

*Original article*

## **A micro-polar theory for binary media with application to phase-transitional flow of fiber suspensions**

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**Abstract.** A phase-transitional flow takes place during the filling stage by injection molding of short-fiber reinforced thermoplastics. The mechanical properties of the final product are highly dependent on the flow-induced distribution and orientation of particles. Therefore, modelling of the flow which allows to predict the formation of fiber microstructure is of particular importance for analysis and design of load bearing components.

The aim of this paper is a discussion of existing models which characterize the behavior of fiber suspensions as well as the derivation of a model which treats the filling process as a phase-transitional flow of a binary medium consisting of fluid particles (liquid constituent) and immersed particles-fibers (solid-liquid constituent). The particle density and the mass density are considered as independent functions in order to account for the phenomenon of sticking of fluid particles to fibers. The liquid constituent is treated as a non-polar viscous fluid, but with a non-symmetric stress tensor. The state of the solid-liquid constituent is described by the antisymmetric stress tensor and the antisymmetric moment stress tensor. The forces of viscous friction between the constituents are taken into account. The equations of motion are formulated for open physical systems in order to consider the phenomenon of sticking. The chemical potential is introduced based on the reduced energy balance equation. The second law of thermodynamics is formulated by means of two inequalities under the assumption that the constituents may have different temperatures. In order to take into account the phase transitions of the liquid-solid type which take place during the flow process a model of compressible fluid and a constitutive equation for the pressure are proposed. Finally, the set of governing equations which should be solved numerically in order to simulate the filling process are summarized. The special cases of these equations are discussed by introduction of restricting assumptions.

**Key words:** injection molding, suspensions, micro-polar theory

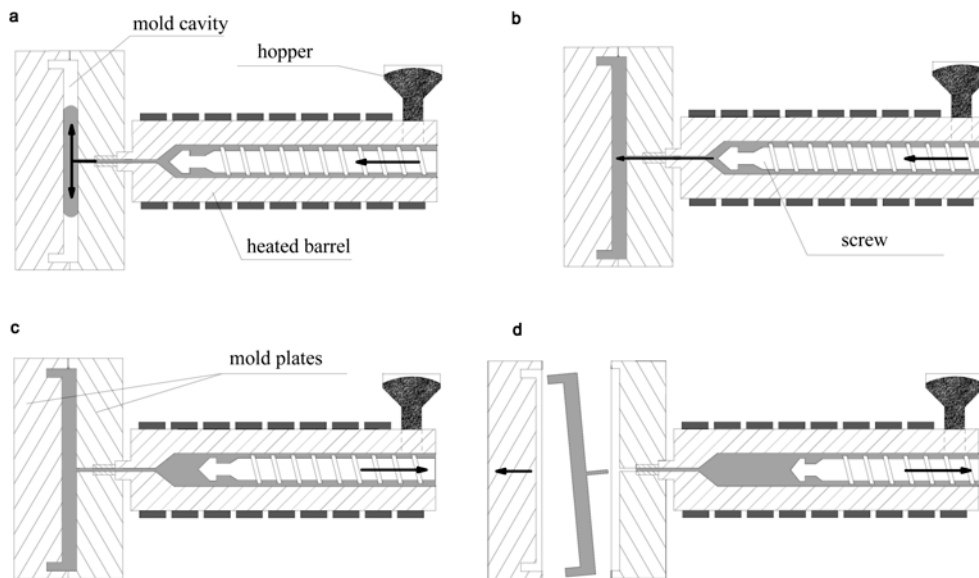
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## 1 Introduction

### 1.1 Motivation

The use of short fiber reinforced thermoplastics (fiber length around 0.1–1 mm, fiber diameter around 0.01 mm, fiber volume fraction 15–40%) has been rapidly increasing during the last years in many industrial branches, e.g. automobile industry, pump industry, etc. [22,30,37,41]. Various load bearing components (usually thin-walled structures) are manufactured from these materials by injection molding. This manufacturing process is of particular interest because of highly automated production, relatively short cycle time and low production costs. Furthermore, the principal advantage of this process over other methods of composites manufacturing is the possibility of mass production of articles with a desired geometrical complexity. However, the mechanical properties of particle reinforced materials are quite poor if compared with those of materials reinforced by continuous fibers. In addition, the stiffness and the strength of short-fiber reinforced composites and thin-walled structures manufactured from these materials are highly dependent on the orientation and the distribution of particles. As show many experimental observations, the fiber orientation microstructure, induced by injection molding has significant spatial variations within the part, e.g. [8,36,41]. The orientation of fibers and the distribution of fibers density depend on many factors including the material type, process conditions and the geometry of the mold cavity, e.g. [24]. Therefore, the key step in the preliminary design of load bearing components lies in the prediction of the fiber orientation pattern for given manufacturing conditions.

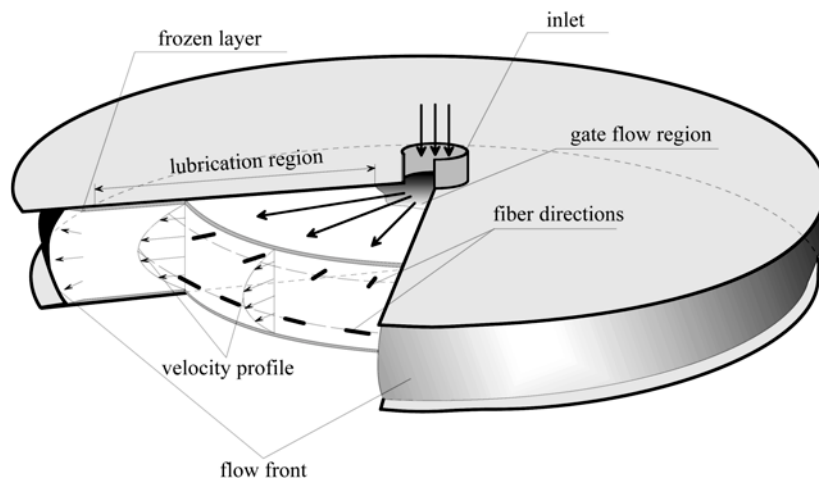
Figure 1 illustrates schematically the basic units of a typical injection molding machine and the main stages of the processing cycle. During the filling stage, Fig. 1a, the rotating screw moves forward and pushes the melt



**Fig. 1.** Basic stages of the injection molding processing cycle. (a) filling stage, (b) packing stage, (c) cooling stage, (d) ejection, for details see [29,30]

into the mold cavity. After the complete filling of the cavity, a pressure is exerted and hold over a period of time by the screw in order to compensate the polymer shrinkage (packing stage), Fig. 1b. During the cooling stage, Fig. 1c, the cavity cools and the material solidifies. At the same time the screw moves backward, a new portion of the material in a granular form is inserted into the barrel from the hopper. Within the heated barrel the material is melted and homogenized by the rotating screw. After the cooling of the cavity the mold opens and the part is ejected, Fig. 1d. The following step is the mold closing, and the beginning of the next cycle. For a detailed description of the injection molding machine, including its various modifications, we refer to the monographs [29,30].

The fiber orientation microstructure is primarily formed during the filling stage and remains unchanged after the solidification. The initial orientation of fibers may be considered to be random as the polymer melt,



**Fig. 2.** Sketch of a radial laminar flow between two parallel plates

homogenized within the barrel, is inserted into the cavity. During the filling stage, the flow of the viscous polymer melt translates and rotates the suspended particles. The micrographs of cross sections of injection molded parts, e.g. [8,36], show that the orientation of fibers exhibits a layered structure. In the mid-surface layer (core layer) the fibers are aligned dominantly perpendicular to the flow direction. On the other hand, in the layers neighboring upon the side walls (skin layers) the fibers lie dominantly parallel to the flow. In addition, the outer surface layers (shell layers) are usually detected having lower fiber concentration and random fiber orientation. As indicated in many works, e.g. [8,24,40], such a microstructure of fiber orientation has a correspondence to the flow behavior of the melt.

During the filling, an unsteady, non-isothermal flow with a moving free surface is observed within a geometrically complex cavity. In order to explain the suspension flow qualitatively let us consider an example of a radial, laminar flow between two parallel plates. The sketch, presented in Fig. 2, and the following comments are partly based on the results of filling simulations for a center-gated disk published in [3,7,35,40] as well as on observations of the flow induced fiber microstructure [8,40]. Because the thickness of the cavity is usually much smaller than other dimensions, one can formally separate the three flow regions [7]: the gate flow region, the lubrication region and the flow front region. Within the assumed lubrication region, Fig. 2, the velocity component in the thickness direction is negligible and the flow may be considered as two-dimensional. Assuming the parabolic velocity profile, one can estimate the kinematics of motion of particles. In the mid-surface layer of the lubrication region the elements of fluid undergo the stretching flow with the maximum strain rate in circumferential direction. Therefore, one can expect that a fiber inserted into the cavity with an arbitrary orientation, will be aligned in the direction perpendicular to the flow. On the other hand, the elements of fluid neighboring the cavity walls are exposed to the shear flow in the planes of radial cross sections. This shearing motion will align the fibers in the radial direction.

The situation is much more complicated in the neighborhood of the flow front. Firstly, as documented in many works, e.g. [7,14], the behavior in the free surface region is governed by the fountain flow, which translates the elements of the fluid from the core zone towards the cavity walls. Secondly, the lower temperature of the cavity walls leads to the formation of the frozen layer (no-flow layer) behind the free surface, Fig. 2. The frozen layer propagates towards the flow front. Particles that enter the frozen layer will be fixed and unaffected by the flow. Thirdly, in the free surface zone one can expect a fiber concentration lower than elsewhere within the flow domain. The flow behavior in the free surface region leads to the formation of the surface layers with lower fiber concentration and random fiber orientation. The discussed processes provide an explanation of principal mechanisms responsible to formation of the short-fiber microstructure. Further details regarding the flow behavior in complex cavities, e.g. the influence of abrupt changes of geometry of the cavity, formation of knit lines, etc., are discussed in the reviewing papers [14,15,39] among others.

In order to focus on the theoretical background of the filling process let us summarize some important features of the flow behavior:

- the flow is non-isothermal with phase transitions,

- the flow is non-steady with a free surface,
- the average fiber volume fraction of suspended particles lies within the range of 15–40%. The local concentration of fibers is affected by the flow and may vary within the flow domain. Therefore, the commonly used concepts of dilute, semi-dilute or concentrated suspensions are in general not suitable for the description of real processes, and
- the mold cavities are usually thin, so that the mold walls have essential influence on the fiber motion.

These factors may have an important influence on the formation of the flow-induced fiber microstructure and should be considered in a theory which allows a description of the filling stage. However, as far as we know, a general theory which is able to consider all the above features of the suspensions flow does not exist at present. The aim of this paper is to discuss the theoretical concepts for the prediction of the fiber orientation microstructure. Firstly, we give a brief review of the models recently proposed for the description of flow of fiber suspensions. Secondly, we develop and discuss a novel theory which treats the filling process as a phase-transitional flow of a binary medium.

### 1.2 Modelling of the flow induced fiber microstructure. State of the art

The motion of an ellipsoidal particle immersed in a viscous fluid was firstly considered by G.B. Jeffery in [26]. The result obtained by Jeffery is most frequently used in the literature on suspended fluids. However, if applying it to injection molding simulations one should take into account a number of important restrictions. Therefore, without discussion regarding the Jeffery solution procedure, let us recall the main result obtained by Jeffery. In what follows we use the so-called direct tensor calculus which is conventional in many books on mechanics and rheology, e.g. [5,21,28,32,38,45] among others. That means that the primary object is a vector  $\mathbf{a}$  rather than a triple of numbers (coordinates). A second rank tensor  $\mathbf{A}$  is any finite sum of the pairs of vectors  $\mathbf{A} = \mathbf{a} \otimes \mathbf{b} + \dots + \mathbf{c} \otimes \mathbf{d}$ . If it is desirable, one can introduce a basis  $\mathbf{g}_i$ . In this case  $\mathbf{a} = a^i \mathbf{g}_i$ ,  $\mathbf{A} = (a^i b^j + \dots + c^i d^j) \mathbf{g}_i \otimes \mathbf{g}_j$ . Below we prefer to operate with vectors and tensors rather than with their coordinates  $a^i$ ,  $A^{ij} = a^i b^j + \dots + c^i d^j$ .

The undisturbed flow (the flow without particle) is supposed by Jeffery to satisfy the following restrictions

$$\begin{aligned} \nabla \mathbf{V}_0(\mathbf{x}) \equiv \mathbf{A} = \text{const}, \quad \text{tr}(\mathbf{A}) \equiv \nabla \cdot \mathbf{V}_0(\mathbf{x}) = 0, \quad \Rightarrow \quad \mathbf{V}_0 = \mathbf{r}_0 \cdot \mathbf{A}, \quad \mathbf{A} = \mathbf{d} - \boldsymbol{\phi} \times \mathbf{E}, \\ \mathbf{d} \equiv \frac{1}{2} (\nabla \mathbf{V}_0 + \nabla \mathbf{V}_0^T), \quad \text{tr}(\mathbf{d}) = 0, \quad \boldsymbol{\phi} \equiv \frac{1}{2} \nabla \times \mathbf{V}_0, \quad \mathbf{V}_0 \cdot \nabla \mathbf{V}_0 = \mathbf{0} \quad \text{or} \quad \mathbf{A} \cdot \mathbf{A} = \mathbf{0}, \end{aligned} \quad (1)$$

where  $\mathbf{V}_0$  is the fluid velocity,  $\mathbf{E}$  is the identity tensor,  $\nabla$  is the nabla operator,

$$\mathbf{r}_0 = x_k \mathbf{i}'_k, \quad \mathbf{i}'_k \cdot \mathbf{i}'_s = \delta_{ks},$$

and the orthonormal unit basis vectors  $\mathbf{i}'_k$  are fixed in the reference system. The last two restrictions in (1) are not given in Jeffery's work in the explicit form. However, they follow immediately from the Navier–Stokes equations

$$-\nabla p + \mu \nabla \cdot \mathbf{A} = \rho \left( \frac{\partial \mathbf{V}_0}{\partial t} + \mathbf{r}_0 \cdot \mathbf{A} \cdot \mathbf{A} \right) \quad \Rightarrow \quad p = p_0 = \text{const}, \quad \mathbf{r}_0 \cdot \mathbf{A} \cdot \mathbf{A} = \mathbf{0} \quad (2)$$

with  $p$  as the pressure,  $\mu$  as the fluid viscosity and  $\rho$  as the fluid density. Only the underlined terms in (2) are not identically zero. Taking into account that the pressure  $p$  must be limited in the space we obtain the above mentioned restrictions. Let us define the ellipsoidal particle by means of the second rank tensor  $\mathbf{A}_0$

$$\mathbf{A}_0 = a^2 \mathbf{i}'_1 \otimes \mathbf{i}'_1 + b^2 \mathbf{i}'_2 \otimes \mathbf{i}'_2 + c^2 \mathbf{i}'_3 \otimes \mathbf{i}'_3 \quad \Rightarrow \quad \mathbf{A}_0^{-1} = a^{-2} \mathbf{i}'_1 \otimes \mathbf{i}'_1 + b^{-2} \mathbf{i}'_2 \otimes \mathbf{i}'_2 + c^{-2} \mathbf{i}'_3 \otimes \mathbf{i}'_3,$$

where the numbers  $a$ ,  $b$ ,  $c$  are the semi-axes of the ellipsoid. Let  $\mathbf{i}_k$  with  $\mathbf{i}_k \cdot \mathbf{i}_s = \delta_{ks}$  be a triplet of the orthogonal unit basis vectors rigidly connected with the particle. Let us introduce the rotation tensor  $\mathbf{P}(t)$  and the angular velocity vector  $\boldsymbol{\omega}(t)$  of the ellipsoidal particle by

$$\mathbf{P}(t) \equiv \mathbf{i}_k(t) \otimes \mathbf{i}'_k, \quad \dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t) \times \mathbf{P}(t) \quad \Rightarrow \quad \dot{\mathbf{m}}(t) = \boldsymbol{\omega}(t) \times \mathbf{m}(t), \quad (3)$$

where  $\mathbf{m}(t) \equiv \mathbf{P}(t) \cdot \mathbf{m}_0$  and  $\mathbf{m}_0$  is an arbitrary vector fixed with respect to the ellipsoid  $\mathbf{A}_0$ . In this case the rotating ellipsoid is determined by the tensor

$$\mathbf{A}(t) = \mathbf{P}(t) \cdot \mathbf{A}_0 \cdot \mathbf{P}^T(t). \quad (4)$$

