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# On nonassembly in the optimal dimensional synthesis of planar mechanisms

**R.J. Minnaar, D.A. Tortorelli and J.A. Snyman**

**Abstract** This paper presents a general method for treating nonassembly in the optimal synthesis of planar mechanisms. The synthesis is performed with gradient based optimization algorithms and the sensitivities are calculated analytically through the method of direct differentiation. The analysis is based on the well-established and general method of computational kinematics. In this study the residuals of the joint constraint equations are minimized rather than equated to zero. This makes it possible to perform the kinematic analysis for any proposed design even though it may not be possible to assemble the mechanism. Several examples are provided.

**Key words** kinematic synthesis, nonassembly, absolute coordinate

## 1 Introduction

In dimensional synthesis the component dimensions that comprise a chosen mechanism are specified to allow the mechanism to perform a given task (Erdman and Sandor 1997). The task may vary but will always be specified by some prescribed points through which, or near which, a tracking point on one of the components must pass. Two different approaches to dimensional synthesis are commonly used. We use the numerical approach rather than the analytical approach. In addition we address the problem of nonassembly.

In the analytical approach, referred to as exact or precision point synthesis, the mechanism will pass through the prescribed points, but only a limited number of points can be specified. The number of points depends on the type of mechanism etc., but is for example limited to 9 points for a path generating four-bar mechanism (Erdman and Sandor 1997).

In the numerical approach, referred to as approximate synthesis, more points may be prescribed but the mechanism no longer passes through all of the points. Rather, the mechanism tracking point trajectory approximates the desired trajectory. The goal of the synthesis is to make the discrepancy between these trajectories as small as possible. This leads to optimal synthesis, where mathematical programming techniques are employed to reduce the trajectory discrepancy. This is not a new approach. As early as 1967 Fox and Willmert used a then newly developed mathematical programming technique for optimal dimensional synthesis. Gabriele (1993) edited a review spanning 40 years that summarizes the development of mechanism synthesis optimization algorithms.

A feature of mechanism synthesis problems that has received little attention is that of nonassembly. This occurs when the dimensions of the mechanism are such that it is not possible for the joint constraints to be satisfied throughout the desired motion range. I.e. the motion is not physically realizable and hence kinematic analysis methods will fail. In optimal dimensional synthesis it often occurs that a proposed design cannot be assembled and consequently the kinematic analysis fails. As a result the cost function (and its gradient) for the synthesis problem can not be evaluated and hence the optimization algorithm fails.

To address the nonassembly issue, constraints which prohibit nonassembly can be incorporated into the optimization problem. However, this approach requires that the optimization algorithm keeps the design in the feasible region. To accomplish this, Paradis and Willmert (1983) developed an optimization technique that can move along linear constraints and introduce the assembly conditions for a four-bar as well as a slider crank mechanism as linear inequality constraints. Makkonen and Persson (1994) enforce the joint constraints with penalty

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functions formulated as barrier functions, so that the design remains in the feasible region.

It is also possible to formulate the optimal synthesis problem to accommodate an infeasible design. Fox and Willmert (1967) specify the vector loop equations of a four-bar mechanism as inequality constraints of the optimization problem, thereby eliminating the need for a separate kinematic analysis. Vucina and Freudenstein (1991), García de Jalón and Bayo (1994) enforced the joint constraints with a penalty function method, hence, they need not consider the kinematic analysis. Krishnamurthy and Turcic (1992) use nonlinear goal programming techniques where priorities are assigned to different objectives. They assign high priority to the assembly of the mechanism, which is described by geometric relations derived for the particular mechanism. Hansen (1992) constrains the square of the errors of the vector loop equations to ensure assembly. Cossalter *et al.* (1992) define a residual that is a measure of the amount by which the mechanism cannot assemble. This residual, which is derived for the considered type of mechanism, penalizes the cost function of the optimal synthesis problem. Vallejo *et al.* (1995) use a strategy whereby the mechanism is deformed to reach the prescribed points and the mechanism that requires the least deformation is the optimum mechanism.

In the method proposed here, the kinematic analysis is formulated to accommodate nonassembly. The method is based on the absolute coordinate kinematic analysis to ensure the method's generality. In the absolute coordinate kinematic analysis, mechanisms are modeled by combining the kinematic constraints that describe a mechanism. A revolute joint is a typical kinematic constraint which would allow the modeling of four-bar and six-bar mechanisms. Translational joints are available to model many more mechanisms. It is this generality of the absolute coordinate kinematic analysis that is an advantage over methods where the kinematic analysis is based on equations that are derived for a particular type of mechanism.

By accommodating nonassembly in the kinematic analysis, the synthesis problem is cast as a standard inequality constrained optimization problem. The advantage of having a standard inequality constrained optimization problem is that it allows conventional constrained optimization algorithms to be employed without modifications.

The two greatest advantages of the proposed method therefore both stem from the particular formulation of the kinematic analysis and are: First, the generality of the kinematic analysis since it is based on the absolute coordinate method. And second, that the synthesis problem is formulated as a simple inequality constrained optimization problem since nonassembly is already accommodated in the kinematic analysis.

The following sections discuss the new formulation of the kinematic analysis as well as a direct differentiation sensitivity analysis to obtain accurate and efficient sensitivity computations. The efficient sensitivity computations is an additional advantage. Finally three numerical optimization examples are presented.

## 2 Analysis

In order to evaluate the objective function of the optimal synthesis problem it is necessary to evaluate the trajectory of the tracking point. One suitable method would be to calculate this trajectory through a kinematic analysis. The proposed method is based on the well-established absolute coordinate method for kinematic analysis (Nikravesh 1988; Haug 1989; Shabana 1994). An attractive feature of this method is that it is general.

