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Original article

Kinematics and elasticity framework for materials with two fiber families

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Abstract. A kinematics framework is developed for materials with two fiber families that are not necessarily orthogonal or mechanically equivalent. These two latter conditions represent important subclasses that are analyzed. To succinctly define the strain, six scalar strain attributes are developed that have direct physical interpretation. In the hyperelastic limit, this approach allows the Cauchy stress $\mathbf{t}$ to be expressed as the sum of six response terms, almost all of which are mutually orthogonal (i.e. 14 of the 15 inner products vanish). For small deformations, the response terms are entirely orthogonal (i.e. all 15 inner products vanish). Experimental advantage is demonstrated for finite strain hyperelastic materials by showing that common tests, for the first time, can directly determine terms in the strain energy function of two fiber composites.

Key words: finite elasticity, solid mechanics, continuum mechanics, composite materials

1 Introduction

Materials reinforced with two families of fibers have tremendous utility in industry, engineering, medicine, and science. Some undergo finite strain (e.g. tires and hoses) while others do not (e.g. epoxy reinforced with carbon fiber cloth). The latter example is very elastic whereas soft biotissues (e.g. arteries) are visco-elastic. Unfortunately, it is because of this wide range of behaviors that the strain in such materials is not defined consistently. Infinitesimal elasticity uses the engineering strain tensor, finite elasticity uses invariants of the right Cauchy-Green deformation tensor $\mathbf{C}$, and visco-elasticity must utilize the velocity gradient in addition to strain measures. However, it is shown herein (via demonstration) that an intrinsic, kinematics framework exists for materials with two distinct fiber directions.

In addition, this framework offers distinct advantage for defining the constitutive behaviors of such materials. Based on the results of Criscione et al. (2002), it is now evident that the conventional approach to anisotropic finite elasticity (as in Spencer, 1984) is experimentally intractable. To see why, recall that the Cauchy stress $\mathbf{t}$ is conventionally given by

$$\mathbf{t} = \frac{1}{J} \sum_{i=1}^{7} \frac{\partial W}{\partial I_i} \breve{\mathbf{A}}_i \,, \tag{1.1}$$

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where $J = \det(\mathbf{F})$ and with the left Cauchy-Green deformation tensor, $\mathbf{B} = \mathbf{F}\mathbf{F}^T$,

$$\breve{\mathbf{A}}_1 = 2\mathbf{B}\,, \quad \breve{\mathbf{A}}_2 = 2\left(I_1\mathbf{B} - \mathbf{B}^2\right)\,, \quad \breve{\mathbf{A}}_3 = 2I_3\mathbf{I} \quad , \tag{1.2$_{1-3}$}$$

$$\breve{\mathbf{A}}_4 = 2I_4\mathbf{m}_1 \otimes \mathbf{m}_1\,, \quad \breve{\mathbf{A}}_5 = 2I_4\left(\mathbf{B}\mathbf{m}_1 \otimes \mathbf{m}_1 + \mathbf{m}_1 \otimes \mathbf{B}\mathbf{m}_1\right)\,, \tag{1.2$_{4-5}$}$$

$$\breve{\mathbf{A}}_6 = 2I_6\mathbf{m}_2 \otimes \mathbf{m}_2\,, \quad \breve{\mathbf{A}}_7 = 2I_6\left(\mathbf{B}\mathbf{m}_2 \otimes \mathbf{m}_2 + \mathbf{m}_2 \otimes \mathbf{B}\mathbf{m}_2\right)\,. \tag{1.2$_{6-7}$}$$

The unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ represent the current directions of the reference orientations $\mathbf{M}_1$ and $\mathbf{M}_2$. Equivalently, $\mathbf{m}_1$ and $\mathbf{m}_2$ are respectively $\mathbf{F}\mathbf{M}_1$ and $\mathbf{F}\mathbf{M}_2$ with normalization to yield unit magnitude. W is the strain energy potential function and it is a function of I_{1-7}, the invariants or integrity basis. The scalar partial derivatives of W, the $\partial W/\partial \mathrm{I}_i$, are called response functions, and they specify the constitutive behavior of a particular hyperelastic material.

As shown in Criscione (2003), it is experimentally ill-conceived to determine $\partial W/\partial I_1$ and $\partial W/\partial I_2$ for rubber because the response terms (i.e. $\partial W/\partial I_1\breve{\mathbf{A}}_1$ and $\partial W/\partial I_2\breve{\mathbf{A}}_2$) are highly covariant. By highly covariant, we mean that the absolute value of the inner contraction $\breve{\mathbf{A}}_1 : \breve{\mathbf{A}}_2$ is nearly equal to the product of the magnitudes, $|\breve{\mathbf{A}}_1||\breve{\mathbf{A}}_2|$, which is the maximum possible value. It directly follows that (1.1) is experimentally ill-conceived as well. Worse yet, most of the tensors in (1.2) are highly covariant. As shown in Criscione (2003), it is necessary to minimize covariance in order to minimize the propagation of experimental error.

Another drawback of the conventional invariants is the need to use a different set of invariants for 2-fiber materials depending on whether the fibers are (1) orthogonal or (2) mechanically equivalent (see Green and Adkins, 1960). In particular, for case (1) the fiber directions are used to define invariants of $\mathbf{C}$. For case (2), the fiber bisectors are used.

This paper is the fourth in a series of articles that report scalar strain attributes[1] with minimal covariance in hyperelasticity. For isotropy see Criscione et al., 2000; for transverse isotropy see Criscione et al., 2001; and for laminae composed of one family of fibers see Criscione et al., 2002. In contrast to the conventional invariants, the six scalar strain attributes ξ_{1-6} developed herein are: (1) similarly defined for all two-fiber materials, (2) physically descriptive of the strain, (3) related in a one-to-one fashion to the components of $\mathbf{C}$ when defined relative to reference material directions, and (4) yield mostly orthogonal response terms in the hyperelastic limit.

Because of condition (3) and the principle of material frame indifference (i.e. $W = W(\mathbf{C})$ for hyperelastic materials, see Gurtin, 1981), the strain energy function W is expressible as[2]:

$$W = W(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6). \tag{1.3}$$

This expression is valid for all 2-fiber, hyperelastic materials whether their behavior is orthotropic or more fully anisotropic (i.e. fibers are neither orthogonal nor mechanically-equivalent). If the material behavior is orthotropic, then it is shown in Sect. 7 how to refine the general class of (1.3) so that W is invariant under the group of symmetry transformations that is possessed by particular materials. For orthogonal fibers, there are mirror-symmetry planes with normals in the fiber directions; whereas for mechanically equivalent fibers, there are mirror-symmetry planes with normals in the directions of the fiber bisectors. For materials with fiber families that are orthogonal AND mechanically equivalent, our approach allows both sets of symmetry constraints to be combined forthwith to yield a highly refined subclass of (1.3).

2 An intrinsic, local deformation gradient

Let the unit vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ be the separate fiber directions in the reference configuration. Moreover, let the material have two distinct fiber directions in the sense that $\mathbf{M}_1$ and $\mathbf{M}_2$ are not parallel (i.e.$\mathbf{M}_1 \cdot \mathbf{M}_2$ is not ± 1).

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[1] As in Criscione et al. (2002), we use the word 'attribute' instead of 'invariant' because some of our strain attributes are not invariant under the symmetry group of orthotropy.

[2] We are liberal in our usage of W to represent arbitrary strain energy functions. For example, $W(I_1, I_2)$ means "W is an arbitrary function of I_1 and I_2" and $W(\xi_1, \xi_2)$ means "W is an arbitrary function of ξ_1 and ξ_2". The functional forms may differ such that W does not depend on ξ_1 and ξ_2 in precisely the same way that W depends on I_1 and I_2.

If the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ respectively represent the current directions of $\mathbf{M}_1$ and $\mathbf{M}_2$ then

$$\mathbf{m}_1 = \frac{\mathbf{F}\mathbf{M}_1}{\sqrt{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1}}\,, \quad \mathbf{m}_2 = \frac{\mathbf{F}\mathbf{M}_2}{\sqrt{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2}}\,. \tag{2.1}$$

Although $\mathbf{M}_1$ and $\mathbf{M}_2$ are not necessarily orthogonal, their bisectors are always orthogonal, and we define an orthonormal triad in the reference configuration as follows:

$$\mathbf{M}_A = \frac{\mathbf{M}_1 + \mathbf{M}_2}{\sqrt{2 + 2\mathbf{M}_1 \cdot \mathbf{M}_2}}\,, \quad \mathbf{M}_B = \frac{-\mathbf{M}_1 + \mathbf{M}_2}{\sqrt{2 - 2\mathbf{M}_1 \cdot \mathbf{M}_2}}\,, \quad \mathbf{N} = \mathbf{M}_A \times \mathbf{M}_B\,. \tag{2.2}$$

The unit vector $\mathbf{N}$ is orthogonal to the plane containing the fibers[3], and $\mathbf{M}_A$ bisects $\mathbf{M}_1$ and $\mathbf{M}_2$ whereas $\mathbf{M}_B$ bisects -$\mathbf{M}_1$ and $\mathbf{M}_2$. Such use of the bisectors as the axes for analysis goes back at least to Rivlin (1955), although Spencer (1984) preferred not to use them.

The fibers themselves may not define a sense for $\mathbf{M}_1$ and $\mathbf{M}_2$. Nonetheless, being vectors, a sense for $\mathbf{M}_1$ and $\mathbf{M}_2$ must be chosen. Upon doing so, $\mathbf{M}_A$ and $\mathbf{M}_B$ are uniquely defined in the reference configuration. Since a change in sense of $\mathbf{M}_1$ or $\mathbf{M}_2$ would cause 90° shifts in $\mathbf{M}_A$ and $\mathbf{M}_B$, it is anticipated that a sense will be chosen such that $\mathbf{M}_A$ and $\mathbf{M}_B$ will have forthright representations in the structure of interest. For example, consider a hose that is composed of concentric lamina, each of which is reinforced with two families of helically wound fibers of equal but opposite pitch. Moreover, let the pitch be different for the separate laminae. If the sense of $\mathbf{M}_1$ and $\mathbf{M}_2$ are chosen in each of the laminae so that $\mathbf{M}_1$ and $\mathbf{M}_2$ have similar projections on the long axis, then $\mathbf{M}_A$ will be parallel to the axis whereas $\mathbf{M}_B$ will be parallel to the hoop direction.

For the current configuration we define an orthonormal triad similar to (2.2) as follows:

$$\mathbf{m}_A = \frac{\mathbf{m}_1 + \mathbf{m}_2}{\sqrt{2 + 2\mathbf{m}_1 \cdot \mathbf{m}_2}}\,, \quad \mathbf{m}_B = \frac{-\mathbf{m}_1 + \mathbf{m}_2}{\sqrt{2 - 2\mathbf{m}_1 \cdot \mathbf{m}_2}}\,, \quad \mathbf{n} = \mathbf{m}_A \times \mathbf{m}_B\,. \tag{2.3}$$

As above, $\mathbf{n}$ is orthogonal to the plane containing the fibers, and $\mathbf{m}_A$ bisects $\mathbf{m}_1$ and $\mathbf{m}_2$ whereas $\mathbf{m}_B$ bisects $-\mathbf{m}_1$ and $\mathbf{m}_2$. However, neither of the bisectors represents a material line segment, whereby $\mathbf{m}_A$ is not necessarily parallel to $\mathbf{F}\mathbf{M}_A$. Nonetheless, note that the handedness[4] of $(\mathbf{n}, \mathbf{m}_A, \mathbf{m}_B)$ and $(\mathbf{N}, \mathbf{M}_A, \mathbf{M}_B)$ is identical, and that $\mathbf{m}_A \cong \mathbf{M}_A$, $\mathbf{m}_B \cong \mathbf{M}_B$, and $\mathbf{n} \cong \mathbf{N}$ whenever $\mathbf{F} \cong \mathbf{I}$.