The surface  $S$  of the rotating ellipsoid is determined by the equation

$$\mathbf{r} \cdot \mathbf{A}^{-1}(t) \cdot \mathbf{r} = 1, \quad \mathbf{r} = x_k \mathbf{i}_k, \quad \mathbf{r}(t) = \mathbf{P}(t) \cdot \mathbf{r}_0. \quad (5)$$

Jeffery considered the following problem: find the solution of quasi-static Navier–Stokes equations

$$-\nabla p + \mu \Delta \mathbf{V} = \mathbf{0}, \quad (6)$$

satisfying the boundary conditions

$$\mathbf{V}(t)|_S = \boldsymbol{\omega}(t) \times \mathbf{r}_S(t), \quad \text{and} \quad \mathbf{V}(t) \rightarrow \mathbf{V}_0 \quad \text{when} \quad |\mathbf{r}| \rightarrow \infty. \quad (7)$$

Jeffery solved (6)–(7) and calculated the moment acting on the particle. Setting this moment equal to zero Jeffery determined the angular velocity  $\boldsymbol{\omega}$  of the particle in the following form

$$\boldsymbol{\omega} = \boldsymbol{\phi} + (\text{tr}(\mathbf{A}) \mathbf{E} - \mathbf{A})^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_{\times}, \quad (\mathbf{a} \otimes \mathbf{b})_{\times} \equiv \mathbf{a} \times \mathbf{b}. \quad (8)$$

This is the invariant form of (36) presented in the Jeffery paper. Let us emphasize that the constant vector  $\boldsymbol{\phi}$  and the constant tensor  $\mathbf{d}$  are defined in terms of the vector  $\mathbf{V}_0$  by means of (1). However, the tensor  $\mathbf{A}$  contains the unknown rotation tensor  $\mathbf{P}(t)$ . In order to find  $\mathbf{P}(t)$  one has to solve the left Darboux problem, see e.g. [43]

$$\dot{\mathbf{P}} = \left[ \boldsymbol{\phi} + (\text{tr}(\mathbf{A}) \mathbf{E} - \mathbf{A})^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_{\times} \right] \times \mathbf{P}.$$

If we multiply this equation by the vector  $\mathbf{m}_0$ , then we can rewrite the Jeffery result for an arbitrary vector  $\mathbf{m}(t)$  rigidly connected with the ellipsoid

$$\dot{\mathbf{m}}(t) = \left[ \boldsymbol{\phi} + (\text{tr}(\mathbf{A}) \mathbf{E} - \mathbf{A})^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_{\times} \right] \times \mathbf{m}(t). \quad (9)$$

A more familiar form of (9) can be obtained in the case when the tensor  $\mathbf{A}(t)$  is transversely isotropic and the vector  $\mathbf{m}(t)$  is the axis of symmetry of  $\mathbf{A}$ . If  $a = b \neq c$  then we have

$$\begin{aligned} \mathbf{A} &= c^2 \mathbf{m} \otimes \mathbf{m} + a^2 (\mathbf{E} - \mathbf{m} \otimes \mathbf{m}) \Rightarrow (\text{tr}(\mathbf{A}) \mathbf{E} - \mathbf{A})^{-1} = \frac{1}{2a^2} \mathbf{m} \otimes \mathbf{m} + \frac{1}{c^2 + a^2} (\mathbf{E} - \mathbf{m} \otimes \mathbf{m}), \\ (\mathbf{A} \cdot \mathbf{d})_{\times} &= (c^2 - a^2) \mathbf{m} \times \mathbf{d} \cdot \mathbf{m}, \quad (\text{tr}(\mathbf{A}) \mathbf{E} - \mathbf{A})^{-1} \cdot (\mathbf{A} \cdot \mathbf{d})_{\times} = \frac{c^2 - a^2}{c^2 + a^2} \mathbf{m} \times \mathbf{d} \cdot \mathbf{m}. \end{aligned}$$

With these relations (9) can be rewritten as follows

$$\dot{\mathbf{m}} = \left( \boldsymbol{\phi} + \frac{c^2 - a^2}{c^2 + a^2} \mathbf{m} \times \mathbf{d} \cdot \mathbf{m} \right) \times \mathbf{m}. \quad (10)$$

Using (1) and eliminating the vector  $\boldsymbol{\phi}$  from (10) we obtain the most popular form of the Jeffery result

$$\dot{\mathbf{m}} = (\mathbf{d} - \boldsymbol{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} (m^2 \mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m}) \mathbf{m}). \quad (11)$$

with  $m = |\mathbf{m}|$ . If  $m = 1$  then instead of (11) we have

$$\dot{\mathbf{m}} = (\mathbf{d} - \boldsymbol{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} (\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m}) \mathbf{m}), \quad \boldsymbol{\Lambda} \equiv \nabla \mathbf{V}_0, \quad 2\mathbf{d} = \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T. \quad (12)$$

This is exactly the form of the Jeffery equation used in many works on fiber suspensions, e.g. [1,4,3,15,25,39]. Note that the vector  $\mathbf{V}_0$  satisfies the very strong restrictions given in (1). It was not proved whether (12) can be used in other cases. To be correct we have to mark that (9)–(12) are not present in the Jeffery paper. However, the necessary theoretical background for the derivation of these equations was well-known at the beginning of the XIX century. In derivation of (9)–(12) the only essential result is the expression (8) for the angular velocity which was found by Jeffery. Thus we may consider (9)–(12) as the Jeffery equations.

Now we are able to discuss the applications of the Jeffery result in the literature on the subject under consideration. Firstly, instead of (12) the following equations are used

$$\dot{\mathbf{m}} = (\mathbf{d} - \mathbf{\Lambda}) \cdot \mathbf{m} + \frac{c^2 - a^2}{c^2 + a^2} (\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m}), \quad \mathbf{\Lambda} \equiv \nabla \mathbf{V}, \quad 2\mathbf{d} = \mathbf{\Lambda} + \mathbf{\Lambda}^T, \quad (13)$$

where the vector  $\mathbf{V}$  is assumed to be the actual flow velocity without any restriction, except  $\nabla \cdot \mathbf{V} = 0$ . Secondly, it is supposed that the solution of (13) with the initial condition  $|\mathbf{m}(0)| = 1$  is a unit vector, see, for example, p. 257 of [25]. However, it is easy to prove that this assumption is not valid for any case. The restriction  $\mathbf{m} \cdot \mathbf{m} = 1$  must be connected with the (13) as an additional condition.

In the pioneering works on the injection molding simulations, e.g. [40], the Jeffery equation is numerically integrated for the known velocity field in order to calculate the fiber directions. The velocity gradient is computed by solving the flow problem of a Newtonian fluid. Such an approach is based on the assumption that interactions between the particles are negligible. In [39], p.165, Tucker and Advani pointed out that “the interaction between the multiple particles appears to be the most significant “non-Jeffery” effect in practical composite material problems”. The common approach in modelling of the filling process is to treat the flow of a fiber suspension as the flow of a single-component anisotropic fluent medium, e.g. [15,39]. Following this approach, the main problem is to find a rheological equation connecting the stress generated by the motion of the fluid with local characteristics of the motion. For a viscous incompressible fluid Batchelor [6] introduced the following equation

$$\boldsymbol{\sigma} = -p\mathbf{E} + \boldsymbol{\mu} \cdot \mathbf{\Lambda}, \quad \mathbf{\Lambda} = \nabla \mathbf{V}, \quad (14)$$

where  $\boldsymbol{\sigma}$  is the stress tensor and  $\boldsymbol{\mu}$  is the fourth rank viscosity tensor determined by the local state of the fluid. Various approaches have been proposed in order to find a particular form of the constitutive equation. Batchelor [6] discussed the volume averaging procedure for a suspension. An important point in his consideration is the assumption that the inertia forces associated with fluctuations about the average motion are small if compared with the viscous forces and that the equation of the motion of a fluid reduces to the linear quasi-static Stokes equation. He found the following expression for the bulk stress in a suspension

$$\boldsymbol{\sigma} = -p\mathbf{E} + \boldsymbol{\mu}(\mathbf{\Lambda} + \mathbf{\Lambda}^T) + \boldsymbol{\sigma}_p. \quad (15)$$

In order to obtain the particle stress  $\boldsymbol{\sigma}_p$  in (15) one should calculate the local velocity and stress fields of the fluid around a particle. Batchelor discussed the averaging procedure for dilute suspensions, i.e. assuming that the flow around each particle is unaffected by the presence of others. Based on the Jeffery solution for an ellipsoidal particle Batchelor obtained an explicit expression for  $\boldsymbol{\sigma}_p$ . Finally, he calculated the viscosity tensor  $\boldsymbol{\mu}$  in (14) for the case of perfectly aligned particles for a given orientation state as well as for randomly oriented particles by means of averaging over all orientations.

The state of art on rheology of fiber suspensions can be found in reviews [15,33,39]. In the literature on rheology one distinguishes between dilute, semi-dilute and concentrated suspensions. Assuming that the immersed particles are slender bodies of revolution with  $a$  and  $c$  as particle dimensions,  $a_p = c/a > 1$  as the particle aspect ratio and  $\xi$  is the fiber volume fraction, one specifies the suspension to be dilute when  $\xi a_p^2 < 1$ ; semi-dilute when  $1 < \xi a_p^2 < a_p$  and concentrated when  $\xi a_p^2 > a_p$ . This classification is made with regard to the kind of interactions between the particles by the flow of fiber suspension, e.g. [15,39]. In the first case one assumes no interactions, in the second case one assumes interaction of hydrodynamic nature and in the third case the interactions may have both hydrodynamic and direct mechanic origins. In fact, the concepts of dilute or non-dilute suspensions are intuitive assumptions rather than approximations of any general constitutive model. Such a model does not exist. The only known fact is that the commercial materials are non-dilute, see, e.g. [39].

In order to account for the fiber-fiber interactions as well as to consider the randomness of the fiber orientation at the inlet zone, the commonly used approach is the orientational averaging. As an example, let us introduce the model proposed by Dinh and Armstrong [12]. The starting point is the expression (15) for the bulk stress. In order to avoid the evaluation of the local flow fields around each particle, the authors considered a single slender body test particle in an effective continuous medium. They used the orientational probability density function  $\psi(\mathbf{m})$ , where  $\mathbf{m}$  is the unit vector associated with a test particle. Further, the influence of the surrounding media on the test particle is considered by means of a surface force, which is determined from the transversely isotropic drag law. The resulting expression can be formulated as follows

$$\boldsymbol{\sigma} = -p\mathbf{E} + \mu(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T) \cdot \left[ {}^{(4)}\mathbf{E} + \frac{nl^2}{12\mu}\zeta_p\mathbf{a}_4 \right], \quad \mathbf{a}_4 = \int_{(S)} \Psi(\mathbf{m})\mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m} dS. \quad (16)$$

Here  $n$  is the number of particles per unit volume,  $l$  is the particle length, and  $\zeta_p$  is the drag coefficient determined by

$$\zeta_p = \frac{2\pi\mu l}{\ln(2h/d)}$$

with  $d$  as the fiber diameter and  $h = (nl)^{-1/2}$  for aligned systems while  $h = (nl^2)^{-1}$  for random systems. In (16)  ${}^{(4)}\mathbf{E}$  is the fourth rank identity tensor and  $\mathbf{a}_4$  denotes the fourth rank structure tensor, which characterizes the actual fiber orientation state and  $dS$  is a differential element on a unit sphere. In order to formulate the evolution equation for the structure tensor one needs an equation for the probability density function. Assuming that the mechanism of fiber-fiber interactions is governed by the rotary Brownian motion, e.g., [39] the Smoluchowski type equation, e.g. [9, 13], is applied

$$\dot{\Psi} + \nabla_s \cdot (\Psi\boldsymbol{\omega} - D_r\nabla_s\Psi) = 0, \quad \boldsymbol{\omega} = \mathbf{m} \times \dot{\mathbf{m}}, \quad (17)$$

where  $(\dot{\dots})$  denotes the material derivative,  $D_r$  is the coefficient of rotary diffusion, and

$$\nabla_s(\dots) = \mathbf{e}_k\epsilon_{ijk}m_i\frac{\partial(\dots)}{\partial m_j}, \quad \mathbf{m} \cdot \mathbf{m} = 1$$

with  $\epsilon_{ijk}$  as the permutation symbol. For modelling the flow of fiber suspensions (17) is modified assuming that  $D_r$  is a scalar valued function of  $\mathbf{d}$ , e.g. [39]. Furthermore,  $\boldsymbol{\omega}$  in (17) is treated as the angular velocity of a single particle. Following [1] the Jeffery's result (11) is inserted into (17). Using the series representation of  $\Psi(\mathbf{m})$  by means of the spherical harmonics and introducing the moments of  $\Psi(\mathbf{m})$  by

$$\mathbf{a}_n = \int_{(S)} \Psi(\mathbf{m})\mathbf{m}^{\otimes n} dS, \quad n = 2, 4, \dots,$$

where  $\mathbf{a}_n$  are termed as  $n$ -th rank structure tensors and  $(\dots)^{\otimes n}$  is the  $n$ -th dyadic product, (17) is replaced by a set of coupled evolution equations for  $\mathbf{a}_n$  [1]. In the injection molding simulations the evolution equation for  $\mathbf{a}_2$

$$\dot{\mathbf{a}}_2 = (\mathbf{a}_2 \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{a}_2) + \lambda(\mathbf{d} \cdot \mathbf{a}_2 + \mathbf{a}_2 \cdot \mathbf{d} - 2\mathbf{a}_4 \cdot \mathbf{d}) - 6D_r \left( \mathbf{a}_2 - \frac{1}{3}\mathbf{E} \right), \quad \mathbf{w} = \boldsymbol{\Lambda} - \mathbf{d} \quad (18)$$

is usually solved while for  $\mathbf{a}_4$  a closure approximation is applied. Various types of closure approximations can be found in [1, 2, 11, 15, 31]. Let us note that the model (16) in connection with the evolution equation of the type (18) is widely used in injection molding simulations, e.g. [4, 10]. The tensor  $\mathbf{a}_2$  (and in some cases  $\mathbf{a}_4$ ) provides the information about the actual state of the fiber orientation.

Finally, let us discuss the anisotropic fluid models developed within the framework of the continuum mechanics. Based on the invariance conditions Ericksen [17] found a simplest form of the constitutive model for a

transversely isotropic single-component incompressible fluid

$$\begin{aligned}\boldsymbol{\sigma} &= -p\mathbf{E} + 2\mu\mathbf{d} + (\mu_1 + \mu_2\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m} \otimes \mathbf{m} + 2\mu_3[\mathbf{m} \otimes (\mathbf{d} \cdot \mathbf{m}) + (\mathbf{d} \cdot \mathbf{m}) \otimes \mathbf{m}], \\ \overset{\circ}{\mathbf{m}} &\equiv \dot{\mathbf{m}} - \mathbf{w} \cdot \mathbf{m} = \lambda(\mathbf{d} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{d} \cdot \mathbf{m})\mathbf{m}), \quad \mathbf{m} \cdot \mathbf{m} = 1.\end{aligned}\quad (19)$$

Here  $\lambda$  and  $\mu, \mu_1, \mu_2, \mu_3$  are constants,  $\mathbf{m}$  is a unit vector and  $(\overset{\circ}{\cdot})$  denotes the co-rotational time derivative. The concept of Ericksen's fluid associates the anisotropic behavior of a fluid with a director  $\mathbf{m}$ , changing with time according to the second equation in (19). Let us note that the second equation in (19) formally coincides with the Jeffery (13) for the single particle by setting  $\lambda = (c^2 - a^2)/(c^2 + a^2)$ . However, it should be noted that (19) and (13) are derived based on two different considerations. Furthermore, the constitutive model (19) must be introduced together with balance laws in continuum mechanics [16]. Eringen [18,19] developed a micro-polar theory of anisotropic fluids and applied it to the flow of fiber suspensions. The important feature of his theory is the modified balance law for the inertia tensor which accounts the phenomenon of sticking of the fluid to suspended particles. Assuming the inertia tensor to be transversely isotropic Eringen derived the evolution equation ((6.4) in [18]) which is similar to (13) and (19).

