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Analysis and optimization of energy flows in structures composed of beam elements – Part I: problem formulation and solution technique

S.V. Sorokin, J.B. Nielsen and N. Olhoff

Abstract The paper addresses the analysis of in-plane coupled flexural and longitudinal vibrations of a structure composed of elastic tubular beams. Emphasis is put on optimization of structural performance in terms of minimization of total power flow over a broad-banded frequency range, i.e. minimization of the emitted/transmitted energy of vibrations. The objective function is chosen as an energy outflow (a structural intensity) integrated within the given frequency range at a given remote point (cross-section) of the tubular structure. To gain insight into the physical mechanisms of energy transportation, the structure is decomposed into a set of elementary dynamical systems. These elementary dynamical systems (subsystems) are chosen as one-dimensional wave-guides carrying either longitudinal or flexural waves. Vibrations of each dynamical subsystem are described by a boundary equation method, a novel method in structural dynamics which ideally fits the substructuring concept since it deals with physical variables at boundaries between substructures, no matter what type of excitation is applied at individual substructures. A system of boundary integral equations is accomplished by continuity conditions at interfaces between subsystems and by boundary conditions. Since Green's functions used in the boundary integral formulation satisfy the radiation principle, the governing system of equations is equally applicable for analysis of vibrations of both finite-length structures and the structures having infinitely long elements jointed with elements of finite length.

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1 Introduction

A control of the energy transmission from an excitation point to remote parts of complex structures is an important issue in many technical applications. A simple example of such an application is the problem of design of a "quiet" pipeline for a house heating system. Analysis of its vibrations at an arbitrary frequency requires the use of sophisticated models of structural dynamics, e.g. thin shell theory or three-dimensional theory of elasticity as well as refined models of fluid-structure interaction. Nevertheless, in a not too high frequency range, the pipeline may adequately be modelled as a structure composed of long tubular beam elements. At the "zero-order" approximation a fluid's role is reduced to its efficient added mass in flexural motions of a structure which may be co-opted into consideration by an "effective" wall thickness of tubular elements. The analysis of vibrations of structures composed of beam elements has a long history and many examples may be found in textbooks on structural dynamics or numerical methods in engineering (see, e.g. Meirovitch 1990; Petyt 1990; Doyle 1997). Any commercial finite element code deals with calculation of eigenfrequencies and eigenmodes of vibration of complex beam structures, see for example Swanson Analysis Systems Inc. (1992), so now this issue does not constitute a scientific problem. However, a detailed description of the transportation of vibration energy from a source of excitation through the structure and of energy exchanges between elements of a structure has gained less attention in the literature. This aspect of a theory of vibrations of complex structures is more relevant to statistical energy analysis (Lyon 1975) and in most cases presents serious difficulties. In particular, it is a practical difficulty that parameters characterising energy flows cannot be easily obtained by direct applying any standard finite element software. A better understanding of energy transportation phenomena is possible if a complex beam structure is decomposed into a set of substructures and their interaction is considered. This approach, common in statistical energy analysis, has been applied for investigations into vibrations of structures of a finite length by several authors (see, e.g. Shankar and Keane 1995a,b; Ren and Beards 1995). They considered structures composed of beam elements and used a standard finite element technique for constructing free-free modes of each substructure. Then a set of free-free modes was used for decomposing dynamic responses of each substructure and analysis of energy flows between them. Unfortunately, analysis of vibrations of substructures by the technique suggested by Shankar and Keane (1995a) and Shankar and Keane (1995b) becomes complicated for complex structures with semi-infinite substructures, because for these substructures the solution has the form of a travelling wave rather than a standing wave as adopted in finite element decomposition. In a standard finite element formulation of structural dynamics, some impedance conditions are imposed on vibrations of a semi-infinite substructure at a certain distance from its interface with the others to deal with travelling waves (see Meirovitch 1990; Petyt 1990). Alternatively, the spectral element method (see Doyle 1997) may also be used for formulating a system of governing equations for such a substructure.

In the present paper, another description of structural dynamics is used. It is based on the boundary integral equations method, which for a structure composed of beam elements makes it possible to set up a simple system of linear algebraic equations that gives an exact solution at each excitation frequency (see Sorokin 1993). This method, which is well-known in structural acoustics, has gained less attention in structural vibrations. However, for structures composed of "simple" substructures (as the model considered here) this method is the most efficient one, not only for calculation of amplitudes of displacements (in a study of forced vibrations), but also for the analysis of energy flows in space and frequency domains. Although it is quite close to the spectral element method in the sense that it utilizes exact analytical solutions for propagating waves, it differs from the above method as the nodal interfacial points are introduced only at "physical" boundaries of the substructures which may be arbitrarily loaded inside each substructure. This feature of the boundary equation method becomes very important in the optimization process when design variables include the location of dampers and stiffeners as well as damping and stiffness parameters themselves. This general approach is illustrated here with a relatively simple model system specified in the following section. The optimization of such a simple system is done by way of an extensive parametric study rather than by use of advanced multi-parameter numerical algorithms of optimization (see, e.g. the comprehensive textbook by Rozvany 1976). This approach is aimed to gain a better physical understanding of energy flows in complex structures.