Since they have the same handedness, the orthonormal systems $(\mathbf{N}, \mathbf{M}_A, \mathbf{M}_B)$ and $(\mathbf{n}, \mathbf{m}_A, \mathbf{m}_B)$ can be related by a rigid rotation. We define $\mathbf{Q}$ such that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, $\det(\mathbf{Q}) = 1$, and

$$\mathbf{n} = \mathbf{Q}\mathbf{N}, \quad \mathbf{m}_A = \mathbf{Q}\mathbf{M}_A, \quad \mathbf{m}_B = \mathbf{Q}\mathbf{M}_B\,. \tag{2.4}$$

To an observer whose rotation is given by $\mathbf{Q}$, the fiber bisectors and the normal to the fiber plane appear fixed, i.e. relative to this rotating frame $\mathbf{n} = \mathbf{N}$, $\mathbf{m}_A = \mathbf{M}_A$, and $\mathbf{m}_B = \mathbf{M}_B$. Moreover, the observed deformation gradient only has six independent components. In particular, two components must vanish, and an identity exists amongst the other seven. Toward this end, the deformation seen by such a rotating observer is

$$\mathbf{f} = \mathbf{Q}^T\mathbf{F}\,. \tag{2.5}$$

It should be evident that $\mathbf{f}$ and $\mathbf{F}$ induce the same Lagrangian strain because $\mathbf{f}^T\mathbf{f} = \mathbf{F}^T\mathbf{F} = \mathbf{C} = \mathbf{U}^2$.

The two vanishing components of $\mathbf{f}$ arise because the reference fiber plane and the deformed fiber plane have the same normal. A reference normal is related to a current normal as follows:

$$\mathbf{n} = \frac{\mathbf{F}^{-T}\mathbf{N}}{|\mathbf{F}^{-T}\mathbf{N}|}\,. \tag{2.6}$$

Whereby with use of (2.4) and (2.5), we obtain,

$$\mathbf{N} = \frac{\mathbf{f}^{-T}\mathbf{N}}{|\mathbf{f}^{-T}\mathbf{N}|}\,. \tag{2.7}$$

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[3] It is straightforward to show that $\mathbf{N}$ points in the $\mathbf{M}_1 \times \mathbf{M}_2$ direction.

[4] In particular, $\mathbf{n} \cdot \mathbf{m}_A \times \mathbf{m}_B = \mathbf{N} \cdot \mathbf{M}_A \times \mathbf{M}_B$.

Substitution of $\mathbf{N}$ above into $\mathbf{N}\cdot\mathbf{fM}_A$ and $\mathbf{N}\cdot\mathbf{fM}_B$ confirms that these components must vanish, and $\mathbf{f}$ is of the following form:

$$[\mathbf{f}]_{N,M_A,M_B} = \begin{bmatrix} f_{NN} & 0 & 0 \\ f_{AN} & f_{AA} & f_{AB} \\ f_{BN} & f_{BA} & f_{BB} \end{bmatrix}. \tag{2.8}$$

The remaining constraint on $\mathbf{f}$ arises because $\mathbf{f}$ must be such that the fiber bisectors remain unchanged. In order to derive this constraint, it is helpful to express $\mathbf{M}_1$ and $\mathbf{M}_2$ as

$$\mathbf{M}_1 = \cos\Theta\mathbf{M}_A - \sin\Theta\mathbf{M}_B, \quad \mathbf{M}_2 = \cos\Theta\mathbf{M}_A + \sin\Theta\mathbf{M}_B\,, \tag{2.9}$$

wherein $\Theta \in (0, \frac{\pi}{2})$ is half of the angle subtended[5] by $\mathbf{M}_1$ and $\mathbf{M}_2$. This is a straightforward geometric result of (2.2) and note that Θ is defined in the reference configuration (i.e. it does not depend on $\mathbf{F}$) and that $\cos 2\Theta = \mathbf{M}_1\cdot\mathbf{M}_2$. Appendix A shows that (2.4) and (2.5) require that

$$f_{BA} = f_{AA}^{-1} f_{AB} f_{BB}\tan^2\Theta\,. \tag{2.10}$$

We now factor $\mathbf{f}$ into distinct modes of deformation that correspond to the six scalar strain attributes that we will utilize. Toward this end, it is helpful to make the following definitions:

$$J = f_{NN}(f_{AA}f_{BB} - f_{AB}f_{BA}), \quad \alpha = J^{-2/3}(f_{AA}f_{BB} - f_{AB}f_{BA}), \tag{2.11$_{1-2}$}$$
$$\beta = (f_{BB}^{-1}f_{AA})^{1/2}, \quad \gamma = J^{-1/3}\alpha^{-1/2}(\beta^{-1}f_{AB} + \beta f_{BA}), \tag{2.11$_{3-4}$}$$
$$\psi_{AN} = \frac{f_{AN}f_{BB} - f_{BN}f_{AB}}{f_{AA}f_{BB} - f_{AB}f_{BA}}, \quad \psi_{BN} = \frac{f_{BN}f_{AA} - f_{AN}f_{BA}}{f_{AA}f_{BB} - f_{AB}f_{BA}}. \tag{2.11$_{5-6}$}$$

Together with the constraint (2.10), the above six scalars specify the seven non-vanishing components of $\mathbf{f}$ as follows:

$$f_{NN} = J^{1/3}\alpha^{-1}, \quad f_{AA} = J^{1/3}\alpha^{1/2}\beta(1+\gamma^2s^2c^2)^{1/2}, \quad f_{BB} = J^{1/3}\alpha^{1/2}\beta^{-1}(1+\gamma^2s^2c^2)^{1/2}, \tag{2.12$_{1-3}$}$$
$$f_{AN} = J^{1/3}\alpha^{1/2}\beta\left(\left(1+\gamma^2c^2s^2\right)^{1/2}\psi_{AN} + \gamma c^2\psi_{BN}\right), \quad f_{AB} = J^{1/3}\alpha^{1/2}\beta\gamma c^2, \tag{2.12$_{4-5}$}$$
$$f_{BN} = J^{1/3}\alpha^{1/2}\beta^{-1}\left(\gamma s^2\psi_{AN} + \left(1+\gamma^2c^2s^2\right)^{1/2}\psi_{BN}\right), \quad f_{BA} = J^{1/3}\alpha^{1/2}\beta^{-1}\gamma s^2, \tag{2.12$_{6-7}$}$$

wherein $c = \cos\Theta$ and $s = \sin\Theta$. With this change in variables, $\mathbf{f}$ can be factored as follows:

$$[\mathbf{f}]_{\mathbf{N},\mathbf{M}_A,\mathbf{M}_B} = J^{1/3}\begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \sqrt{\alpha} & 0 \\ 0 & 0 & \sqrt{\alpha} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1+\gamma^2\gamma s^2c^2} & \gamma c^2 \\ 0 & \gamma s^2 & \sqrt{1+\gamma^2s^2c^2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \psi_{AN} & 1 & 0 \\ \psi_{BN} & 0 & 1 \end{bmatrix}. \tag{2.13}$$

Equivalently, in coordinate-free notation, with the above order preserved

$$\mathbf{f} = J^{1/3}\mathbf{f}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\mathbf{f}_\psi\,, \tag{2.14}$$

wherein the tensors $\mathbf{f}_\alpha$, $\mathbf{f}_\beta$, $\mathbf{f}_\gamma$, and $\mathbf{f}_\psi$ are

$$\mathbf{f}_\alpha = \alpha^{-1}\mathbf{N}\otimes\mathbf{N} + \alpha^{1/2}(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B)\,, \tag{2.15$_1$}$$
$$\mathbf{f}_\beta = \mathbf{N}\otimes\mathbf{N} + \beta\mathbf{M}_A\otimes\mathbf{M}_A + \beta^{-1}\mathbf{M}_B\otimes\mathbf{M}_B\,, \tag{2.15$_2$}$$
$$\mathbf{f}_\gamma = \mathbf{N}\otimes\mathbf{N} + (1+\gamma^2c^2s^2)^{1/2}(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B) + \gamma(s^2\mathbf{M}_B\otimes\mathbf{M}_A + c^2\mathbf{M}_A\otimes\mathbf{M}_B)\,, \tag{2.15$_3$}$$
$$\mathbf{f}_\psi = \mathbf{I} + \psi_{AN}\mathbf{M}_A\otimes\mathbf{N} + \psi_{BN}\mathbf{M}_B\otimes\mathbf{N}\,. \tag{2.15$_4$}$$

[5] Herein we do not consider materials wherein the fiber families are colinear (i.e. the angle subtended by $\mathbf{M}_1$ and $\mathbf{M}_2$ is zero or π).

With scalar multiplication by $J^{1/3}$ representing the dilatation, the tensors $\mathbf{f}_\alpha$, $\mathbf{f}_\beta$, $\mathbf{f}_\gamma$, and $\mathbf{f}_\psi$ are isochoric or purely distortional deformations. Figure 1 depicts the four distinct actions of $\mathbf{f}_\alpha$, $\mathbf{f}_\beta$, $\mathbf{f}_\gamma$, and $\mathbf{f}_\psi$ which comprise the distortion. Note that $\mathbf{f}_\alpha$ uniformly changes the area ratio of the fiber plane, $\mathbf{f}_\beta$ perturbs the angle subtended by the fibers, $\mathbf{f}_\gamma$ differentially changes the lengths of the fibers, and $\mathbf{f}_\psi$ shears one fiber plane over an adjacent fiber plane.

Physically, the kinematic parameters are; $J = \det(\mathbf{f}) = \det(\mathbf{F})$ is the volume ratio, α is the area ratio of the fiber plane due to distortion (i.e. that which is not due to dilatation J), β is a pure shear stretch in the fiber plane that makes the angle subtended by $\mathbf{m}_1$ and $\mathbf{m}_2$ more acute if $\beta > 1$ or less acute if $\beta < 1$, γ is a shear strain in the fiber plane that differentially changes the lengths of $\mathbf{m}_1$ and $\mathbf{m}_2$ yet does not perturb the angle subtended by $\mathbf{m}_1$ and $\mathbf{m}_2$, ψ_{AN} is the tangent of a shear angle in the $(\mathbf{N}, \mathbf{M}_A)$ plane, and ψ_{BN} is the tangent of a shear angle in the $(\mathbf{N}, \mathbf{M}_B)$ plane.

