-1)^r}{r!}\,\xi\left(\frac{n+1}{2}-r\right)\,\alpha^r\right\} \tag{58}$$

that converges absolutely if $\alpha \leq 2\pi$ and holds for *all* values of n.

When $\alpha \to 0$ $I_n^-(\alpha)$ diverges as $\alpha^{-\left|\frac{n-1}{2}\right|}$ if $n < 1$, diverges as $\ln(1/\alpha)$ if $n = 1$ and converges toward

$$I_n^-(0) = \frac{1}{2}\,\Gamma\left(\frac{n+1}{2}\right)\,\xi\left(\frac{n+1}{2}\right) \quad \text{for} \quad n > 1. \tag{59}$$

As indicated also by the differentiation property (21), if $1 < n \leq 3$ the $I_n^-(\alpha)$ has an infinite slope for $\alpha = 0$ although the function itself remains finite.

The curves in Fig. 5 indicates the fractional percentage error calculated with the use of expansion (58) in modelling the behavior of $I_n^-(\alpha)$ for $n = -2, 0, 2, 4, 6, 8$. We note that the series (58) converges rapidly in the neighborhood of $\alpha = 0$ only for positive values of n which are not too large. In particular by the curves on the left of Fig. 5, all relative errors obtained using the first five terms of (58) lies below 10% when $\alpha \leq 0.9$ and never more that 1% for $\alpha \leq 0.5$.

All the errors can be reduced by using additional terms of the expansion, in fact with the first seven terms of (58), the curves on the right indicates excursions never more that 10% for $\alpha \leq 2.4$ and of less than 1% for $\alpha \leq 1.8$.

We remark that to calculate the explicit analytic expressions for $\{\alpha,\, F(z)\}$ in the neighborhood of T_c, we can use only the first two terms (in grown power of α) of expansion (58) for I_2^- and I_4^-. Thus by $(57)_1$ and $(57)_3$ we obtain for $0 \leq \alpha \ll 1$

$$\alpha = \frac{4}{\pi^2}\,\left[I_2^-(0) - \gamma\,\frac{\rho}{T^{3/2}}\right]^2, \quad F(z) = \nu_B - \frac{3}{2}\,\frac{I_2(0)}{I_4(0)}\,\frac{4}{\pi^2}\,\nu_B\,\left[I_2^-(0) - \gamma\,z\right]^2\,. \tag{60}$$

Bose statistics for $T < T_c$ with Bose condensation

For a strongly degenerate Bose gas with $T < T_c$ (i.e., $\alpha = 0$) we have $I_n^-(0) = \infty$ for $n \leq 1$ while $I_n^-(0)$ will be expressed by (59) for $n > 1$, and

$$\rho\,(1-\varphi) = \frac{3}{2}\,\frac{m}{k_B}\,\nu_B\,\frac{I_2^-(0)}{I_4^-(0)}\,T^{3/2}\,, \qquad p = \nu_B\,T^{5/2} \qquad F(z) = \nu_B\,, \tag{61}$$

where $\varphi = \rho_0/\rho$ is the fraction of condensed matter, being ρ_0 the mass density of the condensate phase. We note that $0 \leq \varphi \leq 1$ in particular $T \to T_c$ as $\varphi \to 0$ and $T \to 0\,\mathrm{K}$ as $\varphi \to 1$, besides, by evaluating the coefficients (28–30), we obtain explicitly:

$$\begin{aligned} A = \Bigg\{ & \frac{6}{35}\,\left[-\frac{70}{3}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) + \frac{126}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\right] \\ & \cdot \left[-\frac{10}{3}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) + \frac{14}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\right]^{-1}\Bigg\}\,\frac{1}{\nu_B\,T^{5/2}}\,, \end{aligned} \tag{62}$$

$$A_1 = \frac{14}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\,\frac{k_B}{m}\,T \quad , \quad A_3 = \frac{4}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\,\frac{k_B}{m}\,\frac{1}{\nu_B\,T^{3/2}}\,, \tag{63}$$

$$\begin{aligned} A_2 = \Bigg\{ & -\frac{4}{25}\,\Bigg[84\,\frac{I_6^-(0)}{I_2^-(0)}\,(1-\varphi) + \frac{1764}{25}\,\left(\frac{I_6^-(0)}{I_4^-(0)}\right)^2 - \frac{396}{5}\,\frac{I_8^-(0)}{I_4^-(0)} \\ & -\frac{700}{9}\,\left(\frac{I_4^-(0)}{I_2^-(0)}\right)^2\,(1-\varphi)^2\Bigg] \cdot \left[-\frac{10}{3}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) + \frac{14}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\right]^{-2}\Bigg\}\,\frac{1}{\nu_B\,T^{5/2}}\,, \end{aligned}$$

$$B_0 = 2\,\frac{I_6^-(0)}{I_4^-(0)}\,\frac{k_B}{m}\,\nu_B\,T^{7/2} \quad , \quad B_2 = \frac{7}{15}\,\frac{I_6^-(0)}{I_4^-(0)}\,\frac{k_B}{m}\,\frac{1}{\nu_B\,T^{3/2}}\,, \tag{64}$$

$$\begin{aligned} B_1 = \Bigg\{ & -\frac{1}{25}\,\Bigg[-\frac{4000}{27}\,\left(\frac{I_4^-(0)}{I_2^-(0)}\right)^2\,(1-\varphi)^2 + \frac{560}{3}\,\frac{I_6^-(0)}{I_2^-(0)}\,(1-\varphi) - 264\,\frac{I_8^-(0)}{I_4^-(0)} \\ & + \frac{1176}{5}\,\left(\frac{I_6^-(0)}{I_4^-(0)}\right)^2\Bigg] \cdot \left[-\frac{10}{3}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) + \frac{14}{5}\,\frac{I_6^-(0)}{I_4^-(0)}\right]^{-2}\Bigg\}\,\frac{1}{\nu_B\,T^{5/2}}\,. \end{aligned}$$

6 Characteristic polynomial

All the coefficients of the characteristic polynomial (37) have a form which depends of the order of the series expansion of constitutive functions, and they will be expressed as functions both of two dimensionless variables $\{\widetilde{\sigma},\ \widetilde{q}\}$ and the dimensionless parameter α (or equivalently on the dimensionless parameter φ for the Bose gas with $T < T_c$). We have studied the hyperbolicity zone at first and second order through a numerical computation of the roots of the characteristic polynomial, representing in plane $\{\widetilde{q},\ \widetilde{\sigma}\}$ (by fixing the values both of the parameter α and of parameter φ) the regions with zero, two or four complex conjugate roots. We consider separately two different cases:

6.1 Fermi gas and Bose gas with $T \geq T_c$

By considering the general relations (17–19), (21–23),(27–30) and (32–33), we can calculate explicitly both the matrix J_{AB} and the determinant $det\mathcal{A}^{\mathbf{0}}$ which define the hyperbolicity conditions for the system. In particular, for the Fermi gas and Bose gas with $T \geq T_c$ should be:

$$\det\left(\mathcal{A}^0_{AB}\right) = -6\,\frac{k_B}{m}\,\rho^2\,\left\{\frac{5}{3}\,\frac{I_4^{\pm}(\alpha)}{I_2^{\pm}(\alpha)} - 3\,\frac{I_2^{\pm}(\alpha)}{I_0^{\pm}(\alpha)}\right\} \neq 0\,. \tag{65}$$

