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Original article

Nonextensive statistical mechanics: A brief introduction

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Abstract. Boltzmann–Gibbs statistical mechanics is based on the entropy $S_{\rm BG} = -k\sum_{i=1}^{W} p_i \ln p_i$. It enables a successful thermal approach to ubiquitous systems, such as those involving short-range interactions, markovian processes, and, generally speaking, those systems whose dynamical occupancy of phase space tends to be ergodic. For systems whose microscopic dynamics is more complex, it is natural to expect that the dynamical occupancy of phase space will have a less trivial structure, for example a (multi)fractal or hierarchical geometry. The question naturally arises whether it is possible to study such systems with concepts and methods similar to those of standard statistical mechanics. The answer appears to be *yes* for ubiquitous systems, but the concept of entropy needs to be adequately generalized. Some classes of such systems can be satisfactorily approached with the entropy $S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}$ (with $q \in \mathcal{R}$, and $S_1 = S_{\rm BG}$). This theory is sometimes referred in the literature as *nonextensive statistical mechanics*. We provide here a brief introduction to the formalism, its dynamical foundations, and some illustrative applications. In addition to these, we illustrate with a few examples the concept of *stability* (or *experimental robustness*) introduced by B. Lesche in 1982 and recently revisited by S. Abe.

Key words: nonextensive entropy, nonextensive statistical mechanics, nonlinear dynamics, entropy stability, metastability
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1 Introduction

Thermodynamics is based on two pillars: *energy* and *entropy*. The first one concerns (dynamical or mechanical) *possibilities*; the second one concerns the *probabilities* of those possibilities. The first one is more basic, and clearly depends on the *physical system* (classical, quantum, relativistic, or any other); the second one is more subtle, and reflects the *information* upon the physical system. It is obvious that the first one, the energy, is directly associated with the specific system, characterized for example by its Hamiltonian (which includes possible inertial terms, possible interactions, possible presence of external fields). It is *much less* obvious that the same seems to happen with the second one, the entropy. It was long believed that the microscopic expression of the physical entropy *had* to be universal, i.e., system-independent (apart for the trivial fact that it has to depend on W). More precisely, it *had* to be, for *all* systems, the well known one, namely

$$S_{\rm BG}(\{p_i\}) = -k\sum_{i=1}^{W} p_i \ln p_i \;, \tag{1}$$

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with its celebrated expression for equal probabilities

$$S_{\rm BG}(p_i = 1/W, \forall i) = k \ln W \ , \tag{2}$$

W being the total number of possibilities of the system under overall restrictions such as its total energy, total number of particles, and so on. However, this widespread belief of universality appears to have no rigorous basis. Indeed, it appears nowadays that the concept of physical information, and its microscopic expression in terms of probabilities, must be adapted to the system about which it is providing information, pretty much like impedances of interconnected circuits have to be similar in order to have an efficient and useful communication. Expressions (1) and (2) are so commonly used because most of the systems whose thermal properties are studied belong to the type involving (*strong*) chaos in its microscopic dynamics, i.e., *positive* Lyapunov exponents. There is no fundamental reason for which the *same* expression should necessarily be used for systems involving say a *vanishing* Lyapunov spectrum, i.e., for systems exhibiting *weak* chaos. Indeed, such systems, if isolated, might have serious difficulties in satisfying the ergodic hypothesis during the observational time of physical measures.

It is clear that the above statements about the nonuniversality of the microscopic expression for the entropy are *by no means* self-evident. It is through a variety of recent verifications that we have come to such a possibility. The present paper focuses on some of this growing evidence. If, however, $S_{\rm BG}$ is not universal, how does one generalize it? No logical-deductive path ever existed for proposing a new physical theory, or for generalizing a pre-existing one. In fact, such proposals frequently — perhaps always — occur on a metaphorical level. The analysis of the structure of the BG theory provides us a metaphor for formulating a statistical mechanics which might be more powerful than the one we already have thanks to the genius of Boltzmann, Gibbs, and others. It is perhaps important at this stage to explicitly insist that we are talking of a *generalization* of the BG theory, *by no means of an alternative to it*. Without further delays, let us construct [1] a statistical mechanics based on the following expression for the entropy:

$$S_q(\{p_i\}) = k \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} \quad (q \in \mathcal{R};\ S_1 = S_{\rm BG}) \ , \tag{3}$$

with the following expression for equal probabilities

$$S_q(p_i = 1/W, \forall i) = k \frac{W^{1-q} - 1}{1-q} \equiv k \ln_q W \ \ (\ln_1 W = \ln W). \tag{4}$$

If A and B are two independent systems (i.e., $p_{ij}^{A+B} = p_i^A p_j^B$), we verify that

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q) \frac{S_q(A)}{k} \frac{S_q(B)}{k} \ , \tag{5}$$

hence subextensivity (superextensivity) occurs if $q > 1$ ($q < 1$). Furthermore, it can be shown that the nonnegative entropy S_q is concave (convex) for $q > 0$ ($q < 0$).