In the numerical approach to kinematic analysis, the position and orientation of each body with relation to a global coordinate system are determined. In this study, absolute generalized coordinates $\mathbf{q}_i$ (Shabana 1994) are used to describe the position and orientation of body $i$. For the planar case, each body $i$ has three coordinates

$$\mathbf{q}_i = (x_i, y_i, \phi_i)^T , \tag{1}$$

specifying the origin $(x_i, y_i)$ of the body $i$ fixed coordinate system and the angular orientation $\phi_i$ of this coordinate system relative to the global coordinate system. The coordinates of the $n$ bodies in the mechanism are combined to form the vector of generalized coordinates or state variables,

$$\mathbf{q} = \left(\mathbf{q}_1^T, \mathbf{q}_2^T, \dots, \mathbf{q}_n^T\right)^T . \tag{2}$$

The kinematic analysis problem is formulated as a system of constraint equations. These equations describe the geometric and driving constraints of the mechanism and are functions of the state variables $\mathbf{q}$ and time $t$,

$$\mathbf{\Phi}[\mathbf{q}(t), t] = \mathbf{0} . \tag{3}$$

For any specified time $t = t_i$, (3) represents a nonlinear system of equations in $\mathbf{q}$ which can be solved computationally by various methods to ensure that

$$\|\mathbf{\Phi}\| < \epsilon_a , \tag{4}$$

where $\epsilon_a$ is the analysis convergence tolerance. It is, however, possible that the dimensions of the mechanism are such that it cannot be assembled at time $t = t_i$. In such a case (3) no longer has a solution for $t = t_i$ and any method used to solve (3) will fail.

In optimal synthesis, the problem of nonassembly often occurs. It is possible that the optimization algorithm, in an attempt to find a better mechanism design, will suggest a design that will not assemble for some portion of the motion under consideration. For the instances in time when nonassembly occurs, the solution $\mathbf{q}$ to (3) does not exist and consequently the cost function for the optimal synthesis problem, which is a function of $\mathbf{q}$, cannot be evaluated. Obviously this situation is troublesome for any optimization algorithm.

An alternate formulation for the kinematic analysis that accommodates nonassembly is proposed here to eliminate the problem of nonassembly. In the new kinematic analysis formulation, instead of solving (3), the norm of the constraint equation is minimized,

$$\text{minimize } e[\mathbf{q}(t)] = \frac{1}{2}\|\boldsymbol{\Phi}[\mathbf{q}(t), t]\|^2 \,. \tag{5}$$

Haug (1989) uses a similar strategy to find an assembled initial configuration. The stationary condition for (5) is

$$\nabla e = \boldsymbol{\Phi}_{\mathbf{q}}^T \boldsymbol{\Phi} = \mathbf{0} \,, \tag{6}$$

where $\nabla$ is the gradient operator, the subscript indicates the partial derivative i.e. $\boldsymbol{\Phi}_{\mathbf{q}} = \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{q}}$ and the arguments have been suppressed for conciseness. From (6) it is clear that three conditions will satisfy the stationary condition.[1] The first condition is,

$$\boldsymbol{\Phi}[\mathbf{q}(t), t] = \mathbf{0} \,, \tag{7}$$

which is exactly equal to the original kinematic analysis problem stated in (3). It is therefore clear that the new formulation, stated in (5), is equivalent to the original formulation, stated in (3), when assembly is possible.

The second condition that satisfies (6) is

$$\det \boldsymbol{\Phi}_{\mathbf{q}}[\mathbf{q}(t), t] = 0 \,. \tag{8}$$

This condition occurs when $\boldsymbol{\Phi}_{\mathbf{q}}$ is rank deficient so that there need not be a solution to the original problem stated in (3). Moreover, if Newton's method is used to solve (3) then the update $\mathbf{q}_{j+1} = \mathbf{q}_j + \Delta\mathbf{q}$, calculated by,

$$\boldsymbol{\Phi}_{\mathbf{q}}|_{\mathbf{q}=\mathbf{q}_j} \Delta\mathbf{q} = -\boldsymbol{\Phi}|_{\mathbf{q}=\mathbf{q}_j} \,, \tag{9}$$

where $\Delta\mathbf{q}$ is the Newton update will fail.

The third condition that satisfies (6) is the combination of the first two, i.e. $\boldsymbol{\Phi}(\mathbf{q}(t), t) = \mathbf{0}$ and $\det \boldsymbol{\Phi}_{\mathbf{q}}[\mathbf{q}(t), t] = 0$. In this case, the mechanism is in a singular configuration from which bifurcation or lock-up follows.

The minimization problem stated in (5) is solved by satisfying the stationary conditions expressed in (6). This is done by Newton's method where the update $\Delta\mathbf{q}$ is calculated by the linear equation,

$$\mathbf{H}\Delta\mathbf{q} = \left(\boldsymbol{\Phi}_{\mathbf{q}}^T \bar{\boldsymbol{\Phi}}\right)_q \Delta\mathbf{q} + \boldsymbol{\Phi}_{\mathbf{q}}^T \boldsymbol{\Phi}_{\mathbf{q}} \Delta\mathbf{q} = -\nabla e \,, \tag{10}$$

where $\mathbf{H} = D^2 e$ denotes the Hessian of $e$ and the overbar denotes quantities that are viewed as constants for the differentiation. The only new term in (10) [that is not used in the analysis of (9)] is $\boldsymbol{\Phi}_{\mathbf{qq}}^T \boldsymbol{\Phi}$. However, this term is readily computed using the methods already available from the acceleration analysis with the conventional absolute coordinate method. There the term $(\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}}$, appears (Haug 1989) which is quite similar to our $\left(\boldsymbol{\Phi}_{\mathbf{q}}^T \bar{\boldsymbol{\Phi}}\right)_q \Delta q$ term.

Two points regarding the present analysis should be noted. First, Newton's method does not distinguish between minima, maxima or saddle points of $e$, rather it only ensures that $\nabla e = 0$. Consequently, the solution to (6) does not necessarily provide the solution to (5) as no second order sufficiency conditions are considered. Second, multiple solutions, i.e. minimum, may exist.