By using the velocity (35) and the dimensionless quantities (36), we obtain the characteristic polynomial (37) with a expansion both first-order and second-order. In this case we obtain that

$$c = \left[\frac{10}{9}\,\frac{I_4^{\pm}(\alpha)}{I_2^{\pm}(\alpha)}\,\frac{k_B}{m}\,T\right]^{1/2} = \left[\frac{5}{3}\,\frac{p}{\rho}\right]^{1/2} \tag{66}$$

and, to the first order of expansion, the dimensionless coefficients

$$\tilde{g}_4 = \tilde{g}_0 = 0\,, \quad \tilde{g}_2 = -\frac{96}{25}\,\widetilde{q}\,, \tag{67}$$

$$\tilde{g}_3 = \left\{\left[3\,\frac{I_4^{\pm}}{I_2^{\pm}} + \frac{14}{15}\,\frac{I_6^{\pm}}{I_4^{\pm}} + \frac{21}{5}\left(\frac{I_2^{\pm}}{I_4^{\pm}}\right)^2 \frac{I_6^{\pm}}{I_0^{\pm}} - \frac{62}{5}\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]\widetilde{\sigma} + \left[\frac{252}{125}\left(\frac{I_2^{\pm}}{I_4^{\pm}}\right)^2 \frac{I_6^{\pm}}{I_0^{\pm}} - \frac{63}{25}\,\frac{I_6^{\pm}}{I_4^{\pm}} + 3\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]\right\}\cdot\left[\frac{5}{3}\,\frac{I_4^{\pm}}{I_2^{\pm}} - 3\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]^{-1},$$

$$\tilde{g}_1 = \frac{3}{25}\left\{7\left[2\,\frac{I_6^{\pm}}{I_4^{\pm}} + 9\left(\frac{I_2^{\pm}}{I_4^{\pm}}\right)^2 \frac{I_6^{\pm}}{I_0^{\pm}} - 15\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]\widetilde{\sigma}^2 - 3\left(2\,\widetilde{\sigma} - \frac{3}{5}\right)\cdot\left[21\left(\frac{I_2^{\pm}}{I_4^{\pm}}\right)^2 \frac{I_6^{\pm}}{I_0^{\pm}} - 25\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]\right\}\cdot\left[\frac{5}{3}\,\frac{I_4^{\pm}}{I_2^{\pm}} - 3\,\frac{I_2^{\pm}}{I_0^{\pm}}\right]^{-1},$$

while to the second order of expansion, we have

$$\begin{aligned}
\tilde{g}_4 &= \alpha_{41}\,\widetilde{q}\,, \\
\tilde{g}_3 &= \alpha_{31} + \alpha_{32}\,\widetilde{\sigma} + \alpha_{33}\,\widetilde{\sigma}^2 + \alpha_{34}\,\widetilde{q}^2\,, \\
\tilde{g}_2 &= \alpha_{21}\,\widetilde{q} + \alpha_{22}\,\widetilde{q}\,\widetilde{\sigma} + \alpha_{23}\,\widetilde{q}\,\widetilde{\sigma}^2 + \alpha_{24}\,\widetilde{q}^3\,, \\
\tilde{g}_1 &= \alpha_{11} + \alpha_{12}\,\widetilde{\sigma} + \alpha_{13}\,\widetilde{\sigma}^2 + \alpha_{14}\,\widetilde{\sigma}^3 + \alpha_{15}\,\widetilde{\sigma}^4 + \alpha_{16}\,\widetilde{q}^2 + \alpha_{17}\,\widetilde{q}^2\,\widetilde{\sigma} + \alpha_{18}\,\widetilde{q}^2\,\widetilde{\sigma}^2\,, \\
\tilde{g}_0 &= \alpha_{01}\,\widetilde{q} + \alpha_{02}\,\widetilde{q}\,\widetilde{\sigma} + \alpha_{03}\,\widetilde{q}\,\widetilde{\sigma}^2 + \alpha_{04}\,\widetilde{q}\,\widetilde{\sigma}^3 + \alpha_{05}\,\widetilde{q}^3 + \alpha_{06}\,\widetilde{q}^3\,\widetilde{\sigma}\,,
\end{aligned} \tag{68}$$

where the dimensionless coefficients α_{ij} are very complicated, functions of the integrals $I_n^{\pm}(\alpha)$ that have been obtained within the framework of algebraic computing[1]. It is important to note the presence, in some of the coefficients ($\alpha_{03}, \cdots, \alpha_{06},\ \alpha_{13}, \cdots, \alpha_{18},\ \alpha_{23}, \alpha_{24}, \alpha_{33}, \alpha_{34}$), of term $I_{-2}^{-}(\alpha)/[I_0^{-}(\alpha)]^3$ that implies (see appendix) a discontinuity at $T = T_c$.

In this way, the roots of the characteristic polynomial will be expressed (by fixing the parameter α) as function of the variables $\{\widetilde{q}, \widetilde{\sigma}\}$, both in the equilibrium state ($\widetilde{q} = \widetilde{\sigma} = 0$) and in non-equilibrium state ($\widetilde{q} \neq 0$ and/or $\widetilde{\sigma} \neq 0$).

We remark that in the classic approximation all coefficients $\tilde{g}_i$ are independent by α parameter, in fact by using the (21) and (45) in the relations (67–68), it is possible to show that, with a first-order of expansion, we obtain the usual classic characteristic polynomial [1]

$$\tilde{\lambda}^5 + \left(-\frac{78}{25} + \frac{62}{15}\,\widetilde{\sigma}\right)\tilde{\lambda}^3 - \frac{96}{25}\,\widetilde{q}\,\tilde{\lambda}^2 + \left(\frac{27}{25} - \frac{18}{5}\,\widetilde{\sigma} + \frac{21}{5}\,\widetilde{\sigma}^2\right)\tilde{\lambda} = 0\,, \tag{69}$$

while with a second-order of expansion we can write

$$\begin{aligned}
&\tilde{\lambda}^5 - \frac{184}{15}\,\widetilde{q}\,\tilde{\lambda}^4 + \left(-\frac{78}{25} + \frac{166}{25}\,\widetilde{\sigma} - \frac{12}{5}\,\widetilde{\sigma}^2 + \frac{19684}{675}\,\widetilde{q}^2\right)\tilde{\lambda}^3 + \left(\frac{468}{25}\,\widetilde{q}\right. \\
&\left. - \frac{756}{25}\,\widetilde{q}\,\widetilde{\sigma} - \frac{32}{75}\,\widetilde{q}\,\widetilde{\sigma}^2 - \frac{592}{45}\,\widetilde{q}^3\right)\tilde{\lambda}^2 + \left(\frac{27}{25} - \frac{18}{5}\,\widetilde{\sigma} + 3\,\widetilde{\sigma}^2 - \frac{464}{25}\,\widetilde{q}^2\right. \\
&\left. + \frac{2476}{75}\,\widetilde{q}^2\,\widetilde{\sigma} - \frac{304}{45}\,\widetilde{q}^2\,\widetilde{\sigma}^2\right)\tilde{\lambda} - \frac{36}{25}\,\widetilde{q} + \frac{144}{25}\,\widetilde{q}\,\widetilde{\sigma} - \frac{36}{5}\,\widetilde{q}\,\widetilde{\sigma}^2 + \frac{8}{3}\,\widetilde{q}\,\widetilde{\sigma}^3 = 0\,.
\end{aligned} \tag{70}$$

[1] The program written using the language of the *MAPLE* symbolic mathematics package, is available through the authors.