We are now naturally led to use this entropy expression in order to generalize BG statistical mechanics. The most familiar situation is that of a system in contact with a thermostat (the *canonical ensemble*). To study this ensemble and obtain the thermal (meta)equilibrium distribution we have to optimize S_q with some constraint related to energy (in addition to the trivial one $\sum_{i=1}^{W} p_i = 1$). If we just use the constraint from BG statistics ($\sum_{i=1}^{W} p_i E_i = U$) we soon discover that it is impossible to define a partition function. To overcome this, and other emerging mathematical difficulties, the energy constraint was generalized as follows (see [3]):

$$\langle \mathcal{H} \rangle_q \equiv \frac{\sum_{i=1}^{W} p_i^q E_i}{\sum_{j=1}^{W} p_j^q} = U_q \ , \tag{6}$$

where $\{E_i\}$ are the eigenvalues of the Hamiltonian $\mathcal{H}$ with the appropriate boundary conditions and $p_i^q / \sum_{j=1}^{W} p_j^q$ is referred to as the *escort distribution* [4]. The optimization of the entropic form with these constraints straightforwardly yields

$$p_i = \frac{e_q^{-\beta_q (E_i - U_q)}}{\bar{Z}_q} \ , \tag{7}$$

with

$$\bar{Z}_q \equiv \sum_{j=1}^{W} e_q^{-\beta_q (E_j - U_q)} \ , \tag{8}$$

and

$$\beta_q \equiv \frac{\beta}{\sum_{j=1}^{W} p_j^q} \ , \tag{9}$$

β being the Lagrange parameter associated with the energy constraint and the q-exponential function being defined as the inverse of the $\ln_q x$ function, i.e., $e_q^x \equiv [1+(1-q)x]^{\frac{1}{1-q}}$ ($e_1^x = e^x$). We easily verify that $q=1$ recovers the standard BG weight, $q>1$ implies in a power-law tail at high values of E_i, and $q<1$ implies in a cutoff at high values of E_i.

It is important to make some comments, in addition to those appearing in [3], about the fact that the constraint should be expressed, for the canonical ensemble, as appears in (6). A strong argument has been produced by Abe and Rajagopal [5] who, using the method of steepest descents and considering the q-exponential function property $\mathrm{d}e_q^x/\mathrm{d}x = (e_q^x)^q$, arrive at the same (meta)equilibrium distribution and naturally show the necessity of using p_i^q, and not p_i, for expressing the constraints. We can also point out another relevant observation. If we express the two canonical ensemble constraints in the space of the energy, for high energy they asymptotically behave as indicated:

$$\begin{aligned} \sum_i p_i &\sim \int_{\text{constant}}^{\infty} \mathrm{d}E\, g(E)p(E) \sim \int_{\text{constant}}^{\infty} \mathrm{d}E\, g(E)E^{-1/(q-1)}, \\ \sum_i p_i^q E_i / \sum_j p_j^q &\sim \int_{\text{constant}}^{\infty} \mathrm{d}E\, g(E)E/E^{q/(q-1)} \sim \int_{\text{constant}}^{\infty} \mathrm{d}E\, g(E)E^{-1/(q-1)}, \end{aligned} \tag{10}$$

where $g(E)$ is the density of states. We can observe that the domain where the two constraints are finite is the *same* if we use p_i^q. For example, if we assume the simple case where the high energy approximation is given by $g(E) \sim E^\delta$ ($\delta \in \Re$), the convergence of *both* integrals is guaranteed for $1/(q-1) - \delta > 1$. This feature constitutes an ingredient of consistency within the theory, and reinforces the special form of the energy constraint.

From the preceding results we can also derive the connection to thermodynamics. In fact, the entire Legendre transformation structure of thermodynamics is q-invariant. In particular, it can be proved that

$$k\beta = \frac{\partial S_q}{\partial U_q} \ , \tag{11}$$

as well as

$$F_q \equiv U_q - \frac{S_q}{k\beta} = -\frac{1}{\beta} \ln_q Z_q \ , \tag{12}$$

where

$$\ln_q Z_q = \ln_q \bar{Z}_q - \beta U_q \ . \tag{13}$$

Also, it can be proved that

$$U_q = -\frac{\partial}{\partial \beta} \ln_q Z_q \ . \tag{14}$$

We want to comment that $k\beta$, in contrast with the BG case, is not the inverse of the temperature T_q that, as focused on by S. Abe et al. [6], is defined by the expression

$$T_q = \left(1 + \frac{1-q}{k} S_q\right) \frac{1}{k\beta}. \tag{15}$$

A set of mini-reviews on the subject can be found in [7]. The main properties of the theory, as well as its connection with thermodynamics, are explained there in great detail.

2 Dynamical foundations

It is clear that for the above theory to be complete we need to indicate how the entropic index q can in principle be calculated *a priori* for a given system. Consistent with the ideas of Einstein[1], Krylov [8], Cohen [9], Baranger

[1] A. Einstein: Theorie der Opaleszenz von homogenen Flüssigkeiten und Flüssigkeitsgemischen in der Nähe des kritischen Zustandes. Annalen der Physik **33**, 1275 (1910) [*"Usually W is put equal to the number of complexions... In order to calculate W, one needs a complete (molecular-mechanical) theory of the system under consideration. Therefore it is dubious whether the Boltzmann principle has any meaning without a complete molecular-mechanical theory or some other theory which describes the elementary processes. $S = \frac{R}{N}\log W +$ const. seems without content, from a phenomenological point of view, without giving in addition such an Elementartheorie."* (Translation: Abraham Pais, Subtle is the Lord .... Oxford University Press, 1982)].