When Newton's method converges to a saddle point, the following problem arises. If a mechanism is such that there is nonassembly at $t = t_{i-1}$ but that the mechanism can assemble at $t = t_i$ then our experience shows that Newton's method is likely to converge to a saddle point of $e$ rather than a minimum of $e$ at $t = t_i$. The solution would then continue along this saddle point solution for $t = t_i, \ldots, t_n$. These solutions do not solve (5) since saddle points, although zeros of $\nabla e$, do not minimize $e$. In such cases the associated cost function values of the dimensional synthesis problem would be physically meaningless.

The following strategy is employed to obtain a minimum once a saddle point has been found. It is known that a saddle point occurs when the Hessian matrix $\mathbf{H}$ is indefinite hence $e$ does not attain its minimum zero value and therefore the solution $\mathbf{q}$ only satisfies the condition stated in (8). Therefore we only check whether $\mathbf{H}$ is indefinite when Newton's method (of (10)) has converged to a solution where $\det \boldsymbol{\Phi}_{\mathbf{q}} = 0$ and $\boldsymbol{\Phi} \neq \mathbf{0}$.

If we compute a $\mathbf{q}$ that satisfies (6) and have $\det \boldsymbol{\Phi}_{\mathbf{q}} = 0$ and $\boldsymbol{\Phi} \neq \mathbf{0}$ and if the nonzero eigenvalues of $\mathbf{H}$ are all positive then the solution is a minimum of $e$ as is desired (although in this case the mechanism cannot be assembled). If this is not the case, a minimizing solution must be found. To find this minimum we use the method presented by Gill *et al.* (1981) in which a direction of negative curvature $\mathbf{s}$ is found such that

$$\mathbf{s}^T \mathbf{H} \mathbf{s} < 0 \,, \tag{11}$$

where $\mathbf{s}$ is the eigenvector that is associated with the most negative eigenvalue of $\mathbf{H}$. A line search is performed along $\mathbf{s}$ to determine the $\alpha$ that minimizes $e(\mathbf{q} + \alpha\mathbf{s})$. Note that both $\alpha > 0$ or $\alpha < 0$ yield descent directions as is illustrated in Fig. 1. This implies that the mechanism has multiple configurations, i.e. multiple circuits and/or branches (Chase and Mirth 1993) that satisfy (3) at this instant in time. To choose the correct sign of $\alpha$ so that when $e$ is minimized we remain in a smooth trajectory care must be taken, for we do not want suddenly jump to a different circuit (as seen in Fig. 1). To these ends, we choose the sign of $\alpha$ so that the sign of $\det(\boldsymbol{\Phi}_{\mathbf{q}})$ remains unchanged throughout the trajectory. This approach is sufficient for simple mechanisms. However, if the mechanism has multiple loops or even for double-rocker or rocker-crank four-

[1] We do not consider redundant constraints (Haug 1989)

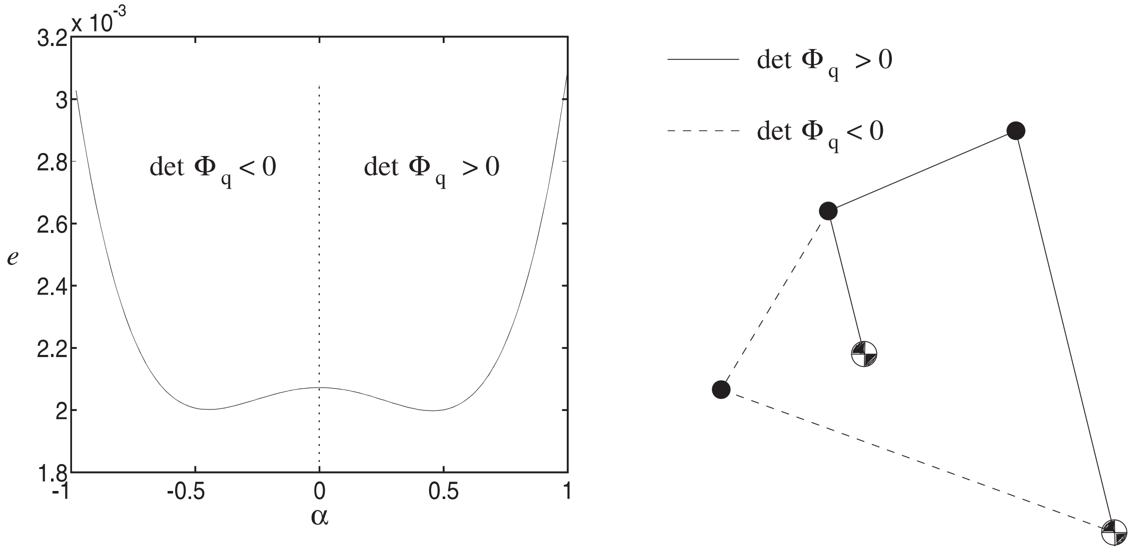

**Fig. 1** Saddle point as plotted along the direction of negative curvature $\mathbf{s}$. Also shown are the two alternative configurations of a four-bar mechanism with their relation to the determinant of the Jacobian matrix

bar mechanism the determinant alone is not a sufficient gage whether or not the mechanism jumps to a different circuit between adjacent time steps (Chase and Mirth 1993).

With $\mathbf{s}$ and the sign of $\alpha$ known, a line search is performed to find the minimum of $e(\mathbf{q}+\alpha\mathbf{s})$ after which Newton's method [see (10)] is restarted. Since this point is in the vicinity of the saddle point it is possible that Newton's method will again converge to a saddle point. To this end a line search is appended to Newton's method once it is restarted to ensure that $e$ is monotonically decreasing. It is emphasized that the line search is only appended to Newton's method to find a minimum after a saddle point solution was found, therefore it is used infrequently.