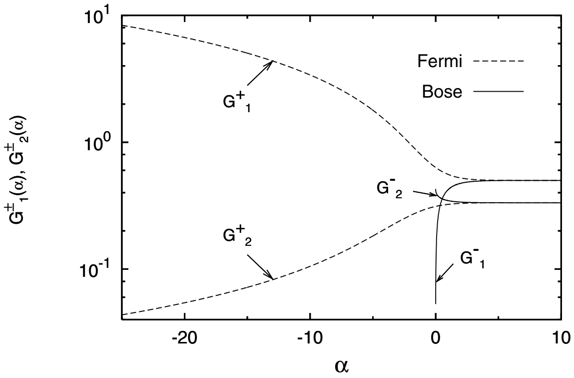

Fig. 6. The concavity of entropy in the equilibrium state implies the positiveness conditions (71). We report the functions $G_1^{\pm}$, $G_2^{\pm}$ vs α parameter for a Fermi gas covering the range $[-25, 10]$ and for the Bose gas in the interval $]0, 10]$

At last we consider the requirement (38) of convexity of $-h(\rho, p)|_E$ in terms of integral functions $I_n^{\pm}(\alpha)$. In this case obtain the two conditions

$$G_1^{\pm}(\alpha) = \frac{I_2^{\pm}(\alpha)}{I_0^{\pm}(\alpha)} > 0 \quad , \quad G_2^{\pm}(\alpha) = \frac{1}{3}\left[\frac{5}{3}\,\frac{I_4^{\pm}(\alpha)}{I_2^{\pm}(\alpha)} - 3\,\frac{I_2^{\pm}(\alpha)}{I_0^{\pm}(\alpha)}\right] > 0\,. \tag{71}$$

We note immediately that the positive condition $(71)_2$ implies the condition (65), and as is well known [1,15], these are violated ($det\mathcal{A}^{\mathbf{0}} = G_2^{+}(\alpha) = 0$ or equivalently $F' = (5/3)(F/z)$) only for a *completely degenerate Fermi gas* (i.e., when $T \to 0\,\mathrm{K}$ and $-\alpha$ increase indefinitely).

In the case of the Bose gas, the condition $(71)_2$ is always satisfied for $T \geq T_c$ while, for $T \to T_c^{+}$ (i.e., $\alpha \to 0$) the condition $(71)_1$ is violated ($G_1^{-}(0) = 0$ or $F'|_{T_c} = 0$) and consequently is violated the requirement of convexity of $-h_E$. Both with a analytic and numeric inspection, it is possible to show that the relations (71) are true for all values of α (with the exception of the previous case). In particular Fig. 6 shows the behavior of $G_1^{\pm}(\alpha)$ and $G_2^{\pm}(\alpha)$ both for a Fermi Gas covering the range of $\alpha \in [-25, 10]$ and for a Bose gas in the interval $\alpha \in]0, 10]$. We remark that i) the functions $G_1^{\pm}$ and $G_2^{\pm}$ are always positive, ii) for the Fermi gas $G_2^{+}(\alpha)$ shows a asymptotic behavior to zero for $-\alpha \gg 0$ iii) for the Bose gas $G_1^{-}(\alpha) \to 0$ for $\alpha \to 0$ ($T \to T_c^{+}$).

In Figs. 7–11 we report on the hyperbolicity zone obtained, with the first and the second order of expansion, through a numerical computation of the roots of the characteristic polynomial (37), by fixing values of α and representing in plane $\{\widetilde{\sigma}, \widetilde{q}\}$ the regions with zero (white zone), two (light gray zone), or four (dark gray zone) complex conjugate roots.

In particular the Fermi integrals $I_n^{+}(\alpha)$ have been calculated numerically in the interval $\alpha \in [-25, 10]$. The figure (7) report on the region of hyperbolicity, calculated using the first order of expansion, only for $\alpha \in [-16, 4]$ because in a different interval the numerical results remain unaltered. We note that the system is always hyperbolic in the neighborhood of equilibrium, and that, in the linearized theory, the hyperbolicity region increases if the α parameter increases reaching its greatest extension in the case of *classic approximation* (see also the figure 8.3 in [1]).

In Figs. 8 and 9 we report the hyperbolicity zone obtained, with the second order of expansion, in the range $\alpha \in [-25, 7]$ because for larger α values the extension of hyperbolicity region remain unaltered. In particular Fig. 8 shows that for strongly degenerate conditions ($-25 \leq \alpha \leq -13$), the system is hyperbolic only in a very small region in the neighborhood of equilibrium. The region of hyperbolicity increases if the α parameter increases, and for intermediate α values ($-3 \leq \alpha \leq 2$), corresponding to the *transition region* for the system, we observe a large variation of hyperbolicity zone that reaches its greatest extension in the case of *classic approximation* (i.e., large α).

In the case of Bose gas, for $T > T_c$, the integrals $I_n^{-}(\alpha)$ have been calculated numerically in the interval $\alpha \in [0.01, 10]$. In Figs. 10 and 11 we report on the regions of hyperbolicity by using respectively, the first order of expansion with $\alpha \in [0.01, 4]$ and the second order of expansion with $\alpha \in [0.01, 6]$ because for larger α values the hyperbolicity zone remain unaltered. We observe a large variation of hyperbolicity for very small α values (strongly degenerate condition) and that, in the linearized theory, for increasing α values the region of hyperbolicity decreases reaching its smaller extension in the case of *classic approximation*. On the contrary, as shown in Fig. 11 by using the non-linear expansion, if the α parameter increases also the corresponding region of hyperbolicity increases reaching its greater extension for large α values (*classic approximation*).