[10], and many others, feel the value of q must be hidden in the microscopic (or mesoscopic) dynamics of the system. A large amount of examples illustrate that it is indeed so. We shall consider here just one of them, a very simple one, namely the family of logistic maps. Many others can be found in [7].

Consider the one-dimensional dissipative map

$$x_{t+1} = 1 - a|x_t|^z \quad (0 \le a \le 2;\ z > 1)\ , \tag{16}$$

with $-1 \le x_t \le 1$ and $t = 0, 1, 2, ...$ A value $a_c(z)$ exists for this map (e.g., $a_c(2) = 1.401155...$) such that for $a > a_c(z)$ ($a < a_c(z)$) the Lyapunov exponent λ_1 tends to be positive (negative). At the edge of chaos, $a = a_c(z)$, the Lyapunov exponent precisely vanishes. For all values of a such that the Lyapunov exponent exponent is different from zero, we have that the sensitivity to the initial conditions $\xi \equiv \lim_{\Delta x(0)\to 0}[\Delta x(t)/\Delta x(0)] = \mathrm{e}^{\lambda_1 t}$. But at $a_c(z)$ we have $\xi = \mathrm{e}_{q_{\mathrm{sen}}}^{\lambda_{q_{\mathrm{sen}}} t}$ [11–13], with $\lambda_{q_{\mathrm{sen}}}(z) > 0$, and $q_{\mathrm{sen}}(z)$ varying from $-\infty$ to almost unity, when z varies from unity to infinity (*sen* stands for *sensitivity*). For example

$$q_{\mathrm{sen}}(2) = 0.2445... \tag{17}$$

This same value can be found through a connection with multifractal geometry. If we denote by $f(\alpha)$ the multifractal function, and $\alpha_{\min}$ and $\alpha_{\max}$ the two values of α at which $f(\alpha)$ vanishes, we can argue [14] that

$$\frac{1}{1-q_{\mathrm{sen}}(z)} = \frac{1}{\alpha_{\min}(z)} - \frac{1}{\alpha_{\max}(z)} = \frac{(z-1)\ln\alpha_F(z)}{\ln 2}\ , \tag{18}$$

where $\alpha_F(z)$ is the z-generalization of the Feigenbaum universal constant.

The same value $q_{\mathrm{sen}}(z)$ can be found through a third method, namely by studying the entropy production per unit time. We partition the $[-1, 1]$ x interval in W little windows. We choose one of them and place (randomly or regularly) M initial conditions inside it. We then follow those points as a function in time, and get the set of occupation numbers $\{M_i(t)\}$ ($\sum_{i=1}^{W} M_i(t) = M$). We then define a probability set through $p_i(t) \equiv M_i(t)/M$ ($\forall i$), and calculate $S_q(t)/k$ using (3), and fixing some value for q. We then average over all initial windows (or a large number of them randomly chosen ones, which turns out to be numerically equivalent), and obtain $\langle S_q\rangle(t)/k$. We finally define the *entropy production per unit time* as follows

$$K_q \equiv \lim_{t\to\infty}\lim_{W\to\infty}\lim_{M\to\infty}\frac{\langle S_q\rangle(t)/k}{t}\ . \tag{19}$$

This concept essentially reproduces the so called Kolmogorov–Sinai entropy although generalized for arbitrary values of q. The equivalent of the usual Kolmogorov–Sinai entropy (which is in fact not an entropy but an entropy production) is recovered for $q = 1$. We can verify for the z-logistic map that K_q vanishes (diverges) for $q > q_{\mathrm{sen}}$ ($q < q_{\mathrm{sen}}$), and that $K_{q_{\mathrm{sen}}}$ is *finite*, and coincident with $\lambda_{q_{\mathrm{sen}}}$ (q-generalization of Pesin theorem [11–13,15]).

It is finally possible to recover q_{sen} through a forth method, namely through *Lebesgue measure shrinking*: see [16,17] for details.

Through this example of the logistic maps we have illustrated many relevant concepts concerning the index q and the dynamical foundations of nonextensive statistical mechanics. One or other of these concepts, as well as a few somewhat different ones, have been shown for quantum chaos [18], cercle maps [19], Henon map [20], standard maps [21,22], lattice Lotka–Volterra model [23], lattice Boltzmann model for the Navier–Stokes equations [24], growth of free-scale networks [25] (the distribution turns out to be a q-exponential), long-range-interacting many-body Hamiltonian classical ferromagnets [26–35], Lennard–Jones clusters [36] (the distributions are in fact q-exponentials), transport in quantum optical lattice [37], correlated [38], Lévy-like [39] and other anomalous diffusions [40,41], multiplicative noise [42] and dichotomic colored noise [43] Langevin-like diffusions, among others.