The analysis is restricted to calculations of structural intensities of a tubular beam structure, strictly speaking, with no fluid loading (or in effect, a tubular structure loaded by an incompressible immobile fluid). As is well-known (see, e.g. Fuller and Fahy 1982; Norton 1986; Pavic 1993), the presence of a flowing fluid enhances dynamics of the system, as does modelling of the structure in the framework of shell theory. However, the references cited are devoted to dynamics of infinitely long straight tubes and no optimization aspects are introduced, whereas in the present paper attention is focused at substructuring of the pipeline and optimization of its performance. It is also appropriate to note that the method used here is equally applicable for analysis of vibrations of fluid conveying tubes in the framework of a general shell theory. These issues will be approached in subsequent publications.

2
A model system and its substructuring

Consider a beam structure composed of several tubular elements connected as shown in Fig. 1. The whole structure may be either of finite length with arbitrary boundary conditions formulated at its edges (Fig. 1a) or it may be of semi-infinite length, as it is shown in Fig. 1b. The analysis is restricted by the planar formulation of a problem, where each element may perform planar flexural and longitudinal motions. These elements (see I, II and III in Fig. 1) constitute physical substructures. A set of four coupling "terminal" points (see 1, 2, 3, 4 in Fig. 1) is introduced in order to model several supports and/or connections of this structure with other elements characterised by their "lumped" stiffness, mass and damping parameters. Actually, at each of these coupling points a "hidden" substructure (treated as a one-degree-of-freedom system) is attached. To describe dynamics of this model structure, six "elementary" subsystems (two subsystems per each tubular element) is introduced as a set of elastic wave-guides carrying either flexural or longitudinal waves. Each of these continuous subsystems is defined by the differential equation of either bending or axial vibrations of a beam element with relevant boundary conditions and continuity conditions at its interface with other substructures.

Thus, each tubular element is modelled by two subsystems. Subsystems nos. 1, 3 and 5 capture flexural vibrations of the elements (substructures) I, II and III, respectively. Their equations of motion are formulated as

$$E_k I_k \left(1 - i\eta_{wk}\omega\right) w_k^{(4)} - \rho_k A_k \omega^2 w_k = q_{wk}\,,$$

$$k = 1, 3, 5\,. \tag{1}$$

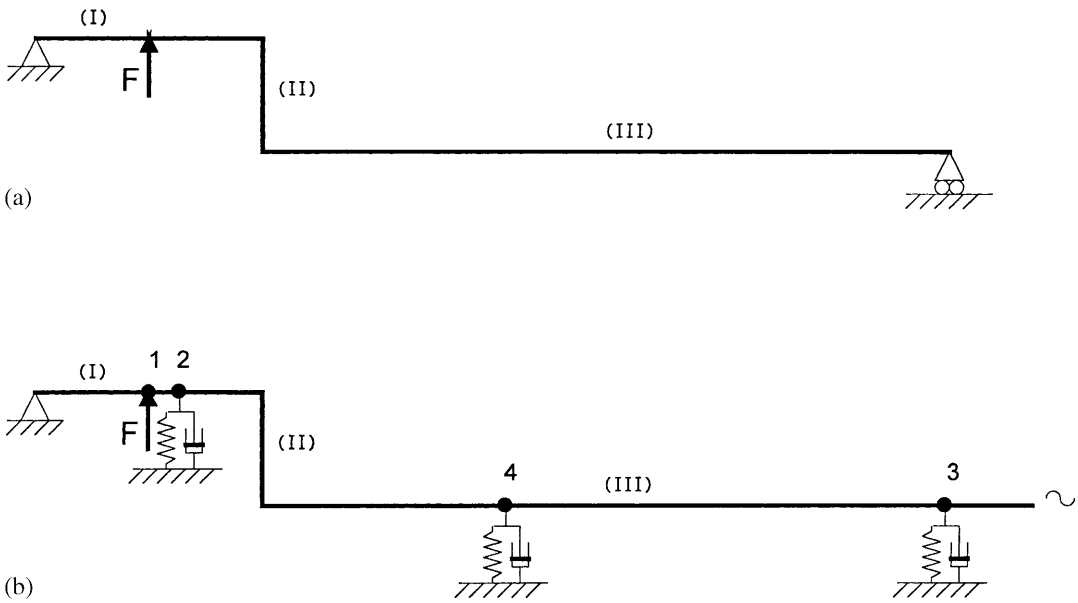

Fig. 1 (a) A model system of finite length. (b) A model system of an infinite length with four terminal points

The time dependence is selected in the form $\exp(-i\omega t)$ and this multiplier is omitted. In (1), $E_k = \rho_k c_k^2$ is Young's modulus of the material of a beam, ρ_k, c_k, η_{wk} are the density, the sound velocity and the internal loss factor in bending vibrations of a beam made of this material, A_k, I_k are the area and the second area moment of inertia of the beam cross-section. Finally, w_k is a flexural displacement and q_{wk} is a transverse driving load.