In the last two decades a large amount of work has been directed to simulations of the injection molding process. For an overview of existing models and their numerical realizations we refer to [14]. Furthermore, various commercial software packages have been developed, e.g. Moldflow<sup>®</sup> [27], which have the purpose to simulate the whole injection molding cycle and to optimize process conditions. Some years ago the commercial filling software has been extended by units allowing the prediction of fiber orientation microstructure. These units are based on the above discussed rheological equations of state and compute the fiber orientation by means of structure tensors. The present state of art on modelling the flow of fiber suspensions is given in [15,39], where a number of open questions is discussed. Regarding the theoretical approach one may note that there is no unified concept in modelling of suspended fluids and as pointed out in [19], p. 117, "at present there is no agreement on any one particular theory".

### 1.3 The scope of the paper

In order to underline the purposes of our approach let us introduce the following definitions. Let  $\eta_1(\mathbf{x}, t)$  be the density of fluid particles and  $\eta_2(\mathbf{x}, t)$  be the density of rigid particles at a given point  $\mathbf{x}$  of an inertial reference system. Specifying by  $dN_1$  and  $dN_2$  the number of particles for the first and the second components in a control volume  $dV$  we can write

$$dN_1 = \eta_1(\mathbf{x}, t)dV, \quad \eta_1 \geq 0; \quad dN_2 = \eta_2(\mathbf{x}, t)dV, \quad \eta_2 \geq 0.$$

The functions  $\eta_1(\mathbf{x}, t)$  and  $\eta_2(\mathbf{x}, t)$  are the principal unknowns in the theory of mixtures. The function  $\eta_2(\mathbf{x}, t)$  is particularly important, because it characterizes the distribution of the rigid particles in a fluid. This distribution will affect the final mechanical properties of the material after the processing. Since the number of fluid particles is much higher than the number of rigid particles we have

$$\eta_1(\mathbf{x}, t) \gg \eta_2(\mathbf{x}, t), \quad m_1 \ll m_2,$$

where  $m_1$  is the mass of one fluid particle and  $m_2$  is the mass of one rigid particle. Therefore, for the mass densities  $\rho_1(\mathbf{x}, t)$  and  $\rho_2(\mathbf{x}, t)$  of both the components in a suspension it is possible that the following equivalence is valid

$$\rho_1(\mathbf{x}, t) \sim \rho_2(\mathbf{x}, t),$$

where  $\sim$  denotes the equality of the orders of magnitude. Such a strong difference between the properties leads to the necessity to introduce different models for each of the two components. The first one is a set of fluid particles characterized by the particle density  $\eta_1$  and the mass density  $\rho_1$ . In what follows this constituent will be termed as the liquid component. The second component is a set of rigid particles immersed in the viscous fluid with  $\eta_2$  and  $\rho_2$  as the particle and mass densities. Below, by making constitutive assumptions, we shall



exclude the possibility that the rigid particles may form a solid body. Thus, we shall assume that the constituent of particles-fibers behaves like a liquid. In what follows the second component will be termed as the solid-liquid component.

The liquid component may be considered as a viscous fluid with some additional properties. For example, the stress tensor of the fluid in our model will be nonsymmetric. For the solid-liquid component it is necessary to consider not only the translation motion but also the rotations. Multi-component mixtures were studied in many works, e.g. [20,34] (see also works cited therein). In the physico-chemical hydrodynamics the diffusion processes play an essential role [34]. Namely, the diffusion determines the relative velocities of constituents by means of the Fick laws. In our case these velocities are determined by the external conditions and by the viscous properties of the fluid. The diffusion can be neglected.

In what follows let us briefly describe the framework of the paper and the distinctive features of our approach:

1. We will assume the particle density and the mass density as independent functions in order to take into account the phenomena of sticking of the fluid particles to the rigid particles. Consequently, the particle balance equations and the mass balance equations are independent from each other.
2. The liquid constituent will be supposed to be a non-polar viscous fluid, but with the non-symmetric stress tensor.
3. The state of the solid-liquid constituent will be described by means of the antisymmetric stress tensor and of the antisymmetric moment stress tensor.
4. Let the vectors  $\mathbf{V}_1(\mathbf{x}, t)$  and  $\mathbf{V}_2(\mathbf{x}, t)$  be the velocities of the particles of liquid and solid-liquid components, respectively. We will assume that  $\mathbf{V}_1(\mathbf{x}, t) \neq \mathbf{V}_2(\mathbf{x}, t)$ , i.e. in our approach, the constituents may slide with respect to each other. Therefore, the forces of friction between components will be taken into account.
5. The equations of motion are given for open physical systems and contain additional terms which are responsible for the phenomenon of sticking.
6. The chemical potential is introduced on the base of the so-called reduced equation of the energy balance. Our definition differs from the definitions proposed in [20,34]. By neglecting the sticking of the fluid particles to fibers, the chemical potential is conserved.
7. We assume that the constituents may have different temperatures. Such a difference may be important if we want to produce the material with desired mechanical properties. Thus, the temperature fields are in general discontinuous. In such a case the conventional form of the second law of thermodynamics, for example the Clausius-Duhem inequality, is not applicable. So, we give an alternative statement of the second law of thermodynamics as a set of two inequalities. The problem connected with the modelling of the heat exchange is discussed.
8. The main purpose of our approach is to describe the real technological process in which the mixture has a stage of solidification. The solidification takes place not only at the final stage of the process, but also during filling near the cavity walls. Therefore, the models of suspensions, based on the assumption of the anisotropic incompressible viscous fluid, are in general not suitable. In this paper we will discuss a model for compressible fluid with phase transitions of the liquid-solid type. The phase transitions are described by means of a proposed constitutive equation for the pressure.

## 2 Kinematical relations

It seems to be evident that in general a continuous medium cannot be modelled as a smooth differentiable manifold. Indeed, as it is known from experience, the particles which are neighbors at a moment of time, do not necessarily occupy neighboring positions at any later time. In such a medium one can expect the occurrence of tangential discontinuities or the nucleation of cavities. For a multi-component medium the situation is more complicated since different components may glide on each other and interact, and the interaction takes place between different species. For instance, let  $A_1, B_1, C_1, \dots$  be some marked parts of a first component and  $A_2, B_2, C_2, \dots$  be the parts of the second one. Let us assume that at a moment of time the parts  $A_1$  and  $A_2$  interact. Then, at any other moment of time another two parts, for example  $A_1$  and  $B_2$  will be in contact. From this follows that the material description, which assumes neighboring particles to be neighbors for all times, is in general not applicable to multi-component media. The only possible way to formulate a theory for such media is the use of the pure spatial description. That means that in contrast to the material description, all principal

operators in the theory must be directly defined within the reference frame rather than over a differentiable manifold. The introduction of these operators is the purpose of the following considerations.

Let us introduce a control volume in the reference frame and assume that the volume at the time  $t$  is filled by the medium. The medium may move with respect to the reference frame, or the medium may be at rest and the reference frame may move with respect to the medium. The difference between these two situations is not essential from the kinematical point of view. The important role plays the velocity field  $\mathbf{V}(\mathbf{x}, t)$ , where the vector  $\mathbf{x}$  defines a point of the reference frame. Thus, the vector  $\mathbf{V}(\mathbf{x}, t)$  characterizes the velocity of that particle which at the given moment of time  $t$  occupies the point  $\mathbf{x}$ . Let  $\mathbf{K}(\mathbf{x}, t)$  be a given field, which can be a tensor of any rank. This field describes a physical quantity of that particle, which is placed at the point  $\mathbf{x}$  of the reference frame at the given moment of time. Let us use the following definition of the material derivative [44]:

*The material derivative of a quantity  $\mathbf{K}(\mathbf{x}, t)$  is the limit of the fraction*

$$\frac{\delta}{\delta t} \mathbf{K}(\mathbf{x}, t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{K}(\mathbf{x} + \Delta \mathbf{s}, t + \Delta t) - \mathbf{K}(\mathbf{x}, t)}{\Delta t}, \quad \Delta \mathbf{s} = \mathbf{V}(\mathbf{x}, t) \Delta t. \quad (20)$$

In this definition  $\Delta \mathbf{s}$  (by neglecting the terms of the second and higher order of magnitude) is the way, passed within the time  $\Delta t$  by that particle, which at the time  $t$  was placed at the point  $\mathbf{x}$ . The nominator in (20) can be rewritten by means of the following expansion

$$\mathbf{K}(\mathbf{x} + \Delta \mathbf{s}, t + \Delta t) = \mathbf{K}(\mathbf{x}, t + \Delta t) + \Delta \mathbf{s} \cdot \nabla \mathbf{K}(\mathbf{x}, t + \Delta t).$$

From the definition (20) follows

$$\frac{\delta}{\delta t} \mathbf{K}(\mathbf{x}, t) = \frac{d}{dt} \mathbf{K}(\mathbf{x}, t) + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \mathbf{K}(\mathbf{x}, t). \quad (21)$$

In the first term of the right hand side in (21) one can formally replace the total time derivative by the partial one. However, such a replacement may lead to difficulties by a change of the reference frame. In several situations the above definition of the material derivative may be not convenient, because the point of observation is assumed to be fixed. Within the conventional Euler's description this is always the case. However, by a change of the reference frame one needs to consider a moving point of observation. Therefore, let us introduce an extended definition of the material derivative. Let  $\mathbf{y}(t)$  be a point of observation, which can move according to any given law. The velocity field  $\mathbf{V}(\mathbf{x}, t)$  is defined in those points of the reference frame, which are occupied by particles of the medium. Therefore, it is also defined in the points  $\mathbf{y}(t)$ . Let us accept the following modification of the definition (20)

$$\frac{\delta}{\delta t} \mathbf{K}(\mathbf{y}(t), t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{K}(\mathbf{y}(t + \Delta t) + \Delta \mathbf{s}, t + \Delta t) - \mathbf{K}(\mathbf{y}(t), t)}{\Delta t}, \quad (22)$$

where

$$\Delta \mathbf{s} = \mathbf{V}_r(\mathbf{y}(t), t) \Delta t, \quad \mathbf{V}_r(\mathbf{y}(t), t) = \mathbf{V}(\mathbf{y}(t), t) - \frac{d\mathbf{y}(t)}{dt}.$$

Here the velocity  $\mathbf{V}_r(\mathbf{y}(t), t)$  is the relative velocity of the material point with respect to the moving point  $\mathbf{y}(t)$ .

Now instead of (21) we have

$$\frac{\delta}{\delta t} \mathbf{K}(\mathbf{y}(t), t) = \frac{d}{dt} \mathbf{K}(\mathbf{y}(t), t) + \left( \mathbf{V}(\mathbf{y}(t), t) - \frac{d\mathbf{y}}{dt} \right) \cdot \nabla \mathbf{K}(\mathbf{y}(t), t). \quad (23)$$

The proposed definition (23) does not coincide with the conventional definition of the material derivative. However, one can rewrite (23) as follows

$$\frac{\delta}{\delta t} \mathbf{K}(\mathbf{y}(t), t) = \frac{\partial}{\partial t} \mathbf{K}(\mathbf{y}(t), t) + \mathbf{V}(\mathbf{y}(t), t) \cdot \nabla \mathbf{K}(\mathbf{y}(t), t). \quad (24)$$

The last expression just coincides with the commonly used material derivative, the difference is only  $\mathbf{y}(t)$ . However, in (24) the relative character of the velocity  $\mathbf{V}_r(\mathbf{y}(t), t)$  is hidden. The expression (24) looks like the total derivative  $d/dt$ . Therefore, in the literature the notation  $d/dt$  is usually preferred. However, in the general case we have to distinguish between  $d/dt$  and  $\delta/\delta t$ . In fact, the total time derivative is

$$\frac{d}{dt}\mathbf{K}(\mathbf{y}(t), t) = \frac{\partial}{\partial t}\mathbf{K}(\mathbf{y}(t), t) + \frac{d\mathbf{y}(t)}{dt} \cdot \nabla\mathbf{K}(\mathbf{y}(t), t), \quad \frac{d\mathbf{y}(t)}{dt} \neq \mathbf{V}.$$

The last expression coincides with the material derivative if and only if the point of observation  $\mathbf{y}(t)$  coincides with the position vector of a fixed particle. In the case of multi-components media such a situation is impossible because in a point of observation may be several different particles with different velocities.

For the material derivative all rules of differentiation are valid. For example

$$\frac{\delta}{\delta t}(\mathbf{a} \otimes \mathbf{b}) = \frac{\delta\mathbf{a}}{\delta t} \otimes \mathbf{b} + \mathbf{a} \otimes \frac{\delta\mathbf{b}}{\delta t}.$$

On the other hand, it is known that

$$\frac{d}{dt}\nabla = \nabla\frac{d}{dt}, \quad \frac{\delta}{\delta t}\nabla \neq \nabla\frac{\delta}{\delta t}.$$

In binary mixtures one usually assumes that one point of a reference frame can be simultaneously occupied by the particles of both species, e.g. [34]. The mixture considered in this work consists of two components. The first one includes particles of the viscous fluid. The second one is built up from small rigid bodies – fibers, which can be considered as ellipsoids of revolution. Let us introduce the notations:  $\mathbf{V}_1(\mathbf{x}, t)$  is the velocity vector of that particle of the fluid, which at the given time  $t$  is placed in  $\mathbf{x}$  of the reference frame;  $\mathbf{V}_2(\mathbf{x}, t)$  is the velocity vector of that particle-fiber, which at the given time  $t$  occupies the place  $\mathbf{x}$  of the reference frame. The vector  $\mathbf{V}_2(\mathbf{x}, t)$  will be treated as the velocity vector of the center of mass of a particle-fiber. Let us note that in theories of mixtures one usually assumes  $\mathbf{V}_1(\mathbf{x}, t) = \mathbf{V}_2(\mathbf{x}, t)$  [34].

By taking the material derivative of the velocity vector  $\mathbf{V}_1(\mathbf{x}, t)$ , we obtain the acceleration vector of a fluid particle

$$\mathbf{W}_1(\mathbf{x}(t), t) = \frac{d}{dt}\mathbf{V}_1(\mathbf{x}(t), t) + \left( \mathbf{V}_1(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt} \right) \cdot \nabla\mathbf{V}_1(\mathbf{x}(t), t).$$

The material derivative of the velocity vector  $\mathbf{V}_2(\mathbf{x}, t)$  yields the velocity of the center of mass of a particle-fiber

$$\mathbf{W}_2(\mathbf{x}(t), t) = \frac{d}{dt}\mathbf{V}_2(\mathbf{x}(t), t) + \left( \mathbf{V}_2(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt} \right) \cdot \nabla\mathbf{V}_2(\mathbf{x}(t), t).$$

Here we used the definition (23) assuming a moving point of observation. One can examine the difference between (23) and (24) by calculating the acceleration vectors for the case

$$\mathbf{V} \rightarrow \mathbf{V} + \mathbf{V}_0, \quad \mathbf{V}_0 = \text{const.}$$

Such a replacement is conventional, if one needs to use the Galilei relativity principle or to change the reference frame. It is clear that such a transformation should not change the accelerations. This will be the case if we use (23). But when applying (24) one must be careful to avoid mistakes.