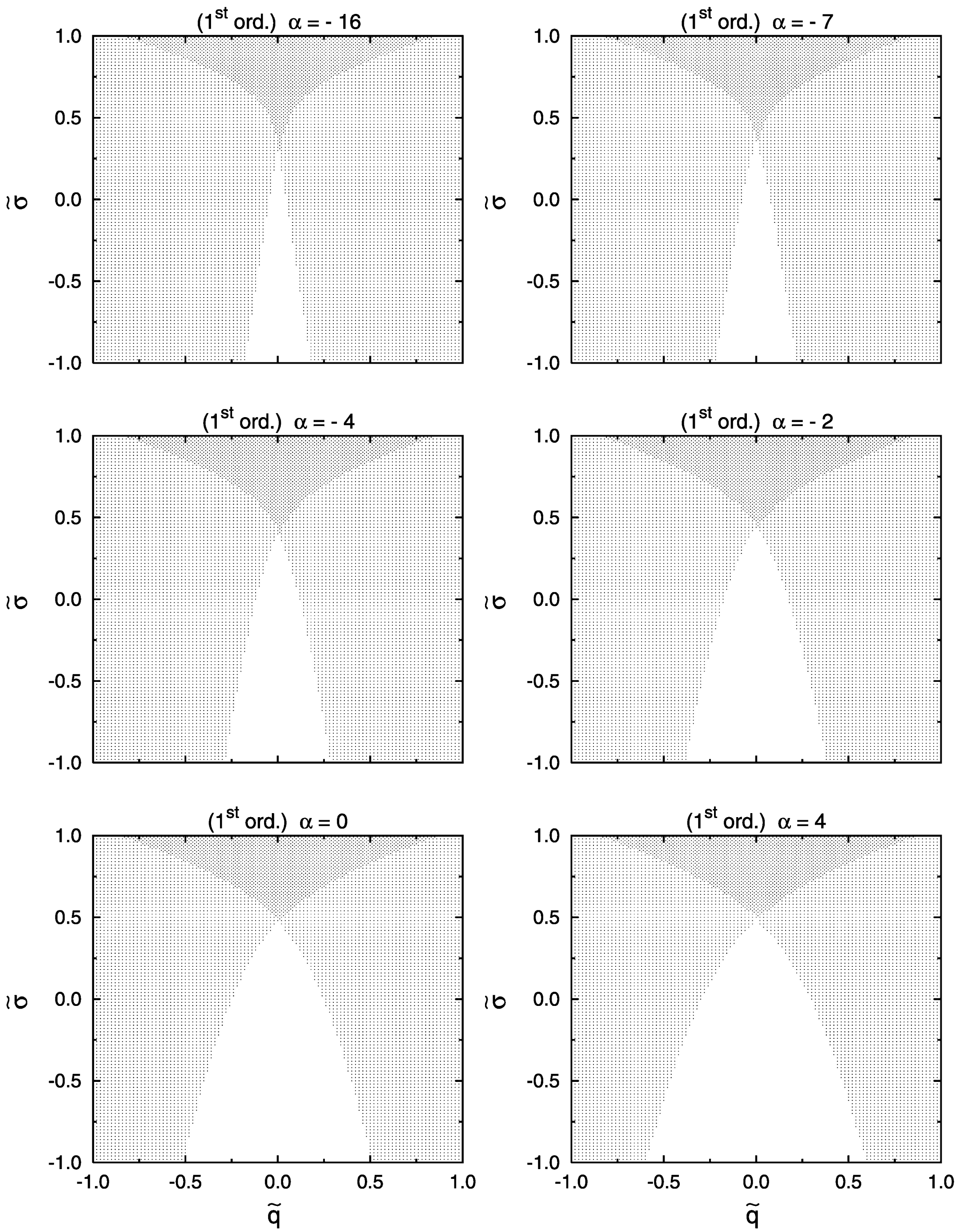

Fig. 7. Hyperbolicity zone for a *Fermi gas* obtained, with a first-order expansion, by fixing different values of α and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

To conclude we note that, both for the Fermi and Bose gases, by using the second order of expansion the region of hyperbolicity is, in general, larger with respect to the corresponding one obtained using the first order of expansion. Besides for $\alpha \gg 1$ the numeric region of hyperbolicity, related to degenerate Fermi and Bose gases, coincides with that obtained in the *classic case* by using different orders of approximation for the constitutive relations.

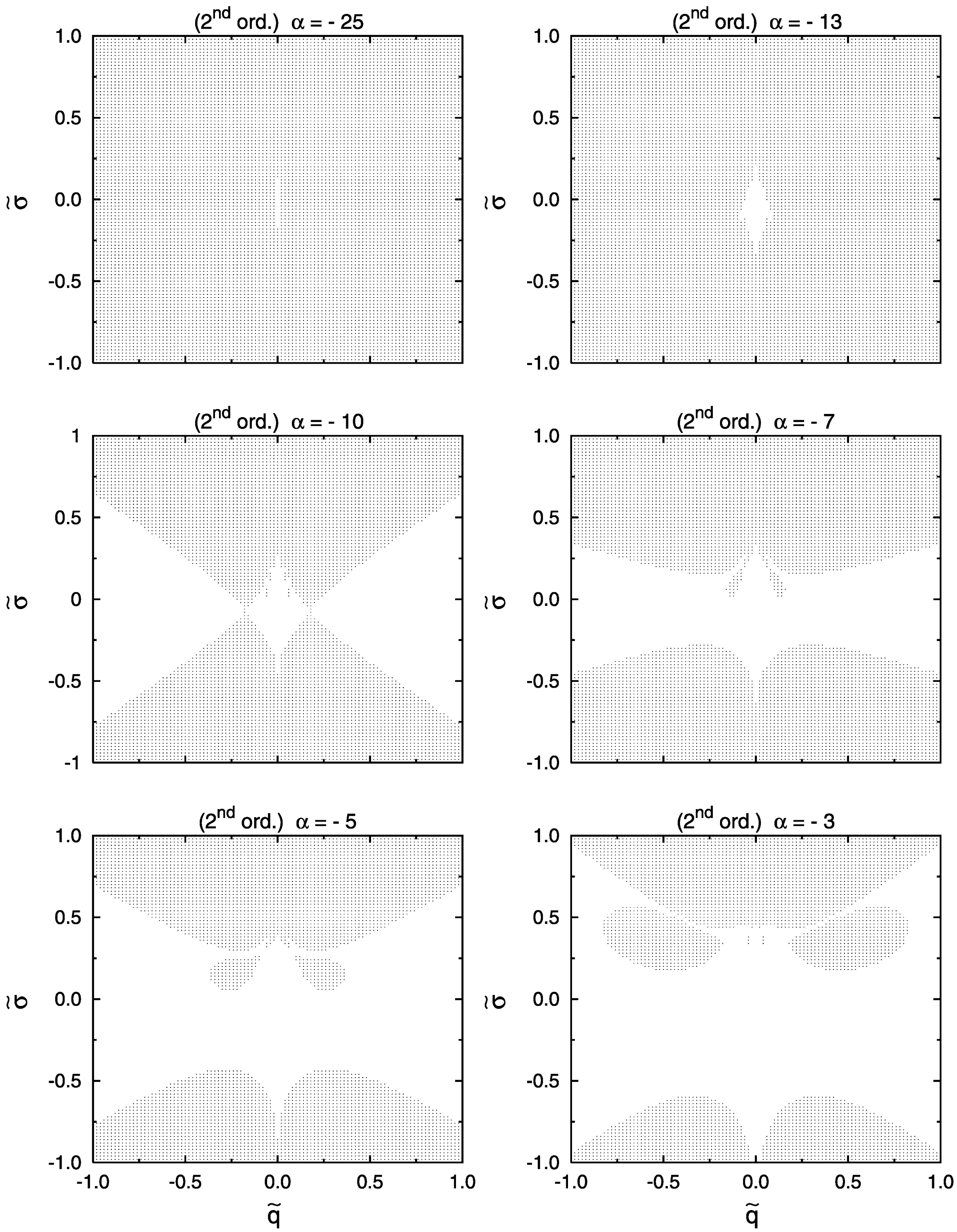

Fig. 8. Hyperbolicity zone for a *Fermi gas* obtained, with a second-order expansion, by fixing different values of $\alpha \in [-25, -3]$ and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

6.2 Strongly degenerate Bose gas with $T < T_c$

For strongly degenerate Bose gases with $T \leq T_c$ one of the conditions associated with convexity of $-h_E$ is always violated being $F'(z) = 0$, while for the remaining condition $F'(z) < 5/3\,(F(z)/z)$ we have

$$\frac{5}{3}\,\frac{F(z)}{z} = \frac{10}{9}\,\frac{k_B}{m}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) > 0\,. \tag{72}$$