Longitudinal vibrations of substructures I, II, III are captured by subsystems 2, 4 and 6. Their equations of motions are formulated as

$$E_k A_k \left(1 - i\eta_{uk}\omega\right) u_k'' + \rho_k A_k \omega^2 u_k = -q_{uk} ,$$
$$k = 2, 4, 6 . \tag{2}$$

Here in addition to notations introduced earlier u_k is an axial displacement, q_{uk} is an axial driving load and η_{uk} is the internal loss factor in longitudinal vibrations of a beam made of this material.

At the left edge of the whole structure, three boundary conditions related to substructure I are

$$\chi_{11} u_1(0) + \chi_{12} u_1'(0) = 0 , \quad \chi_{21} w_1(0) + \chi_{22} w_1'''(0) = 0 ,$$

$$\chi_{31} w_1'(0) + \chi_{32} w_1''(0) = 0 . \tag{3}$$

Here arbitrary conditions from a clamped edge to a completely unconstrained edge can be formulated by a proper choice of the set of coefficients χ_{kj}.

The subsystems introduced by differential equations (1)–(2) are connected to each other at the interfacial points, where a set of the continuity conditions for displacements is formulated along with conditions of balance of forces and moments

$$w_1(\ell_1) = -u_2(0) , \; u_1(\ell_1) = w_2(0) , \; w_1'(\ell_1) = w_2'(0) , \tag{4}$$

$$E_1 A_1 u_1'(\ell_1) = -E_2 A_2 w_2'''(0) ,$$

$$E_1 I_1 w_1'''(\ell_1) = E_2 A_2 u_2'(0) ,$$

$$E_1 I_1 w_1''(\ell_1) = E_2 I_2 w_2''(0) , \tag{5}$$

$$w_2(\ell_2) = u_3(0) , \; u_2(\ell_2) = -w_3(0) , \; w_2'(\ell_2) = w_3'(0) , \tag{6}$$

$$E_2 A_2 u_2'(\ell_2) = E_3 I_3 w_3'''(0) ,$$

$$E_2 I_2 w_2'''(\ell_2) = -E_3 A_3 u_3'(0) ,$$

$$E_2 I_2 w_2''(\ell_2) = E_3 I_3 w_3''(0) . \tag{7}$$

If a structure of finite length is considered, then at its right edge boundary conditions (related to substructure III) are formulated similarly to those at the left edge

$$\chi_{41} u_3(\ell_3) + \chi_{42} u_3'(\ell_3) = 0 ,$$

$$\chi_{51} w_3(\ell_3) + \chi_{52} w_3'''(\ell_3) = 0 ,$$

$$\chi_{61} w_3'(\ell_3) + \chi_{62} w_3''(\ell_3) = 0 . \tag{8}$$

In a case when the third beam element is semi-infinite, equations (8) are replaced by Sommerfeld conditions for travelling flexural and longitudinal waves.

A set of four "terminal" coupling points is introduced. These "terminal" points serve to model linking the whole structure with some other substructures, which are not specified in much detail. Thus, it is assumed that dynamic properties of each of these "hidden" substructures at the coupling points are characterised by their "lumped" masses M_k, stiffness K_k and damping coefficients C_k, $k = 1, 2, 3, 4$. For definiteness, two of them are located at the first beam element and two other are placed at the third beam element (see Fig. 1b). Motion of each terminal substructure is governed by a simple equation

$$\left[K_j - i\omega C_j - \omega^2 M_j\right] v_j = \Phi_j \, . \tag{9}$$

Here ν_j is the amplitude of displacement of a terminal point $x = x_{j0}$ and Φ_j is a force transmitted from the considered structure to a terminal substructure.

The model structure defined by (1)–(9) is exposed to a set of transverse and axial driving forces which may be located at its arbitrary cross-sections. Stationary vibrations are considered and initial conditions do not influence the motion of the structure. In a general case, each force should be specified by its frequency spectrum (the continuous frequency-dependent amplitude). In addition, phase relations between forces should also be given with one of them selected as a reference level. Due to linearity of the problem formulation, there is no interaction of motions of a structure occurring at different frequencies. Thus, to formulate the structural response it is sufficient to obtain an analytical solution at a single frequency and then to integrate the results in a frequency domain.

3
Formulation of a problem in optimization

As has been discussed, the structure specified in the previous Section models a pipeline of a heating system. The analysis of vibrations of this structure should be aimed to seek possibilities of reduction of the energy input from the sources of excitation to the pipeline and, therefore, reduction of the structural intensity in its remote part. This is the formulation of an optimization problem with an objective function characterising the stationary energy flows in the structure in a given frequency range (ω_-, ω_+):

$$\Theta = \int_{\omega_-}^{\omega_+} N(\omega) \, \mathrm{d}\omega \, . \tag{10}$$

In (10) $N \left[\frac{\mathrm{Nm}}{\mathrm{s}}\right]$ is the power flow through the control cross-section of the structure at an arbitrary frequency ω.