3 Some applications

The above theory has been applied in the literature to a large variety of systems, in addition to the ones we have just mentioned. Some of those attempts use various degrees of phenomenology, the q index being obtained from fitting of the available experimental or computational data. This is of course not an intrinsic necessity, but rather the consequence of the ignorance of the exact microscopic or mesoscopic dynamics of the natural or artificial system that is considered.

Examples that have been analyzed include re-association in folded proteins [44], fluxes of cosmic rays [45], turbulence [46–48], finance and economics [49], electron–positron annihilation [50], motion of *Hydra* cells [51], epilepsy [52], linguistics [53], nuclear physics [54], astrophysics [55], distributions in music [56], urban agglomerations [57], internet phenomena [58], among others.

Finally, the numerous applications that have been done [59] in algorithms for global optimization (e.g., the *generalized simulated annealing*) and related computational methods deserve a special mention.

4 Stability or experimental robustness

Two decades ago Lesche introduced [60] a quite interesting property that $S_{\rm BG}$ satisfies. He named this property *stability*. However, in a recent private conversation, one of us (CT) proposed to him the use of the denomination *experimental robustness* instead. He agreed that such denomination reflected better the physical content of the mathematical property he had himself introduced. In addition to this, the expression "stability" might be confused with "thermodynamical stability", which has to do with the concavity of the entropy, a completely different and independent property. We shall therefore use either *stability* or *experimental robustness* interchangeably. This property was further discussed recently by Abe [61,62]. In the present Section, we shall illustrate the property in a few typical cases, and will even start analyzing the possibility of enlarging it to metrics other than the one originally used in [60]. The property consists in a specific kind of continuity. If two probability distributions are close to each other (which corresponds to slightly different realizations of a given experiment), this is expected to correspond to a relatively small discrepancy in any physical functional of the probability distribution, in particular in the entropy.

The mathematical expression of this basic property follows. We first introduce a *distance* $d_\alpha \equiv ||p-p'||_\alpha$ between two probability sets $\{p_i\}$ and $\{p'_i\}$ associated with the same system:

$$d_\alpha \equiv \Big[\sum_{i=1}^{W} |p_i - p'_i|^\alpha\Big]^{1/\alpha} \quad (\alpha \geq 1)\,. \tag{20}$$

Lesche used $\alpha = 1$; we shall enlarge here his definition (e.g., $\alpha = 2$ corresponds to the Pythagorean distance). The interval $0 < \alpha < 1$ also defines a quantity which could be temptatively thought of as measuring distances. However it does not constitute a metric, because it violates the triangular inequality. We shall therefore not consider that possibility at the present stage.

We define now the *relative discrepancy* R of the entropy as follows:

$$R \equiv \frac{S(\{p_i\}) - S(\{p'_i\})}{S_{\rm max}}\ , \tag{21}$$

where $S_{\rm max}$ is the maximal value that the entropy under consideration can attain. For instance, the maximal value for $S_{\rm BG}$ is $k \ln W$.

We will say that S is α-stable (or experimentally α-robust) if and only if, for any given $\epsilon > 0$, a $\delta_\epsilon > 0$ exists such that, independently from W,

$$d_\alpha \leq \delta_\epsilon \implies |R| < \epsilon\ . \tag{22}$$

This implies, in particular, that $\lim_{d_\alpha \to 0} \lim_{W\to\infty} R(d_\alpha, W) = 0$ ($\lim_{W\to\infty} \lim_{d_\alpha\to 0} R(d_\alpha, W)$ always vanishes).

Let us apply definition (22) to the present nonextensive entropy S_q, to the Renyi entropy $S_q^R \equiv \ln \sum_{i=1}^W p_i^q/(q-1) = \ln[1+(1-q)S_q/k]/(1-q)$, to the normalized nonextensive entropy [63] $S_q^N \equiv S_q/\sum_{i=1}^W p_i^q = S_q/[1+(1-q)S_q/k]$, and to the escort entropy S_q^E defined as follows [3]:

$$S_q^E(\{p_i\}) = k\frac{1-(\sum_{j=1}^W p_j^{1/q})^{-q}}{q-1}\ . \tag{23}$$

This entropy emerges from $S_q(\{p_i\})$ where we do the escort transformation, namely $p_i = P_i^{1/q}/\sum_{j=1}^W P_j^{1/q}$, and then, for convenience, rename $P_i \to p_i$.

Lesche showed [60] that $S_{\rm BG}$ is 1-stable and that S_q^R is not for any $q \neq 1$. Abe showed [61,62] that S_q is 1-stable for all $q > 0$, whereas S_q^N is not for any value of $q \neq 1$. We shall illustrate here these facts, including the case $\alpha > 1$. In addition to this, we shall also illustrate that neither S_q^E is 1-stable. To do so we shall consider, following Abe, two typical cases, from now on referred to as *quasi-certainty* (QC) and as *quasi-equal-probabilities* (QEP).