3 An intrinsic set of scalar strain attributes

The six scalars $\{J, \alpha, \beta, \gamma, \psi_{NA}, \psi_{NB}\}$ are the kinematic parameters that form the basis for our set of six strain attributes. First, to develop expressions for these kinematic parameters we need an equation for $\mathbf{F}$. Toward this end, note that (2.5) and (2.14) yield,

$$\mathbf{F} = \mathbf{Q} J^{1/3} \mathbf{f}_\alpha \mathbf{f}_\beta \mathbf{f}_\gamma \mathbf{f}_\psi \, . \tag{3.1}$$

With use of (2.4) and (2.15) we obtain

$$\mathbf{F} = J^{1/3} \begin{pmatrix} \alpha^{-1}\mathbf{n} \otimes \mathbf{N} + \sqrt{\alpha}\sqrt{1+\gamma^2 s^2 c^2}\left(\beta \mathbf{m}_A \otimes \mathbf{M}_A + \beta^{-1}\mathbf{m}_B \otimes \mathbf{M}_B\right) \\ +\sqrt{\alpha}\,\beta\left(\gamma c^2 \mathbf{m}_A \otimes \mathbf{M}_B + \left(\sqrt{1+\gamma^2 s^2 c^2}\,\psi_{AN} + \gamma c^2 \psi_{BN}\right)\mathbf{m}_A \otimes \mathbf{N}\right) \\ +\sqrt{\alpha}\beta^{-1}\left(\gamma s^2 \mathbf{m}_B \otimes \mathbf{M}_A + \left(\gamma s^2 \psi_{AN} + \sqrt{1+\gamma^2 s^2 c^2}\,\psi_{BN}\right)\mathbf{m}_B \otimes \mathbf{N}\right) \end{pmatrix} . \tag{3.2}$$

The kinematic parameters as defined in (2.11) can now be obtained directly from $\mathbf{F}$ as follows:

$$J = \mathbf{n} \cdot \mathbf{FN}\left((\mathbf{m}_A \cdot \mathbf{FM}_A)(\mathbf{m}_B \cdot \mathbf{FM}_B) - (\mathbf{m}_A \cdot \mathbf{FM}_B)(\mathbf{m}_B \cdot \mathbf{FM}_A)\right) = \det(\mathbf{F}) \, , \tag{3.3$_1$}$$

$$\alpha = J^{-2/3}\left((\mathbf{m}_A \cdot \mathbf{FM}_A)(\mathbf{m}_B \cdot \mathbf{FM}_B) - (\mathbf{m}_A \cdot \mathbf{FM}_B)(\mathbf{m}_B \cdot \mathbf{FM}_A)\right) \, , \tag{3.3$_2$}$$

$$\beta = (\mathbf{m}_B \cdot \mathbf{FM}_B)^{-1/2}(\mathbf{m}_A \cdot \mathbf{FM}_A)^{1/2} \, , \tag{3.3$_3$}$$

$$\gamma = J^{-1/3}\alpha^{-1/2}(\beta^{-1}\mathbf{m}_A \cdot \mathbf{FM}_B + \beta \mathbf{m}_B \cdot \mathbf{FM}_A) \, , \tag{3.3$_4$}$$

$$\psi_{AN} = \frac{(\mathbf{m}_A \cdot \mathbf{FN})(\mathbf{m}_B \cdot \mathbf{FM}_B) - (\mathbf{m}_B \cdot \mathbf{FN})(\mathbf{m}_A \cdot \mathbf{FM}_B)}{(\mathbf{m}_A \cdot \mathbf{FM}_A)(\mathbf{m}_B \cdot \mathbf{FM}_B) - (\mathbf{m}_B \cdot \mathbf{FM}_A)(\mathbf{m}_A \cdot \mathbf{FM}_B)} \, , \tag{3.3$_5$}$$

$$\psi_{BN} = \frac{(\mathbf{m}_B \cdot \mathbf{FN})(\mathbf{m}_A \cdot \mathbf{FM}_A) - (\mathbf{m}_A \cdot \mathbf{FN})(\mathbf{m}_B \cdot \mathbf{FM}_A)}{(\mathbf{m}_A \cdot \mathbf{FM}_A)(\mathbf{m}_B \cdot \mathbf{FM}_B) - (\mathbf{m}_B \cdot \mathbf{FM}_A)(\mathbf{m}_A \cdot \mathbf{FM}_B)} \, . \tag{3.3$_6$}$$

The scalars J, α, and β are stretch-like (i.e. unity when **F=I**), yet we prefer arguments for the strain energy function that are strain-like (i.e. null when **F=I**). Consequently, the set of strain attributes utilized herein is as follow:

$$\xi_1 = \ln J, \quad \xi_2 = 3\ln\alpha, \quad \xi_3 = 2\ln\beta, \quad \xi_4 = \gamma, \quad \xi_5 = \psi_{AN}, \quad \xi_6 = \psi_{BN} \, . \tag{3.4}$$

The coefficients in the ξ_2 and ξ_3 definitions are chosen so that linear and quadratic terms in W have meaningful representations for infinitesimal deformations (see Sect. 5). Upon inspection of (3.4), (2.5), and (2.13), it should be evident that a physically realizable $\mathbf{F}$ is obtained for all possible values of the ξ_i that are real. In other words, all of the strain attributes have domains that are $(-\infty, \infty)$. More importantly, the domain of any one of them is entirely unaffected if the others are held at prescribed values. In contrast, the domain constraints of each of the principal invariants of $\mathbf{C}$ are non-linear functions of the other invariants (see Criscione et al., 2000).

Upon recalling the physical interpretations at the end of Sect. 2, note that ξ_1 is the volume strain; ξ_2 is the area strain of the fiber plane due to distortion (i.e. that which is not due to dilatation); ξ_3 indicates change of the

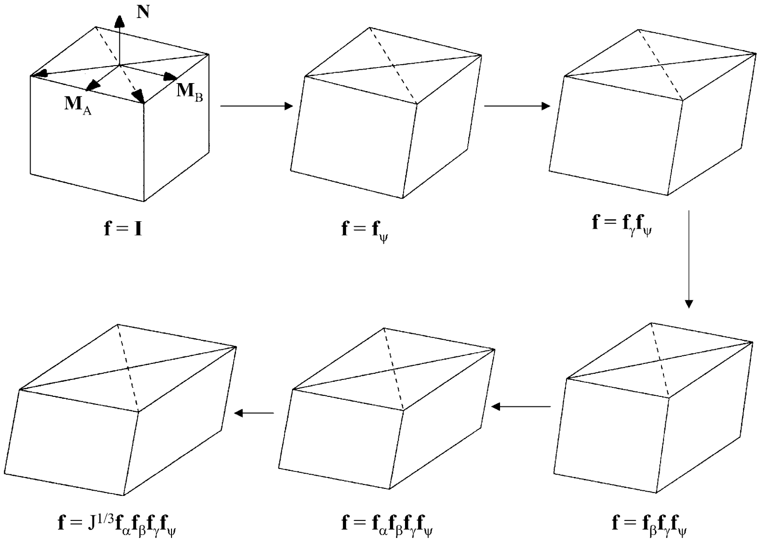

Fig. 1. This sequence depicts the separate actions of the distinct parts of $\mathbf{f}$ which bring the reference configuration (top-left) to the strained configuration (bottom-left). The two fiber directions ($\mathbf{M}_1$ is solid and $\mathbf{M}_2$ is dashed) are shown in each panel with the senses only indicated in the reference. Also shown in the reference are $\mathbf{M}_A$, $\mathbf{M}_B$, and $\mathbf{N}$ which are respectively, the bisector of $\mathbf{M}_1$ and $\mathbf{M}_2$, the bisector of $-\mathbf{M}_1$ and $\mathbf{M}_2$, and the normal to the fiber plane. Although not drawn in the other configurations, the bisectors of the fibers and the normal to the fiber plane are colinear with those shown in the reference – this is the view of an observer that rotates by $\mathbf{Q}$, see Sect. 2. $\mathbf{f}_\psi$ shears adjacent fiber planes along one another yet leaves the fiber planes themselves undistorted. $\mathbf{f}_\gamma$ distorts the fiber planes in a manner that differentially changes the lengths of the fibers (here $\mathbf{M}_1$ is elongated and $\mathbf{M}_2$ is shortened) while keeping the angle subtended by the fibers constant. $\mathbf{f}_\beta$ changes the angle subtended by the fibers by differentially stretching $\mathbf{M}_A$ and $\mathbf{M}_B$ (here the angle subtended by $\mathbf{M}_1$ and $\mathbf{M}_2$ is decreased). Note that $\mathbf{f}_\beta$ will uniformly change the lengths of $\mathbf{M}_1$ and $\mathbf{M}_2$ so that the ratio of $\mathbf{M}_1$ length to $\mathbf{M}_2$ length is as induced by $\mathbf{f}_\gamma$. $\mathbf{f}_\alpha$ uniformly changes the area ratio of the fiber plane and acts normal to the fibers so as to be isochoric. Note that uniform area change does not perturb the ratio of fiber lengths or the angle subtended by them. The prior panels in the sequence all have the same volume, and any dilatation is accomplished by scalar multiplication with $J^{1/3}$ as shown

angle subtended by $\mathbf{m}_1$ and $\mathbf{m}_2$ (i.e. the angle is more acute if $\xi_3 > 0$ and less acute if $\xi_3 < 0$); ξ_4 indicates how the fiber lengths have been changed differentially (i.e. $\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2$ is greater than $\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1$ if $\xi_4 > 0$ and vice-versa if $\xi_4 < 0$, see (B.2)$_4$ in Appendix B); ξ_5 represents shear strain in the $(\mathbf{N}, \mathbf{M}_A)$ plane; and ξ_6 represents shear strain in the $(\mathbf{N}, \mathbf{M}_B)$ plane.

It is important that this set of strain attributes forms a complete set. By a complete set we mean that the strain attributes uniquely define $\mathbf{C}$ provided $\mathbf{M}_1$ and $\mathbf{M}_2$ are given (which uniquely specify $\mathbf{M}_A$, $\mathbf{M}_B$, and Θ). Appendix B develops the one-to-one relationship[6] between ξ_{1-6} and $\mathbf{C}$. Furthermore, with $\mathbf{M}_1$ and $\mathbf{M}_2$ defined by (2.9), then (B.1) yields the following inner products of $\mathbf{M}_1 \otimes \mathbf{M}_1$, $\mathbf{M}_2 \otimes \mathbf{M}_2$, and $\mathbf{M}_1 \otimes \mathbf{M}_2$ with $\mathbf{C}$,

$$\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1 = J^{2/3}\alpha\left(\beta^2 c^2 + \beta^{-2} s^2\right)\left(\sqrt{1+\gamma^2 s^2 c^2} - \gamma s c\right)^2 , \tag{3.5$_1$}$$

$$\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2 = J^{2/3}\alpha\left(\beta^2 c^2 + \beta^{-2} s^2\right)\left(\sqrt{1+\gamma^2 s^2 c^2} + \gamma s c\right)^2 , \tag{3.5$_2$}$$

$$\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_2 = J^{2/3}\alpha\left(\beta^2 c^2 - \beta^{-2} s^2\right) . \tag{3.5$_3$}$$

Note that the squares of the fiber stretches (i.e. $\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1$ and $\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2$) each have four separate factors. The effects of pure dilatation (J) and distortional area change of the fiber plane (α) do not depend on Θ because these two deformations stretch all material segments in the fiber plane by the same amount. In contrast, the effects of the pure shear and simple shear in the fiber plane (i.e. the effects of β and γ, respectively) depend on Θ. The β factor affects the stretch of $\mathbf{M}_1$ and $\mathbf{M}_2$ equally whereas the γ factor does so differentially. To see this, use (3.5)

[6] Given $\mathbf{C}$, the ξ_i are uniquely defined, and given the ξ_i, $\mathbf{C}$ is uniquely defined.

to obtain

$$\sqrt{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1}\sqrt{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2} = J^{2/3}\alpha\left(\beta^2 c^2 + \beta^{-2} s^2\right) , \tag{3.6$_1$}$$

$$\frac{\sqrt{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2}}{\sqrt{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1}} = \frac{\sqrt{1+\gamma^2 s^2 c^2} + \gamma s c}{\sqrt{1+\gamma^2 s^2 c^2} - \gamma s c} . \tag{3.6$_2$}$$

Note that the product of the fiber stretches does not depend on γ whereas their ratio only depends on the shear associated with γ. The current angle subtended by the fibers is 2θ and

$$\cos 2\theta = \frac{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_2}{\sqrt{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1}\sqrt{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2}} = \frac{\beta^2 \cos^2\Theta - \beta^{-2}\sin^2\Theta}{\beta^2 \cos^2\Theta + \beta^{-2}\sin^2\Theta} . \tag{3.7}$$

Given that Θ is a reference quantity, the current angle only depends on β in the sense that the motions associated with J, α, γ, ψ_{AN}, and ψ_{BN} will not perturb θ whatsoever. Moreover, $\theta = \Theta$ iff $\beta = 1$ because we do not consider materials wherein $\mathbf{M}_1$ and $\mathbf{M}_2$ are colinear (i.e. we do not consider materials with $\cos^2\Theta = 0$ or $\sin^2\Theta = 0$). If $\beta > 1$ then $\cos 2\theta > \cos 2\Theta$ (i.e. $\theta < \Theta$) and vice-versa. Similarly, $\theta > \Theta$ iff $\beta < 1$.