The objective function $\Theta \left[\frac{\mathrm{Nm}}{\mathrm{s}}\right]$ is subjected to minimization. The choice of position of the control cross-section for the structure shown in Fig. 1b could be rather arbitrary. For example, it could be placed fairly close to an excitation point (when a structure is loaded as shown in Fig. 1 by a force only). Alternatively, the control cross-section could be positioned at the third beam element sufficiently far from the junction point between beams 2 and 3. As will be shown in several examples, almost all the power supplied by a driving force is transported to infinity by travelling waves, so that minimization of the power flow sufficiently far from an excitation zone is basically equivalent to the minimization of the power input. Design variables in an optimization problem may then be selected as co-ordinates and stiffness/mass/damping characteristics of certain intermediate supports ("terminal" points). As is typical for optimization problems, a set of constraints should be imposed on values of design variables. They are related to strength and reliability of a pipeline as well as to technological aspects of its mounting.

It is clear that the efficiency of numerical algorithms used in energy flow calculations becomes highly important when the optimization problem is addressed. Both the finite element method and the spectral element method (briefly discussed in the Introduction) have disadvantages, which make difficult their application to analysis and optimization of energy flows in a system consisting of several substructures and having several "terminal" points. In the finite element analysis, there are basic difficulties of updating the finite element mesh to adequately describe the shape of the vibrations subject to a growth in the excitation frequency. Another complication in the finite element modal analysis is related to the necessity to keep an increasing number of free-free modes as the frequency grows. If spectral elements are used, then within each one of them an exact standing wave solution is utilized to link amplitudes of nodal displacements and amplitudes of nodal forces. Semi-infinite spectral elements are also available, that make it possible to utilize an exact analytical formulation of outgoing waves. However, if several "terminal" coupling points are introduced and their location, mass, stiffness and damping parameters are subjected to optimization, then a mesh of spectral elements should also be updated at each optimization step since no intermediate points are allowed in a spectral element formulation (see Doyle 1997). A boundary equation method is free from these disadvantages and therefore it proves to be particularly efficient in an optimization process when all the above-mentioned design variables are included.

4
Boundary equations

As has been discussed, boundary equations are very convenient to analyse dynamics of a considered model system at each excitation frequency. They are easily derived by use of a reciprocity theorem, which is briefly outlined below. Consider the k-th $(k = 1, 3, 5)$ subsystem to be an infinitely long beam exhibiting flexural vibrations and let

$W_k(x,\xi)$ be a solution to the following problem:

$$E_kI_k\left(1-i\eta_{wk}\omega\right)\frac{\partial^4 W_k(x,\xi)}{\partial x^4}-\rho_kA_k\omega^2W_k(x,\xi)=$$

$$\delta_{wk}(x-\xi)\,. \tag{11}$$

In (11), $\delta_{wk}(x-\xi)$ is a Dirac delta-function (representing the concentrated transverse force) and $W_k(x,\xi)$ is a Green function (it is derived in explicit form in the Appendix). Then multiplication of (11) by an unknown function $w_k(x)$ [or, which is exactly the same, multiplication of (1) by the Green function $W_k(x,\xi)$] and integration by parts gives the following formula (referred to as Somigliana's identity in analogous problems in theory of elasticity)

$$w_k(\xi)=\int_0^{\ell_k} q_{wk}(x)W_k(x,\xi)\,\mathrm{d}x-$$

$$\Bigg[E_kI_kw_k''(x)W_k(x,\xi)-E_kI_kw_k''(x)\frac{\partial W_k(x,\xi)}{\partial x}+$$

$$E_kI_kw_k'(x)\frac{\partial^2 W_k(x,\xi)}{\partial x^2}-E_kI_kw_k(x)\frac{\partial^3 W_k(x,\xi)}{\partial x^3}\Bigg]\Bigg|_{x=0}^{x=\ell_k}. \tag{12}$$

This identity holds for an arbitrary position of the "observation point" ξ including boundaries.

Another identity should be formulated for a slope $w_k'(x)$ (see Sorokin 1993, for details). It is obtained by differentiating (12) with respect to the co-ordinate ξ of the observation point, which gives

$$w_k'(\xi)=\int_0^{\ell_k} q_{wk}(x)\frac{\partial W_k(x,\xi)}{\partial \xi}\,\mathrm{d}x-$$

$$\Bigg[E_kI_kw_k'''(x)\frac{\partial W_k(x,\xi)}{\partial \xi}-E_kI_kw_k''(x)\frac{\partial^2 W_k(x,\xi)}{\partial x\partial \xi}+$$

$$E_kI_kw_k'(x)\frac{\partial^3 W_k(x,\xi)}{\partial x^2\partial \xi}-E_kI_kw_k(x)\frac{\partial^4 W_k(x,\xi)}{\partial x^3\partial \xi}\Bigg]\Bigg|_{x=0}^{x=\ell_k}. \tag{13}$$