Let us assume that the fluid particle and the particle-fiber, occupying at a given time  $t$  the point  $\mathbf{x}$ , at  $t_0 \leq t$  were located at  $\mathbf{x}_0$  and  $\mathbf{x}_0^*$ , respectively. The displacement vectors  $\mathbf{u}_1(\mathbf{x}, t) = \mathbf{x} - \mathbf{x}_0$  and  $\mathbf{u}_2(\mathbf{x}, t) = \mathbf{x} - \mathbf{x}_0^*$  are determined from the velocities by means of the following relations

$$\mathbf{V}_i(\mathbf{x}, t) = \frac{d}{dt}\mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i(\mathbf{x}, t) \cdot \nabla\mathbf{u}_i(\mathbf{x}, t) \quad \Rightarrow \quad \frac{d}{dt}\mathbf{u}_i(\mathbf{x}, t) = \mathbf{V}_i(\mathbf{x}, t) \cdot \mathbf{g}_i(\mathbf{x}, t) \quad (25)$$

with the notations

$$\mathbf{g}_i(\mathbf{x}, t) \equiv (\mathbf{E} - \nabla \mathbf{u}_i(\mathbf{x}, t)), \quad \det \mathbf{g}_i(\mathbf{x}, t) > 0. \quad (26)$$

By calculating the gradient of both parts of the second equation in (25) and taking into account the permutability of the gradient operator and the total time derivative, we obtain

$$\begin{aligned} \frac{d}{dt} \nabla \mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla \nabla \mathbf{u}_i(\mathbf{x}, t) &= \nabla \mathbf{V}_i(\mathbf{x}, t) \cdot \mathbf{g}_i(\mathbf{x}, t) \Rightarrow \\ \nabla \mathbf{V}_i(\mathbf{x}, t) &= \left( \frac{d}{dt} \nabla \mathbf{u}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla \nabla \mathbf{u}_i(\mathbf{x}, t) \right) \cdot \mathbf{g}_i^{-1}(\mathbf{x}, t). \end{aligned} \quad (27)$$

Equations similar to (27) can be found in [32]. The last equation can be rewritten in the equivalent form

$$\nabla \mathbf{V}_i(\mathbf{x}, t) = - \left( \frac{d}{dt} \mathbf{g}_i(\mathbf{x}, t) + \mathbf{V}_i \cdot \nabla \mathbf{g}_i(\mathbf{x}, t) \right) \cdot \mathbf{g}_i^{-1}(\mathbf{x}, t). \quad (28)$$

Equations (28) will be used later for the formulation of the reduced energy balance equation.

Till now, we did not make any distinction between the fluid particles and the particle-fibers. Let us introduce the rotations of particle-fibers, which will define their orientation in the reference frame. The determination of this orientation is one of the main purposes of the theory. Let us presume that at each point  $\mathbf{x}$  of the reference frame a triple  $\mathbf{d}_k$  with  $\mathbf{d}_k \cdot \mathbf{d}_m = \delta_{km}$  is given. Let us introduce the proper orthogonal tensor  $\mathbf{P}(\mathbf{x}, t)$ , which describes the rotation of the particle-fiber, located at the point  $\mathbf{x}$  at the time  $t$  with respect to the triple  $\mathbf{d}_k$ . Further, let us calculate the angular velocity of the rigid particle. Within the framework of the rigid body dynamics one can apply the Poisson equation [43]

$$\frac{d}{dt} \mathbf{P} = \boldsymbol{\omega} \times \mathbf{P}, \quad (29)$$

where  $\boldsymbol{\omega}$  is the angular velocity vector of a point in the body. It is clear that the definition (29) is not applicable to our case, since at different instances of time the point  $\mathbf{x}$  of the reference frame is occupied by different particles. Therefore, the time derivative in (29) cannot be treated as a characteristic of a particle. Instead of the definition (29) we have to use the following modification of the Poisson equation

$$\frac{d}{dt} \mathbf{P}(\mathbf{x}(t), t) + \left( \mathbf{V}_2(\mathbf{x}(t), t) - \frac{d\mathbf{x}(t)}{dt} \right) \cdot \nabla \mathbf{P}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}(t), t) \times \mathbf{P}(\mathbf{x}(t), t). \quad (30)$$

Here the subscripts for the rotation tensor and for the angular velocity are dropped since these quantities are defined for the particle-fibers only. Let us prove whether the definition (30) corresponds to our intuitive considerations. Consider two motions of the same particle and assume that these two motions have different translation parts, but the same rotations. Let  $\mathbf{x}_A(t)$  and  $\mathbf{y}_A(t)$  be two translations of the particle  $A$  so that

$$\mathbf{y}_A(t) = \mathbf{x}_A(t) + \mathbf{f}_A(t), \quad \mathbf{P}(\mathbf{x}_A, t) = \mathbf{P}(\mathbf{y}_A, t).$$

Making use of (23) it can be shown that  $\boldsymbol{\omega}(\mathbf{x}_A, t) = \boldsymbol{\omega}(\mathbf{y}_A, t)$ . That means that the angular velocity of the particle does not depend on its translation motion.

### 3 Particle balance and mass balance equations

Let us consider three different cases. In the first one, we assume that the total number of particles in both the components remains unchanged. It seems to be evident that such a strongly restricting assumption is not satisfied in the reality. In the second one, we suppose that the total number of fibers remains constant, while the mass of fibers may vary due to the sticking of the fluid particles to the fibers. In this case the number of fluid particles is changing, i.e. the density of fluid particles  $\eta_1$  is not constant. On the other hand, the density of particle-fibers  $\eta_2$  remains constant, while the mass density of particle-fibers is changing. Finally, in the third situation both

the density of fluid particles and the density of particle-fibers are changing. That means that not only the fluid particles can stick to the fibers but also the fibers may stick to each other. The sticking of fibers may lead to the formation of grains-clusters, which must be treated as new particles. Evidently, the last case is more realistic for concentrated suspensions. It is difficult to verify, how important the effects of sticking are for short-time filling processes. The quantitative influence of the sticking effects on the flow process seems to be insignificant. Nevertheless, let us discuss all the three situations separately.

**Liquid and solid-liquid constituents have constant compositions.** Let  $V$  be a control volume in the reference frame and the boundary of  $V$  be a closed surface  $S = \partial V$ . Then, for each of the introduced species we can formulate the following particle balance equations

$$\frac{d}{dt} \int_{(V)} \eta_1(\mathbf{x}, t) dV = - \int_{(S)} \eta_1 \mathbf{n} \cdot \mathbf{V}_1 dS = - \int_{(V)} \nabla \cdot (\eta_1 \mathbf{V}_1) dV, \quad (31)$$

where  $\mathbf{V}_1(\mathbf{x}, t)$  is the velocity of fluid particles,

$$\frac{d}{dt} \int_{(V)} \eta_2(\mathbf{x}, t) dV = - \int_{(S)} \eta_2 \mathbf{n} \cdot \mathbf{V}_2 dS = - \int_{(V)} \nabla \cdot (\eta_2 \mathbf{V}_2) dV, \quad (32)$$

where  $\mathbf{V}_2(\mathbf{x}, t)$  is the velocity of particle-fibers. Note that in the case of the moving point of observation one should replace  $\mathbf{V}_i$  by  $\mathbf{V}_i - \frac{d\mathbf{x}}{dt}$  in (31) and (32).

In the local form (31) and (32) can be written as

$$\frac{d\eta_1}{dt} + \nabla \cdot (\eta_1 \mathbf{V}_1) = 0, \quad \frac{d\eta_2}{dt} + \nabla \cdot (\eta_2 \mathbf{V}_2) = 0. \quad (33)$$

Analogously to (33), we can formulate the mass balance equations

$$\frac{d\rho_1}{dt} + \nabla \cdot (\rho_1 \mathbf{V}_1) = 0, \quad \frac{d\rho_2}{dt} + \nabla \cdot (\rho_2 \mathbf{V}_2) = 0, \quad (34)$$

where  $\rho_1$  and  $\rho_2$  are mass densities of the liquid and the solid-liquid components, respectively.

Let us introduce the following notations for the material derivatives

$$\frac{\delta_1 f}{\delta t} \equiv \frac{df}{dt} + \left( \mathbf{V}_1 - \frac{d\mathbf{x}}{dt} \right) \cdot \nabla f, \quad \frac{\delta_2 f}{\delta t} \equiv \frac{df}{dt} + \left( \mathbf{V}_2 - \frac{d\mathbf{x}}{dt} \right) \cdot \nabla f, \quad (35)$$

where  $f$  is an arbitrary scalar function (or any tensor-valued function). With the introduced notations (33) and (34) take the following form

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \quad (36)$$

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = 0. \quad (37)$$

**The liquid constituent has a variable composition.** In this situation, the density of the liquid component in the selected reference frame may change not only due to the flow, but also as a consequence of the sticking of the fluid particles to the particle-fibers. The fluid particles connected to fibers cannot be considered as fluid particles anymore. They should be related to the mass of fibers. The particle density of fibers may only change with regard to the motion of particle-fibers.

The particle balance equation for the liquid component (31) should be modified as follows

$$\frac{d}{dt} \int_{(V)} \eta_1(\mathbf{x}, t) dV = \int_{(V)} \chi_1(\mathbf{x}, t) dV - \int_{(S)} \eta_1 \mathbf{n} \cdot \mathbf{V}_1 dS = \int_{(V)} [\chi_1(\mathbf{x}, t) - \nabla \cdot (\eta_1 \mathbf{V}_1)] dV, \quad (38)$$

where the function  $\chi_1$  is the rate of production (destruction) of fluid particles at a point of the reference frame.

The particle balance equation for the fibers remains unchanged. Therefore, (36) take now the form

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = \chi_1, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0. \quad (39)$$

The equations of the mass balance should be modified as follows

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = \chi_{1m}, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = \chi_{2m}, \quad (40)$$

where the functions  $\chi_{1m}$  and  $\chi_{2m}$  characterize the rates of mass production (destruction) of fluid particles and particle-fibers, respectively. Because the total mass density  $\rho = \rho_1 + \rho_2$  does not change, the equation of the mass balance for the considered binary medium can be written down in the integral form

$$\frac{d}{dt} \int_{(V)} \rho(\mathbf{x}, t) dV = - \int_{(S)} \rho \mathbf{n} \cdot \mathbf{V}_m dS = - \int_{(V)} \nabla \cdot (\rho \mathbf{V}_m) dV, \quad \rho \mathbf{V}_m = \rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2. \quad (41)$$

The local form of (41) is

$$\frac{\delta_m \rho}{\delta t} + \rho \nabla \cdot \mathbf{V}_m = 0, \quad \frac{\delta_m f}{\delta t} \equiv \frac{df}{dt} + \left( \mathbf{V}_m - \frac{d\mathbf{x}}{dt} \right) \cdot \nabla f. \quad (42)$$

Here the point of observation  $\mathbf{x}(t)$  is selected to be the same for both components. If we add (40) and then subtract from the result (41), we obtain

$$\chi_{2m} = -\chi_{1m}, \quad (43)$$

i.e. the amount of mass acquired per unit time by the solid-liquid component per unit time is equal to the amount of mass lost by the liquid component.

**Liquid and solid-liquid components have variable compositions.** The particle balance equations can be formulated according to the above discussed procedure

$$\frac{\delta_1 \eta_1}{\delta t} + \eta_1 \nabla \cdot \mathbf{V}_1 = \chi_1, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = \chi_2, \quad (44)$$

where the function  $\chi_2$  characterizes the production rate of particle-fibers. The mass balance equations remain the same

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = \chi_{1m}, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = \chi_{2m}, \quad \chi_{2m} = -\chi_{1m}. \quad (45)$$

Let us emphasize that the introduced particle densities and the mass densities are independent functions. Consequently, (44) and (45) are independent. However, the functions  $\chi_1$  and  $\chi_{1m}$  can be assumed to be connected by means of equation  $\chi_{1m} = m \chi_1$ , where  $m$  characterizes the mass of one fluid particle. The last assumption is evident, since the fluid particles cannot form clusters.

Equations (44) and (45) can be rewritten in a scalar form. From (28) follows

$$\nabla \cdot \mathbf{V}_i(\mathbf{x}, t) = -\mathbf{g}_i^{-1}(\mathbf{x}, t) \cdot \left( \frac{\delta_i}{\delta t} \mathbf{g}_i(\mathbf{x}, t) \right). \quad (46)$$

In order to transform the above equation, one can use the following formula, which is valid for any nonsingular tensor  $\mathbf{g}_i$

$$\mathbf{g}_i^{-1} = \frac{1}{I_3(\mathbf{g}_i)} \left( \frac{\partial I_3(\mathbf{g}_i)}{\partial \mathbf{g}_i} \right)^T, \quad I_3(\mathbf{g}_i) = \det(\mathbf{g}_i). \quad (47)$$

After inserting (47) into (46) and performing some transformations we obtain

$$\nabla \cdot \mathbf{V}_i = - \frac{1}{I_3(\mathbf{g}_i)} \left( \frac{\delta_i I_3(\mathbf{g}_i)}{\delta t} \right) \quad (i = 1, 2), \quad (48)$$

where we do not imply summation by repeated subscripts. Using the formulae (48) we rewrite the particle balance equations (44) and the mass balance equations (45) as follows

$$\frac{\delta_i}{\delta t} \left( \frac{\eta_i}{I_3(\mathbf{g}_i)} \right) = \frac{\chi_i}{I_3(\mathbf{g}_i)}, \quad \frac{\delta_i}{\delta t} \left( \frac{\rho_i}{I_3(\mathbf{g}_i)} \right) = \frac{\chi_{im}}{I_3(\mathbf{g}_i)} \quad (i = 1, 2). \quad (49)$$

In what follows we shall primarily discuss the second situation assuming that the number of the fluid particles is not conserved, while the number of the rigid particles is constant. That means that we shall use (39)–(40). In this case it is necessary to formulate constitutive equations concerning the functions  $\chi_1, \chi_{1m}, \chi_{2m}$ . The functions  $\eta_1$  and  $\rho_1$  are connected by the relation  $m \eta_1 = \rho_1$ , where  $m$  is the mass of one fluid particle. Thus we have

$$m \chi_1 \equiv -\chi, \quad \chi_{1m} \equiv -\chi, \quad \chi_{2m} \equiv \chi, \quad \chi_2 = 0.$$

The three independent equations from (39)–(40) take a form

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = \chi$$

or

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2}{\delta t} \ln \frac{\rho_2}{\eta_2} = \frac{\chi}{\rho_2}. \quad (50)$$

The fraction  $\rho_2/\eta_2$  specifies the variable mass of one particle-fibre.

#### 4 The laws of dynamics

The fundamental laws in the spatial description must be formulated for open systems, i.e. for systems, which interchange with the surrounding medium mass, momentum, kinetic moment, energy, etc. The momentum of particles for a control volume  $V$  is defined as follows

$$\mathbf{K}_1 = \int_{(V)} (\rho_1(\mathbf{x}, t) \mathbf{V}_1(\mathbf{x}, t) + \rho_2(\mathbf{x}, t) \mathbf{V}_2(\mathbf{x}, t)) dV(\mathbf{x}) = \int_{(V)} \rho(\mathbf{x}, t) \mathbf{V}_m(\mathbf{x}, t) dV(\mathbf{x}). \quad (51)$$

**The Euler first law of dynamics** is the following statement: *The rate of change of the momentum for an arbitrary physical system is equal to the external force acting on the system plus the external supply of momentum into the system.*

The mathematical form of the first law of dynamics is

$$\frac{d}{dt} \int_{(V)} \rho \mathbf{V}_m dV = \int_{(V)} \rho \mathbf{F} dV + \int_{(S)} \mathbf{T}_{(n)} dS - \int_{(S)} [\rho_1 (\mathbf{n} \cdot \mathbf{V}_1) \mathbf{V}_1 + \rho_2 (\mathbf{n} \cdot \mathbf{V}_2) \mathbf{V}_2] dS, \quad (52)$$

where the last integral on the right-hand side is the external supply of momentum into the control volume  $V$ .