Small strain limit with large rotations: Firstly use (2.1–2.3) to determine the reference and current bisectors, then to first order, the strain attributes are

$$\xi_1 = \mathbf{m}_A \cdot \mathbf{F}\mathbf{M}_A + \mathbf{m}_B \cdot \mathbf{F}\mathbf{M}_B + \mathbf{n} \cdot \mathbf{F}\mathbf{N} - 3 , \tag{3.8$_1$}$$

$$\xi_2 = \mathbf{m}_A \cdot \mathbf{F}\mathbf{M}_A + \mathbf{m}_B \cdot \mathbf{F}\mathbf{M}_B - 2\mathbf{n} \cdot \mathbf{F}\mathbf{N} , \tag{3.8$_2$}$$

$$\xi_3 = \mathbf{m}_A \cdot \mathbf{F}\mathbf{M}_A - \mathbf{m}_B \cdot \mathbf{F}\mathbf{M}_B , \tag{3.8$_3$}$$

$$\xi_4 = \mathbf{m}_B \cdot \mathbf{F}\mathbf{M}_A + \mathbf{m}_A \cdot \mathbf{F}\mathbf{M}_B , \tag{3.8$_4$}$$

$$\xi_5 = \mathbf{m}_A \cdot \mathbf{F}\mathbf{N} , \tag{3.8$_5$}$$

$$\xi_6 = \mathbf{m}_B \cdot \mathbf{F}\mathbf{N} . \tag{3.8$_6$}$$

These equations can be verified via substitution of $\mathbf{F}$ with (3.2) followed by use of (3.4) and neglecting higher order terms in ξ_{1-6}. The fiber strains are, to first order,

$$\sqrt{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1} - 1 = \frac{\xi_1}{3} + \frac{\xi_2}{6} + \frac{\xi_3}{2}\cos 2\Theta - \frac{\xi_4}{2}\sin 2\Theta , \tag{3.9$_1$}$$

$$\sqrt{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2} - 1 = \frac{\xi_1}{3} + \frac{\xi_2}{6} + \frac{\xi_3}{2}\cos 2\Theta + \frac{\xi_4}{2}\sin 2\Theta . \tag{3.9$_2$}$$

These equations were obtained by substituting (3.4) into $(3.5)_{1,2}$, neglecting high order terms, and approximating $\sqrt{(1+x)}$ with $1 + x/2$.

4 Strain-power

To derive an expression for $\mathbf{t}$ for hyperelasticity we use the stress power and an expression for the velocity gradient to derive an expression for the strain-power (i.e. the time derivative of strain energy). This method is analogous to those in Criscione et al. (2001) and Criscione et al. (2002). To begin, conservation of strain energy yields

$$\dot{W} = J\mathbf{t} : \dot{\mathbf{F}}\mathbf{F}^{-1} . \tag{4.1}$$

To develop an expression for the velocity gradient $\dot{\mathbf{F}}\mathbf{F}^{-1}$ in terms of the $\dot{\xi}_i$, differentiation of $\mathbf{F}$ in (3.1) with respect to time yields

$$\dot{\mathbf{F}} = \begin{pmatrix} \dot{\mathbf{Q}} J^{1/3}\mathbf{f}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\mathbf{f}_\psi + \mathbf{Q}\frac{1}{3}J^{-2/3}\dot{J}\mathbf{f}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\mathbf{f}_\psi + \mathbf{Q}J^{1/3}\dot{\mathbf{f}}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\mathbf{f}_\psi \\ +\mathbf{Q}J^{1/3}\mathbf{f}_\alpha\dot{\mathbf{f}}_\beta\mathbf{f}_\gamma\mathbf{f}_\psi + \mathbf{Q}J^{1/3}\mathbf{f}_\alpha\mathbf{f}_\beta\dot{\mathbf{f}}_\gamma\mathbf{f}_\psi + \mathbf{Q}J^{1/3}\mathbf{f}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\dot{\mathbf{f}}_\psi \end{pmatrix} . \tag{4.2}$$

Also, (3.1) gives

$$\mathbf{F}^{-1} = J^{-1/3}\mathbf{f}_\psi^{-1}\mathbf{f}_\gamma^{-1}\mathbf{f}_\beta^{-1}\mathbf{f}_\alpha^{-1}\mathbf{Q}^T\,, \tag{4.3}$$

and with (2.15) we obtain

$$\mathbf{f}_\alpha^{-1} = \alpha\mathbf{N}\otimes\mathbf{N} + \alpha^{-1/2}(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B)\,, \qquad (4.4)_1$$
$$\mathbf{f}_\beta^{-1} = \mathbf{N}\otimes\mathbf{N} + \beta^{-1}\mathbf{M}_A\otimes\mathbf{M}_A + \beta\mathbf{M}_B\otimes\mathbf{M}_B\,, \qquad (4.4)_2$$
$$\mathbf{f}_\gamma^{-1} = \mathbf{N}\otimes\mathbf{N} + (1+\gamma^2c^2s^2)^{1/2}(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B) - \gamma(s^2\mathbf{M}_B\otimes\mathbf{M}_A + c^2\mathbf{M}_A\otimes\mathbf{M}_B)\,, \qquad (4.4)_3$$
$$\mathbf{f}_\psi^{-1} = \mathbf{I} - \psi_{AN}\mathbf{M}_A\otimes\mathbf{N} - \psi_{BN}\mathbf{M}_B\otimes\mathbf{N}\,. \qquad (4.4)_4$$

Post multiplication of (4.2) by (4.3) yields

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{Q}}\mathbf{Q}^T + \tfrac{1}{3}\mathbf{I}\,J^{-1}\dot{J} + \mathbf{Q}\begin{pmatrix}\dot{\mathbf{f}}_\alpha\mathbf{f}_\alpha^{-1} + \mathbf{f}_\alpha\dot{\mathbf{f}}_\beta\mathbf{f}_\beta^{-1}\mathbf{f}_\alpha^{-1} + \mathbf{f}_\alpha\mathbf{f}_\beta\dot{\mathbf{f}}_\gamma\mathbf{f}_\gamma^{-1}\mathbf{f}_\beta^{-1}\mathbf{f}_\alpha^{-1} \\ +\mathbf{f}_\alpha\mathbf{f}_\beta\mathbf{f}_\gamma\dot{\mathbf{f}}_\psi\mathbf{f}_\psi^{-1}\mathbf{f}_\gamma^{-1}\mathbf{f}_\beta^{-1}\mathbf{f}_\alpha^{-1}\end{pmatrix}\mathbf{Q}^T\,. \tag{4.5}$$

Since $\mathbf{N}, \mathbf{M}_A$, and $\mathbf{M}_B$ are reference directions, they are fixed and cannot change with time. Likewise, Θ does not vary with time. Upon differentiating (2.15) with respect to time and with use of (3.4), we obtain

$$\dot{\mathbf{f}}_\alpha = \left(-\alpha^{-1}\mathbf{N}\otimes\mathbf{N} + \tfrac{1}{2}\alpha^{1/2}\left(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B\right)\right)\tfrac{1}{3}\dot{\xi}_2\,, \qquad (4.6)_1$$
$$\dot{\mathbf{f}}_\beta = \left(\beta\mathbf{M}_A\otimes\mathbf{M}_A - \beta^{-1}\mathbf{M}_B\otimes\mathbf{M}_B\right)\tfrac{1}{2}\dot{\xi}_3\,, \qquad (4.6)_2$$
$$\dot{\mathbf{f}}_\gamma = \left(\gamma c^2s^2\left(1+\gamma^2c^2s^2\right)^{-1/2}\left(\mathbf{M}_A\otimes\mathbf{M}_A + \mathbf{M}_B\otimes\mathbf{M}_B\right) + s^2\mathbf{M}_B\otimes\mathbf{M}_A + c^2\mathbf{M}_A\otimes\mathbf{M}_B\right)\dot{\xi}_4\,, \qquad (4.6)_3$$
$$\dot{\mathbf{f}}_\psi = \mathbf{M}_A\otimes\mathbf{N}\,\dot{\xi}_5 + \mathbf{M}_B\otimes\mathbf{N}\,\dot{\xi}_6\,. \qquad (4.6)_4$$

Using much algebra, substitute (4.6) and (4.4) into (4.5). Then use of (2.4) yields

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{Q}}\mathbf{Q}^T + \sum_{i=1}^{6}\dot{\xi}_i\mathbf{L}_i\,, \tag{4.7}$$

with

$$\mathbf{L}_1 = \tfrac{1}{3}\mathbf{I}\,, \qquad (4.8)_1$$
$$\mathbf{L}_2 = -\tfrac{1}{3}\mathbf{n}\otimes\mathbf{n} + \tfrac{1}{6}\left(\mathbf{m}_A\otimes\mathbf{m}_A + \mathbf{m}_B\otimes\mathbf{m}_B\right)\,, \qquad (4.8)_2$$
$$\mathbf{L}_3 = \tfrac{1}{2}\left(\mathbf{m}_A\otimes\mathbf{m}_A - \mathbf{m}_B\otimes\mathbf{m}_B\right)\,, \qquad (4.8)_3$$
$$\mathbf{L}_4 = \frac{1}{\sqrt{1+\gamma^2s^2c^2}}\left(\beta^2c^2\mathbf{m}_A\otimes\mathbf{m}_B + \beta^{-2}s^2\mathbf{m}_B\otimes\mathbf{m}_A\right)\,, \qquad (4.8)_4$$
$$\mathbf{L}_5 = \alpha^{3/2}\left(\beta\sqrt{1+\gamma^2s^2c^2}\;\mathbf{m}_A\otimes\mathbf{n} + \beta^{-1}\gamma s^2\,\mathbf{m}_B\otimes\mathbf{n}\right)\,, \qquad (4.8)_5$$
$$\mathbf{L}_6 = \alpha^{3/2}\left(\beta\gamma c^2\,\mathbf{m}_A\otimes\mathbf{n} + \beta^{-1}\sqrt{1+\gamma^2s^2c^2}\;\mathbf{m}_B\otimes\mathbf{n}\right)\,. \qquad (4.8)_6$$

Note that the $\mathbf{L}_i$ are mostly orthogonal to one another with $\mathbf{L}_i : \mathbf{L}_j$ typically vanishing when $i \neq j$. Indeed, the only inner product that is potentially non-vanishing is $\mathbf{L}_5 : \mathbf{L}_6$ which vanishes nonetheless when γ is negligible.