The quasi-certainty case refers to the following:

$$p_1 = 1;\ p_i = 0\ (\forall i \neq 1)\ , \tag{24}$$

and

$$p_1' = 1 - \frac{\delta}{2};\ p_i' = \frac{\delta}{2}\frac{1}{W-1}\ (\forall i \neq 1)\ . \tag{25}$$

Hence

$$d_\alpha = \frac{\delta}{2}\left[1 + \frac{1}{(W-1)^{\alpha-1}}\right]^{1/\alpha} \tag{26}$$

and consequently

$$\delta = \frac{2\,d_\alpha}{\left[1 + \frac{1}{(W-1)^{\alpha-1}}\right]^{1/\alpha}}\ . \tag{27}$$

We verify that $d_1 = \delta$ does not depend on W, whereas, for $\alpha > 1$, d_α depends on W, and $\lim_{W\to\infty} d_\alpha = \delta/2$ ($\forall\alpha > 1$).

The quasi-equal-probabilities case refers to the following:

$$p_1 = 0;\ p_i = \frac{1}{W-1}\ (\forall i \neq 1)\ , \tag{28}$$

and

$$p_1' = \frac{\delta}{2};\ p_i' = \left(1 - \frac{\delta}{2}\right)\frac{1}{W-1}\ . \tag{29}$$

Hence

$$d_\alpha = \frac{\delta}{2}\left[1 + \frac{1}{(W-1)^{\alpha-1}}\right]^{1/\alpha} \tag{30}$$

and consequently

$$\delta = \frac{2\,d_\alpha}{\left[1 + \frac{1}{(W-1)^{\alpha-1}}\right]^{1/\alpha}}\ . \tag{31}$$

Once again we verify that $d_1 = \delta$ is independent of W, whereas, for $\alpha > 1$, d_α depends on W, and $\lim_{W\to\infty} d_\alpha = \delta/2$ ($\forall\alpha > 1$). Notice that, for this QEP case, the results in (30) and (31) turn out to exactly coincide with those corresponding to the QC case, namely (26) and (27).

We may now calculate, as functions of (δ, W, α, q) and for both quasi-certainty and quasi-equal-probabilities cases, the quantities R_q, R_q^R, R_q^N and R_q^E, respectively associated with the entropic functionals S_q, S_q^R, S_q^N and S_q^E. We obtain the following results:

$$R_q = -\frac{(1-\delta/2)^q + (\delta/2)^q(W-1)^{1-q} - 1}{W^{1-q} - 1}\quad (QC),$$

$$R_q = \frac{(W-1)^{1-q} - (\delta/2)^q - (1-\delta/2)^q(W-1)^{1-q}}{W^{1-q} - 1}\quad (QEP),$$

$$R_q^R = -\frac{\ln\left[(1-\delta/2)^q + (\delta/2)^q(W-1)^{1-q}\right]}{(1-q)\ln W}\quad (QC),$$

$$R_q^R = \frac{(1-q)\ln(W-1) - \ln\left[(\delta/2)^q + (1-\delta/2)^q(W-1)^{1-q}\right]}{(1-q)\ln W}\quad (QEP),$$

$$R_q^N = -\frac{1 - 1/\left[(1-\delta/2)^q + (\delta/2)^q(W-1)^{1-q}\right]}{1 - W^{q-1}}\quad (QC),$$

$$R_q^N = \frac{1/\left[(\delta/2)^q + (1-\delta/2)^q(W-1)^{1-q}\right] - (W-1)^{q-1}}{1 - W^{q-1}}\quad (QEP),$$

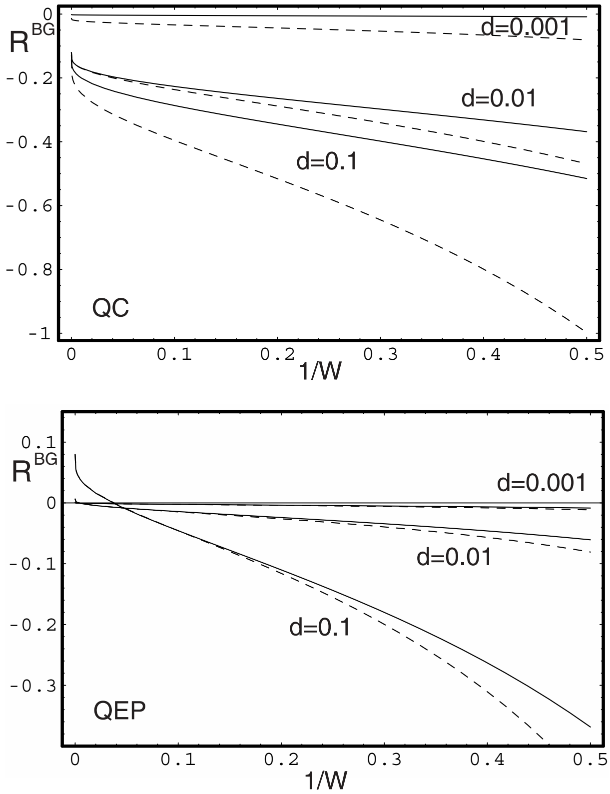

Fig. 1. Typical examples (QEP and QC) for the stability of the $S_{\rm BG}$. The dashed (continuous) curves correspond to the $\alpha = 1$ ($\alpha = 2$) metric

$$R_q^E = -\frac{1 - [(1-\delta/2)^{1/q} + (W-1)(\delta/2)^{1/q}(W-1)^{-1/q}]^{-q}}{W^{1-q} - 1} \quad (QC),$$

$$R_q^E = \frac{(W-1)^{1-q} - [(\delta/2)^{1/q} + (W-1)(1-\delta/2)^{1/q}(W-1)^{-1/q}]^{-q}}{W^{1-q} - 1} \quad (QEP).$$

All these ratios can be expressed as explicit functions of (d_α, W, α, q) if we use (27, 31).