Using the standard arguments one may introduce the stress tensor  $\mathbf{T}$  and derive the Cauchy formulae

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T} \Rightarrow \int_{(S)} \mathbf{T}_{(n)} dS = \int_{(V)} \nabla \cdot \mathbf{T} dV. \quad (53)$$

Equation (52) takes the form

$$\int_{(V)} [(\rho \mathbf{V}_m)' - \rho \mathbf{F} + \nabla \cdot (\rho_1 \mathbf{V}_1 \otimes \mathbf{V}_1 + \rho_2 \mathbf{V}_2 \otimes \mathbf{V}_2) - \nabla \cdot \mathbf{T}] dV = \mathbf{0}.$$

Taking into account (40), the local form of (52) can be obtained

$$\nabla \cdot \mathbf{T} + \rho \mathbf{F} = \rho_1 \frac{\delta_1 \mathbf{V}_1}{\delta t} + \rho_2 \frac{\delta_2 \mathbf{V}_2}{\delta t} + \chi_{1m} (\mathbf{V}_1 - \mathbf{V}_2). \quad (54)$$

In addition, for the mass density of the external force  $\mathbf{F}$  we may write

$$\rho \mathbf{F} = \rho_1 \mathbf{F}_1 + \rho_2 \mathbf{F}_2,$$

where the force densities  $\mathbf{F}_1$  and  $\mathbf{F}_2$  may be of different nature. For example, the rigid particles may be charged.

It is convenient to rewrite (54) in a form of two equations

$$\nabla \cdot \mathbf{T}' + \rho_1 \mathbf{F}_1 + \mathbf{Q} = \rho_1 \frac{\delta_1 \mathbf{V}_1}{\delta t} + \chi_{1m} \mathbf{V}_1, \quad \nabla \cdot \mathbf{T}'' + \rho_2 \mathbf{F}_2 - \mathbf{Q} = \rho_2 \frac{\delta_2 \mathbf{V}_2}{\delta t} + \chi_{2m} \mathbf{V}_2, \quad (55)$$

where  $\mathbf{Q}$  is the force of interaction between the fluid and the solid-liquid components and

$$\mathbf{T} = \mathbf{T}' + \mathbf{T}'' \quad (56)$$

is postulated. Assuming that

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_m \equiv \mathbf{V},$$

one can obtain the conventional form of (54)

$$\nabla \cdot \mathbf{T} + \rho \mathbf{F} = \rho \frac{\delta \mathbf{V}}{\delta t}. \quad (57)$$

**The Euler second law of dynamics** was established in 1771 and at present is known as the following statement: *The rate of change of the kinetic moment for an arbitrary physical system is equal to the external moment acting on the system plus the external supply of the kinetic moment into the system.*

Let us introduce the kinetic moment of the binary medium

$$\mathbf{K}_2 = \int_{(V)} \rho \mathcal{K}_2 dV = \int_{(V)} \left[ \underline{\mathbf{x} \times (\rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2)} + \rho_2 \mathbf{J} \cdot \boldsymbol{\omega} \right] dV, \quad (58)$$

where  $\rho_2 \mathbf{J}$  is the volume density of the inertia tensor of the rigid particles. The underlined term in (58) is called the moment of momentum.

In the integral form the second law of dynamics can be written as

$$\begin{aligned} \frac{d}{dt} \int_{(V)} \rho \mathcal{K}_2 dV &= \int_{(V)} (\rho \mathbf{x} \times \mathbf{F} + \rho_2 \mathbf{L}) dV + \int_{(S)} (\mathbf{x} \times \mathbf{T}_{(n)} + \mathbf{M}_{(n)}) dS \\ &\quad - \int_{(S)} \mathbf{n} \cdot [\rho_1 \mathbf{V}_1 \otimes (\mathbf{x} \times \mathbf{V}_1) + \rho_2 \mathbf{V}_2 \otimes (\mathbf{x} \times \mathbf{V}_2 + \mathbf{J} \cdot \boldsymbol{\omega})] dS. \end{aligned} \quad (59)$$



In (58) and (59)  $\mathcal{K}_2$  and  $\mathbf{L}$  denote the densities of the kinetic moment and the external moment, respectively.

Introducing the moment stress tensor  $\mathbf{M}$  and the Cauchy formulae

$$\mathbf{M}_{(n)} = \mathbf{n} \cdot \mathbf{M} \Rightarrow \int_{(S)} \mathbf{M}_{(n)} dS = \int_{(V)} \nabla \cdot \mathbf{M} dV \quad (60)$$

and taking into account the first law (54), the local form of the second law can be obtained as follows

$$\nabla \cdot \mathbf{M} + \mathbf{T}_\times + \rho_2 \mathbf{L} = \rho_2 \frac{\delta_2}{\delta t} (\mathbf{J} \cdot \boldsymbol{\omega}) + \chi_{2m} \mathbf{J} \cdot \boldsymbol{\omega}. \quad (61)$$

$\mathbf{J}$  is the mass density of the inertia tensor in the actual state. Let  $\mathbf{J}_0$  be the inertia tensor in the reference state. Then we have

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x}, t) \cdot \mathbf{J}_0 \cdot \mathbf{P}^T(\mathbf{x}, t). \quad (62)$$

Let us assume that in the reference state all particle-fibers are transversally isotropic and have the same inertia properties. Thus, we may accept

$$\mathbf{J}_0 = \lambda \mathbf{e} \otimes \mathbf{e} + \mu (\mathbf{E} - \mathbf{e} \otimes \mathbf{e}), \quad (63)$$

where the constants  $\lambda$  and  $\mu$  are the moments of inertia of the rigid particles and the unit vector  $\mathbf{e}$  determines the axis of isotropy of the particles in the reference frame.

The reference direction of the vector  $\mathbf{e}$  is arbitrary and may be selected to be the same in all points of the reference frame including points which are not occupied by particles-fibers at  $t = 0$ . Then the reference distribution of the rigid particles can be given by

$$\mathbf{J}(\mathbf{x}_0, 0) = \mathbf{P}_0(\mathbf{x}_0) \cdot \mathbf{J}_0 \cdot \mathbf{P}_0^T(\mathbf{x}_0), \quad \mathbf{P}_0(\mathbf{x}_0) \equiv \mathbf{P}(\mathbf{x}_0, 0), \quad (64)$$

where the rotation tensor  $\mathbf{P}_0(\mathbf{x}_0)$  determines the initial orientation of the rigid particles. Let us note that for the considered manufacturing process the distribution of the initial orientations is a random function. Therefore, after solving of a deterministic problem for a given distribution of the tensor  $\mathbf{P}_0(\mathbf{x}_0)$  one should solve the problem of the statistical averaging of the results.

If we use (63) the relation (62) may be rewritten as

$$\mathbf{J}(\mathbf{x}, t) = \mu \mathbf{E} + (\lambda - \mu) \mathbf{e}' \otimes \mathbf{e}', \quad \mathbf{e}'(\mathbf{x}, t) \equiv \mathbf{P}(\mathbf{x}, t) \cdot \mathbf{e}. \quad (65)$$

Let us discuss the behavior of the volume density of the inertia tensor  $\rho_2(\mathbf{x}, t) \mathbf{J}(\mathbf{x}, t)$  within the flow process. If we take into account the phenomenon of sticking of the fluid particles to the fibers then this tensor is varying. There are two reasons leading to the change of the tensor  $\rho_2(\mathbf{x}, t) \mathbf{J}(\mathbf{x}, t)$ . The main reason is the variation of the mass density  $\rho_2(\mathbf{x}, t)$ . However, from the theoretical point of view it is possible to assume that the mass density  $\mathbf{J}(\mathbf{x}, t)$  of the inertia tensor is changing too, including both the symmetry properties and the moments of inertia. It seems to be obvious that this second factor is not very important for the considered technological processes. In what follows we shall assume that the volume density of the inertia tensor  $\rho_2(\mathbf{x}, t) \mathbf{J}(\mathbf{x}, t)$  may change only due to the changing mass density  $\rho_2(\mathbf{x}, t)$ . In such a case with respect to (62) and (30) we have

$$\frac{\delta_2}{\delta t} \mathbf{J}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t) \times \mathbf{J}(\mathbf{x}, t) - \mathbf{J}(\mathbf{x}, t) \times \boldsymbol{\omega}(\mathbf{x}, t). \quad (66)$$

Another approach is discussed by Eringen [18, 19], who proposed instead of (66) the following equation

$$\frac{\delta_2}{\delta t} \mathbf{J}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t) \times \mathbf{J}(\mathbf{x}, t) - \mathbf{J}(\mathbf{x}, t) \times \boldsymbol{\omega}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t),$$

where the function  $\mathbf{f}(\mathbf{x}, t)$  accounts for the sticking of the fluid particles to the rigid particles. This function must be defined by a constitutive equation. In this work we will prefer (66) which is broadly used in the dynamics

of rigid bodies and, in essence, was established by Euler. Let us note that it is possible to take into account the phenomenon of sticking even by use of (66).

## 5 Energy balance equation

Each of the fundamental laws introduces a new concept. The first law of dynamics introduces the concept of forces, the second law treats the moments, which are, in general case, not determined through the concept of forces. The third fundamental law in mechanics is the energy balance equation. Within the framework of continuum mechanics this law plays the most important role, but its formulation is much more difficult in comparison with the first and the second law. The energy balance equation introduces a lot of new concepts. The most important of them is the concept of internal energy. The general formulation of the energy balance equation includes the new concept of the total energy. However, the total energy can be conveniently represented as a sum of the kinetic energy, which has been already defined, and the internal energy, which absorbs all the new concepts contained in the concept of the total energy. One of the principal assumptions within continuum mechanics is the statement that the total energy of a system is an additive function of mass and according to the Radon-Nikodym theorem from the theory of sets, e.g. [23], can be presented as an integral over the mass, where the mass is considered to be a measure. The kinetic energy is, according to its definition, an additive function of mass. Therefore, the additivity of the total energy leads to the additivity of the internal energy. Generally speaking, the additivity of the internal energy is provided only for absolutely continuous systems. However, the known physical world is discrete. Therefore, the assumption about the additivity of the internal energy is a strong restriction. The attempts to relax this restriction are usually based on the concepts of the surface energy or the binding energy. In this work we will follow the traditional assumption about the additivity of the internal energy. In this case the total energy of the binary system can be considered by

$$E = \int_{(V)} \left[ \frac{1}{2} (\rho_1 \mathbf{V}_1 \cdot \mathbf{V}_1 + \rho_2 \mathbf{V}_2 \cdot \mathbf{V}_2) + \frac{1}{2} \rho_2 \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \rho_1 U_1 + \rho_2 U_2 + \rho U_{12} \right] dV,$$

where  $U_1$  and  $U_2$  are the mass densities of the internal energy of the fluid and the solid-fluid constituents, respectively.  $U_{12}$  is the energy of the interaction between the constituents of the binary mixture.

**The energy balance equation or the first law of thermodynamics** is the following statement: *The rate of change of the total energy of any physical system is equal to the power of external actions on the system plus the rate of the energy supply of the “non-mechanical” nature, usually in the form of heat.* It is difficult to give a general and strict definition of the concept for the energy of the “non-mechanical” nature. Therefore let us restrict ourself by an ambiguous statement that the energy of the “non-mechanical” nature is that part of the energy which is supplied into the system not through the power of external actions.

The energy balance equation may be formulated as follows

$$\begin{aligned} \frac{dE}{dt} = & \int_{(V)} [\rho_1 \mathbf{F}_1 \cdot \mathbf{V}_1 + \rho_2 \mathbf{F}_2 \cdot \mathbf{V}_2 + \rho_2 \mathbf{L} \cdot \boldsymbol{\omega} + \rho q] dV + \\ & \int_{(S)} \left( \mathbf{T}'_{(n)} \cdot \mathbf{V}_1 + \mathbf{T}''_{(n)} \cdot \mathbf{V}_2 + \mathbf{M}_{(n)} \cdot \boldsymbol{\omega} + h_{(n)} \right) dS - \int_{(S)} \mathbf{n} \cdot \left[ \rho_1 \mathbf{V}_1 \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 + U_1 \right) + \right. \\ & \left. \rho_2 \mathbf{V}_2 \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + U_2 \right) + \rho \mathbf{V}_m U_{12} \right] dS. \quad (67) \end{aligned}$$

In (67) the decomposition of the total vector of tractions  $\mathbf{T}_{(n)}$  is used

$$\mathbf{T}_{(n)} = \mathbf{T}'_{(n)} + \mathbf{T}''_{(n)} = \mathbf{n} \cdot \mathbf{T}' + \mathbf{n} \cdot \mathbf{T}''.$$

$q$  is the rate of production of the energy at a point  $\mathbf{x}$  of the reference frame and  $h_{(n)}$  is the rate of energy supply through the surface  $S$ . The last one can be written using the Stokes rule

$$h_{(n)} = \mathbf{n} \cdot \mathbf{h}, \quad (68)$$

where  $\mathbf{h}$  is the vector of the energy flux, which contains all kinds of energy which are not included in the power of external forces and moments. Let us note that in many works the vector  $(-\mathbf{h})$  is used instead of the vector  $\mathbf{h}$ .

Taking into account (40), (42), (55) and (61) the energy balance equation (67) may be written down in the local form as

$$\begin{aligned} \rho_{1m} \frac{\delta_1 U_1}{\delta t} + \rho_{2m} \frac{\delta_2 U_2}{\delta t} + \rho_m \frac{\delta_m U_{12}}{\delta t} = \\ \mathbf{T}'^T \cdot \cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}''^T \cdot \cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}^T \cdot \cdot \nabla \boldsymbol{\omega} + \mathbf{Q} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ \nabla \cdot \mathbf{h} + \rho_m q + \chi_{1m} \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - U_1 \right) + \chi_{2m} \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2 \right). \end{aligned} \quad (69)$$

The right-hand side of (69) contains the power of forces and moments. A part of this power serves for the change of the internal energy. The remaining part of the power partly conserves within the body in the form of heat and partly radiates into the external medium. In order to separate these parts let us introduce the following decompositions

$$\mathbf{T}' = \mathbf{T}'_e + \mathbf{T}'_f, \quad \mathbf{T}'' = \mathbf{T}''_e + \mathbf{T}''_f, \quad \mathbf{M} = \mathbf{M}_e + \mathbf{M}_f, \quad \mathbf{Q} = \mathbf{Q}_e + \mathbf{Q}_f, \quad (70)$$

where the subscript “e” denotes the part which does not depend on the velocities and the subscript “f” denotes the remaining part. In what follows the quantities with subscript “e” will be termed elastic stresses. These elastic stresses always affect the internal energy. The quantities with the subscript “f” may have an influence on the internal energy but only by means of additional parameters like entropy. These parameters will be introduced later.

Substituting (70) into (69) one may obtain

$$\begin{aligned} \rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} = \\ \mathbf{T}'_e{}^T \cdot \cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}''_e{}^T \cdot \cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_e^T \cdot \cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_e \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ \mathbf{T}'_f{}^T \cdot \cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}''_f{}^T \cdot \cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_f^T \cdot \cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_f \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ \nabla \cdot \mathbf{h} + \rho_m q + \chi_{1m} \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - U_1 \right) + \chi_{2m} \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2 \right). \end{aligned} \quad (71)$$

Such a form of the energy balance equation is useless. Below we shall transform (71) in order to obtain the so-called reduced energy balance equation. The idea of such a transformation was discussed in [44].

## 6 Basic constitutive assumptions

Let us assume that the pressure is moderate, i.e. we will not consider neither the super high nor the super low pressure. That means, that we will exclude the phase transitions of the solid–solid and the liquid–gas types. However, we have to take into account the solid–liquid type phase transitions. In such a case we can assume

$$\mathbf{T}'_e = -p_1(\mathbf{x}, t) \mathbf{E}, \quad \mathbf{T}''_e = -p_2(\mathbf{x}, t) \mathbf{E}, \quad \mathbf{M}_e = \mathbf{0}, \quad \mathbf{Q}_e = \mathbf{0}. \quad (72)$$

From these assumptions it follows that the rigid particles are not able to form the solid body without strong external loads. Otherwise, we have to take into account the deviatoric part of the stress tensor. Thus, within these assumptions the constituent of particle-fibers behaves like a liquid.