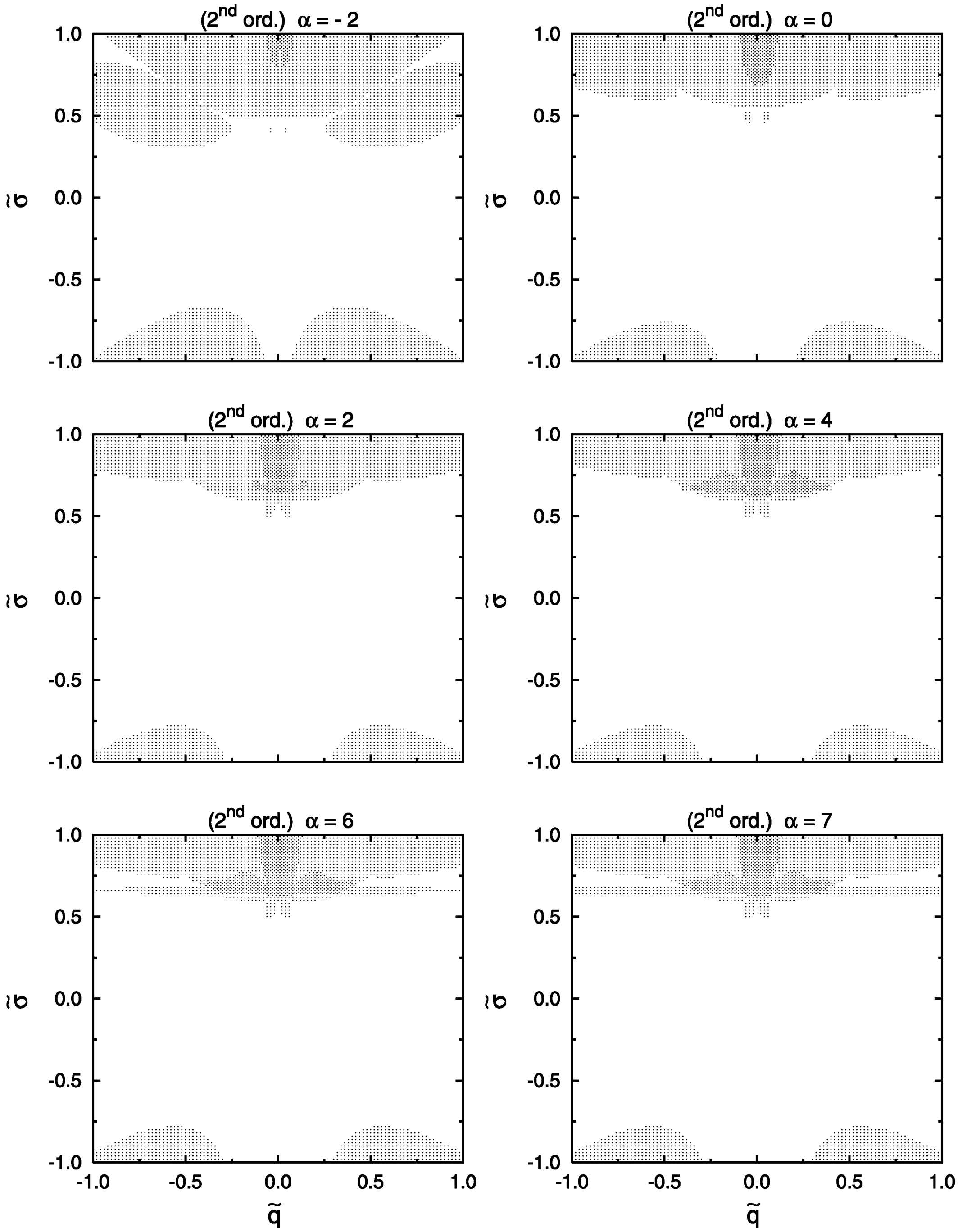

Fig. 9. Hyperbolicity zone for a *Fermi gas* obtained, with a second-order expansion, by fixing different values of $\alpha \in [-2, 7]$ and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

Analogously, we can rewrite (33) in the form

$$\det\left(\mathcal{A}^0_{AB}\right) = -10\,\frac{k_B}{m}\,\rho^2\;\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi) \neq 0\,, \tag{73}$$

and both the relations (72–73) are violated only for $\varphi \to 1$ (i.e., $T \to 0\,^o$K).

We shall prove that the system is hyperbolic only for particular values of the non-equilibrium variables q and σ. In fact in the equilibrium state ($q = \sigma = 0$), we obtain the root $\tilde{\lambda} = 0$ with multiplicity $r = 3$ and only two

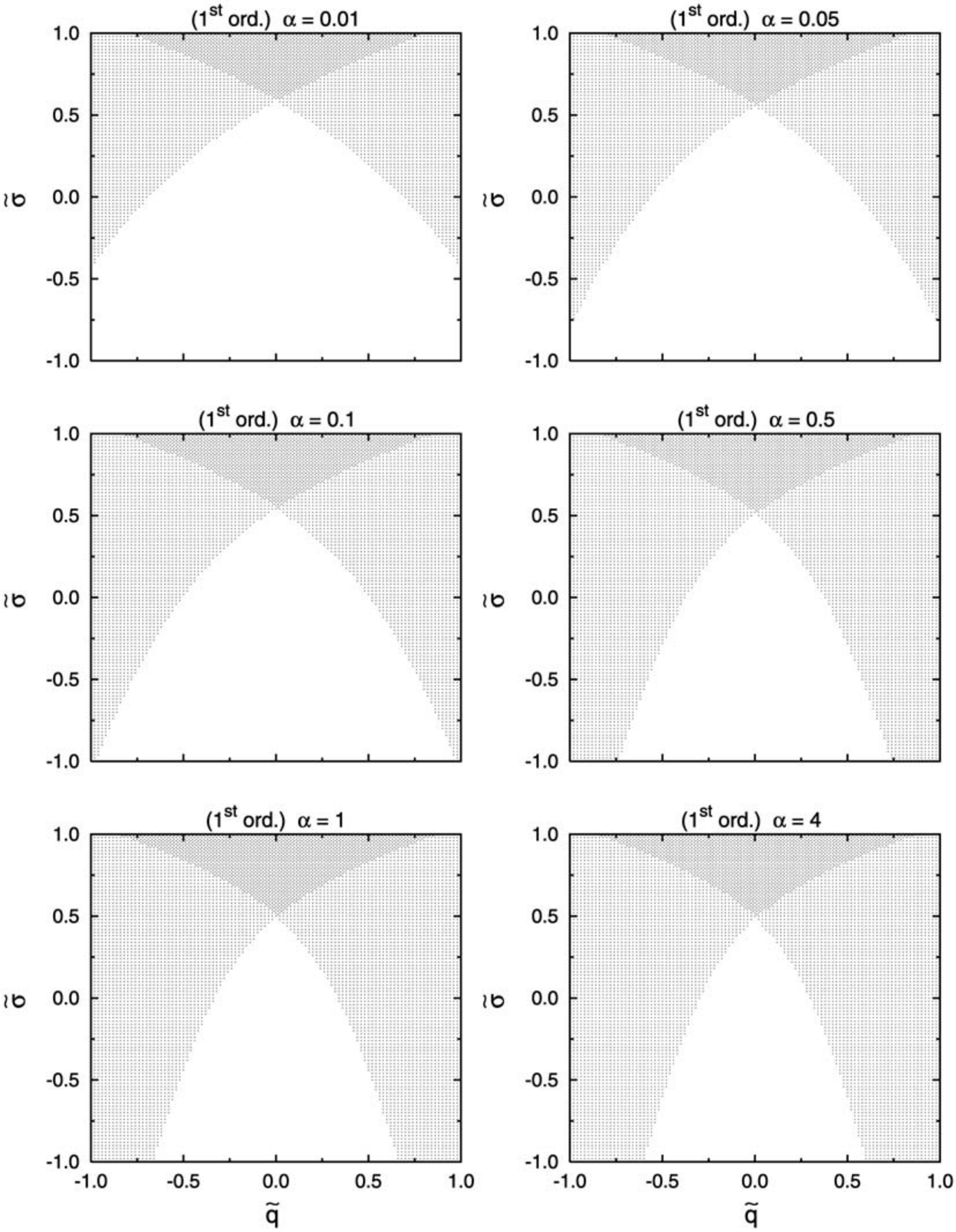

Fig. 10. Hyperbolicity zone for a *Bose gas* ($T > T_c$) obtained, with a first-order expansion, by fixing different values of α and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

independent eigenvectors [15]. In the non-equilibrium state, there are no vanish values of q and/or σ for which we obtain five real and distinct roots of the characteristic polynomial.