Upon substitution of (4.7) into (4.1), we obtain the following strain-power law,

$$\dot{W} = J\left((\mathbf{L}_1:\mathbf{t})\,\dot{\xi}_1 + (\mathbf{L}_2:\mathbf{t})\,\dot{\xi}_2 + (\mathbf{L}_3:\mathbf{t})\,\dot{\xi}_3 + (\mathbf{L}_4:\mathbf{t})\,\dot{\xi}_4 + (\mathbf{L}_5:\mathbf{t})\,\dot{\xi}_5 + (\mathbf{L}_6:\mathbf{t})\,\dot{\xi}_6\right)\,, \tag{4.9}$$

wherein the $\mathbf{Q}$ term vanishes because it is skew whereas $\mathbf{t}$ is symmetric. Since $\dot{W}$ is an exact differential, it should be evident that each ξ_i response function is simply

$$\frac{\partial W}{\partial \xi_i} = J\left(\mathbf{L}_i \colon \mathbf{t}\right) . \tag{4.10}$$

Hence, if $\mathbf{t}$ and $\mathbf{F}$ are known then the response functions can be evaluated forthwith. Moreover, note from (4.8) that the symmetric parts[7] of the $\mathbf{L}_i$ are always linearly independent in the sense that a particular sym($\mathbf{L}_i$) is never a linear combination of the others. Consequently, the six equations represented by (4.10) are six linearly independent equations for $\mathbf{t}$. These 6 equations are nontrivial (i.e. each sym($\mathbf{L}_i$) is never $\mathbf{0}$, the zero tensor). Since $\mathbf{t}$ only has six unknowns, a symmetric $\mathbf{t}$ that satisfies (4.10) is the true stress for hyperelastic materials in static equilibrium.

5 Hyperelastic constitutive behavior

To define the true stress, let $\mathbf{t}$ be given by

$$\mathbf{t} = \frac{1}{J}\sum_{i=1}^{6} \frac{\partial W}{\partial \xi_i} \varXi_i \,, \tag{5.1}$$

with the $\boldsymbol{\Xi}_i$ kinematic tensors given as follows:

$$\varXi_1 = \mathbf{I} \,, \tag{5.2$_1$}$$

$$\varXi_2 = -2\mathbf{n}\otimes\mathbf{n} + \mathbf{m}_A\otimes\mathbf{m}_A + \mathbf{m}_B\otimes\mathbf{m}_B \,, \tag{5.2$_2$}$$

$$\varXi_3 = \left(\mathbf{m}_A\otimes\mathbf{m}_A - \mathbf{m}_B\otimes\mathbf{m}_B\right) \,, \tag{5.2$_3$}$$

$$\varXi_4 = \frac{\sqrt{1+\gamma^2 s^2 c^2}}{\beta^2 c^2 + \beta^{-2} s^2}\left(\mathbf{m}_A\otimes\mathbf{m}_B + \mathbf{m}_B\otimes\mathbf{m}_A\right) \,, \tag{5.2$_4$}$$

$$\varXi_5 = \alpha^{-3/2}\left(\beta^{-1}\sqrt{1+\gamma^2 s^2 c^2}\left(\mathbf{m}_A\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}_A\right) - \beta\gamma c^2\left(\mathbf{m}_B\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}_B\right)\right) \,, \tag{5.2$_5$}$$

$$\varXi_6 = \alpha^{-3/2}\left(-\beta^{-1}\gamma s^2\left(\mathbf{m}_A\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}_A\right) + \beta\sqrt{1+\gamma^2 s^2 c^2}\left(\mathbf{m}_B\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m}_B\right)\right) \,. \tag{5.2$_6$}$$

Recall that $s = \sin\varTheta$ and $c = \cos\varTheta$ where $2\varTheta$ is the angle subtended by $\mathbf{M}_1$ and $\mathbf{M}_2$ in the reference configuration. Upon substitution, it should be evident that $\mathbf{t}$ above is the true stress because it is symmetric and it satisfies (4.10) for each $i = 1 - 6$.

Furthermore, the response terms are mostly orthogonal in the sense that $\boldsymbol{\Xi}_i : \boldsymbol{\Xi}_j = 0$ when $i \neq j$ except when $i = 5$ and $j = 6$ or vice-versa. Indeed, fourteen of the fifteen inner products vanish, and

$$\boldsymbol{\Xi}_5 : \boldsymbol{\Xi}_6 = -2\alpha^{-3}\left(\beta^2 c^2 + \beta^{-2} s^2\right)\gamma\sqrt{1+\gamma^2 s^2 c^2} \,, \tag{5.3}$$

which is only non-negligible for large deformation with γ finite.

In addition to being mostly orthogonal, the stress response due to each strain attribute is physically distinct. The ξ_1 response is the pressure, and the ξ_2 response specifies the average deviatoric stress in the fiber plane. Whereas the ξ_1 and ξ_2 responses have equivalent normal stresses in $\mathbf{m}_A$ and $\mathbf{m}_B$, the ξ_3 response provides the difference of the normal stresses in $\mathbf{m}_A$ and $\mathbf{m}_B$. The ξ_4 response specifies a simple shear stress in the fiber plane. The ξ_5 and ξ_6 responses are complimentary with the ξ_5 response primarily specifying simple shear stress in the $(\mathbf{n}, \mathbf{m}_A)$ plane and secondarily specifying simple shear stress in the $(\mathbf{n}, \mathbf{m}_B)$ plane. In vice-versa fashion, ξ_6 has a primary response in the $(\mathbf{n}, \mathbf{m}_B)$ plane and a secondary response in the $(\mathbf{n}, \mathbf{m}_A)$ plane.

To further refine W, let us assume that the dependence of $\mathbf{t}$ on $\mathbf{F}$ is smooth. Whereby, (4.10) requires that the derivatives of W be smooth because the $\mathbf{L}_i$ smoothly vary with $\mathbf{F}$. With its derivatives smooth, W is analytical and thus expressible as a power series. Also, to make the transition from linear to nonlinear material behavior

[7] The symmetric part, i.e. sym($\mathbf{L}_i$), is specified because only the symmetric part can contribute to contraction with $\mathbf{t}$.

forthright, consider a form of W with the constant, linear, and quadratic terms represented directly as follows:

$$W = W_{\mathrm{ref}} + \sum_{i=1}^{6} \left(q_i \xi_i + \tfrac{1}{2} \sum_{j=1}^{6} g_{ij} \xi_i \xi_j \right) + G\left(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\right) \,, \tag{5.4}$$

where W_{ref}, q_i, and g_{ij} are constants and the function[8] G depends on cubic orders or higher of ξ_{1-6}. Since $\xi_i \xi_j = \xi_j \xi_i$ there are only 21 quadratic coefficients in (5.4), and to enforce this constraint we require $g_{ij} = g_{ji}$. To justify the form of W in (5.4), first consider deformation without strain such that $\mathbf{F} = \mathbf{Q}$ (i.e. $\mathbf{f} = \mathbf{I}$). Since ξ_{1-6} all vanish, $W = W_{\mathrm{ref}}$ when $\mathbf{F} = \mathbf{Q}$. Furthermore, the $\partial G / \partial \xi_i$ must vanish when $\mathbf{F} = \mathbf{Q}$ because G must be at least of cubic order. Consequently, $\mathbf{t}$ becomes

$$\mathbf{t}\,|_{(\mathbf{F}=\mathbf{Q})} = \begin{pmatrix} q_1 \mathbf{I} + q_2 \left(-2\mathbf{n} \otimes \mathbf{n} + \mathbf{m}_A \otimes \mathbf{m}_A + \mathbf{m}_B \otimes \mathbf{m}_B\right) \\ +q_3 \left(\mathbf{m}_A \otimes \mathbf{m}_A - \mathbf{m}_B \otimes \mathbf{m}_B\right) + q_4 \left(\mathbf{m}_A \otimes \mathbf{m}_B + \mathbf{m}_B \otimes \mathbf{m}_A\right) \\ +q_5 \left(\mathbf{m}_A \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}_A\right) + q_6 \left(\mathbf{m}_B \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}_B\right) \end{pmatrix} \,, \tag{5.5}$$

wherein $\mathbf{m}_A = \mathbf{Q}\mathbf{M}_A$, $\mathbf{m}_B = \mathbf{Q}\mathbf{M}_B$, and $\mathbf{n} = \mathbf{Q}\mathbf{N}$. It should be evident from (5.5) that the q_i specify six orthogonal components of the initial stress.

Upon noting that $|\xi_i| \ll 1$ for infinitesimal strain (i.e. $\mathbf{f} \cong \mathbf{I}$), the 21 quadratic coefficients are elasticity moduli with the function G being negligible because it has all higher order terms. To see this, let the reference configuration be stress free and strain energy free so that W_{ref} and the q_i vanish in (5.4). Unlike classical infinitesimal strain analyses, however, let the rotation be finite. Upon neglecting all terms higher than first order and utilizing the condition $g_{ij} = g_{ji}$,

$$\mathbf{t} = \sum_{i=1}^{6} \begin{pmatrix} g_{1i} \xi_i \mathbf{I} + g_{2i} \xi_i \left(-2\mathbf{n} \otimes \mathbf{n} + \mathbf{m}_A \otimes \mathbf{m}_A + \mathbf{m}_B \otimes \mathbf{m}_B\right) \\ +g_{3i} \xi_i \left(\mathbf{m}_A \otimes \mathbf{m}_A - \mathbf{m}_B \otimes \mathbf{m}_B\right) + g_{4i} \xi_i \left(\mathbf{m}_A \otimes \mathbf{m}_B + \mathbf{m}_B \otimes \mathbf{m}_A\right) \\ +g_{5i} \xi_i \left(\mathbf{m}_A \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}_A\right) + g_{6i} \xi_i \left(\mathbf{m}_B \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}_B\right) \end{pmatrix} . \tag{5.6}$$

Recall that $\mathbf{n}$, $\mathbf{m}_A$, and $\mathbf{m}_B$ are easily computed with (2.1) and (2.3), and the ξ_i for small strain are given by (3.8).

6 Incompressibility

Since many materials that undergo finite deformation exhibit behavior that is nearly incompressible, our strain attributes were developed with incompressibility in mind. Indeed, ξ_{2-6} only depend on distortion in the sense that they do not depend on dilatation whatsoever.