From the intuitive point of view the first three assumptions in (72) seem to be quite reasonable. The last assumption in (72) is related to the elastic interaction between the fluid and the solid-liquid constituents. Let us note that (69) includes the quantity  $\mathbf{Q} = \mathbf{Q}_e + \mathbf{Q}_f$ . The force  $\mathbf{Q}_e$  characterizes the elastic interaction and should

be determined in such a way that the following equation

$$\mathbf{Q}_e \cdot (\mathbf{V}_1 - \mathbf{V}_2) = \frac{d\mathcal{P}}{dt}, \quad (73)$$

is satisfied, i.e. the elastic force should have a potential. The elastic interaction is present in the nature of the considered phenomenon. Indeed, a moving force field connected with moving particles should be treated using equations of the type (73). In the case of a binary medium the right hand side of (73) should be expressed in terms of two different material derivatives with respect to the velocities  $\mathbf{V}_1$  and  $\mathbf{V}_2$ . This problem requires additional investigation. In this paper we will neglect the elastic interaction assuming  $\mathbf{Q}_e = \mathbf{0}$ .

With assumptions (72) and taking into account (50) the energy balance equation (71) may be rewritten as follows

$$\begin{aligned} \rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} &= \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \psi \frac{\delta_2 z}{\delta t} + \nabla \cdot \mathbf{h} + \rho q + \\ &\mathbf{T}_f'^T \cdot \cdot (\nabla \mathbf{V}_1 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{T}_f''^T \cdot \cdot (\nabla \mathbf{V}_2 + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}_f^T \cdot \cdot \nabla \boldsymbol{\omega} + \mathbf{Q}_f \cdot (\mathbf{V}_2 - \mathbf{V}_1), \end{aligned} \quad (74)$$

where

$$z \equiv \ln \frac{\rho_2 \eta_2^0}{\rho_2^0 \eta_2}, \quad \psi \equiv \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - \frac{p_2}{\rho_2} - U_2 \right) - \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \frac{p_1}{\rho_1} - U_1 \right) \quad (75)$$

and  $\eta_2^0$  and  $\rho_2^0$  are the reference density and the mass density of particle-fibers, respectively.

In addition, we have to introduce constitutive assumptions with respect to the forces and the moments of viscous friction. Let us use the conventional notations

$$\mathbf{d} = \frac{1}{2} (\nabla \mathbf{V}_1 + \nabla \mathbf{V}_1^T), \quad \mathbf{D} = \frac{1}{2} \left( \nabla \mathbf{V}_1 + \nabla \mathbf{V}_1^T - \frac{2}{3} (\nabla \cdot \mathbf{V}_1) \mathbf{E} \right).$$

For the tensor  $\mathbf{T}_f'$  let us postulate the following constitutive equation

$$\mathbf{T}_f' = 2\boldsymbol{\mu} \cdot \mathbf{D} + \mathbf{t}' \times \mathbf{E}, \quad \mathbf{t}' = \eta_2 \boldsymbol{\mu}_1 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right). \quad (76)$$

Here the vector  $\mathbf{t}'$  characterizes the viscous friction between the solid particles and the fluid. In the first equation in (76) the viscosity fourth rank tensor  $\boldsymbol{\mu}$  must satisfy the following restrictions

$$\mathbf{a} \cdot \boldsymbol{\mu} \cdot \mathbf{a} \geq 0, \quad \mathbf{a} \cdot \boldsymbol{\mu} = \boldsymbol{\mu} \cdot \mathbf{a}, \quad \mathbf{c} \cdot \boldsymbol{\mu} = \mathbf{0}, \quad \mathbf{E} \cdot \boldsymbol{\mu} = \mathbf{0}, \quad \forall \mathbf{a}, \mathbf{c} \text{ with } \mathbf{c} = -\mathbf{c}^T, \quad (77)$$

where  $\mathbf{a}$  and  $\mathbf{c}$  are second rank tensors. Furthermore, if the particle density  $\eta_2$  vanishes, then the tensor  $\boldsymbol{\mu}$  must be isotropic. In the majority of works on suspensions the tensor  $\boldsymbol{\mu}$  is supposed to be a transversely isotropic function of  $\mathbf{e}'$  and  $\mathbf{D}$ , where the vector  $\mathbf{e}'$  is defined by equation (65). Furthermore, the traditional approach assumes that the difference between the suspension and the ordinary fluid lies in the structure of the tensor  $\boldsymbol{\mu}$  (see the introduction to this paper). In this work we do not deny the possibility that the tensor  $\boldsymbol{\mu}$  may depend on  $\mathbf{e}'$  and  $\mathbf{D}$ . However, such a dependence is not crucial in our approach. From the physical point of view it seems to be reasonable to assume the tensor  $\boldsymbol{\mu}$  to be isotropic.

The viscosity second rank tensor  $\boldsymbol{\mu}_1$  in (76) must satisfy the restrictions

$$\mathbf{a} \cdot \boldsymbol{\mu}_1 \cdot \mathbf{a} \geq 0, \quad \mathbf{a} \cdot \boldsymbol{\mu}_1 = \boldsymbol{\mu}_1 \cdot \mathbf{a}, \quad \forall \mathbf{a} \text{ with } |\mathbf{a}| \neq 0, \quad (78)$$

where  $\mathbf{a}$  is a vector. Besides, if the particle density  $\eta_2$  vanishes, then the vector  $\mathbf{t}'$  must be zero. We assume that the tensor  $\boldsymbol{\mu}_1$  is transversely isotropic

$$\boldsymbol{\mu}_1 = \mu_1^1 \mathbf{e}' \otimes \mathbf{e}' + \mu_1^2 (\mathbf{E} - \mathbf{e}' \otimes \mathbf{e}'), \quad \mu_1^1 \geq 0, \quad \mu_1^2 \geq 0, \quad (79)$$

where the vector  $\mathbf{e}'$  is defined by (65).

The constitutive equation for the viscous stresses in the solid-liquid constituent may be written as

$$\mathbf{T}_f'' = \mathbf{t}'' \times \mathbf{E}, \quad \mathbf{t}'' = \eta_2 \boldsymbol{\mu}_2 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right), \quad (80)$$

where the second rank tensor  $\boldsymbol{\mu}_2$  has the same form as in (79). The vector  $\mathbf{t}''$  describes the viscous friction between the solid particles. It is clear that the vector  $\mathbf{t}''$  must vanish if  $\eta_2 = 0$ .

The constitutive equation for the viscous moment stress tensor may be accepted in a simple form

$$\mathbf{M}_f = \mathbf{m} \times \mathbf{E}, \quad \mathbf{m} = -\eta_2 \mu_3 (\nabla \times \boldsymbol{\omega}), \quad \mu_3 \geq 0. \quad (81)$$

Finally, we assume the constitutive equation for the force  $\mathbf{Q}_f$  in the following form

$$\mathbf{Q}_f = 2\eta_2 \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1), \quad \boldsymbol{\mu}_{12} = \mu_{12}^1 \mathbf{e}' \otimes \mathbf{e}' + \mu_{12}^2 (\mathbf{E} - \mathbf{e}' \otimes \mathbf{e}'), \quad \mu_{12}^1 \geq 0, \mu_{12}^2 \geq 0. \quad (82)$$

The substitution of (76)–(82) into (74) leads to the following form of the energy balance equation

$$\begin{aligned} \rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} &= \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \psi \frac{\delta_2 z}{\delta t} + \\ &\nabla \cdot \mathbf{h} + \rho q + 2\mathbf{D} \cdot \boldsymbol{\mu} \cdot \mathbf{D} + 2\eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ &\eta_2 \sum_{i=1}^2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) \cdot \boldsymbol{\mu}_i \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) + \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2. \end{aligned} \quad (83)$$

## 7 The heat conduction equation. The second law of thermodynamics

In order to state the so-called reduced equation of the energy balance we need to define the concepts of temperature, entropy and chemical potential. As a rule, all these concepts are supposed to be well-known [38]. However, in fact there are no satisfactory definitions for them in continuum mechanics. The problem is that it is impossible to prove that the temperature introduced in thermodynamics or in statistical physics coincides with the temperature in continuum mechanics. The same may be said with respect to the entropy and chemical potential. In what follows we will use the approach discussed in [44]. Let us introduce the new variables  $\vartheta_1$ ,  $\vartheta_2$ ,  $H_1$ , and  $H_2$  such that

$$\begin{aligned} \nabla \cdot \mathbf{h} + \rho q + 2\mathbf{D} \cdot \boldsymbol{\mu} \cdot \mathbf{D} + 2\eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2 + \\ \eta_2 \sum_{i=1}^2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) \cdot \boldsymbol{\mu}_i \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_i \right) = \rho_1 \vartheta_1 \frac{\delta_1 H_1}{\delta t} + \rho_2 \vartheta_2 \frac{\delta_2 H_2}{\delta t}, \end{aligned} \quad (84)$$

where the parameters  $\vartheta_1$  and  $\vartheta_2$  will be called temperatures of liquid and solid-liquid species, respectively, and the parameters  $H_1$  and  $H_2$  will be called entropies of the species. The functions  $\vartheta_1$  and  $\vartheta_2$  are supposed to be measurable by means of some experimental procedure. The functions  $H_1$  and  $H_2$  must be defined by means of constitutive equations in such a manner that the temperatures found theoretically coincide with the temperatures found experimentally. From this it follows that the entropy itself has not the meaning of any objective (measurable) quantity. If we change the meaning of temperature, then the meaning of the entropy will be changed as well. Thus we see that (84) is the true equality rather than an additional assumption. In some sense one may say that the right-hand side of (84) is the notation for the left-hand side of (84). Equation (84) is termed the heat conduction equation.

Let us rewrite (84) in an equivalent form using the decomposition

$$\mathbf{h} = \mathbf{h}' + \mathbf{h}''.$$

In such a case we have

$$\begin{aligned} \nabla \cdot \mathbf{h}' + \rho_1 q_1 + \mathcal{Q} + 2\mathbf{D} \cdot \boldsymbol{\mu} \cdot \mathbf{D} + \eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ \eta_2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) \cdot \boldsymbol{\mu}_1 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) = \rho_1 \vartheta_1 \frac{\delta_1 H_1}{\delta t}, \end{aligned}$$

$$\begin{aligned} \nabla \cdot \mathbf{h}'' + \rho_2 q_2 - \mathcal{Q} + \eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \\ \eta_2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) \cdot \boldsymbol{\mu}_2 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) + \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2 = \rho_2 \vartheta_2 \frac{\delta_2 H_2}{\delta t}, \quad (85) \end{aligned}$$

where the quantity  $\mathcal{Q}$  is termed the heat exchange between the liquid and the solid-liquid constituent. Equation (84) follows from (85). The equivalence of (84) and (85) is determined by the presence of the undefined quantity  $\mathcal{Q}$ . With the separation of (84) into two equations (85) we can state the second law of thermodynamics in a form of two inequalities [42]. The amount of the heat accumulated in each constituent is determined by the heat exchange  $\mathcal{Q}$ . For the heat fluxes we apply the Fourier-Stokes law

$$\mathbf{h}' = \kappa_1 \nabla \vartheta_1, \quad \mathbf{h}'' = \kappa_2 \nabla \vartheta_2, \quad \mathcal{Q} = -\kappa (\vartheta_1 - \vartheta_2), \quad \kappa_1 \geq 0, \kappa_2 \geq 0, \kappa \geq 0, \quad (86)$$

where  $\kappa_1$ ,  $\kappa_2$  and  $\kappa$  are the heat conductivities. The latter inequalities do not contradict the second law of thermodynamics which can be formulated as a set of two inequalities of the Clausius-Duhem type [44]

$$\frac{d}{dt} \int_{(V)} \rho_1 H_1 dV - \int_{(V)} \left[ \frac{\rho_1 q_1}{\vartheta_1} + \frac{\mathcal{Q}}{\vartheta_2} \right] dV - \int_{(S)} \mathbf{n} \cdot \left[ \frac{\mathbf{h}'}{\vartheta_1} - \rho_1 \mathbf{V}_1 H_1 \right] dS \geq 0, \quad (87)$$

$$\frac{d}{dt} \int_{(V)} \rho_2 H_2 dV - \int_{(V)} \left[ \frac{\rho_2 q_2}{\vartheta_2} - \frac{\mathcal{Q}}{\vartheta_1} \right] dV - \int_{(S)} \mathbf{n} \cdot \left[ \frac{\mathbf{h}''}{\vartheta_2} - \rho_2 \mathbf{V}_2 H_2 \right] dS \geq 0. \quad (88)$$

In the local form the inequalities (87)–(88) may be written as follows

$$\rho_1 \frac{\delta_1 H_1}{\delta t} - \frac{1}{\vartheta_1} (\nabla \cdot \mathbf{h}' + \rho_1 q_1 + \mathcal{Q}) + \mathcal{Q} \left( \frac{1}{\vartheta_1} - \frac{1}{\vartheta_2} \right) - \chi H_1 + \frac{1}{\vartheta_1^2} \mathbf{h}' \cdot \nabla \vartheta_1 \geq 0, \quad (89)$$

$$\rho_2 \frac{\delta_2 H_2}{\delta t} - \frac{1}{\vartheta_2} (\nabla \cdot \mathbf{h}'' + \rho_2 q_2 - \mathcal{Q}) + \mathcal{Q} \left( \frac{1}{\vartheta_1} - \frac{1}{\vartheta_2} \right) + \chi H_2 + \frac{1}{\vartheta_2^2} \mathbf{h}'' \cdot \nabla \vartheta_2 \geq 0. \quad (90)$$

Making use of (85) we obtain

$$\begin{aligned} 2\mathbf{D} \cdot \boldsymbol{\mu} \cdot \mathbf{D} + \eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \eta_2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) \cdot \boldsymbol{\mu}_1 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) + \\ \mathcal{Q} \left( \frac{1}{\vartheta_1} - \frac{1}{\vartheta_2} \right) - \chi H_1 + \frac{1}{\vartheta_1^2} \mathbf{h}' \cdot \nabla \vartheta_1 \geq 0, \quad (91) \end{aligned}$$

$$\begin{aligned} \eta_2 (\mathbf{V}_2 - \mathbf{V}_1) \cdot \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \eta_2 \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) \cdot \boldsymbol{\mu}_2 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) + \\ \eta_2 \mu_3 |\nabla \times \boldsymbol{\omega}|^2 + \mathcal{Q} \left( \frac{1}{\vartheta_1} - \frac{1}{\vartheta_2} \right) + \chi H_2 + \frac{1}{\vartheta_2^2} \mathbf{h}'' \cdot \nabla \vartheta_2 \geq 0. \quad (92) \end{aligned}$$

Inequalities (91) and (92) are necessary restrictions which must be valid always and for all processes. If we neglect the sticking of the fluid particles to the solid particles, i.e. assume  $\chi = 0$ , then the above introduced restrictions for the viscosities (77), (78) and (81) as well as for the heat conductivities (86), are sufficient conditions to satisfy the inequalities (91) and (92). If  $\chi \neq 0$ , then the inequalities (91) and (92) contain the terms  $\chi H_1$  and

$\chi H_2$ . From the formal point of view it is not obvious that these inequalities are always satisfied. Nevertheless, we guess that even in this case the inequalities (91) and (92) should be valid without essential restrictions.