A similar identity is valid for longitudinal vibrations. Consider the k-th ($k=2,4,6$) subsystem to be an infinitely long rod and let $U_k(x,\xi)$ be a solution to the following problem:

$$E_kA_k\left(1-i\eta_{uk}\omega\right)\frac{\partial^2 U_k(x,\xi)}{\partial x^2}-\rho_kA_k\omega^2U_k(x,\xi)=$$

$$-\delta_{wk}(x-\xi)\,.$$

In this equation, $\delta_{wk}(x-\xi)$ is a Dirac delta-function representing now the concentrated longitudinal force and $U_k(x,\xi)$ is a Green function which is also derived in explicit form in the Appendix. The differential operator in this equation is of the second order and integration by parts is performed only twice to obtain the formula

$$u_k(\xi)=\int_0^{\ell_k} q_{uk}(x)U_k(x,\xi)\,\mathrm{d}x+$$

$$\Bigg[E_kA_ku_k'(x)U_k(x,\xi)-E_kA_ku_k(x)\frac{\partial U_k(x,\xi)}{\partial x}\Bigg]\Bigg|_{x=0}^{x=\ell_k}. \tag{14}$$

This identity is held for an arbitrary position of the "observation point" ξ including boundaries.

As the observation point ξ tends firstly to the edge $x=0$ and then to $x=\ell_k$, six algebraic equations follow from identities (12)–(14) applied to each beam element. These equations contain twelve unknown quantities: $u_k(x)$, $x=0$, $x=\ell_k$, $u_k'(x)$, $x=0$, $x=\ell_k$, $w_k(x)$, $x=0$, $x=\ell_k$, $w_k'(x)$, $x=0$, $x=\ell_k$, $w_k''(x)$, $x=0$, $x=\ell_k$, $w_k'''(x)$, $x=0$, $x=\ell_k$, $k=1,2,3$. In a case when no terminal points are introduced, (12)–(14) completely describe forced vibrations of a structure excited by a set of given driving loads at a frequency ω. These equations are equally applicable for analysis of forced vibrations of both finite structures and structures of an infinite extent as well as for calculations of eigenfrequencies of finite structures.

5 System of governing equations

The integral terms in (12)–(14) should now be specified provided that the structure is loaded by concentrated driving forces modelled by Dirac functions, like $q_{w1}(x)=F_1\delta(x-x_1)$ and also by concentrated coupling forces at the terminal points. Unlike driving forces, these coupling forces are unknown and additional compatibility equations should be formulated to determine them.

The force acting from a terminal substructure to the "host" structure is given as $-\Phi_j$ and continuity of displacements is formulated as

$$v_j=w_k(x_{j0})\,. \tag{15}$$

Thus, integral terms in (12) and (13) are decomposed to

$$\int_0^{\ell_k} q_{wk}(x)W_k(x,\xi)\,\mathrm{d}x=\sum_m F_mW_k(x_m,\xi)-$$

$$\sum_j\left(K_j-i\omega C_j-\omega^2M_j\right)w_k(x_{j0})W_k(x_{j0},\xi)\,,$$

$$\int_0^{\ell_k} q_{wk}(x)\frac{\partial W_k(x,\xi)}{\partial \xi}\,\mathrm{d}x=\sum_m F_m\frac{\partial W_k(x_m,\xi)}{\partial \xi}-$$

$$\sum_j\left(K_j-i\omega C_j-\omega^2M_j\right)w_k(x_{j0})\frac{\partial W_k(x_{j0},\xi)}{\partial \xi}\,,$$

To formulate continuity conditions for displacements, Somigliana's identity (12) should be applied at each "terminal" point. Hence, in addition to boundary equations at the edges, the following equations are formulated to define amplitudes of displacement $w_k(x_{j0})$

$$w_k(\xi_{j0}) \sum_m F_m W_k(x_m, \xi_{j0}) -$$

$$\sum_j \left(K_j - i\omega C_j - \omega^2 M_j \right) w_k(x_j) W_k(x_j, \xi_{j0}) -$$

$$\Bigg[E_k I_k w_k''(x) W_k(x, \xi_{j0}) - E_k I_k w''(x) \frac{\partial W_k(x, \xi_{j0})}{\partial x} +$$

$$E_k I_k w_k'(x) \frac{\partial^2 W_k(x, \xi_{j0})}{\partial x^2} -$$

$$E_k I_k w_k(x) \frac{\partial^3 W_k(x, \xi_{j0})}{\partial x^3} \Bigg] \Bigg|_{x=0}^{x=\ell_k} . \tag{16}$$

It should be pointed out that the problem formulation by boundary equations permits us to change easily not only the locations of "terminal" points, but also the number of these points. In a case where an additional "terminal" point is introduced, the system of boundary equations is simply extended by an additional equation of the same type as (16). Simultaneously, one more algebraic unknown is introduced, that is the amplitude of displacement at the additional "terminal" point. Accordingly, if the terminal point is removed, this only results in elimination of one column and one row of the original system of boundary equations.