We present typical examples in Fig. 1 for $S_{\rm BG}$, and in Figs. 2 and 3 for its four generalizations. We notice in Figs. 2 and 3 that *only* R_q satisfies (22) for *all* the examples; R_q^R violates (22) for QC with $q < 1$, and for QEP with $q > 1$; R_q^N violates (22) for QC with $q < 1$, and for QEP with $q > 1$; R_q^E violates (22) for QEP with $q < 1$, and for QC with $q > 1$.

At this point, let us make a remark about the $\alpha > 1$ metrics. This class of distances satisfies, as already mentioned, the triangular inequality. This is why we have considered it until now. However, it presents an inconvenient peculiarity. Let us illustrate this through the following example (assuming W is even):

$$p_i = 0 \left(i = 1, 2, ..., \frac{W}{2} \right), \; p_i = \frac{2}{W} \left(i = \frac{W}{2} + 1, \frac{W}{2} + 2, ..., W \right), \tag{40}$$

and

$$p'_1 = \frac{\delta}{2} \frac{2}{W} \left(i = 1, 2, ..., \frac{W}{2} \right), \; p'_i = \left(1 - \frac{\delta}{2} \right) \frac{2}{W} \left(i = \frac{W}{2} + 1, \frac{W}{2} + 2, ..., W \right), \tag{41}$$

hence

$$d_\alpha = \delta W^{(1-\alpha)/\alpha} . \tag{42}$$

As for the QC and QEP cases considered above, $d_1 = \delta$. *But*, for $\alpha > 1$ and $W \to \infty$, $d_\alpha \to 0$. Consequently, although $\{p_i\} \neq \{p'_i\}$, their distance d_α vanishes, a quite inconvenient feature. We thus verify that d_1 is very special indeed, since it does not depend on W.

Finally, in order that it becomes clear that stability is a property independent from concavity, we represent, in Fig. 4, typical cases (namely those associated with the $W = 2$ system) of all four generalizations of $S_{\rm BG}$ considered here. We notice that *only* S_q satisfies concavity for *all* values of $q > 0$; S_q^R and S_q^N are concave for $0 < q \leq 1$, but not for $q > 1$; S_q^E is concave for $q \geq 1$, but not for $0 < q < 1$.