### 8 The reduced energy balance equation. The Cauchy–Green relations

Using (84) one may rewrite (83) as follows

$$\rho_1 \frac{\delta_1 U_1}{\delta t} + \rho_2 \frac{\delta_2 U_2}{\delta t} + \rho \frac{\delta_m U_{12}}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} + \rho_1 \vartheta_1 \frac{\delta_1 H_1}{\delta t} + \rho_2 \vartheta_2 \frac{\delta_2 H_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t}. \quad (93)$$

The equation of the energy balance written in the form of (93) is termed the reduced energy balance equation. From (93) it is obvious how to define the internal energies  $U_1$ ,  $U_2$  and  $U_{12}$ . The corresponding constitutive equations may be assumed in the simplest form

$$U_1 = U_1(\rho_1, H_1), \quad U_2 = U_2(\rho_2, H_2, z), \quad U_{12} = \text{const.} \quad (94)$$

After substituting (94) into (93) one can derive the Cauchy–Green relations

$$p_1 = \rho_1^2 \frac{\partial U_1}{\partial \rho_1}, \quad p_2 = \rho_2^2 \frac{\partial U_2}{\partial \rho_2}, \quad \vartheta_1 = \frac{\partial U_1}{\partial H_1}, \quad \vartheta_2 = \frac{\partial U_2}{\partial H_2}, \quad \Psi = \frac{\partial U_2}{\partial z}. \quad (95)$$

Instead of internal energies let us introduce the free energies

$$F_1(\rho_1, \vartheta_1) = U_1 - \vartheta_1 H_1, \quad F_2(\rho_2, \vartheta_2, z) = U_2 - \vartheta_2 H_2. \quad (96)$$

The reduced energy balance equation (93) takes now the form

$$\rho_1 \frac{\delta_1 F_1}{\delta t} + \rho_2 \frac{\delta_2 F_2}{\delta t} = \frac{p_1}{\rho_1} \frac{\delta_1 \rho_1}{\delta t} + \frac{p_2}{\rho_2} \frac{\delta_2 \rho_2}{\delta t} - \rho_1 H_1 \frac{\delta_1 \vartheta_1}{\delta t} - \rho_2 H_2 \frac{\delta_2 \vartheta_2}{\delta t} + \rho_2 \Psi \frac{\delta_2 z}{\delta t}. \quad (97)$$

The Cauchy–Green relations (95) can be transformed as follows

$$p_1 = \rho_1^2 \frac{\partial F_1}{\partial \rho_1}, \quad p_2 = \rho_2^2 \frac{\partial F_2}{\partial \rho_2}, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2}, \quad \Psi = \frac{\partial F_2}{\partial z}. \quad (98)$$

From the last equation in (98) we may conclude that the function  $\Psi$  plays the role of the chemical potential. In addition, from the last equations in (95) and (98) it is obvious that the entropy  $H_2$  does not depend on the variable  $z$ . In fact, we have

$$\Psi = \frac{\partial F_2}{\partial z} = \frac{\partial U_2}{\partial z} \Rightarrow \frac{\partial \vartheta_2 H_2}{\partial z} = 0 \Rightarrow \frac{\partial H_2}{\partial z} = 0.$$

Let us note that the representation of the function  $\Psi$  as the derivative of the internal energy is in fact a restriction imposed on the dependence of the internal energy on the variable  $z$ . Indeed, accordingly to (75) and (95) we have

$$\frac{\partial U_2}{\partial z} = \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - \rho_2 \frac{\partial U_2}{\partial \rho_2} - U_2 \right) - \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \rho_1 \frac{\partial U_1}{\partial \rho_1} - U_1 \right). \quad (99)$$

Equation (99) is the partial differential equation for the internal energy. It may be rewritten in the following equivalent form

$$\frac{\partial U_2}{\partial z} + \frac{\partial U_2}{\partial x} = \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2 \right) - \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \rho_1 \frac{\partial U_1}{\partial \rho_1} - U_1 \right), \quad (100)$$

where  $x \equiv \ln(\rho_2/\rho_2^0)$ . If we introduce the new variables

$$\alpha = \frac{z+x}{2} = \ln \left[ \frac{\rho_2}{\rho_2^0} \sqrt{\frac{\eta_2^0}{\eta_2}} \right], \quad \beta = \frac{z-x}{2} = \ln \sqrt{\frac{\eta_2^0}{\eta_2}}, \quad z = \ln \frac{\rho_2 \eta_2^0}{\rho_2^0 \eta_2}, \quad x = \ln \frac{\rho_2}{\rho_2^0}, \quad (101)$$

then instead of (100) we obtain

$$\frac{\partial U_2}{\partial \alpha} = \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} - U_2 \right) - \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \rho_1 \frac{\partial U_1}{\partial \rho_1} - U_1 \right), \quad (102)$$

where the internal energy  $U_2$  must be considered as a function of  $\alpha$ ,  $\beta$ ,  $H_2$ . Furthermore, this function must satisfy the condition (102). Now the Cauchy–Green relations (95) take the form

$$p_1 = \rho_1^2 \frac{\partial U_1}{\partial \rho_1}, \quad p_2 = \rho_2 \frac{\partial U_2}{\partial \alpha}, \quad \vartheta_1 = \frac{\partial U_1}{\partial H_1}, \quad \vartheta_2 = \frac{\partial U_2}{\partial H_2}, \quad \Psi = \frac{1}{2} \frac{\partial U_2}{\partial \alpha} + \frac{1}{2} \frac{\partial U_2}{\partial \beta}. \quad (103)$$

In terms of the free energy the Cauchy–Green relations may be rewritten as

$$p_1 = \rho_1^2 \frac{\partial F_1}{\partial \rho_1}, \quad p_2 = \rho_2 \frac{\partial F_2}{\partial \alpha}, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2}, \quad \Psi = \frac{1}{2} \frac{\partial F_2}{\partial \alpha} + \frac{1}{2} \frac{\partial F_2}{\partial \beta}. \quad (104)$$

Let us recall that the pressure  $p_2$  characterizes the interaction between the fibers. For suspensions under consideration we can assume  $p_2 = 0$ . In this case we can observe from (104) that  $U_2$  and  $F_2$  do not depend on  $\alpha$ . Consequently, the internal energy  $U_2$  can be found from (102) as follows

$$U_2 = \left( \frac{1}{2} \mathbf{V}_2 \cdot \mathbf{V}_2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} \right) - \left( \frac{1}{2} \mathbf{V}_1 \cdot \mathbf{V}_1 - \rho_1 \frac{\partial U_1}{\partial \rho_1} - U_1 \right). \quad (105)$$

Furthermore, instead of (104) we have

$$p_1 = \rho_1^2 \frac{\partial F_1}{\partial \rho_1}, \quad p_2 = 0, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2}, \quad \Psi = \frac{1}{2} \frac{\partial F_2}{\partial \beta}. \quad (106)$$

Here the function  $\Psi$  has the exact meaning of the chemical potential. Let us recall that the chemical potential in physics is defined as the derivative of the free energy with respect to the number of the particles in the system under consideration. However, from (75) and (102) we see that the chemical potential  $\Psi$  is negligibly small. Therefore, the effect of sticking of the fluid particles to the fibres can be ignored. Of course, such a conclusion is valid only within the assumption that the solid particles by itself cannot form a solid body. That means that the distances between the fibres are too large so that the inter-particle forces may be ignored. Thus, we can assume the following representations

$$F_1 = F_1(\rho_1, \vartheta_1), \quad F_2 = F_2(\vartheta_2). \quad (107)$$

Let us introduce a new variable

$$\zeta = \frac{\rho_1^0}{\rho_1} - b, \quad b \equiv \frac{\rho_1^0}{\rho_1^*} \simeq 0.7 \div 0.9; \quad \zeta = 0 \quad \Rightarrow \quad \rho_1 = \rho_1^*, \quad (108)$$

where  $b$  is an empirical constant,  $\rho_1^*$  is the upper limit of the mass density of the first constituent and  $\rho_1^0$  is the corresponding equilibrium mass density at  $p = 0$  and  $\vartheta_1 = 0$ . Of course, we do not take into account the quantum effects. That means that we consider the case when the temperature is far from the absolute zero. With the variable  $\zeta$  (106) can be written as follows

$$p_1 = -\frac{\partial \rho_1^0 F_1}{\partial \zeta}, \quad p_2 = 0, \quad H_1 = -\frac{\partial F_1}{\partial \vartheta_1}, \quad H_2 = -\frac{\partial F_2}{\partial \vartheta_2}, \quad \Psi = 0. \quad (109)$$

The parameter  $\rho_1^0$  will be discussed in more details in the next section. Finally, let us emphasize that from our conclusion it does not follow that the function  $\chi$  in (50) may be ignored.

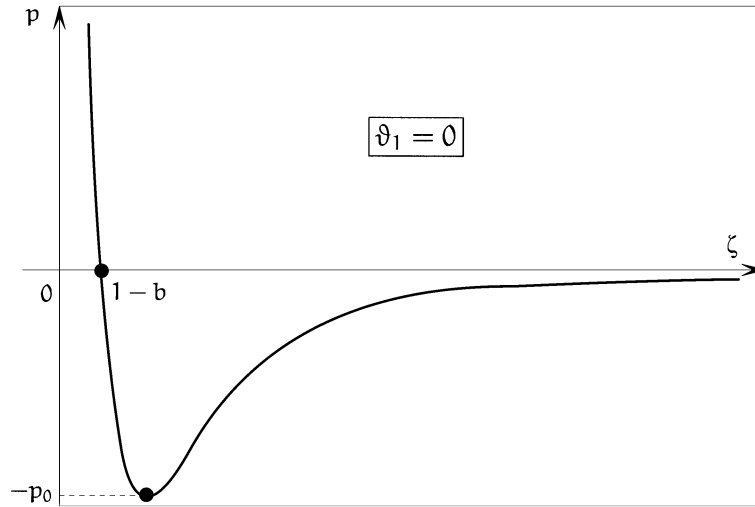


### 9 Constitutive equation for the pressure

The constitutive equation for the pressure inside the fluid must be formulated based on known experimental facts. The commonly used approach in models of the suspension flow by the filling is the application of the incompressibility condition, e.g. [4,3,7,10,15,39], among others. Furthermore, as far as we know, the stage of the solidification was not examined in theoretical works. If we desire to consider solidification, then we should modify the model of an incompressible fluid. The starting point is the constitutive equation for the pressure proposed in [44]

$$p = p_0 \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ \left(\frac{1-b}{\zeta}\right)^m - \left(\frac{1-b}{\zeta}\right)^n \right] + \frac{c\vartheta_1}{\zeta^k}, \quad 1 < k < n < m, \quad p_0 > 0, \quad (110)$$

where  $p \equiv p_1$  is the pressure in the species 1 and the constant parameters  $p_0, m, n, k, c, b$  should be identified experimentally. From the physical point of view it is clear that the constants  $m, n, k$  must be odd integers.



**Fig. 3.** Qualitative variation of the pressure in the solid phase

Let us discuss the main features of the constitutive equation (110). First of all if the temperature  $\vartheta_1 = 0$ , then for the pressure  $p = 0$  we have  $\zeta = 1 - b$  or  $\rho_1 = \rho_1^0$ , see Fig. 3. Thus, the mass density  $\rho_1^0$  corresponds to the stable equilibrium state of the material in the solid state. The meaning of the constant  $p_0$  in (110) follows from the expression

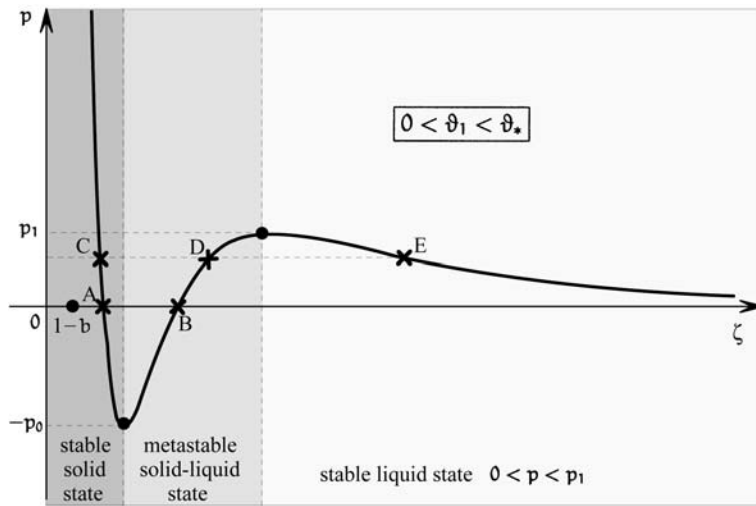
$$p_{min} = -p_0, \quad \vartheta_1 = 0.$$

Thus,  $p_0$  is the tensile strength of the material in the solid state at low temperature. It is important to note that the material under consideration has a finite tensile strength. If the value of the pressure is less than  $(-p_0)$ , then the material fails. If the value is higher than  $(-p_0)$ , then the material may exist only in the solid state.

Let us examine the case when  $0 < \vartheta_1 < \vartheta_*$ , where the temperature  $\vartheta_*$  will be introduced later. The pressure diagram corresponding to (110) is shown in Fig. 4. Here we have two equilibrium states with the normalized densities  $\zeta_1$  and  $\zeta_2$  denoted by the points A and B, respectively,  $(1 - b < \zeta_1 < \zeta_2)$ .  $\zeta_1$  and  $\zeta_2$  can be calculated as the roots of the following equation

$$p_0 \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ \left(\frac{1-b}{\zeta}\right)^m - \left(\frac{1-b}{\zeta}\right)^n \right] + \frac{c\vartheta_1}{\zeta^k} = 0, \quad 0 < \vartheta_1 < \vartheta_*. \quad (111)$$

The first root  $\zeta_1$  corresponds to a stable equilibrium state of the material in the solid phase. The second one  $\zeta_2$  corresponds to an unstable equilibrium state of the material. The first zone in Fig. 4 corresponds to the stable solid phase of the material. The pressure within this zone is determined by the constitutive equation (110) and the dependence of the pressure on the density should be verified experimentally. The second zone in Fig. 4



**Fig. 4.** Qualitative variation of the pressure for  $0 < \vartheta_1 < \vartheta_*$

corresponds to the so-called metastable state of the material. Within this zone the material behavior is determined by the equations of motion rather than by the constitutive equation. Let us underline that within this zone there is no static solution or, what is the same, there is an infinite number of static solutions. Furthermore, within this zone we have a mixture of two phase states of the material: the liquid and the solid one. The third zone in Fig. 4 corresponds to the stable liquid phase of the material. In this phase the material can exist only for the pressure lying within the interval  $0 < p < p_1$ , where  $p_1$  is marked in Fig. 4. Let us assume that the diagram presented in Fig. 4 corresponds to the temperature of polymerization  $\vartheta_p$  at the pressure  $p_1$ . Let the temperature  $\vartheta_p$  be constant and the pressure  $p$  is less than  $p_1$ . In such a case we have three equilibrium states denoted by points C, D and E in Fig. 4. Two of these states (points C, E) are stable and the equilibrium state in D is unstable. What state will be realized depends on the initial conditions.

If the temperature  $\vartheta_1$  increases from 0 up to a value  $\vartheta_1 < \vartheta_*$ , then the tensile strength  $p_\vartheta$  of the material decreases to

$$p_\vartheta = -\frac{p_0}{k} \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ (m-k) \left(\frac{1-b}{\zeta_s(\vartheta_1)}\right)^m - (n-k) \left(\frac{1-b}{\zeta_s(\vartheta_1)}\right)^n \right], \quad (112)$$

where  $\zeta_s(\vartheta_1)$  is the least root of the following equation

$$p_0 \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ m \left(\frac{1-b}{\zeta_s}\right)^m - n \left(\frac{1-b}{\zeta_s}\right)^n \right] + k \frac{c\vartheta_1}{\zeta_s^k} = 0, \quad 0 < \vartheta_1 < \vartheta_*. \quad (113)$$

The tensile strength  $p_\vartheta$  of the material should be found experimentally. The polymerization of the suspended fluid is possible only if  $\vartheta_1 \leq \vartheta_*$ .