The second problem, i.e. nonuniqueness, occurs when, at $t = t_{i-1}$, the mechanism is in a singular configuration i.e. when $\mathbf{\Phi} = \mathbf{0}$ and $\det \mathbf{\Phi_q} = 0$. For cases where the mechanism cannot be assembled immediately beyond this time i.e. when $\mathbf{\Phi} \neq \mathbf{0}$ for $t = t_i$, a unique solution to (5), that satisfies (8), is found. However, if the mechanism can assemble for time $t = t_i$ then we are in the presence of a bifurcation. We arbitrarily seek the trajectory that retains the sign of $\det \mathbf{\Phi_q}$. Similarly, if the trajectory at $t = t_{i-1}$ is near a singular configuration and passes through the singular configuration during the next time step then, if the mechanism can assemble at $t = t_i$, we desire the solution that retains the sign of $\det \mathbf{\Phi_q}$. If, for $t = t_i$, Newton's method converges to an undesirable minimum (according to the determinant criteria stated above) then the desired minimum is found by first finding the saddle point according to the method of Smith (1990) and using the method discussed in the previous paragraphs.

In this second problem for which $\mathbf{\Phi} = \mathbf{0}$ and $\det \mathbf{\Phi_q} = 0$ we see that $\mathbf{H}$ must be rank deficient, cf. (10). In practice, this condition is never precisely encountered however, if it were, we could use some alternative method, eg. singular value decomposition (Strang 1988) to solve (10).

The new formulation will yield results that are consistent with those of the original kinematic analysis problem when assembly is possible. Furthermore, since the norm of $\mathbf{\Phi}$ is minimized, we obtain the best possible solution if the mechanism cannot assemble i.e. the configuration where the violation of (3) has been minimized.[2] This implies that the cost function for the optimal synthesis problem can always be evaluated and consequently that gradient-based optimization methods can be employed to solve the optimal synthesis problem. Moreover, to ensure that the optimized mechanism design can be assembled, the norm of $\mathbf{\Phi}$ i.e. (4), can be used as an inequality constraint in the design optimization to define the feasible region.

## 3 Optimization

We now have nested minimization problems. In the inner-problem, i.e. analysis, a trajectory is determined for a specific mechanism design by minimizing the function $e$. In the outer-problem, i.e. design optimization, a cost function is minimized subject to constraints. Iterations are performed in which a design is specified by the outer-loop and the inner-problem is solved to evaluate trajectory which is needed to compute the values of the cost and constraint functions. Critical to the success of the outer-loop minimization is the ability to compute sensitivities of the cost and constrain functions with respect to the design variables. Such capabilities, to be discussed, are augmented to the inner-loop analysis problem.

[2] Of course the solution in this case is physically meaningless

For the optimal synthesis problem, the cost function $F$ quantifies the discrepancy between a prescribed path and the trajectory of the tracking point of the mechanism that is defined by the design variables $\mathbf{b}$, i.e. the link dimensions. Constraints, derived from (4), ensure that the optimized mechanism can be assembled.

The cost function is stated as follows:

$$F(\mathbf{b}) = \sum_{i=1}^{n} f_i[\mathbf{q}(\mathbf{b}, t_i), \mathbf{b}, t_i] \,, \tag{12}$$

where we note the dependence of the state $\mathbf{q}$ on $\mathbf{b}$ and $f_i$ is the measure of trajectory discrepancy defined as

$$f_i = \sqrt{(x_{i_{pre}} - x_{i_{tp}})^2 + (y_{i_{pre}} - y_{i_{tp}})^2} \,, \tag{13}$$

where $(x_{i_{pre}}, y_{i_{pre}})$ are the coordinates of points on the prescribed trajectory and $(x_{i_{tp}}, y_{i_{tp}})$ are the computed coordinates of the tracking point at $t = t_i$.

From (13) it is clear that the discrepancy is calculated on a point-by-point basis. Although more sophisticated methods are available (Hansen and Tortorelli 1996, Ullah and Kota 1997) the point-by-point comparison is still widely used due to its simplicity. This comparison does however, have one drawback. Since distinct points on the prescribed and calculated trajectories are compared, it is now necessary to add additional design variables to parameterize, in time, the rates at which the driving links are driven. In this way, links may be driven at fast or slow rates to help minimize $F$. Although these additional design variables have no influence on the dimensions of the mechanism, they are vital for the point-by-point comparison to be successful.

To ensure that the synthesis produces a design that can be assembled, the following inequality constraints are defined:

$$c_i = \frac{1}{2}\|\mathbf{\Phi}|_{t=t_i}\|^2 - \epsilon_a \leq 0 \,, \tag{14}$$

where there is one constraint $c_i$ for every time step $t = t_i$ in the analysis. The tolerance $\epsilon_a > 0$ is that which is traditionally used to verify the solution of (3).

The design sensitivity analysis of planar mechanisms has been studied extensively (Hansen and Tortorelli 1996, Bruns 1992, Haug and Sohoni 1984, Etman 1997). In this study the design sensitivities are obtained by the direct differentiation method rather than the finite difference or adjoint methods. This method requires $n$ efficient pseudo analyzes whereas the finite difference method requires $n$ complete analyzes with $n$ the number of design variables in $\mathbf{b}$.

The sensitivities of the cost function $F$ are given by

$$\frac{\mathrm{d}F}{\mathrm{d}\mathbf{b}} = \sum_{i=1}^{n} \left( f_{i_\mathbf{q}} \frac{\partial \mathbf{q}}{\partial \mathbf{b}} + f_{i_\mathbf{b}} \right)\Bigg|_{t=t_i} \,, \tag{15}$$

and similarly, from (14) the sensitivities of the inequality constraints $c_i$ are given by

$$\frac{\partial c_i}{\partial \mathbf{b}} = \mathbf{\Phi}^T \left( \mathbf{\Phi}_\mathbf{q} \frac{\partial \mathbf{q}}{\partial \mathbf{b}} + \mathbf{\Phi}_\mathbf{b} \right) |_{t=t_i} \,, \tag{16}$$

where the response derivative $\frac{\partial \mathbf{q}}{\partial \mathbf{b}}$ is not readily available due to the implicit dependence of $\mathbf{q}$ on $\mathbf{b}$. Indeed, for a given $\mathbf{b}$ we solve (5) to evaluate $\mathbf{q}$.