By using the dimensionless quantities (36) and the relations (27), (62–64), we obtain explicitly the characteristic polynomial (37). In this case we consider as a suitable velocity c a quantity that is proportional to the thermal velocity associated to the fraction of particle that are not in the condensed state, i.e.,

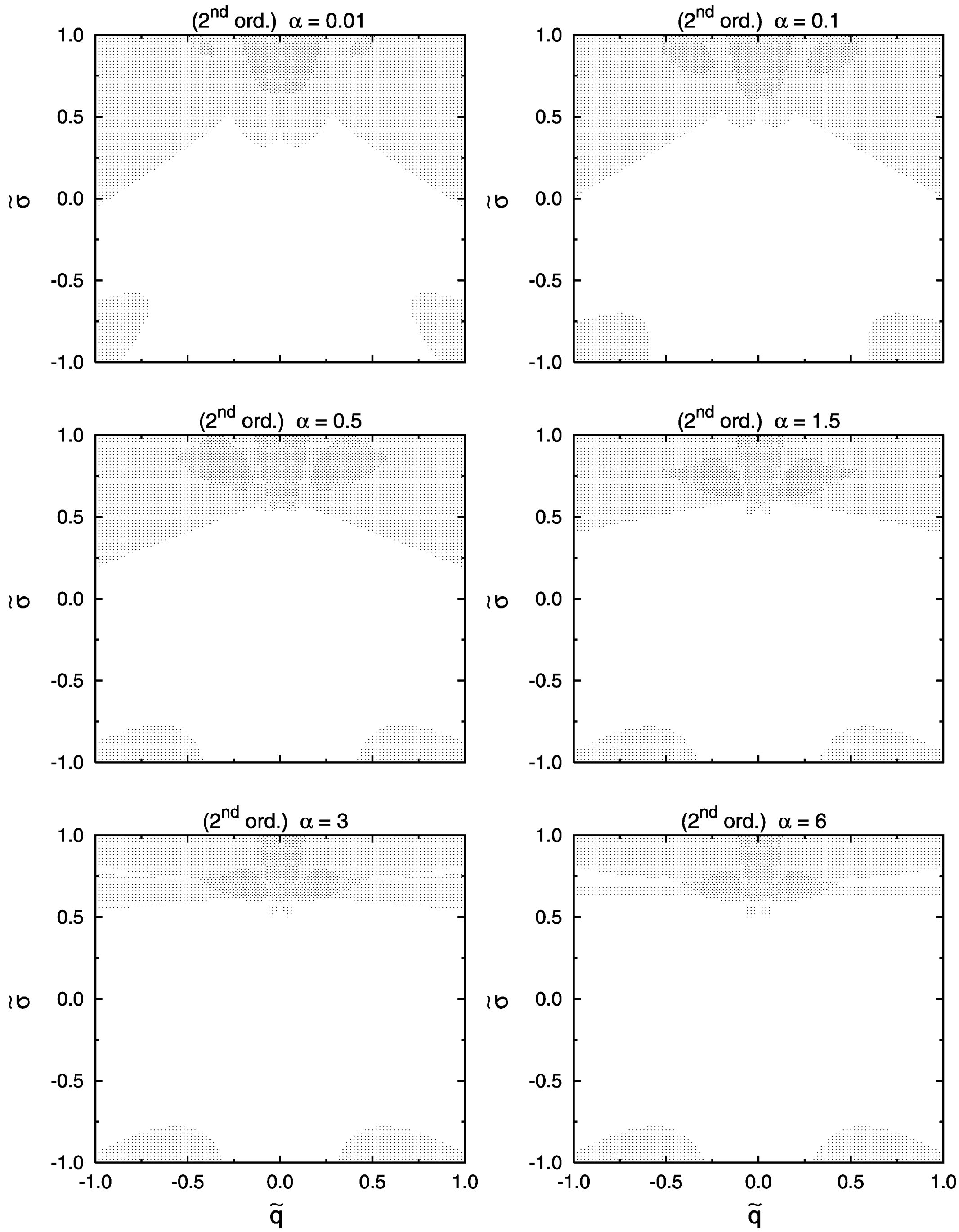

Fig. 11. Hyperbolicity zone for a *Bose gas* ($T > T_c$) obtained, with a second-order expansion, by fixing different values of α and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

$$c = \left[\frac{10}{9}\,\frac{I_4^-(0)}{I_2^-(0)}\,\frac{k_B}{m}\,T\right]^{1/2} = \left[\frac{5}{3}\,\frac{p}{\rho\,(1-\varphi)}\right]^{1/2} \tag{74}$$

and, evaluating the first-order of expansion, we have

$$\tilde{\lambda}^5 + \left\{ -\frac{189}{125}\, \frac{I_2^-(0)\, I_6^-(0)}{\left(I_4^-(0)\right)^2} + \left[\frac{9}{5} + \frac{14}{25}\, \frac{I_2^-(0)\, I_6^-(0)}{\left(I_4^-(0)\right)^2}\, \frac{1}{1-\varphi} \right] \widetilde{\sigma} \right\} \tilde{\lambda}^3 \tag{75}$$
$$-\frac{96}{25}\, \widetilde{q}\, \tilde{\lambda}^2 + \left\{ \frac{126}{125}\, \frac{I_2^-(0)\, I_6^-(0)}{\left(I_4^-(0)\right)^2}\, \frac{\widetilde{\sigma}^2}{1-\varphi} \right\} \tilde{\lambda} = 0\,,$$

while to the second order of expansion, we obtain for the coefficients $\{\tilde{g}_0,\, \tilde{g}_1,\, \tilde{g}_2,\, \tilde{g}_3,\, \tilde{g}_4\}$ of the relations analogous to the expressions (68), where the dimensionless quantities α_{ij} will be, in this case, complicated functions of the φ parameter.

In particular for $\widetilde{\sigma} = \widetilde{q} = 0$, we obtain the equilibrium solutions

$$\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = 0, \quad \text{and} \quad \tilde{\lambda}_{4,5} = \pm \left[\frac{189}{125}\, \frac{I_2^-(0)\, I_6^-(0)}{\left(I_4^-(0)\right)^2} \right]^{1/2} \approx \pm 2.0302$$

where the eigenvalue $\tilde{\lambda} = 0$, with $r = 3$, has only two independent eigenvectors and consequently (in accordance with the condition $F'(z) = 0$) the hyperbolicity of the system is violate [15].

For the sake of simplicity, we consider only the first order of expansion for the constitutive relations, and in Fig. 12 we report the region of hyperbolicity for $T \leq T_c$ in non-equilibrium conditions ($q \neq 0$ and $\sigma \neq 0$) for different φ values, while we remark that the equilibrium state ($q = \sigma = 0$) is out of the hyperbolic region. We observe that for increasing φ values the region of hyperbolicity decreases and for $\varphi \to 1$ (i.e., $T \to 0$ K) the hyperbolicity of system is lost (because, being $\det \mathcal{A}^0 = 0$, the condition (73) is violate).

All the coefficients of characteristic polynomial (37), in the linearized theory, are continuous functions at $T = T_c$. Consequently the region of hyperbolicity varies with continuity in proximity of critical temperature T_c, and its extension for $\varphi \to 0$ (i.e., $T \to T_c^-$) coincides with that obtained for $\alpha \to 0$ (i.e. $T \to T_c^+$).

Contrary to the linear theory, using the second-order expansion, the continuity of coefficients of characteristic polynomial is violated at $T = T_c$. In fact some of dimensionless coefficients α_{ij}, in the relations (68), have a kink at $T = T_c$ (see appendix) and consequently the region of hyperbolicity do not vary with continuity in proximity of critical temperature.