Now let us determine the critical temperature  $\vartheta_*$ . The case  $\vartheta = \vartheta_*$  is presented in Fig. 5. Here we have only one equilibrium state at the zero pressure. The material has three different liquid phases. If the pressure  $p$  lies within the range  $0 < p < p_1$ , then the material has two different liquid states. The first and the third zone in Fig. 5 correspond to the two different stable liquid phases. The second, intermediate zone characterizes an unstable state which corresponds to a mixture of two different liquid phases. If the pressure  $p$  is higher than  $p_1$ , then we have only one liquid phase. In order to find the density  $\zeta_*$  corresponding to this state and the critical temperature  $\vartheta_*$  we have to solve the following system of equations

$$\begin{aligned} p_0 \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ \left(\frac{1-b}{\zeta_*}\right)^m - \left(\frac{1-b}{\zeta_*}\right)^n \right] + \frac{c\vartheta_*}{\zeta_*^k} &= 0, \\ p_0 \frac{n}{m-n} \left(\frac{m}{n}\right)^{\frac{n}{m-n}} \left[ m \left(\frac{1-b}{\zeta_*}\right)^m - n \left(\frac{1-b}{\zeta_*}\right)^n \right] + k \frac{c\vartheta_*}{\zeta_*^k} &= 0. \end{aligned} \quad (114)$$

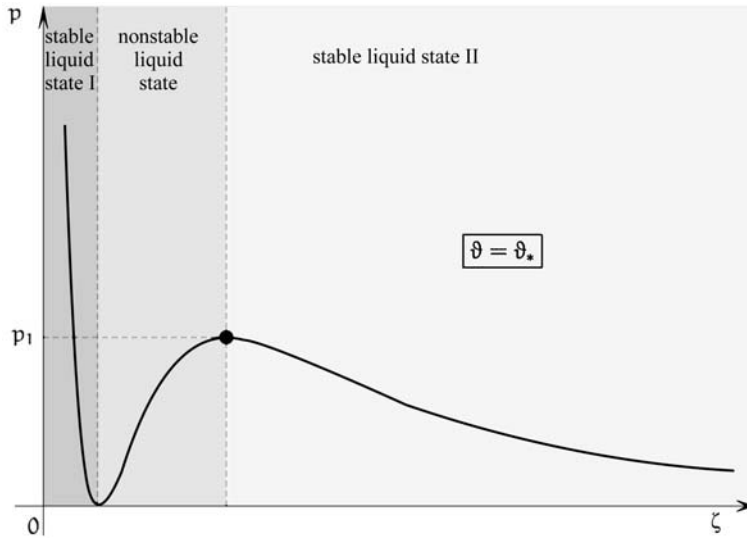


Fig. 5. Qualitative variation of the pressure for  $\vartheta = \vartheta_*$

The solution of this system may be found as

$$\zeta_* = (1 - b) \left( \frac{m - k}{n - k} \right)^{\frac{1}{m-n}}, \quad \frac{c\vartheta_*}{(1 - b)^k} = p_0 \frac{n}{m - k} \left( \frac{m}{n} \right)^{\frac{n}{m-n}} \left( \frac{n - k}{m - k} \right)^{\frac{n-k}{m-n}}. \quad (115)$$

Equations (115) can be used in order to find the constants  $m, n, k$  since the quantities  $\zeta_*$  and  $\vartheta_*$  are experimentally measurable. Let us emphasize that if the temperature  $\vartheta_1$  is higher than  $\vartheta_*$ , then according to the constitutive equation (110) the solidification of the material is impossible. The critical temperature  $\vartheta_*$  may be termed as the melting temperature.

Figure 6 shows the pressure dependence for the case  $\vartheta_* < \vartheta_1 < \vartheta_{**}$ . Within this temperature range the material may exist in two liquid phase states. If the temperature  $\vartheta_1$  is higher than  $\vartheta_{**}$ , then the material has only one liquid phase. Let us recall that we do not consider here the gaseous phase of the material. The difference between gas and liquid is that for gas the attractive force decreases more slowly with increasing of  $\zeta$ . The temperature  $\vartheta_{**}$  may be found from the equations

$$\frac{dp}{d\zeta} = 0, \quad \frac{d^2p}{d\zeta^2} = 0.$$

By use of (109) and (110) we obtain the expression of the free energy

$$\rho_1^0 F_1 = F_0 \left[ -\frac{\zeta}{m-1} \left( \frac{1-b}{\zeta} \right)^m + \frac{\zeta}{n-1} \left( \frac{1-b}{\zeta} \right)^n \right] - \frac{1}{k-1} \frac{c\vartheta_1}{\zeta^{k-1}} + \psi(\vartheta_1), \quad (116)$$

where the function  $\psi(\vartheta_1)$  must be specified for the given material. One may find the function  $\psi(\vartheta_1)$  from the following equation

$$-\frac{\partial\psi(\vartheta_1)}{\partial\vartheta_1} = c_\varepsilon \ln \frac{\vartheta_1}{\vartheta_0},$$

where  $c_\varepsilon$  is the heat capacity at constant strains.

## 10 The final system of equations

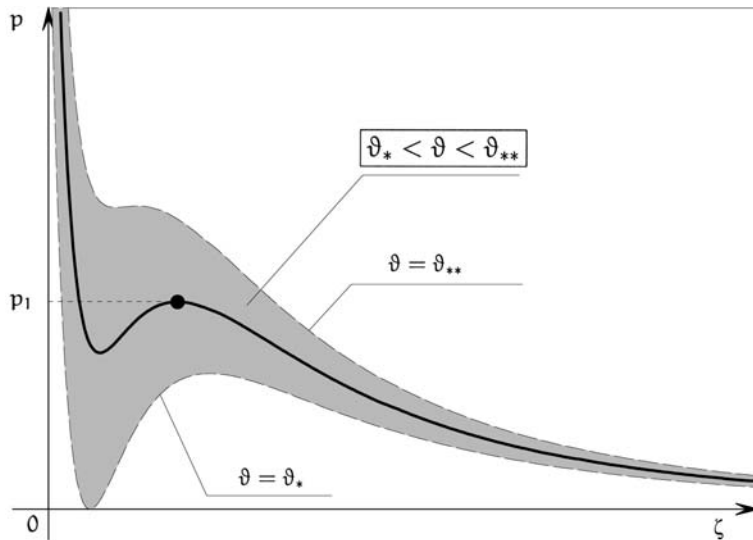
The basic unknowns of the considered problem are  $\rho_1$  and  $\eta_2$ ,  $\rho_2$ . For these functions we have the set of equations (50)

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = -\chi, \quad \frac{\delta_2 \eta_2}{\delta t} + \eta_2 \nabla \cdot \mathbf{V}_2 = 0, \quad \frac{\delta_2}{\delta t} \ln \frac{\rho_2}{\eta_2} = \frac{\chi}{\rho_2}, \quad (117)$$

where the function  $\chi$  should be specified. In the simplest case it is possible to assume that  $\chi = 0$ .

The equations of motion of the liquid constituent are given by the first equation from (55). Taking into account the constitutive equation (76) we obtain

$$-\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \mathbf{D}) + \nabla \times \left[ \eta_2 \boldsymbol{\mu}_1 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) \right] + \eta_2 \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_2 - \mathbf{V}_1) + \rho_1 \mathbf{F}_1 = \rho_1 \frac{\delta_1 \mathbf{V}_1}{\delta t} - \chi \mathbf{V}_1. \quad (118)$$



**Fig. 6.** Qualitative variation of the pressure for  $\vartheta_* < \vartheta_1 < \vartheta_{**}$

The equations of the translation motion of the solid-liquid constituent take the form

$$-\nabla p_2 + \nabla \times \left[ \eta_2 \boldsymbol{\mu}_2 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) \right] + \eta_2 \boldsymbol{\mu}_{12} \cdot (\mathbf{V}_1 - \mathbf{V}_2) + \rho_2 \mathbf{F}_2 = \rho_2 \frac{\delta_2 \mathbf{V}_2}{\delta t} + \chi \mathbf{V}_2, \quad (119)$$

where the partial pressure  $p_2$  may be assumed to be zero, or may be defined by a constitutive equation like (110).

Equation (61) for the spinor motion of fibres can be written as follows

$$\begin{aligned} -\eta_2 \boldsymbol{\mu}_3 \nabla \times (\nabla \times \boldsymbol{\omega}) - 2\eta_2 \boldsymbol{\mu}_2 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_2 \right) - 2\eta_2 \boldsymbol{\mu}_1 \cdot \left( \boldsymbol{\omega} - \frac{1}{2} \nabla \times \mathbf{V}_1 \right) + \rho_2 \mathbf{L} \\ = \rho_2 \frac{\delta_2}{\delta t} (\mathbf{J} \cdot \boldsymbol{\omega}) + \chi (\mathbf{J} \cdot \boldsymbol{\omega}). \end{aligned} \quad (120)$$

To the above equations (117)–(120) we have to add the heat conduction equations (85) together with constitutive equations (86) in which it is possible to assume that  $\kappa_2 = 0$ .

In order to obtain the final statement of the basic equations we have to determine the volume forces  $\rho_1 \mathbf{F}_1$ ,  $\rho_2 \mathbf{F}_2$  and the volume moment  $\rho_2 \mathbf{L}$ . Partly they are determined by the external fields, usually it is the gravity

field, which does not create the external moment  $\rho_2 \mathbf{L}$ . Besides, the volume forces and moments may arise due to the boundary walls. Let us suppose, that the boundary of the mold cavity  $V$  is bounded by the surface  $S$ . Let this surface consists of two parts  $S = S_0 \cup S_1$ , where  $S_0$  is the rigid wall and  $S_1$  is an inlet of the cavity through which the mixture flows into the cavity  $V$ . Let  $\mathbf{n}$  be a unit normal vector to  $S$ , directed towards the domain  $V$ . Let  $s$  be a distance along the normal  $\mathbf{n}$ . Let us assume that the influence of the wall may be described in terms of the external force field which must be considered by means of the volume forces. Let us specify the volume forces as follows

$$\mathbf{L} = \mathbf{0}, \quad \mathbf{F}_1 = \mathbf{F}_2 = \mathbf{g} + F_0 \left[ \left( \frac{s}{l} \right)^{-p} - \left( \frac{s}{l} \right)^{-q} \right] \mathbf{n}, \quad p > q > 0, \quad s \geq 0, \quad (121)$$

where  $l > 0$  is a very small constant having the dimension of length. In general, the wall creates the moment acting on the rigid particles. The mixture in fluid state is inserted into the cavity through the inlet  $S_1$  and occupies some domain  $V_*$  which is changing in time. The boundary of  $V_*$  is the surface  $S_*$  that consists of three parts  $S_* = S_1 \cup S_0^* \cup S_f$ , where  $S_0^*$  is those part of  $S_0$  which is in contact with the mixture and  $S_f$  is the free surface of the mixture. For all these surfaces we have to state the boundary conditions which may be prescribed in the conventional form for the velocities and the pressure.

## 11 Discussion

The aim of this paper was to derive the governing equations describing the flow of fiber suspensions within the framework of the micro-polar model of a binary medium. In a forthcoming paper the numerical solution of the proposed equations as well as some illustrating examples will be presented.

As a discussion let us compare our approach with the existing theories. For this purpose we will consider simplifying assumptions as being made within the existing theories. Let us proceed by several steps.

The first step is the simplest one. Let us assume that

$$\vartheta_1 = \vartheta_2 = \text{const}, \quad \chi = 0. \quad (122)$$

In this case from (117) follows

$$\frac{\delta_1 \rho_1}{\delta t} + \rho_1 \nabla \cdot \mathbf{V}_1 = 0, \quad \frac{\delta_2 \rho_2}{\delta t} + \rho_2 \nabla \cdot \mathbf{V}_2 = 0 \quad \Rightarrow \quad \frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{V}_m) = 0, \quad (123)$$

where  $\rho = \rho_1 + \rho_2$  and  $\rho \mathbf{V}_m = \rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2$ .

The next assumption of the conventional theory is that there is no sliding between the rigid particles and the fluid. From the physical point of view that means that the norms of the viscous friction tensors strive to infinity

$$\|\boldsymbol{\mu}_1\| \rightarrow \infty, \quad \|\boldsymbol{\mu}_2\| \rightarrow \infty, \quad \|\boldsymbol{\mu}_{12}\| \rightarrow \infty. \quad (124)$$

Since the norms of vectors  $\mathbf{t}'$ ,  $\mathbf{t}''$  and  $\mathbf{Q}$  must be limited then from the constitutive equations (76), (80) and (82) we obtain the following restrictions

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}_m \equiv \mathbf{V}, \quad 2\boldsymbol{\omega} = \nabla \times \mathbf{V}, \quad (125)$$

which are usually assumed except, may be, the last condition. However, this condition must be valid too. In this case instead of equations of motion (118) – (120) we shall get

$$-\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \mathbf{D}) + \nabla \times \mathbf{t}' + \mathbf{Q} + \rho_1 \mathbf{F}_1 = \rho_1 \frac{\delta \mathbf{V}}{\delta t},$$

$$-\nabla p_2 + \nabla \times \mathbf{t}'' - \mathbf{Q} + \rho_2 \mathbf{F}_2 = \rho_2 \frac{\delta \mathbf{V}}{\delta t}, \quad p_2 \simeq 0,$$

$$-\eta_2\mu_3\nabla \times (\nabla \times \boldsymbol{\omega}) - 2\mathbf{t} + \rho_2\mathbf{L} = \rho_2\frac{\delta}{\delta t}(\mathbf{J} \cdot \boldsymbol{\omega}), \quad (126)$$

where the vectors  $\mathbf{t}'$ ,  $\mathbf{t}''$ ,  $\mathbf{Q}$  are not defined by constitutive equations any more. We may rewrite this system as a set of two equations

$$\begin{aligned} -\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \mathbf{D}) + \nabla \times \mathbf{t} + \rho\mathbf{F} &= \rho\frac{\delta\mathbf{V}}{\delta t}, \quad 2\boldsymbol{\omega} = \nabla \times \mathbf{V}, \\ -\eta_2\mu_3\nabla \times (\nabla \times \boldsymbol{\omega}) - 2\mathbf{t} + \rho_2\mathbf{L} &= \rho_2\frac{\delta}{\delta t}(\mathbf{J} \cdot \boldsymbol{\omega}). \end{aligned} \quad (127)$$

The latter equation allows to express the vector  $\mathbf{t}$  in terms of the velocity  $\mathbf{V}$  and its derivatives. Generally speaking, further simplifications are not possible. However, (127) are more complicated than those used in the conventional theory.

To proceed with one more step forward to the conventional theory we have to assume that

$$\mu_3 = 0, \quad \mathbf{L} = \mathbf{0}, \quad \mathbf{J} = \mathbf{0} \quad \Rightarrow \quad \mathbf{t} = \mathbf{0}. \quad (128)$$

The first of these restrictions is the hypothesis of the dilute mixture. The third one is the hypothesis of the inertialess particles. Now we have the following equations

$$-\nabla p_1 + 2\nabla \cdot (\boldsymbol{\mu} \cdot \mathbf{D}) + \rho\mathbf{F} = \rho\frac{\delta\mathbf{V}}{\delta t}, \quad \boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{V}, \quad (\nabla \cdot \boldsymbol{\omega} = 0). \quad (129)$$

If we introduce a unit vector  $\mathbf{m}(t)$  connected to the fiber symmetry axis, then we obtain

$$\frac{\delta\mathbf{m}(t)}{\delta t} = \frac{1}{2}(\nabla \times \mathbf{V}) \times \mathbf{m}(t) = \frac{1}{2}(\mathbf{m} \cdot \nabla \mathbf{V} - \nabla \mathbf{V} \cdot \mathbf{m}). \quad (130)$$

The last, but not least restriction is that of incompressibility

$$\nabla \cdot \mathbf{V} = 0, \quad \frac{\delta\rho}{\delta t} = 0. \quad (131)$$

This condition means that the pressure is not defined by the constitutive equation anymore. It must be found from the equations of motion. Thus, the phase transitions in the mixture must be ignored. Let us note, that the condition  $\rho = \text{const}$  does not follow from (131).

Thus, after all assumptions we have obtained the set of equations (129)–(131). We have to add to these equations the statement that the viscous tensor  $\boldsymbol{\mu}$  is a transversely isotropic function of the unit vector  $\mathbf{m}$  and the tensor  $\mathbf{D}$ . We do not discuss this question since the solution of this problem is well-known. It is easy to see that the set of equations (129)–(131) does not correspond exactly to that in the conventional theory (see the Introduction to this paper). The only difference is that (130) does not coincide with the (13). In order to see this difference more clearly let us rewrite (13) in the equivalent form

$$\dot{\mathbf{m}} = \left( \frac{1}{2}\nabla \times \mathbf{V} + \frac{c^2 - a^2}{c^2 + a^2}\mathbf{m} \times \mathbf{d} \cdot \mathbf{m} \right) \times \mathbf{m}, \quad \mathbf{d} = \frac{1}{2}(\nabla\mathbf{V} + \nabla\mathbf{V}^T). \quad (132)$$

What equation ((130) or (132)) is more right? We do not know the exact answer. On one hand, (132) seems to be more preferable since it contains the parameters of the particle. On the other hand, this fact seems to be strange if we assume that there is no sliding between the solid particle and the fluid. Consequently the rotations of the solid particle and the fluid must be the same. That means that the shape of a particle should not be important. This is true for (130) but it is not true for (132). Let us note that (130) may be derived by many ways.

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