Thus, for the beam structure of finite length shown in Fig. 1a, the set of algebraic unknowns consists of 36 boundary displacements and their derivatives and 4 displacements at the "terminal" points. They are easily found from six boundary conditions (3) and (8), twelve continuity conditions at the interfacial points (4)–(7), four continuity conditions at the "terminal" points (16) and eighteen boundary equations (12)–(14). If the structure is semi-infinite, the number of algebraic unknowns reduces to thirty, the number of boundary conditions reduces to three and the number of boundary equations reduces to fifteen. This modification is most straightforward: all six unknowns at $x = \ell_3$ are discarded. Three boundary conditions (8) and three boundary equations at this point are also omitted. This transformation is easily performed because complex-valued Green functions (see the Appendix) used in (10)–(12) satisfy Sommerfeld's radiation condition automatically, so that they are equally applicable for the analysis of vibrations in the form of standing waves and for the analysis of propagation of travelling waves. As this system of linear algebraic equations is solved, the shape of vibrations is easily reconstructed. Displacements at any point of beam elements are straightforwardly calculated by use of (10)–(12).

6 Power flow calculation

Analysis of energy flows between subsystems may easily be performed within the framework of a boundary equation formulation of the problem. To find the power input into a system, the Somigliana's identity should be used to calculate amplitudes of displacements at the loading points. The power input into the system is defined by the formula (see Doyle 1997; Norton 1986)

$$N_{\text{input}} = \frac{1}{2T} \int_0^T \text{Re} \left[\sum_m F_m \overline{v}(x_m) \right] \mathrm{d}t =$$

$$\frac{1}{2} \omega \sum_m \{ F_m \cos \alpha_m \text{Im}[w(x_m)] - F_m \sin \alpha_m \text{Re}[w(x_m)] \} . \tag{17}$$

Here T is the period of forced vibrations, F_m is the amplitude of the m-th driving force (it is a pure real quantity), α_m is the phase of this force (for definiteness, the phase of the first driving force is put to zero, $\alpha_1 = 0$), and $\overline{v}(x_m)$ is the complex conjugate of the velocity of the beam at the loading point calculated via Somigliana's formula (12).

The energy flow (structural intensity) of the axial deformation in a stationary case is formulated as

$$N_{\text{axial}} = \frac{1}{2T} \int_0^T \text{Re} \left[(-EAu') \cdot \overline{\dot{u}} \right] \mathrm{d}t . \tag{18}$$

Here EAu' is the axial force and $\overline{\dot{u}}$ is the complex conjugate of the axial velocity. As integration in time is performed, this formula is reduced to

$$N_{\text{axial}} = -\frac{E_k A_k}{2} \omega \Bigg\{ \left(\text{Re} \left[\frac{\mathrm{d}u}{\mathrm{d}x} \bigg|_{x=x_k} \right] \right) (\text{Im}[u(x_k)]) -$$

$$(\text{Re}[u(x_k)]) \left(\text{Im} \left[\frac{\mathrm{d}u}{\mathrm{d}x} \bigg|_{x=x_k} \right] \right) \times$$

$$\omega\eta (\text{Re}[u(x_k)]) \left(\text{Re} \left[\frac{\mathrm{d}u}{\mathrm{d}x} \bigg|_{x=x_k} \right] \right) +$$

$$\omega\eta (\text{Im}[u(x_k)]) \left(\text{Im} \left[\frac{\mathrm{d}u}{\mathrm{d}x} \bigg|_{x=x_k} \right] \right) \Bigg\} . \tag{19}$$

The energy flow (the structural intensity) of the bending deformation is formulated as

$$N_{\text{bending}} = \frac{1}{2T} \int_0^T \text{Re} \left[(EIw''') \cdot \overline{\dot{w}} - (EIw'') \cdot \overline{\dot{w}}' \right] \mathrm{d}t .$$

As integration in time is performed, this formula is reduced to

$$N_{\text{bending}} = -\frac{E_k I_k}{2}\omega\left\{\left(\mathrm{Im}[w(x_k)]\right)\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right|_{x=x_k}\right]\right)-\left(\mathrm{Re}[w(x_k)]\right)\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right|_{x=x_k}\right]\right)\right\}-$$
$$\frac{E_k I_k}{2}\omega\left\{\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=x_k}\right]\right)\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=x_k}\right]\right)-\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=x_k}\right]\right)\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=x_k}\right]\right)\right\}+$$
$$\frac{E_k I_k}{2}\omega^2\eta\left\{\left(\mathrm{Re}[w(x_k)]\right)\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right|_{x=x_k}\right]\right)+\left(\mathrm{Im}[w(x_k)]\right)\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\right|_{x=x_k}\right]\right)\right\}-$$
$$\frac{E_k I_k}{2}\omega^2\eta\left\{\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=x_k}\right]\right)\left(\mathrm{Im}\left[\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=x_k}\right]\right)+\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=x_k}\right]\right)\left(\mathrm{Re}\left[\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=x_k}\right]\right)\right\}. \quad (20)$$

The structural intensities are defined by (19)–(20) for an arbitrary point of a beam structure. The intensities of bending and axial vibrations at the interfaces are computed directly via boundary displacements and forces found as a solution to the system of boundary integral equations. A structural intensity at a certain arbitrarily chosen point of the whole structure is also calculated via (19)–(20). To be able to perform this calculation, it is necessary to calculate displacements and their derivatives at these points. Displacements are readily available via (12)–(14), whereas to obtain their derivatives, these equations should be differentiated with respect to the coordinate of an observation point ξ. This procedure is used to calculate the structural intensity that is transported to infinity by propagating waves in the semi-infinite element of the structure.