The evaluation of $\frac{\partial \mathbf{q}}{\partial \mathbf{b}}$ uses the stationary condition of the kinematic analysis problem, as stated in (6), but repeated here with all of the dependencies shown,

$$\mathbf{\Phi}_\mathbf{q}^T[\mathbf{q}(\mathbf{b}, t), \mathbf{b}, t]\mathbf{\Phi}[\mathbf{q}(\mathbf{b}, t), \mathbf{b}, t] = \mathbf{0} \,. \tag{17}$$

The chain rule of differentiation is used to differentiate (17) with respect to a single design variable $b_k$ to obtain, after some rearrangement,

$$\mathbf{H}\frac{\partial \mathbf{q}}{\partial b_k} = -\mathbf{\Phi}_{\mathbf{q}b_k}\mathbf{\Phi} - \mathbf{\Phi}_\mathbf{q}^T \mathbf{\Phi}_{b_k} \,. \tag{18}$$

The above linear system is solved for the derivative $\frac{\partial \mathbf{q}}{\partial b_k}$ where it is noted that $\mathbf{H}$ has already been assembled and decomposed to solve (6) [see (10)]. Therefore the calculations of the sensitivities $\frac{\partial \mathbf{q}}{\partial b_k}$ and hence $\frac{\partial F}{\partial b_k}$ and $\frac{\partial c_i}{\partial b_k}$ are very efficient as compared to the analysis itself. The pseudo analysis of (18) is done in tandem with the the analysis of (6). That is, in the inner-loop, at each time step $t = t_i$, we first solve (10) via Newton's method and then, using the decomposed Hessian $\mathbf{H}$, we solve (18) $n$ times for each of the design variables $b_k$, $k = 1, 2, \ldots, n$.

The Sequential Quadratic Programming (SQP) algorithm of *DOT* (Vanderplaats, Miura & Associates 1992) was used in this study to solve the optimization problem

$$\text{minimize} \quad F(\mathbf{b}) \quad \text{subject to} \quad c_i < \epsilon_a \,. \tag{19}$$

The fact that the synthesis problem is cast as a standard inequality constrained optimization problem allows the use of any constrained optimization algorithm.

Due to the nonlinearities of the kinematic analysis, a somewhat accurate initial configuration guess, i.e. $\mathbf{q}^0(0, \mathbf{b})$, must be supplied for each analysis. For this reason the design changes between successive optimization iterations, i.e. the move limits, are limited so that the initial configuration from the previous design iteration given by $\mathbf{b}_{j-1}$ represents a suitable initial configuration guess for the current design iteration given by $\mathbf{b}_j$, i.e $\mathbf{q}^0(0, \mathbf{b}_j) = \mathbf{q}(0, \mathbf{b}_{j-1})$.

## 4 Numerical results

Three path generating dimensional synthesis problems are solved using the proposed method. The three paths, shown in Fig. 2, vary in complexity and are each described by 50 equally spaced points where the ⊗ indicates

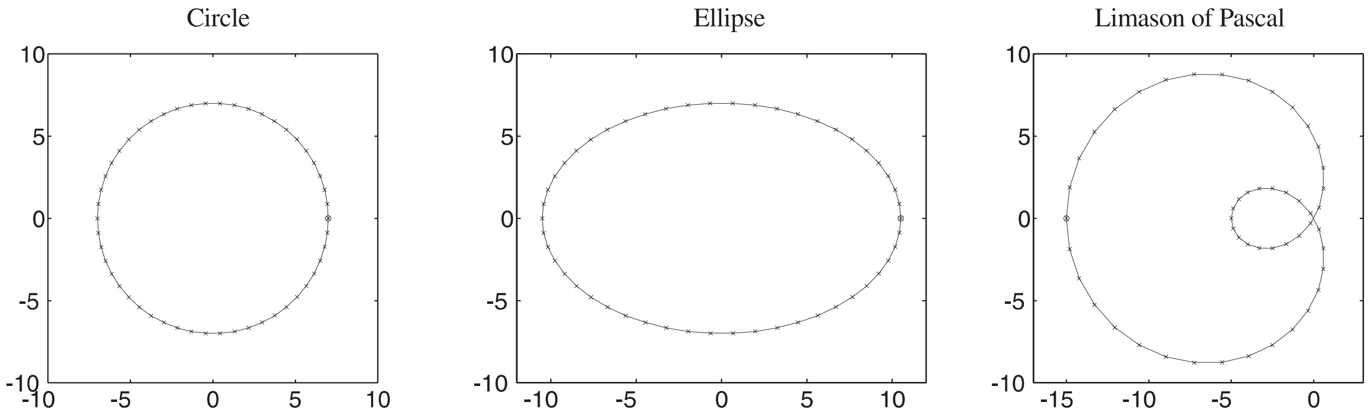

**Fig. 2** Prescribed paths

the first point. The tracking point trajectory is calculated with 50 fixed time steps i.e. $\Delta t$ is constant. In this way only one driving parameter, i.e. the initial orientation, was required for the optimization [cf. the discussion following (13)]. The tolerance $\epsilon_a$ that is used to verify the convergence of the solution to (6) and is used in the definition of the constraint equations (14) is set to $\epsilon_a = 10^{-6}$.

Four-bar mechanisms are used to obtain the prescribed paths. Ten design variables parameterize specified for the four-bar mechanism as shown in Fig. 3. The design variable $b_8$ is the initial orientation of the driven link.