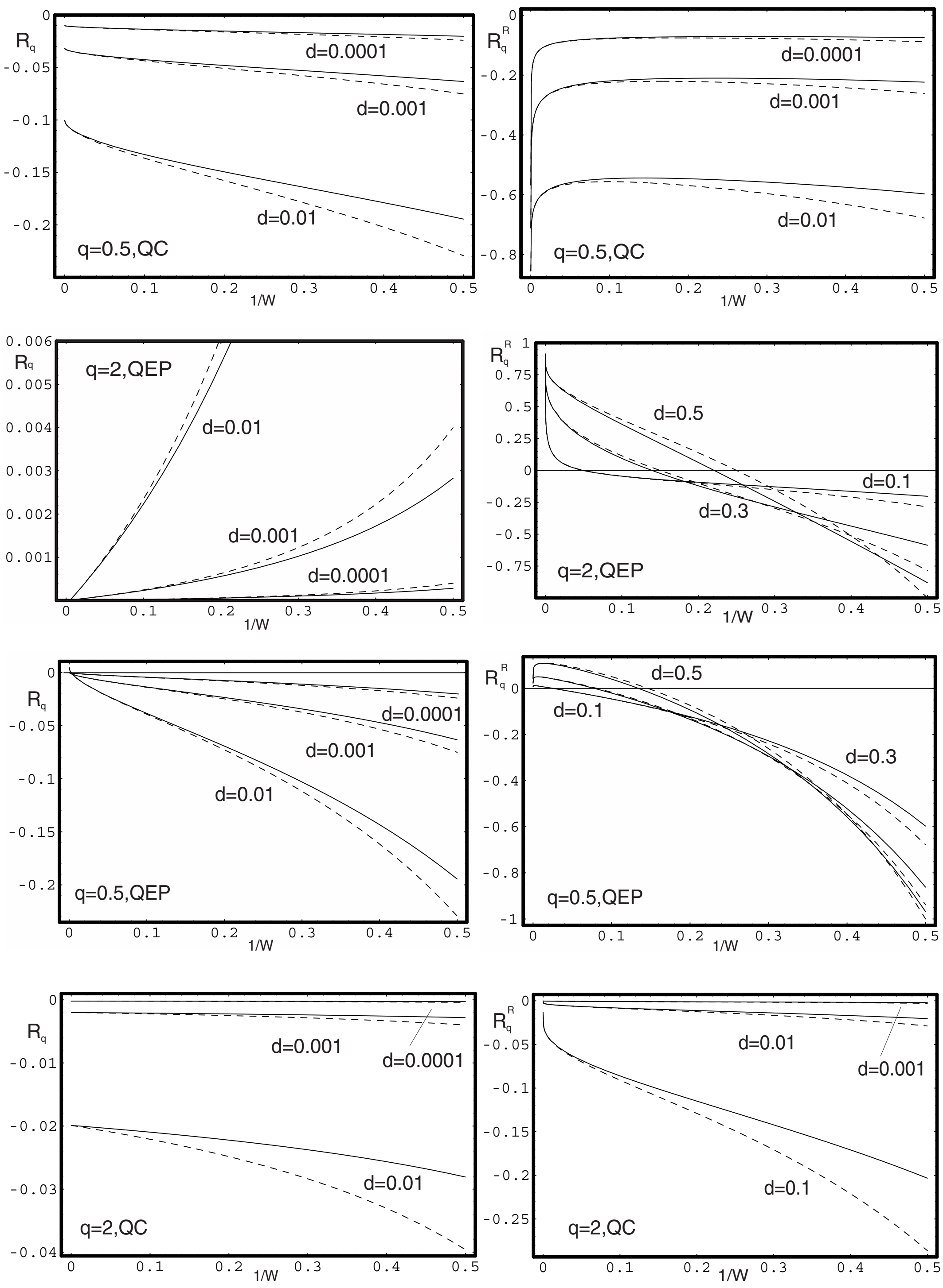

Fig. 2. Typical examples (QEP and QC, $q < 1$ and $q > 1$) for the stability (or lack of stability) of the S_q and S_q^R entropies. The dashed (continuous) curves correspond to the $\alpha = 1$ ($\alpha = 2$) metric