With $J \equiv 1$, or equivalently $\xi_1 \equiv 0$, the incompressibility constraint requires that $\dot{\xi}_1$ vanish. Hence, the strain-power law (4.9) becomes,

$$\dot{W} = J\left(\left(\mathbf{L}_2 : \mathbf{t}\right) \dot{\xi}_2 + \left(\mathbf{L}_3 : \mathbf{t}\right) \dot{\xi}_3 + \left(\mathbf{L}_4 : \mathbf{t}\right) \dot{\xi}_4 + \left(\mathbf{L}_5 : \mathbf{t}\right) \dot{\xi}_5 + \left(\mathbf{L}_6 : \mathbf{t}\right) \dot{\xi}_6 \right) . \tag{6.1}$$

Since $\mathbf{L}_{2-6}$ are all deviatoric, the pressure can assume any value without perturbing W. Boundary conditions and the equilibrium equations must be used to determine the pressure. Nonetheless, the five deviatoric unknowns of $\mathbf{t}$ are uniquely determined by (4.10) with $i = 2-6$.

As in Sect. 5, let us propose a $\mathbf{t}$ as follows:

$$\mathbf{t} = -p\,\mathbf{I} + \sum_{i=2}^{6} \frac{\partial W}{\partial \xi_i} \Xi_i \,, \tag{6.2}$$

[8] Throughout this manuscript, $G(a, b, c)$, for example, means to consider G as a function of a, b, and c. Hence, $G(d, e, f)$ does not imply that G depends on d, e, and f in the precise fashion that G depends on a, b, and c. When a specific G is required, the equation number is given.

where the $\boldsymbol{\Xi}_{2-6}$ are given by (5.2)$_{2-6}$ and p is the indeterminate pressure. Since (6.2) satisfies (4.10) with $i = 2 - 6$, it is an admissible constitutive law for incompressible, hyperelastic materials with two distinct fiber directions.

Following the reasoning in Sect. 5, $W(\xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ is analytical if $\mathbf{t}$ depends on $\mathbf{F}$ in a smooth manner. For incompressible behavior it can be expressed as,

$$W = W_{\text{ref}} + \sum_{i=2}^{6} \left(q_i \xi_i + \tfrac{1}{2} \sum_{j=2}^{6} g_{ij} \xi_i \xi_j \right) + G\left(\xi_2, \xi_3, \xi_4, \xi_5, \xi_6\right) \,, \tag{6.3}$$

where W_{ref}, q_i, and g_{ij} are constants and G depends on cubic orders or higher of ξ_{2-6}. As before, we require $g_{ij} = g_{ji}$, however, instead of 21 moduli there are only 15 because dependence on $\xi_1 \equiv 0$ vanishes.

7 Refinement of *W* for material symmetries

For many materials with two families of fibers, the reference configuration has one, three, or five mirror symmetry planes, and subsequently, the form of W is reducible. Throughout this section we define a reflection tensor $\mathbf{P}$ which is a member of the symmetry group of interest and we require that $W(\mathbf{F}) = W(\mathbf{FP})$ so that the behavior is invariant of $\mathbf{P}$. After determining how the ξ_i change when $\mathbf{F} \rightarrow \mathbf{FP}$, a refined form of W is found. Since it is trivial, the identity tensor is neglected in the symmetry groups here.

The first 4 symmetry groups considered are associated with orthotropy, as exemplified by two families of mechanically equivalent fibers reinforcing an isotropic matrix. For such a material, the angle subtended by the fiber families in the reference configuration is unconstrained. However, for the mirror symmetries of groups 5-8 below, it is necessary that the fiber families be orthogonal in the reference configuration (i.e. $\Theta = \pi/4$) because there is a mirror plane that is normal to at least one fiber direction. For these later cases, it is assumed that Θ has been set to $\pi/4$ in the derivations and results stated.

Group 1 – Fiber plane is a mirror plane[9]*:* If the fiber plane is a mirror-symmetry plane in the reference configuration then the reflection tensor $\mathbf{P} = (\mathbf{I} - 2\mathbf{N} \otimes \mathbf{N})$ is an allowable symmetry transformation. For this case $\mathbf{PN} = -\mathbf{N}$ whereas $\mathbf{PM}_A = \mathbf{M}_A$, $\mathbf{PM}_B = \mathbf{M}_B$. With use of (2.9) it follows that $\mathbf{PM}_1 = \mathbf{M}_1$ and $\mathbf{PM}_2 = \mathbf{M}_2$. When $\mathbf{F} \rightarrow \mathbf{FP}$ then $\mathbf{C} \rightarrow \mathbf{P}^T\mathbf{CP}$. Upon substitution of $\mathbf{C} \rightarrow \mathbf{P}^T\mathbf{CP}$ into (B.2), note that J, α, β, and γ retain their prior values whereas ψ_{AN} and ψ_{BN} change sign. In other words, ξ_1, ξ_2, ξ_3, and ξ_4 are invariant under the symmetry transformation $\mathbf{P}$ whereas ξ_5 and ξ_6 both change sign.

Hence, the form $W(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^2, \xi_6^2, \xi_5\xi_6)$ does not violate symmetry because $W(\mathbf{F}) = W(\mathbf{FP})$. In order to prove that this form is sufficient, consider a power series expansion of W in terms of ξ_5 and ξ_6 with coefficients that are functions of ξ_{1-4}. Upon discarding all terms that violate symmetry (i.e. those that change sign when $\mathbf{F} \rightarrow \mathbf{FP}$), the aforementioned form of W is obtained. In particular, (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + \sum\limits_{i=1}^{4} \left(q_i \xi_i + \frac{1}{2} \sum\limits_{j=1}^{4} g_{ij} \xi_i \xi_j \right) + \frac{1}{2} g_{55} \xi_5^2 + \frac{1}{2} g_{66} \xi_6^2 + g_{56} \xi_5 \xi_6 \\ + G\left(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5^2, \xi_6^2, \xi_5 \xi_6\right) \end{pmatrix} . \tag{7.1}$$

Recall that $g_{ij} = g_{ji}$, and instead of 21 infinitesimal elasticity moduli, only 13 are needed. If the material is incompressible such that $\xi_1 \equiv 0$ then (7.1) can be reduced further (simply neglect the ξ_1 terms, see Sect. 6), and only 9 moduli are needed. Since the reference configuration has a mirror symmetry plane with a normal in $\mathbf{N}$, then the stress in the reference must possess a mirror plane with normal $\mathbf{N}$. Whereby, $\mathbf{N}$ must be a principal direction of $\mathbf{t}$ when $\mathbf{F} = \mathbf{I}$, and rightfully, q_5 and q_6 in (5.5) vanish.

Group 2 – Plane normal to $\mathbf{M}_A$ *is a mirror plane*[10]*:* For this case, the reflection tensor $\mathbf{P} = (\mathbf{I} - 2\mathbf{M}_A \otimes \mathbf{M}_A)$ is an allowable symmetry transformation and $\mathbf{PM}_A = -\mathbf{M}_A$ whereas $\mathbf{PM}_B = \mathbf{M}_B$ and $\mathbf{PN} = \mathbf{N}$. With

[9] If the fibers are neither orthogonal nor mechanical equivalent, this symmetry will be the only possible one.

[10] Since the fibers must be mechanically equivalent for a bisector to be normal to a mirror symmetry plane, it seems unlikely that only this symmetry would be present. As an example of a material with only symmetry group 2, consider a material with mechanically equivalent fibers that has an initial shear stress in the $(\mathbf{N}, \mathbf{M}_B)$ plane in its reference configuration, i.e. $\mathbf{t}(\mathbf{F} = \mathbf{I}) = t_0(\mathbf{N} \otimes \mathbf{M}_B + \mathbf{M}_B \otimes \mathbf{N})$. In similar fashion, symmetry groups 3, 5, and 6 are unlikely to be solely present unless residual shear or a processing method induces anisotropy in an otherwise isotropic matrix surrounding the fibers.

use of (2.9), note that $\mathbf{PM}_1 = -\mathbf{M}_2$ and $\mathbf{PM}_2 = -\mathbf{M}_1$. Now with substitution of $\mathbf{C} \to \mathbf{P}^T\mathbf{CP}$ into (B.2), it follows that ξ_1, ξ_2, ξ_3, and ξ_6 are invariant under $\mathbf{P}$ whereas ξ_4 and ξ_5 change sign. Hence, the form $W(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5^2, \xi_4\xi_5, \xi_6)$ is admissible. Sufficiency can be proven with a power series method analogous to that in the above paragraph. W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + \sum\limits_{i=1,-3,6} \left(q_i\xi_i + \frac{1}{2} \sum\limits_{j=1,-3,6} g_{ij}\xi_i\xi_j \right) + \frac{1}{2}g_{44}\xi_4^2 + \frac{1}{2}g_{55}\xi_5^2 + g_{45}\xi_4\xi_5 \\ +G\left(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5^2, \xi_4\xi_5, \xi_6\right) \end{pmatrix}. \tag{7.2}$$

As above, 13 infinitesimal elasticity moduli are needed in general, 9 for incompressible materials. Similarly, $\mathbf{M}_A$ must be a principal direction of $\mathbf{t}$ when $\mathbf{F} = \mathbf{I}$. Rightfully, q_4 and q_5 in (5.5) vanish.

Group 3 – Plane normal to $\mathbf{M}_B$ *is a mirror plane:* The reflection tensor $\mathbf{P} = (\mathbf{I} - 2\mathbf{M}_B \otimes \mathbf{M}_B)$ is an allowable symmetry transformation. In similar fashion, it follows that ξ_1, ξ_2, ξ_3, and ξ_5 are invariant under $\mathbf{P}$ whereas ξ_4 and ξ_6 change sign. The form $W(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5, \xi_6^2, \xi_4\xi_6)$ is sufficient, and (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + \sum\limits_{i=1,-3,5} \left(q_i\xi_i + \frac{1}{2} \sum\limits_{j=1,-3,5} g_{ij}\xi_i\xi_j \right) + \frac{1}{2}g_{44}\xi_4^2 + \frac{1}{2}g_{66}\xi_6^2 + g_{46}\xi_4\xi_6 \\ +G\left(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5, \xi_6^2, \xi_4\xi_6\right) \end{pmatrix}. \tag{7.3}$$

Again, 13 infinitesimal elasticity moduli are needed in general, 9 for incompressible materials. When $\mathbf{F} = \mathbf{I}$, $\mathbf{M}_B$ is a principal direction of $\mathbf{t}$, and rightfully, q_4 and q_6 in (5.5) vanish.

Group 4 – Orthotropy with mechanically equivalent fiber families: An isotropic matrix that is reinforced with two separate fiber families that are mechanically equivalent has a symmetry group that contains all three of the above mirror symmetries. It should be evident that the form $W(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5^2, \xi_6^2, \xi_4\xi_5\xi_6)$ is sufficient, and W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + q_1\xi_1 + q_2\xi_2 + q_3\xi_3 + \frac{1}{2}g_{11}\xi_1^2 + \frac{1}{2}g_{22}\xi_2^2 + \frac{1}{2}g_{33}\xi_3^2 \\ +\frac{1}{2}g_{44}\xi_4^2 + \frac{1}{2}g_{55}\xi_5^2 + \frac{1}{2}g_{66}\xi_6^2 + g_{12}\xi_1\xi_2 + g_{13}\xi_1\xi_3 + g_{23}\xi_2\xi_3 \\ +G\left(\xi_1, \xi_2, \xi_3, \xi_4^2, \xi_5^2, \xi_6^2, \xi_4\xi_5\xi_6\right) \end{pmatrix}. \tag{7.4}$$

For this case, 9 infinitesimal elasticity moduli are needed in general, and for incompressible materials there are only 6. Moreover, the principal directions of the initial stress must coincide with normals to the three orthogonal mirror planes such that q_4, q_5, and q_6 in (5.5) vanish.