7 The Navier-Stokes and Fourier equations

The closed HD system for the variables $(\rho,\, p,\, m_{<ij>},\, q_i)$ can be used to derive general closure in a reduced system of balance equations more akin to standard ordinary thermodynamics. In fact, using an approximated procedure analogous to the *Maxwellian iteration*, we provide a generalization of the phenomenological constitutive equations of Fourier and Navier–Stokes for the heat flow and viscous stress. Through the first iteration, it is possible to rewrite the set of balance equations for the variables $\{\sigma_{\langle ij\rangle},\, q_i\}$ in the form [1]

$$q_i = -\frac{\tau_q}{2} \left\{ \frac{\partial\, m_{ikll}|_E}{\partial\, x_k} - 5\, \frac{p}{\rho}\, \frac{\partial\, p}{\partial\, x_i} \right\}\,, \qquad \sigma_{\langle ij\rangle} = -2\tau_\sigma\, p\, \frac{\partial\, v_{\langle i}}{\partial\, x_{j\rangle}} \tag{76}$$

where $\tau_q > 0$ and $\tau_\sigma > 0$ are the relaxation time for q_i and $\sigma_{\langle ij\rangle}$, respectively.

Thus the previous relations (76) can be expressed in the usual way

$$q_i = -\kappa\, \frac{\partial\, T}{\partial\, x_i}\,, \qquad \sigma_{\langle ij\rangle} = -2\, \eta\, \frac{\partial\, v_{\langle i}}{\partial\, x_{j\rangle}} \tag{77}$$

being κ and η the thermal conductivity and viscosity, respectively.

In this way by using the relations (17),(18), (19),(22),(23),(61) we consider separately two different cases:

7.1 Degenerate Fermi gas

$$\kappa = \frac{\tau_q}{3} \left(\frac{k_B}{m} \right)^2 \rho T\, \frac{I_4^+(\alpha)}{I_2^+(\alpha)} \left[7\, \frac{I_6^+(\alpha)}{I_4^+(\alpha)} - \frac{25}{3}\, \frac{I_4^+(\alpha)}{I_2^+(\alpha)} \right]\,, \qquad \eta = \frac{2}{3}\, \tau_\sigma\, \frac{k_B}{m}\, \rho T\, \frac{I_4^+(\alpha)}{I_2^+(\alpha)}\,, \tag{78}$$

where κ and η are always positive quantities to exception of the case $T \to 0\,$K (i.e., CDF gas).

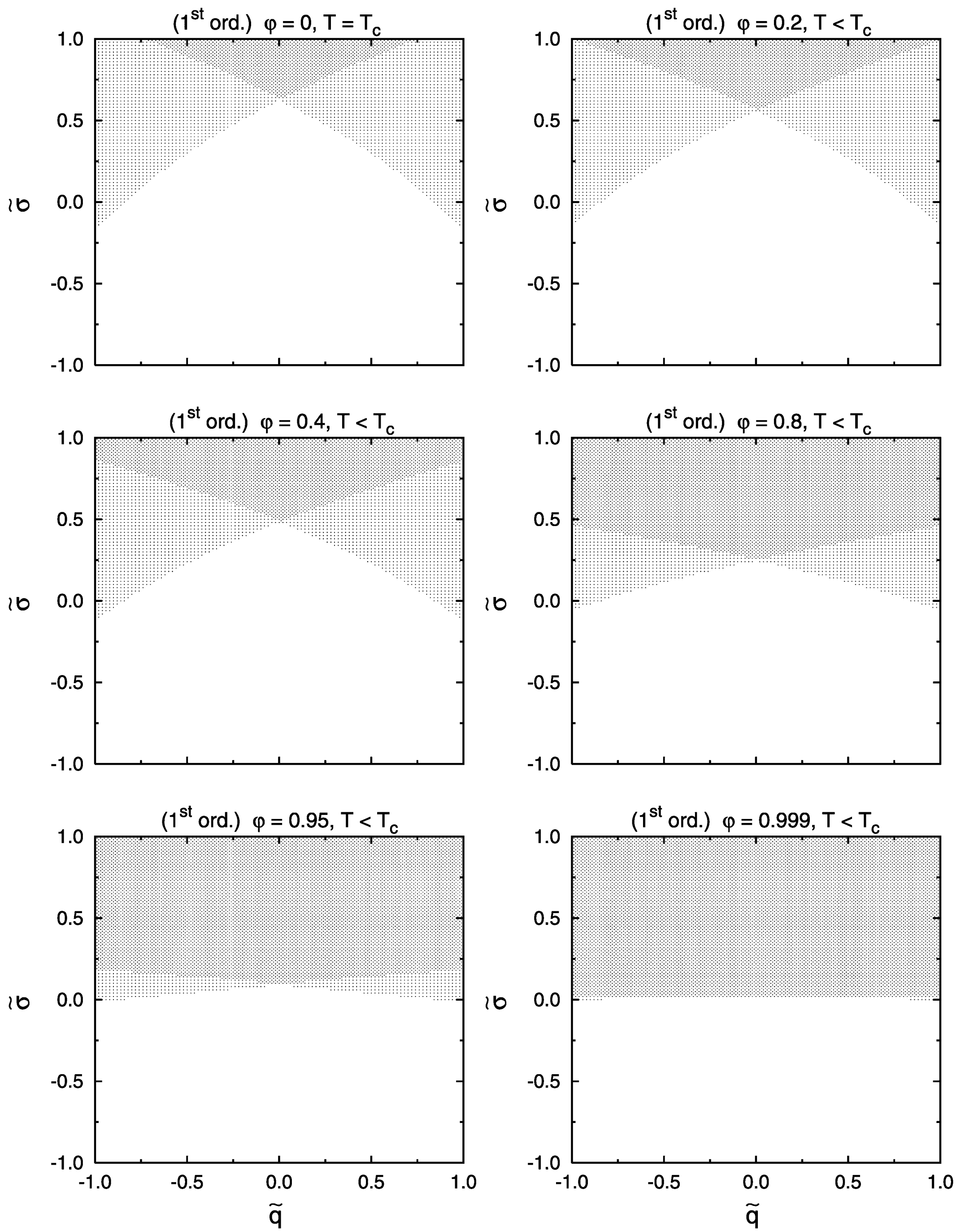

Fig. 12. Hyperbolicity zone for a *strongly degenerate Bose gas* ($T \leq T_c$) obtained, with a first-order expansion, by fixing different values of φ and representing in the plane $\{\tilde{q}, \tilde{\sigma}\}$ the regions with zero (white zone), two (light gray zone), and four (dark gray zone) complex conjugate roots of the characteristic polynomial (37)

7.2 Degenerate Bose gas

$$\kappa = \frac{\tau_q}{2}\,\frac{k_B}{m}\,\nu_B\,T^{5/2}\,\frac{I_4^-(\alpha)}{I_4^-(0)}\left[7\,\frac{I_6^-(\alpha)}{I_4^-(\alpha)} - \frac{25}{3}\,\frac{I_4^-(\alpha)}{I_2^-(\alpha)}\right], \qquad \text{for} \qquad T \geq T_c \tag{79}$$

$$\kappa = \frac{\tau_q}{2}\,\frac{k_B}{m}\,\nu_B\,T^{5/2}\left[7\,\frac{I_6^-(0)}{I_4^-(0)} - \frac{25}{3}\,\frac{I_4^-(0)}{I_2^-(0)}\,(1-\varphi)\right], \qquad \text{for} \qquad T \leq T_c \tag{80}$$

$$\eta = \tau_\sigma \, \nu_B \, T^{5/2} \, \frac{I_4^-(\alpha)}{I_4^-(0)} \, , \qquad \text{for} \qquad T \geq T_c \, , \tag{81}$$

$$\eta = \tau_\sigma \, \nu_B \, T^{5/2} \, , \qquad \text{for} \qquad T \leq T_c \, , \tag{82}$$

where κ and η are always positive quantities to exception of the case $T \to 0$ K.