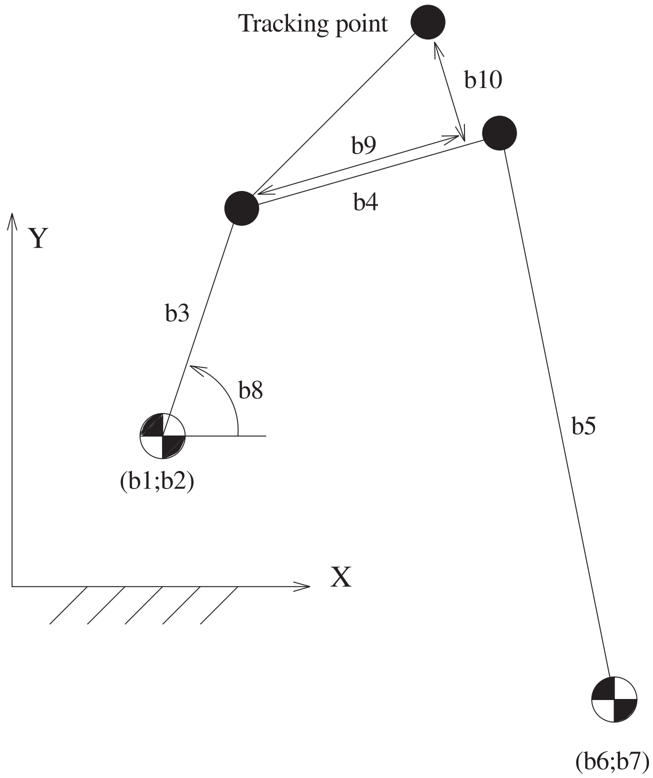

**Fig. 3** Four-bar mechanism with the design variables

**Table 1** The initial designs with the bounds on the design variables

| b | Lower | Upper | Input1 | Input2 | Input3 | Input4 |
|---|---|---|---|---|---|---|
| $b_1$ | −20.0 | 20.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $b_2$ | −20.0 | 20.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $b_3$ | 5.0 | 60.0 | 5.0 | 5.0 | 40.0 | 40.0 |
| $b_4$ | 0.1 | 60.0 | 10.0 | 10.0 | 10.0 | 10.0 |
| $b_5$ | 0.1 | 60.0 | 15.0 | 15.0 | 40.0 | 40.0 |
| $b_6$ | −20.0 | 20.0 | 10.0 | 10.0 | 10.0 | 10.0 |
| $b_7$ | −20.0 | 20.0 | −10.0 | −10.0 | 0.0 | 0.0 |
| $b_8$ | −10.0 | 10.0 | 0.1257 | 0.3539 | 0.1257 | 0.1257 |
| $b_9$ | −60.0 | 60.0 | 5.0 | 5.0 | 5.0 | 5.0 |
| $b_{10}$ | −60.0 | 60.0 | 1.0 | 1.0 | 1.0 | −35.0 |

**Table 2** Results for circle

| Input<br>] File | Initial<br>$F$ value | Optimal<br>$F$ value | Function<br>Evaluations |
|---|---|---|---|
| Input1 | 404.116505 | 0.003380 | 177 |
|  | 275.645133 | 0.003100 | 135 |
| Input3 | 2010.218841 | 0.009658 | 151 |
| Input4 | 1495.604740 | 0.056589 | 172 |

**Table 3** Final designs for circle

| b | Input1 | Input2 | Input3 | Input4 |
|---|---|---|---|---|
| $b_1$ | −8.611E−5 | 7.750E−6 | −2.665E−4 | −5.730E−4 |
| $b_2$ | −1.320E−5 | 2.625E−5 | −3.521E−5 | 2.094E−4 |
| $b_3$ | 7.000 | 7.000 | 11.470 | 10.808 |
| $b_4$ | 10.358 | 12.637 | 12.413 | 6.298 |
| $b_5$ | 17.014 | 11.145 | 4.707 | 9.778 |
| $b_6$ | 10.136 | 3.585 | 5.519E−4 | −9.616E−4 |
| $b_7$ | −9.897 | −3.734 | −1.416E−4 | −1.235E−3 |
| $b_8$ | 3.519E−6 | −1179E−5 | 2.390E−1 | 2.108E−1 |
| $b_9$ | 8.375E−6 | −6.342E−6 | 4.948 | 3.082 |
| $b_{10}$ | −8.361E−6 | −5.988E−5 | −2.369E−1 | −2.889 |

It is known that optimal synthesis problems might have local minima and for this reason four different initial designs were used for each of the three optimizations. Side constraints were also placed on all the design variables as specified in Table 1. Also listed in this table are

**Table 4** Results for ellipse

| Input File | Initial $F$ value | Optimal $F$ value | Function Evaluations | Side Constraints |
|---|---|---|---|---|
| Input1 | 475.407002 | 39.477200 | 176 | $-7$ |
| Input2 | 310.308966 | 26.628608 | 95 | $-7$ |
| Input3 | 2019.153771 | 143.526336 | 108 | – |
| Input4 | 1530.844523 | 87.500640 | 71 | – |

the values of the design variables for each of the initial designs. It should be noted that the only difference between Input1 and Input2 is the initial value of $b_8$ and between Input3 and Input4 the only difference is the initial value of $b_{10}$.

## 4.1 Circle

The first and simplest of the prescribed paths is the circle. The circle's centre point coincides with the origin of the global axis system and the circle has a diameter of 7 units. One obvious solution to this problem would be a crank with a length of 7 units, pivoting around the origin of the global axis system and with the tracking point at the end of the crank, i.e. $b_1 = b_2 = b_9 = b_{10} = 0$ and $b_3 = 7$.

The results obtained for the four different initial designs are listed in Tables 2 and 3. For the initial designs Input1 and Input2 the solution described above with the crank length $b_3$ at 7 units, is found (see Table 3). The initial and final trajectories yielded by Input1 are shown in Fig. 4. The initial designs Input3 and Input4 are impractical because the ground points are almost coincident, i.e. $b_1 \approx b_2 \approx b_6 \approx b_7 = 0$. In these mechanisms we now have two links which pivot around the origin of the global axis system. All designs satisfy the assembly and side constraints.