The energy dissipation at each terminal point is formulated via amplitudes of displacements already found from the system of boundary equations

$$N_j^{\text{diss}} = \frac{1}{2}\omega^2 C_j\left\{\left(\mathrm{Re}\left[w(x_{j0})\right]\right)^2+\left(\mathrm{Im}\left[w(x_{j0})\right]\right)^2\right\}. \quad (21)$$

The energy conservation in absence of internal damping is simply formulated as

$$N_{\text{input}} = \sum_j N_j^{\text{diss}} + N_{\text{axial}}^{\text{out}} + N_{\text{bending}}^{\text{out}}\,. \quad (22)$$

This means that an energy, which is not dissipated at the terminal substructures is transported by travelling flexural and longitudinal waves to infinity.

7 Conclusions

A formulation of a problem of in-plane coupled flexural and longitudinal vibrations of a planar structure composed by tubular elements of both a finite and an infinite length is presented. The purpose of the study is an optimization of energy flows from a source to a remote zone of a structure for a given frequency range and given excitation conditions by means of varying the location, stiffness, damping and mass parameters of the attached terminal points. A boundary integral equation method is used to set up a system of governing equations describing forced stationary vibrations of the structure. The algorithm based on use of boundary equations is also used to analyse energy flows between subsystems and to compare contributions to the structural intensity from flexural and longitudinal waves. The objective function is selected as the power input into the system and the optimization strategy is set up as reduction of the power input because it simultaneously gives the minimization of an energy outflow.

Detailed analysis of performance of a model structure is given in Part II of this paper (Sorokin *et al.* 2001).

References

Swanson Analysis Systems Inc. 1992: *ANSYS user's manual for revision 5.0*

Doyle, J.F. 1997: *Wave propagation in structures*. Berlin, Heidelberg, New York: Springer

Fuller, C.R.; Fahy, F.J. 1982: Characteristics of wave propagation and energy distribution in cylindrical elastic shells filled with fluid. *J. Sound & Vib.* **81**, 501–518

Lyon, R.H. 1975: *Statistical energy analysis of dynamical systems: theory and applications*. Cambridge, MA: MIT Press

Meirovitch, L. 1990: *Dynamics and control of structures*. New York: John Wiley & Sons

Norton, M.P. 1986: *Fundamentals of noise and vibration analysis for engineers*. Cambridge, UK: Cambridge University Press

Pavic, G. 1993: Vibroacoustical energy flow through straight pipes. *J. Sound & Vib.* **154**, 411–429

Petyt, M. 1990: *Introduction to finite element vibration analysis*. Cambridge, UK: Cambridge University Press

Ren, Y.; Beards, C.F. 1995: On substructure synthesis with FRF data. *J. Sound & Vib.* **185**, 845–866

Rozvany, G.I.N. 1976: *Optimal design of flexural systems*. Oxford: Pergamon Press

Shankar, K.; Keane, A.J. 1995a: A study of the vibrational energies of two coupled beams by finite element and Green function (receptance) methods. *J. Sound & Vib.* **181**, 801–838

Shankar, K.; Keane, A.J. 1995b: Energy flow predictions in a structure of rigidly joined beams using receptance theory. *J. Sound & Vib.* **185**, 867–890

Sorokin, S.V. 1993: Asymptotic analysis and numerical solution of the two-level boundary equations of a plane problem of stationary hydroelasticity. *J. Appl. Math. & Mech.* **57**, 105–115

Sorokin, S.V.; Nielsen, J.B.; Olhoff, N. 2001: Analysis and optimization of energy flows in structures composed of beam elements. Part II: Examples and discussion. *Struct. Multidisc. Optim.* **22**, 12–23

Appendix. Green's functions for axial and flexural vibrations of a beam with material damping

Axial vibration

Green's function for axial vibrations of a straight rod of uniform cross-sectional area is a solution to the following differential equation

$$E_k A_k (1 - i\eta_{wk}\omega) \frac{\partial^2 U_k(x,\xi)}{\partial x^2} + \rho_k A_k \omega^2 U_k(x,\xi) = -\delta(x-\xi)\,. \quad (23)$$

The function $U_k(x,\xi)$ formulates a longitudinal wave propagating along the rod and generated by the point force of unit intensity applied at $x=\xi$. To construct this function, it is more convenient to reformulate (23) as

$$E_k A_k (1 - i\eta_{uk}\omega) \frac{\partial^2 U_k(x,\xi)}{\partial x^2} + \rho_k A_k \omega^2 U_k(x,\xi) = 0\,. \quad (24)$$

This equation should be solved with boundary condition at $x=\xi$

$$E_k A_k (1 - i\eta_{uk}\omega) \frac{\partial U_k(x,\xi)}{\partial x} = -\frac{1}{2}\,\mathrm{sign}\,(x-\xi)\,. \quad (25)$$