Group 5 – Plane normal to $\mathbf{M}_1$ *is a mirror plane:* For this case, $\mathbf{P} = (\mathbf{I} - 2\mathbf{M}_1 \otimes \mathbf{M}_1)$ is an allowable symmetry transformation, and it follows that $\mathbf{PM}_1 = -\mathbf{M}_1$ and $\mathbf{PM}_2 = \mathbf{M}_2$, because $\mathbf{M}_1 \cdot \mathbf{M}_2 = 0$. Moreover, with (2.2) we obtain $\mathbf{PM}_A = \mathbf{M}_B$, $\mathbf{PM}_B = \mathbf{M}_A$, and $\mathbf{PN} = \mathbf{N}$. Although the end results are straightforward, the derivations below are more complicated. To clarify these derivations, let ξ_1^*, for example, be ξ_1 for the deformation $\mathbf{FP}$, whereas let ξ_1 without the '*' be ξ_1 for the deformation $\mathbf{F}$. With this notation, the effect of $\mathbf{P}$ on the ξ_i or other kinematic quantities can be discussed in a forthright manner. Upon setting $\tan\theta = 1$, then with use of $(\text{B.2})_3$ note that β^* multiplied by β is unity. Whereby, $(3.4)_3$ gives $\xi_3^* = -\xi_3$. Also, it should be evident that $\xi_5^* = \xi_6$ and $\xi_6^* = \xi_5$ and that ξ_1, ξ_2, and ξ_4 are invariant under $\mathbf{P}$ (i.e. $\xi_1^* = \xi_1$, $\xi_2^* = \xi_2$ and $\xi_4^* = \xi_4$).

Hence, the form $W(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5 + \xi_6, \xi_3(\xi_5 - \xi_6), \xi_5\xi_6)$ does not violate this symmetry. Similar to above, for sufficiency use a power series for W expanded in terms of ξ_3, ξ_5, and ξ_6 with coefficients that are functions of ξ_1, ξ_2, and ξ_4. Symmetry thus requires many coefficients to vanish, and in addition, the coefficient of ξ_5^{2n} must be equal to that of ξ_6^{2n} whereas the coefficient of $\xi_3^{2m+1}\xi_5^{2n+1}$ must be equal but opposite to that of $\xi_3^{2m+1}\xi_6^{2n+1}$. Sufficiency follows upon grouping terms and using the fact that $\xi_5^{2n} + \xi_6^{2n}$, for example, can be expressed in terms of $\xi_5 + \xi_6$ and $\xi_5\xi_6$ factors. W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + \sum\limits_{i=1,2,4} \left(q_i\xi_i + \frac{1}{2} \sum\limits_{j=1,2,4} g_{ij}\xi_i\xi_j \right) + q_5(\xi_5 + \xi_6) + \frac{1}{2}g_{33}\xi_3^2 + g_{55}\left(\xi_5^2 + \xi_6^2\right) \\ +g_{56}\xi_5\xi_6 + g_{15}\xi_1\left(\xi_5 + \xi_6\right) + g_{25}\xi_2\left(\xi_5 + \xi_6\right) + g_{45}\xi_4\left(\xi_5 + \xi_6\right) \\ +g_{35}\xi_3\left(\xi_5 - \xi_6\right) + G\left(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5 + \xi_6, \xi_3(\xi_5 - \xi_6), \xi_5\xi_6\right) \end{pmatrix}. \tag{7.5}$$

Again, 13 infinitesimal elasticity moduli are needed in general, 9 for incompressible materials. With $q_3 = 0$ and $q_5 = q_6$ in (5.5), it can be verified that the initial stress is such that $\mathbf{M}_1$ (recall that θ must be $\pi/4$) is a principal stress direction when $\mathbf{F} = \mathbf{I}$.

Group 6 – Plane normal to $\mathbf{M}_2$ *is a mirror plane:* For this case, $\mathbf{P} = (\mathbf{I} - 2\mathbf{M}_2 \otimes \mathbf{M}_2)$, $\mathbf{PM}_1 = \mathbf{M}_1$, $\mathbf{PM}_2 = -\mathbf{M}_2$, $\mathbf{PM}_A = -\mathbf{M}_B$, $\mathbf{PM}_B = -\mathbf{M}_A$, and $\mathbf{PN} = \mathbf{N}$. With an approach similar to the above, it should be evident that $\xi_3^* = -\xi_3$, $\xi_5^* = -\xi_6$ and $\xi_6^* = -\xi_5$. As above, ξ_1, ξ_2, and ξ_4 are invariant under $\mathbf{P}$. Hence, $W(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5 - \xi_6, \xi_3(\xi_5 + \xi_6), \xi_5\xi_6)$ does not violate this symmetry. Sufficiency can be proven as outlined in the previous symmetry group. W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + \sum\limits_{i=1,2,4} \left(q_i\xi_i + \frac{1}{2} \sum\limits_{j=1,2,4} g_{ij}\xi_i\xi_j \right) + q_5(\xi_5 - \xi_6) + \frac{1}{2} g_{33}\xi_3^2 + g_{55}\left(\xi_5^2 + \xi_6^2\right) \\ +g_{56}\xi_5\xi_6 + g_{15}\xi_1\left(\xi_5 - \xi_6\right) + g_{25}\xi_2\left(\xi_5 - \xi_6\right) + g_{45}\xi_4\left(\xi_5 - \xi_6\right) \\ +g_{35}\xi_3\left(\xi_5 + \xi_6\right) + G\left(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5 - \xi_6, \xi_3(\xi_5 + \xi_6), \xi_5\xi_6\right) \end{pmatrix}. \tag{7.6}$$

Again, 13 infinitesimal elasticity moduli are needed in general, 9 for incompressible materials. With $q_3 = 0$ and $q_5 = -q_6$ in (5.5), note that $\mathbf{M}_2$ is a principal stress direction of the initial stress.

Group 7 – Orthotropy wherein the fiber families are orthogonal: If all three planes with normals $\mathbf{N}, \mathbf{M}_1$, and $\mathbf{M}_2$ are planes of mirror symmetry then the behavior is orthotropic with respect to the reference configuration. The prototypical example of this type of material is an isotropic matrix reinforced with two distinct families of fibers that are orthogonal. It should be evident that the form $W(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5^2 + \xi_6^2, \xi_3(\xi_5^2 + \xi_6^2), \xi_5\xi_6)$ is invariant of these reflection symmetries. To prove sufficiency, use a power series approach as above. W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + q_1\xi_1 + q_2\xi_2 + q_4\xi_4 + \frac{1}{2} g_{11}\xi_1^2 + \frac{1}{2} g_{22}\xi_2^2 + \frac{1}{2} g_{33}\xi_3^2 + \frac{1}{2} g_{44}\xi_4^2 \\ +\frac{1}{2} g_{55}\left(\xi_5^2 + \xi_6^2\right) + g_{12}\xi_1\xi_2 + g_{14}\xi_1\xi_4 + g_{24}\xi_2\xi_4 + g_{56}\xi_5\xi_6 \\ +G\left(\xi_1, \xi_2, \xi_3^2, \xi_4, \xi_5^2 + \xi_6^2, \xi_3(\xi_5^2 - \xi_6^2), \xi_5\xi_6\right) \end{pmatrix}. \tag{7.7}$$

As is consistent with orthotropic symmetry, only 9 infinitesimal elasticity moduli are needed in general, 6 for incompressible materials. Rightfully, q_3, q_5, and q_6 in (5.5) vanish, whereby $\mathbf{N}$, $\mathbf{M}_1$, and $\mathbf{M}_2$ are principal directions of the initial stress.

Group 8 – Fiber families are orthogonal and mechanically equivalent: If the fibers are orthogonal and mechanically equivalent in the reference configuration then the reference configuration has five mirror symmetry planes with normals in $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_A$, $\mathbf{M}_B$, and $\mathbf{N}$. With an approach similar to those above, it follows that $W(\xi_1, \xi_2, \xi_3^2, \xi_4^2, \xi_5^2 + \xi_6^2, \xi_4\xi_5\xi_6)$ is sufficient, and W in (5.4) becomes,

$$W = \begin{pmatrix} W_{\text{ref}} + q_1\xi_1 + q_2\xi_2 + \frac{1}{2} g_{11}\xi_1^2 + \frac{1}{2} g_{22}\xi_2^2 + g_{12}\xi_1\xi_2 + \frac{1}{2} g_{33}\xi_3^2 \\ +\frac{1}{2} g_{44}\xi_4^2 + \frac{1}{2} g_{55}\left(\xi_5^2 + \xi_6^2\right) + G\left(\xi_1, \xi_2, \xi_3^2, \xi_4^2, \xi_5^2 + \xi_6^2, \xi_4\xi_5\xi_6\right) \end{pmatrix}. \tag{7.8}$$

In general, only 6 infinitesimal elasticity moduli are needed. For incompressibility, only 4 moduli are needed. Since the initial stress must have principal directions $\mathbf{N}$, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_A$, and $\mathbf{M}_B$, then rightfully, q_{3-6} in (5.5) vanish.

8 Experimental utility

With the approach herein there is experimental advantage with regard to determining the functional form of W for materials with two families of fibers. In particular, we show that specific terms in W can be found from biaxial stretching on incompressible materials with orthotropic symmetry. No other hyperelasticity formulation, to our knowledge, allows such definiteness for these materials. With prior approaches, a form for W has had to be assumed *a priori* with experimental data being used to determine (with a non-linear regression method) material parameters of the assumed form.

Biaxial stretching on orthotropic materials is presently done with the orthotropic material directions corresponding to principal stretch directions. For such tests, symmetry in the current configuration constrains $\mathbf{t}$ to be coaxial to $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, and only normal tractions are needed to deform the specimen. Hence, consider a sheet of incompressible material with mechanically equivalent fibers undergoing biaxial stretch with $\mathbf{M}_A$ and $\mathbf{M}_B$ as the in-plane stretching directions. Let the deformation be given by,

$$\mathbf{F} = \lambda_A \mathbf{M}_A \otimes \mathbf{M}_A + \lambda_B \mathbf{M}_B \otimes \mathbf{M}_B + \lambda_A^{-1}\lambda_B^{-1}\mathbf{N} \otimes \mathbf{N} \,. \tag{8.1}$$

Whereby, $\mathbf{m}_A = \mathbf{M}_A$, $\mathbf{m}_B = \mathbf{M}_B$, and $\mathbf{n} = \mathbf{N}$. As for the kinematic parameters of (3.3): $J = 1$, $\alpha = (\lambda_A\lambda_B)^{1/2}$, $\beta = (\lambda_A/\lambda_B)^{1/2}$, $\gamma = 0$, $\psi_{AN} = 0$, and $\psi_{BN} = 0$. For this test, four arguments (ξ_1, ξ_4, ξ_5, and ξ_6) of W vanish, and $\xi_2 = (3/2)(\ln\lambda_A + \ln\lambda_B)$ and $\xi_3 = (\ln\lambda_1 - \ln\lambda_2)$. As for the measured stress,

$$\mathbf{t} = t_A \mathbf{M}_A \otimes \mathbf{M}_A + t_B \mathbf{M}_B \otimes \mathbf{M}_B \,, \tag{8.2}$$

where it is assumed that the sheet surface is traction free.