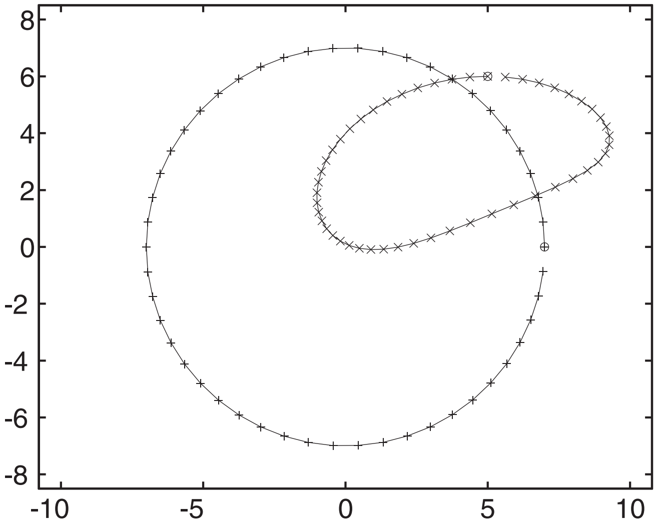

**Fig. 4** Initial (×) and final (+) trajectories obtained for the the prescribed circle with the initial design Input1

## 4.2 Ellipse

The second prescribed path is an ellipse described by

$$\theta_i = \frac{2\pi}{50} i\,, \quad i = 0, \dots, 49\,,$$

$$(x_{i_{pre}}, y_{i_{pre}}) = \left[\left(7 + \frac{7}{2}\right)\cos\theta_i\,, 7\sin\theta_i\right]\,. \tag{20}$$

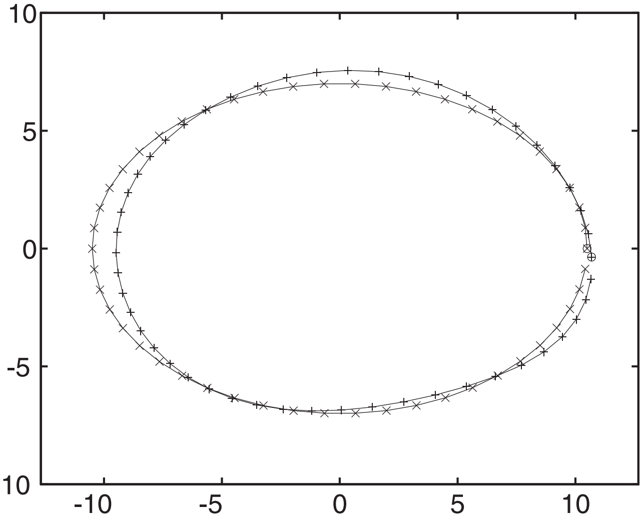

**Fig. 5** The prescribed Limason of Pascal (×) and the final trajectory obtained for the initial design Input4 (+)

**Table 5** Final designs for ellipse

| b | Input1 | Input2 | Input3 | Input4 |
|---|---|---|---|---|
| $b_1$ | $-1.471$ | $-3.266$ | 1.924E$-$1 | 5.714E$-$3 |
| $b_2$ | $-1.313$ | $-1.197$E$-$1 | 2.108E$-$1 | $-1.025$E$-$2 |
| $b_3$ | 8.882 | 9.978 | 19.304 | 28.598 |
| $b_4$ | 10.237 | 14.554 | 26.709 | 9.321 |
| $b_5$ | 18.503 | 18.998 | 8.212 | 28.568 |
| $b_6$ | $-8.995$E$-$1 | 8.930 | 7.512E$-$1 | 2.278E$-$3 |
| $b_7$ | $-20.000$ | $-19.994$ | $-3.769$E$-$1 | $-1.058$E$-$2 |
| $b_8$ | 1.266E$-$1 | 2.993E$-$2 | $-1.570$E$-$1 | 7.958E$-$1 |
| $b_9$ | 1.481 | 3.838 | 10.459 | 9.903 |
| $b_{10}$ | 2.311 | 9.625E$-$1 | $-2.331$ | $-21.126$ |

**Table 6** Results for the Limason of Pascal

| Input File | Initial $F$ value | Optimal $F$ value | Function Evaluations | Active Side Constraints |
|---|---|---|---|---|
| Input1 | 544.343052 | 250.592783 | 86 | $-3$ |
| Input2 | 556.976071 | 20.939624 | 355 | 3 |
| Input3 | 2071.161562 | 275.076024 | 56 | – |
| Input4 | 1549.821596 | 16.022744 | 76 | 5 |

The results obtained for the ellipse are listed in Tables 4 and 5. Again, all of the designs satisfy the assembly and side constraints.

The final trajectory yielded by Input2, the best design, along with the prescribed trajectory are shown in Fig. 5. It is clear that although there are discrepancies, the final trajectory is similar to the prescribed trajectory. It should be noted that $b_7$ is at its lower bound and that it is possible that a better design might result if the side constraint imposed on $b_7$ is relaxed.

The results for the remaining 3 initial designs vary greatly with all three final designs being unacceptable. The variation in the "optimal" design due to different initial designs emphasizes the need for multiple starting designs to ensure that a good design is found.

## 4.3 Limason of Pascal

The last prescribed path, a Limason of Pascal, is presented by Cossalter *et al.* (1992). The path is described by the following relations:

$$\begin{aligned}
&\alpha_i = \frac{2\pi}{50} i \qquad i = 0, \ldots, 49\,, \\
&r_i = \rho \cos\alpha_i + \frac{\rho}{2}\,, \\
&(x_{i_{pre}}, y_{i_{pre}}) = (-r_i \cos\alpha_i, -r_i \sin\alpha_i)\,, \qquad (21)
\end{aligned}$$

where $\rho = 10$. The results obtained for this path are listed in Tables 6 and 7. Here it is interesting to note the difference that the additional variable (the initial orientation $b_8$) makes for the initial designs Input1 and Input2. For Input1 the crank length $b_3$ is decreased to its lower limit where a local minimum is found but this is a poor solution. For Input2 however, an acceptable final design is found. For the initial design Input3 a poor design is generated in which the crank length $b_3$ is again decreased rather than increased as is the case for the two acceptable designs. None of the final designs violate any assembly constraints.