Solution is sought as $U_k(x-\xi) = A\,\exp(\lambda|x-\xi|)$.The characteristic equation is

$$E_k A_k (1 - i\eta_{uk}\omega)\lambda^2 = -\rho_k A_k \omega^2\,. \quad (26)$$

Only one of two roots of (26)

$$\lambda = i\frac{\omega}{c_0}\left(1 + \frac{1}{2} i\eta\omega\right) \quad (27)$$

satisfies the radiation condition $c_{\mathrm{phase}} = \frac{\omega}{\lambda} > 0$ as well as the decay condition for $\eta_{uk} \neq 0$. It is assumed that material losses are fairly small and that the following condition holds: $(\eta_k\omega)^2 \ll 1$. Thus, only terms that are linear in $\eta_k\omega$ are retained hereafter. The boundary condition (25) defines an amplitude propagating wave as $A_0 = \frac{i}{2}\frac{c_0}{\omega}(1 + \frac{1}{2} i\eta_k\omega)$. Thus, Green's function for longitudinal vibrations of an infinitely long beam becomes

$$U_k(x,\xi) = \frac{i}{2E_kA_k}\frac{\omega}{c_0}\left(1 + \frac{1}{2}i\eta_k\omega\right)\exp\left[i\frac{\omega}{c_0}\left(1 + \frac{1}{2}i\eta_k\omega\right)\right]. \quad (28)$$

Flexural vibrations

Green's function for flexural vibrations of a straight beam of uniform cross-sectional area is a solution to the following differential equation:

$$E_k I_k (1 - i\eta_{wk}\omega) \frac{\partial^4 W_k(x,\xi)}{\partial x^4} - \rho_k A_k \omega^2 W_k(x,\xi) = \delta(x-\xi)\,. \quad (29)$$

The function $W_k(x,\xi)$ formulates a flexural wave propagating along the beam and generated by the point force of unit intensity applied at $x=\xi$. Then it is more convenient to reformulate (29) as

$$E_k I_k (1 - i\eta_{wk}\omega) \frac{\partial^4 W_k(x,\xi)}{\partial x^4} - \rho_k A_k \omega^2 W_k(x,\xi) = 0\,, \quad (30)$$

with two boundary conditions at $x=\xi$

$$E_k I_k (1 - i\eta_{wk}\omega) \frac{\partial^3 W_k(x,\xi)}{\partial x^3} = \frac{1}{2}\,\mathrm{sign}\,(x-\xi)\,, \quad (31)$$

$$\frac{\partial W_k(x,\xi)}{\partial x} = 0\,. \quad (32)$$

The first condition models a concentrated force acting at the beam, the second condition formulates symmetry of the bending wave with respect to the loading point.

Solution is sought as $W_k(x-\xi) = A_0\,\exp(\lambda|x-\xi|)$. The characteristic equation is

$$E_k I_k (1 - i\eta_{wk}\omega)\lambda^4 = \rho_k A_k \omega^2\,, \quad (33)$$

Only two of four roots for (33),

$$\lambda_1 = ik_0 - \frac{1}{4}k_0\eta_{wk}\omega\,, \tag{34}$$

$$\lambda_{22} = -k_0 + \frac{i}{4}k_0\eta_{wk}\omega\,, \quad k_0 = \sqrt[4]{\frac{\rho_k A_k \omega_k^2}{E_k I_K}} \tag{35}$$

satisfy the radiation condition $c_{\text{phase}} = \frac{\omega}{\lambda_j} > 0,\ j = 1, 2.$ Similarly to the case of axial vibrations, it is assumed that material losses are fairly small and that the following condition holds: $(\eta_{wk}\omega)^2 \ll 1$. Green's function is presented as $W_k(x-\xi) = A_{01}\ \exp\,(\lambda_1|x-\xi|) + A_{02}\ \exp\,(\lambda_2|x-\xi|.)$ Coefficients A_{01} and A_{02} are found from the two algebraic equations

$$\left(-k_0 - \frac{1}{4}ik_0\eta_{wk}\omega\right)A_{01} + \left(ik_0 - \frac{1}{4}k_0\eta_{wk}\omega\right)A_{02} = 0\,,$$

$$\left(1 - i\eta_{wk}\omega\right)\left[\left(-k_0^3 - \frac{3}{4}ik_0^3\eta_{wk}\omega\right)A_{01} + \left(ik_0^3 - \frac{3}{4}k_0^3\eta_{wk}\omega\right)A_{02}\right] = \frac{1}{2}\frac{1}{E_k I_k}\,.$$

Thus, Green's function for flexural vibrations of an infinitely long beam with material losses becomes

$$W(x,\xi) = -\frac{1}{4k_0^3}\left(1 + \frac{1}{4}i\eta_{wk}\omega\right) \times \exp\left[-\left(1 + \frac{1}{4}i\eta_{wk}\omega\right)k_0|x-\xi|\right]\frac{1}{E_k I_k} + \frac{i}{4k_0^3}\left(1 + \frac{1}{4}i\eta_{wk}\omega\right)\exp\left[\left(i - \frac{1}{4}\eta_{wk}\omega\right)k_0|x-\xi|\right]\frac{1}{E_k I_k}\,. \tag{36}$$