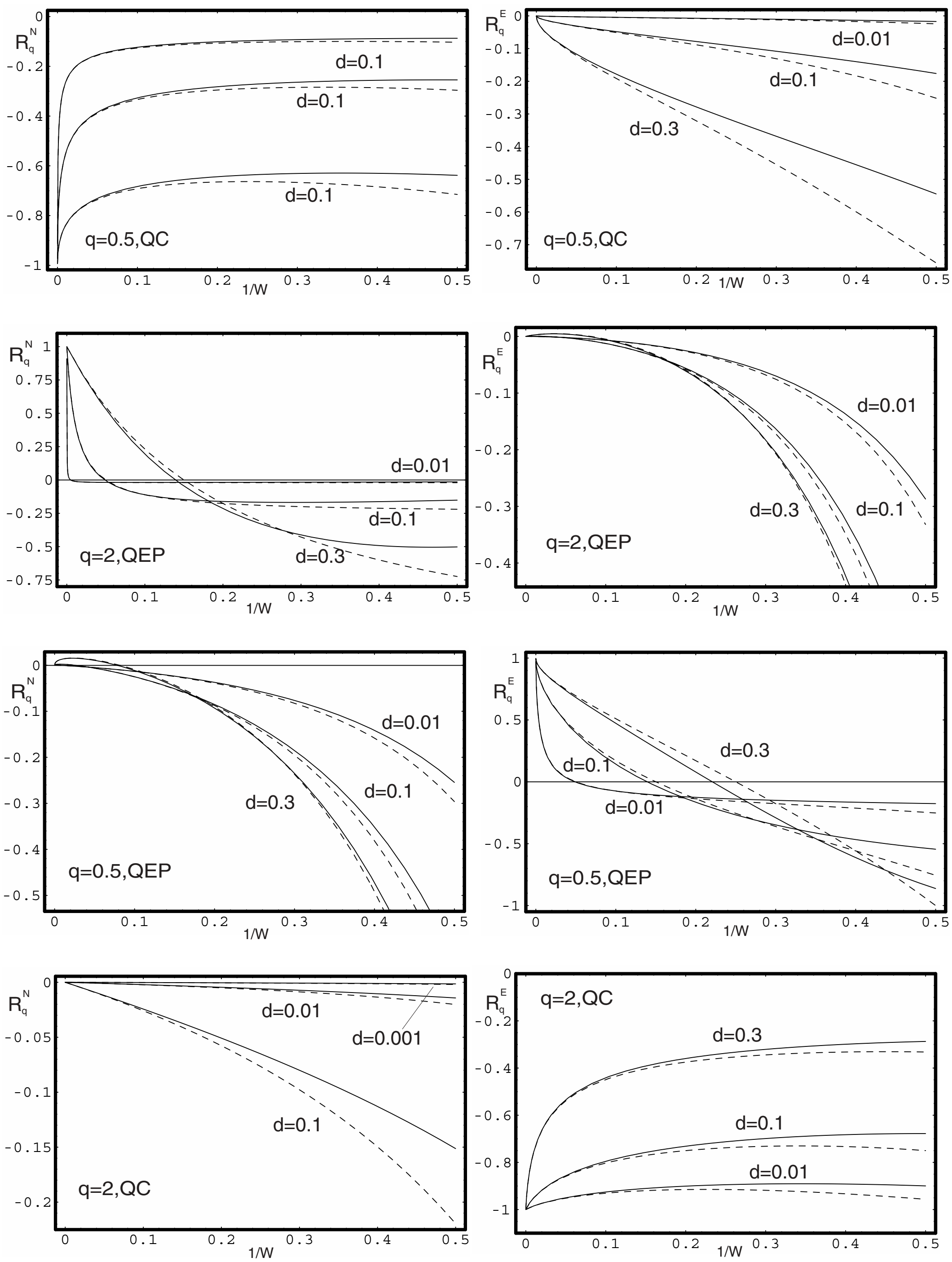

Fig. 3. The same as in Fig. 3, for the S_q^N and S_q^E entropies

5 Conclusion

We have provided a brief introduction to nonextensive statistical mechanics and to its connections to thermodynamics, as well as a few explanations about its dynamical foundations. Various applications have been mentioned, and the reader is referred to the appropriate references for further details. Many interesting open questions remain to be investigated further. Among them, let us mention the subtle, and not yet completely clarified, connection between the present ideas and long-range-interacting Hamiltonians.

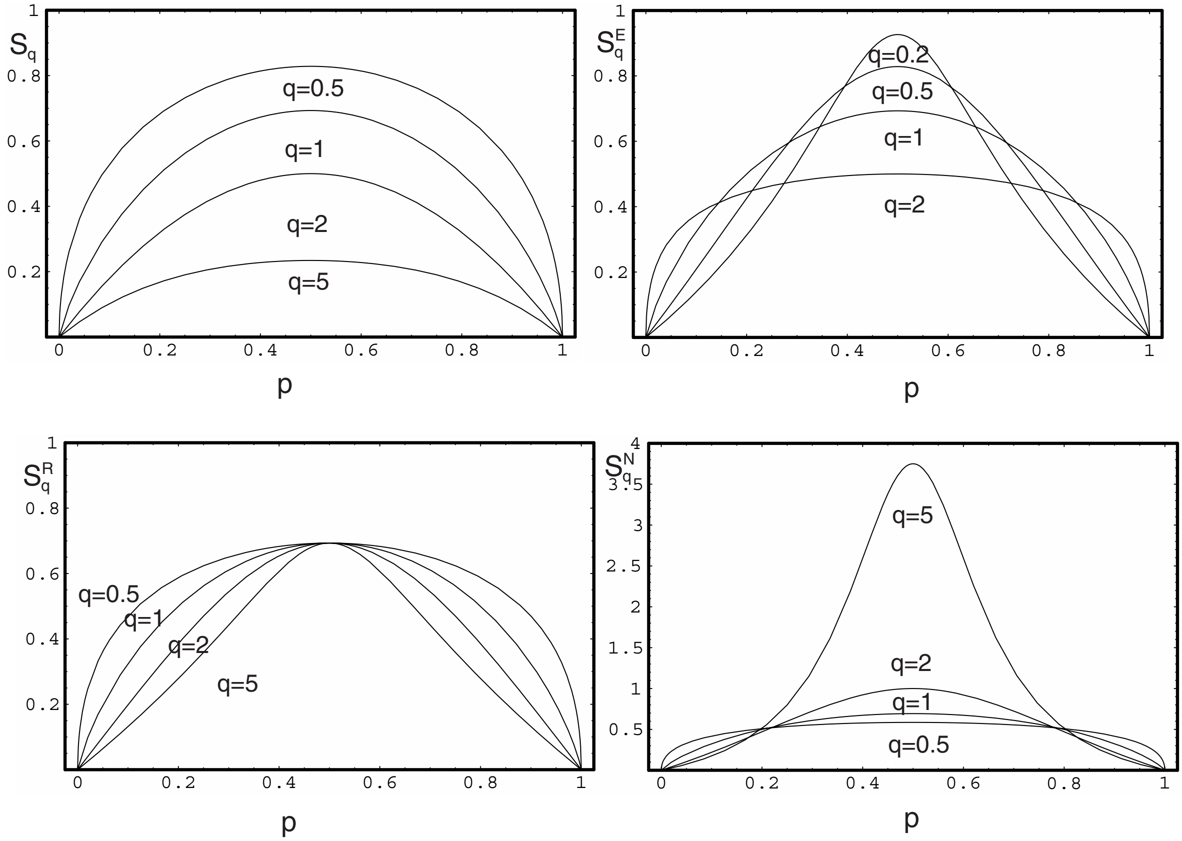

Fig. 4. Typical examples exhibiting the concavity (or the lack of concavity) of the four entropies discussed here

In Sect. 4 we have addressed the property of stability (or experimental robustness) introduced by Lesche in 1982. We have illustrated, for typical cases and metrics, the generic validity of this property for the nonextensive entropy S_q, as well as its nonvalidity for the (extensive) Renyi entropy S_q^R, for the normalized nonextensive entropy S_q^N, and for the escort nonextensive entropy S_q^E. In addition to this, we have also illustrated the validity or nonvalidity of the concavity property for the same entropic forms.

Last but not least, the history of sciences teaches us that all attempts of innovation in scientific theories have been the subject of objections or critical comments. The present attempt is no exception. We may mention the critique in [64], counterbalanced in [65–67], the critique in [68], counterbalanced in [32,33], and the critique in [69], counterbalanced in [40,49].

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