**Table 7** Final designs for the Limason of Pascal

| **b** | Input1 | Input2 | Input3 | Input4 |
|---|---|---|---|---|
| $b_1$ | $-2.895$ | $-4.052$ | $-2.576$ | $-4.045\text{E}{-1}$ |
| $b_2$ | $-9.356\text{E}{-1}$ | 4.834 | $-4.666\text{E}{-1}$ | 1.886 |
| $b_3$ | 5.003 | 59.995 | 5.928 | 59.692 |
| $b_4$ | 19.066 | 49.256 | 13.283 | 23.430 |
| $b_5$ | 20.274 | 38.098 | 6.722 | 59.992 |
| $b_6$ | 10.691 | $-1.452$ | 6.838 | $-1.154\text{E}{-1}$ |
| $b_7$ | 12.980 | $-2.322$ | $8.894\text{E}{-1}$ | $-1.961$ |
| $b_8$ | 3.184 | 5.329 | 2.967 | 1.844 |
| $b_9$ | 1.618 | 51.839 | $-5.030\text{E}{-1}$ | 6.091 |
| $b_{10}$ | 1.665 | $-36.523$ | $-1.139$ | $-58.645$ |

The results obtained for the Limason of Pascal are compared to those obtained by Cossalter *et al.* (1992) according to the following two measures:

$$\begin{aligned}
\epsilon_{xy_{mean}} &= \frac{1}{n} \sum_{i=1}^{n} \sqrt{(x_{i_{pre}} - x_{i_{tp}})^2 + (y_{i_{pre}} - y_{i_{tp}})^2}\,, \\
\epsilon_{xy_{max}} &= \max \sqrt{(x_{i_{pre}} - x_{i_{tp}})^2 + (y_{i_{pre}} - y_{i_{tp}})^2}\,. \qquad (22)
\end{aligned}$$

**Table 8** Comparison between the results yielded by the initial design Input4 and the results reported by Cossalter *et al.* (1992)

| | Input4 | Cossalter |
|---|---|---|
| $\epsilon xy_{\text{mean}}$ | 0.2983 | 0.250 |
| $\epsilon xy_{\text{max}}$ | 0.5129 | 0.441 |

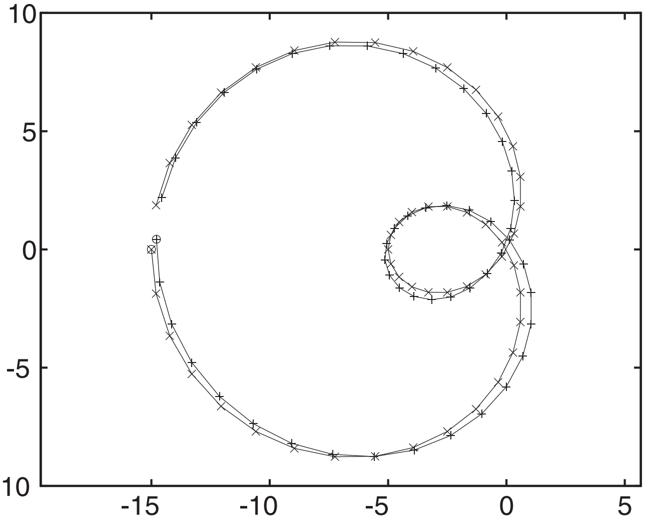

**Fig. 6** The prescribed ellipse (×) and final trajectory obtained for the initial design Input2 (+)

In Table 8 the best design obtained, i.e. Input4 (see Fig. 6), is compared to the best design reported by Cossalter *et al.* (1992). The results published by Cossalter *et al.* (1992) are marginally better. However, two points should be noted. First, their published result is the best design from 50 different initial designs that were analyzed as opposed to only four initial designs analyzed in this study. Second, 2353 iterations of the simplex method, implying a total of 2353 function evaluations, were needed to arrive at the their best design (for a single initial design) whereas a maximum of 355 function evaluations were needed to arrive at a final design in this study. On average, the designs presented here required 138 iterations to converge.

### 4.4 Discussion

For each of the prescribed paths our proposed method yielded at least one acceptable design. Furthermore, all the final designs were in the feasible region. Thus it appears that the formulation is able to produce feasible designs even if infeasible designs are encountered during the optimization. The dependence of the optimized design on the initial design is apparent for all examples and highlights the need for multiple initial designs to ensure that a good design is found.

## 5 Conclusions

A general procedure for addressing nonassembly issues in the optimal dimensional synthesis of planar mechanisms is presented. The procedure is based on an alternative formulation of the computational kinematic analysis problem in which the norm of the kinematic constraint vector $\mathbf{\Phi}$ is not equated to zero but is minimized. This ensures that the kinematic analysis can be performed even though the mechanism cannot assemble, which implies that the cost function and its sensitivity for the optimal synthesis problem is always defined. Inequality constraints for the optimal synthesis problem are defined to ensure that the final design is feasible but there is no requirement that the mechanism must remain in the feasible region during the intermediate iterations of the optimal synthesis problem. The design sensitivities are calculated analytically to make efficient use of a gradient based optimization algorithm. Three numerical examples are presented and it is shown that the method not only finds good designs but converges to these designs within an acceptable number of function evaluations. The method is general and can be expanded to allow for the optimal dimensional synthesis of any planar or spatial mechanism and include constraints on velocities, accelerations and reaction forces and torques.

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