8 Conclusions

We have determined the hyperbolicity region for degenerate gases through analytical and numerical techniques. It turns out that the determination of the hyperbolicity region for degenerate gases requires precise numerical calculations of the Fermi and Bose integral functions since their approximated values can not give satisfactory results. For degenerate gases it was shown that, both for ET_{13}^1 and for ET_{13}^2, there exists a neighborhood of the equilibrium state in which the universal principles of Extended Thermodynamics remain valid, and we have obtained quantitative results about the range of the allowed values for the non-equilibrium variables (heat flux and shear stress). For Fermi gases the hyperbolicity region turns out to be empty only in the completely degenerate case of $T = 0$ K, while for the completely degenerate Bose gas the hyperbolicity is lost when $T \leq T_c$ in the equilibrium state due to the coexistence of the normal phase and the condensed one. Nevertheless it also remains for $T \leq T_c$ a small non-equilibrium region in which the hyperbolicity is verified.

A Appendix

From expansion (58) one can verify that $I_{-2}^-(\alpha)/[I_0^-(\alpha)]^3 \to -4/\pi^2$ for $\alpha \to 0$, consequently by using the relation $(24)_2$ with $\gamma = (2\, k_B \, I_4^-(0))/(3\, \nu_B \, m)$ we obtain

$$\lim_{T \to T_c^+} F'' - \lim_{T \to T_c^-} F'' = -\frac{16}{3} \left(\frac{k_B}{m}\right)^2 \frac{I_2^-(0)\, I_4^-(0)}{\pi^2 \, \nu_B} \, , \tag{83}$$

Analogously, if we calculate the second derivatives of h_E with respect to $\{\rho, T\}$

$$\left(\frac{\partial^2 h_E}{\partial \rho^2}\right)_T = \frac{3}{2} \frac{1}{T^{3/2}} \left[F''(z) - \frac{2}{3} \frac{F'(z)}{z}\right] \tag{84}$$

$$\left(\frac{\partial^2 h_E}{\partial T^2}\right)_\rho = \frac{9}{8} \frac{\rho}{T^2} \left[3\, z \, F''(z) - 3\, F'(z) + \frac{5}{3} \frac{F(z)}{z}\right] \, ,$$

$$\frac{\partial^2 h_E}{\partial \rho \, \partial T} = -\frac{9}{4} \frac{\rho}{T^{5/2}} \left[F''(z) - \frac{2}{3} \frac{F'(z)}{z}\right] \, ,$$

therefore, for the discontinuity of F'', also these quantities have a kink at $T = T_c$

$$\lim_{T \to T_c^+} \left(\frac{\partial^2 h_E}{\partial \rho^2}\right)_T - \lim_{T \to T_c^-} \left(\frac{\partial^2 h_E}{\partial \rho^2}\right)_T = -\frac{8}{\pi^2} \left(\frac{k_B}{m}\right)^2 \frac{I_2^-(0)\, I_4^-(0)}{\nu_B \, T_c^{3/2}} \, , \tag{85}$$

$$\lim_{T \to T_c^+} \left(\frac{\partial^2 h_E}{\partial T^2}\right)_\rho - \lim_{T \to T_c^-} \left(\frac{\partial^2 h_E}{\partial T^2}\right)_\rho = -\frac{81}{2} \frac{\nu_B}{\pi^2} \frac{1}{T_c^{1/2}} \frac{\left[I_2^-(0)\right]^3}{I_4^-(0)} \, ,$$

$$\lim_{T \to T_c^+} \frac{\partial^2 h_E}{\partial \rho \, \partial T} - \lim_{T \to T_c^-} \frac{\partial^2 h_E}{\partial \rho \, \partial T} = \frac{18}{\pi^2} \frac{k_B}{m} \frac{\left[I_2^-(0)\right]^2}{T_c} \, ,$$

If we consider, in the case of Bose gas, the dimensionless coefficients α_{ij} contained in the relations (68) the presence, in some of this coefficients, of term $I_{-2}^-(\alpha) \, / [I_0^-(\alpha)]^3$ implies a discontinuity at $T = T_c$. In this way if we define

$$\lim_{T \to T_c^+} \alpha_{ij} - \lim_{T \to T_c^-} \alpha_{ij} = \triangle \alpha_{ij}|_{T_c} \, , \tag{86}$$

we obtain

$$\triangle \alpha_{03}|_{T_c} = -\frac{18}{125} \frac{1}{\pi^2} \left[\frac{I_2^-(0)}{I_4^-(0)}\right]^3 \left[25 \, \left(I_4^-(0)\right)^2 - 27\, I_2^-(0)\, I_6^-(0)\right] \, , \quad \triangle \alpha_{04}|_{T_c} = -\frac{5}{3} \triangle \alpha_{03}|_{T_c} \, , \tag{87}$$

$$\triangle\alpha_{05}|_{T_c} = \frac{144}{25}\,\frac{1}{\pi^2}\,\frac{\left[I_2^-(0)\right]^3}{I_4^-(0)}\,\frac{25\,\left(I_4^-(0)\right)^2 - 27\,I_2^-(0)\,I_6^-(0)}{25\,\left(I_4^-(0)\right)^2 - 21\,I_2^-(0)\,I_6^-(0)} \quad , \quad \triangle\alpha_{06}|_{T_c} = -\frac{5}{3}\,\triangle\alpha_{05}|_{T_c}\ ,$$

$$\triangle\alpha_{13}|_{T_c} = \frac{27}{125}\,\frac{1}{\pi^2}\left[\frac{I_2^-(0)}{I_4^-(0)}\right]^3 \left[25\,\left(I_4^-(0)\right)^2 - 21\,I_2^-(0)\,I_6^-(0)\right] \quad , \quad \triangle\alpha_{14}|_{T_c} = -\triangle\alpha_{13}|_{T_c}\ ,$$

$$\triangle\alpha_{15}|_{T_c} = \triangle\alpha_{03}|_{T_c},\ \ \triangle\alpha_{16}|_{T_c} = -\frac{216}{25}\,\frac{1}{\pi^2}\,\frac{\left[I_2^-(0)\right]^3}{I_4^-(0)},\ \ \triangle\alpha_{17}|_{T_c} = -\triangle\alpha_{16}|_{T_c},\ \ \triangle\alpha_{18}|_{T_c} = \triangle\alpha_{05}|_{T_c}\ ,$$

$$\triangle\alpha_{23}|_{T_c} = -\triangle\alpha_{03}|_{T_c},\quad \triangle\alpha_{24}|_{T_c} = -\triangle\alpha_{05}|_{T_c},\quad \triangle\alpha_{33}|_{T_c} = -\frac{5}{9}\,\triangle\alpha_{13}|_{T_c},\quad \triangle\alpha_{34}|_{T_c} = -\frac{5}{9}\,\triangle\alpha_{16}|_{T_c}\ ,$$

while the remaining coefficients $\{\alpha_{01},\ \alpha_{02},\ \alpha_{11},\ \alpha_{12},\ \alpha_{21},\ \alpha_{22},\ \alpha_{31},\ \alpha_{32},\ \alpha_{41}\}$ are continue functions at $T = T_c$.

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