To obtain expressions for the ξ_2 and ξ_3 response functions in terms of the measured principal stresses, substitute (8.2) into (4.10) with i respectively set to 2 and 3. With W and G given by (7.4), we obtain

$$\frac{\partial G(0,\xi_2,\xi_3,0,0,0,0)}{\partial \xi_2} + g_{22}\xi_2 + g_{23}\xi_3 = \tfrac{1}{3}\left(t_A + t_B\right) \,, \tag{8.3$_1$}$$

$$\frac{\partial G(0,\xi_2,\xi_3,0,0,0,0)}{\partial \xi_3} + g_{23}\xi_2 + g_{33}\xi_3 = \tfrac{1}{2}\left(t_A - t_B\right) \,. \tag{8.3$_2$}$$

One particularly useful test is pure shear in the fiber plane (i.e. $\xi_2 = 0$), whereby $G(0,0,\xi_3,0,0,0,0)$, g_{23}, and g_{33} can be found with $(8.3)_2$. Similarly, equibiaxial stretch (i.e. $\xi_3 = 0$) can determine $G(0,\xi_2,0,0,0,0,0)$, g_{22}, and g_{23}. Furthermore, $G(0,\xi_2,\xi_3,0,0,0,0)$ can be completely characterized by performing a series of biaxial stretching tests with ξ_2 or ξ_3 held constant because $G(0,\xi_2,\xi_3,0,0,0,0)$ is only a two parameter function. Note that tests with ξ_2 held constant are forthright because the area of the fiber plane (or the product $\lambda_A\lambda_B$) is merely held constant. For tests with ξ_3 constant, hold the ratio λ_A/λ_B constant.

For incompressible materials with orthogonal fiber families, consider a biaxial test such that $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{N}$ are the principal stretch directions. For this test, ξ_2 and ξ_4 vary whereas ξ_1, ξ_3, ξ_5, and ξ_6 vanish. Consequently, with W and G given by (7.7), then g_{22}, g_{24}, g_{44}, and $G(0,\xi_2,0,\xi_4,0,0)$ can be completely characterized. Moreover, if an incompressible material with orthogonal families has mechanically equivalent fibers, then with W and G given by (7.8), the two biaxial testing orientations can be combined to determine g_{22}, g_{33}, g_{44}, $G(0,\xi_2,0,\xi_4^2,0,0)$, and $G(0,\xi_2,\xi_3^2,0,0,0)$.

9 Conclusions

The approach herein has distinct advantage over prior invariant approaches because common tests on high strain, 2-fiber materials can determine terms in W for the first time (see Sect. 8). Since covariance (among the parameters being optimized) can cause oscillation and instability in an optimization algorithm, this approach (with its reduced covariance amongst response terms) may enhance the speed and precision of inverse finite element analyses that attempt to estimate material properties from tests with a heterogeneous deformation and an indeterminate stress field.

For linearized small strain elasticity (see Sect. 5), all terms that are cubic order or higher are simply neglected such that the transition from small strain to large strain is forthright. Moreover, the response terms are entirely orthogonal for small strain, and the elasticity moduli have direct physical meaning. As shown in Criscione (2003), orthogonality, being the absolute minimum of covariance, is of paramount importance for the determination of material properties from mechanical tests.

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Appendix A

This appendix derives equation (2.10) which is a condition on the components of $\mathbf{f}$, the local deformation gradient as seen by an observer who rotates such that the bisectors of the fibers appear fixed. Hence, the current bisector of $\mathbf{fM}_1$ and $\mathbf{fM}_2$ must be colinear to $\mathbf{M}_A$, the reference bisector of $\mathbf{M}_1$ and $\mathbf{M}_2$. Upon dividing $\mathbf{fM}_1$ and $\mathbf{fM}_2$ by their respective lengths (i.e. by their respective fiber stretches, λ_1 and λ_2) and summing the resulting unit vectors, we obtain

$$\lambda_1^{-1}\mathbf{fM}_1 + \lambda_2^{-1}\mathbf{fM}_2 = k_A \mathbf{M}_A\,, \tag{A.1}$$

where k_A is a scaling factor that is needed because the sum of two unit vectors is not necessarily a unit vector. Since $\mathbf{M}_A$ and $\mathbf{M}_B$ are orthogonal, take the dot product of both sides of (A.1) with $\lambda_1\lambda_2\mathbf{M}_B$ to obtain

$$\lambda_2 \mathbf{M}_B \cdot \mathbf{fM}_1 + \lambda_1 \mathbf{M}_B \cdot \mathbf{fM}_2 = 0\,. \tag{A.2}$$

In likewise fashion, the current bisector of $-\mathbf{fM}_1$ and $\mathbf{fM}_2$ must be colinear to $\mathbf{M}_B$, the reference bisector of $-\mathbf{M}_1$ and $\mathbf{M}_2$, whereby

$$-\lambda_1^{-1}\mathbf{fM}_1 + \lambda_2^{-1}\mathbf{fM}_2 = k_B \mathbf{M}_B\,, \tag{A.3}$$

where k_B is a scaling factor similar to k_A. Upon taking the inner product of (A.3) with $\lambda_1\lambda_2\mathbf{M}_A$,

$$-\lambda_2 \mathbf{M}_A \cdot \mathbf{fM}_1 + \lambda_1 \mathbf{M}_A \cdot \mathbf{fM}_2 = 0\,. \tag{A.4}$$

With $\mathbf{M}_1$ and $\mathbf{M}_2$ given by (2.9) and $\mathbf{f}$ given by (2.8), then (A.2) and (A.4) respectively yield

$$\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{f_{BA}\cos\Theta}{f_{BB}\sin\Theta}\,, \tag{A.5$_1$}$$

$$\frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{f_{AB}\sin\Theta}{f_{AA}\cos\Theta}\,. \tag{A.5$_2$}$$

Whereby, equation (2.10) should be evident.

Appendix B

This appendix develops the one-to-one mapping between the components of $\mathbf{C}$ and the strain attributes ξ_{1-6}. To begin, premultiply (3.2) by its transpose to obtain $\mathbf{C}$ which is given by the following six components:

$$\mathbf{N}\cdot\mathbf{CN} = J^{2/3}\begin{pmatrix} \alpha^{-2} + \alpha\beta^2\left(\sqrt{1+\gamma^2 s^2 c^2}\psi_{AN} + \gamma c^2 \psi_{BN}\right)^2 \\ +\alpha\beta^{-2}\left(\gamma s^2 \psi_{AN} + \sqrt{1+\gamma^2 s^2 c^2}\psi_{BN}\right)^2 \end{pmatrix}, \tag{B.1$_1$}$$

$$\mathbf{M}_A\cdot\mathbf{CM}_A = J^{2/3}\alpha\left(\beta^2\left(1+\gamma^2 s^2 c^2\right) + \beta^{-2}\gamma^2 s^4\right), \tag{B.1$_2$}$$

$$\mathbf{M}_B\cdot\mathbf{CM}_B = J^{2/3}\alpha\left(\beta^{-2}\left(1+\gamma^2 s^2 c^2\right) + \beta^{2}\gamma^2 c^4\right), \tag{B.1$_3$}$$

$$\mathbf{M}_B\cdot\mathbf{CM}_A = J^{2/3}\alpha\gamma\sqrt{1+\gamma^2 s^2 c^2}\left(\beta^2 c^2 + \beta^{-2} s^2\right), \tag{B.1$_4$}$$

$$\mathbf{M}_A\cdot\mathbf{CN} = J^{2/3}\alpha\begin{pmatrix} \beta^2\sqrt{1+\gamma^2 s^2 c^2}\left(\sqrt{1+\gamma^2 s^2 c^2}\psi_{AN} + \gamma c^2\psi_{BN}\right) \\ +\beta^{-2}\gamma s^2\left(\gamma s^2\psi_{AN} + \sqrt{1+\gamma^2 s^2 c^2}\psi_{BN}\right) \end{pmatrix}, \tag{B.1$_5$}$$

$$\mathbf{M}_B\cdot\mathbf{CN} = J^{2/3}\alpha\begin{pmatrix} \beta^{-2}\sqrt{1+\gamma^2 s^2 c^2}\left(\sqrt{1+\gamma^2 s^2 c^2}\psi_{BN} + \gamma s^2\psi_{AN}\right) \\ +\beta^{2}\gamma c^2\left(\gamma c^2\psi_{BN} + \sqrt{1+\gamma^2 s^2 c^2}\psi_{AN}\right) \end{pmatrix}. \tag{B.1$_6$}$$

With (3.5) and much algebra and trigonometry, it can be verified that

$$J = \sqrt{\det(\mathbf{C})}\,, \tag{B.2$_1$}$$

$$\alpha = \sqrt{\frac{(\mathbf{M}_1\cdot\mathbf{CM}_1)(\mathbf{M}_2\cdot\mathbf{CM}_2) - (\mathbf{M}_1\cdot\mathbf{CM}_2)^2}{J^{4/3}\sin^2 2\Theta}}\,, \tag{B.2$_2$}$$

$$\beta = \sqrt{\tan\Theta \frac{\sqrt{(\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1)(\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2)} + (\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_2)}{\sqrt{(\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1)(\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2) - (\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_2)^2}}}\,, \tag{B.2$_3$}$$

$$\gamma = \frac{\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2 - \mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1}{\sin 2\Theta \sqrt{(\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1)(\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2)}} \left(2 + \frac{\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1 + \mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2}{\sqrt{(\mathbf{M}_1 \cdot \mathbf{C}\mathbf{M}_1)(\mathbf{M}_2 \cdot \mathbf{C}\mathbf{M}_2)}}\right)^{-1/2}, \tag{B.2$_4$}$$

$$\psi_{AN} = \frac{(\mathbf{N} \cdot \mathbf{C}\mathbf{M}_A)(\mathbf{M}_B \cdot \mathbf{C}\mathbf{M}_B) - (\mathbf{N} \cdot \mathbf{C}\mathbf{M}_B)(\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_B)}{(\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_A)(\mathbf{M}_B \cdot \mathbf{C}\mathbf{M}_B) - (\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_B)^2}\,, \tag{B.2$_5$}$$

$$\psi_{BN} = \frac{(\mathbf{N} \cdot \mathbf{C}\mathbf{M}_B)(\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_A) - (\mathbf{N} \cdot \mathbf{C}\mathbf{M}_A)(\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_B)}{(\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_A)(\mathbf{M}_B \cdot \mathbf{C}\mathbf{M}_B) - (\mathbf{M}_A \cdot \mathbf{C}\mathbf{M}_B)^2}\,. \tag{B.2$_6$}$$

Now, it should be evident that for every $\mathbf{C}$ there is one set of values for $\{J, \alpha, \beta, \gamma, \psi_{AN}, \psi_{BN}\}$, and for every set $\{J, \alpha, \beta, \gamma, \psi_{AN}, \psi_{BN}\}$ there is one $\mathbf{C}$. Since (3.4) gives the one-to-one correspondence between ξ_{1-6} and the set $\{J, \alpha, \beta, \gamma, \psi_{AN}, \psi_{BN}\}$, it follows that there is a one-to-one relationship between the ξ_i and $\mathbf{C}$. Although (B.2) with (3.4) could be used to compute ξ_{1-6}, it is faster to use (3.3) instead of (B.2) because $\mathbf{m}_A$ and $\mathbf{m}_B$ have to be computed for $\mathbf{t}$ anyway.

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