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On design-dependent constraints and singular topologies*

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Abstract A historical perspective of design-dependent constraints and singular topologies is presented and their theoretical background as well as fundamental features discussed, together with methods for treating computational difficulties. This note contains some rather surprising new facts about singular topologies and it is hoped that it will provide both a comprehensive review and additional insights.

1 Introduction

The problem of singular topologies caused by design dependent constraints is reviewed in greater detail in this Special Issue, because they involve severe computational difficulties in topology optimization. Moreover, they may result in a complete breakdown of FSD/stress ratio type methods (Rozvany 2001).

Simple but important examples of design dependent constraints are limits on stress values. These are only valid if the cross-sectional area of a member is nonzero and become ineffective for members vanishing from the ground structure.

In this text, the history, theoretical background and basic features of singular topologies are examined, together with the treatment of computational difficulties caused by them.

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2 Historical perspective and theoretical background

2.1 Discovery of the weight reducing effect of member elimination (Sved and Ginos 1968)

The first observation of the effect of singular topologies is due to an ingenious Hungarian-born Australian researcher George Sved – the author's close friend who passed away recently. Sved demonstrated that

- in optimizing a redundant truss for stress constraints and multiple load conditions, the weight can be reduced further if some members are allowed to vanish (together with their stress constraints), and
- such an optimal solution cannot be reached through a conventional iterative procedure.

Sved and Ginos (1968) illustrated this phenomenon with a simple example. The structure under consideration was a three-bar truss (Fig. 1a), with three alternative load conditions and different permissible stresses for the members. Using mathematical programming, Schmit (1960) obtained an optimal three-bar solution for this problem (Fig. 1b), with a weight of $W = 15.985$. The optimal two-bar solution (Fig. 1c) derived by Sved and Ginos (1968) has a weight of $W = 12.814$, a weight difference of *24.7 per cent*(!)

2.2 Extensive search technique by Sheu and Schmit (1972)

Determining the optimal set of member eliminations is a topological problem involving 0-1 type optimization. To reduce the size of the problem, Sheu and Schmit (1972) established bounds on the optimal weight and thereby a reduced number of candidate designs. However, a separate optimization of all these potential topologies and selection of the best design by weight comparison is still

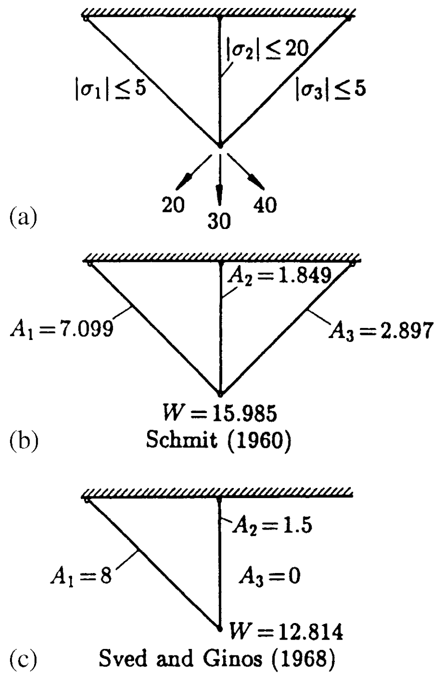

Fig. 1 (a) Three-bar truss with three alternative loads and member-dependent permissible stresses; (b) three-bar solution by Schmit (1960); (c) improved two-bar solution by Sved and Ginos (1968)

prohibitively expensive for large ground structures used in practice.

2.3 Pioneering contributions by Hajela (1982)

These findings are of considerable historic and conceptual importance and came to the author's attention only recently[1]. Hajela reached the following correct conclusions in his doctoral thesis.

- In iterative procedures (nonlinear programming) for redundant trusses with side constraints and/or multiple loading, the design "moves away from a fully stressed configuration" and converges to an "incorrect optimum".
- The cause of this is "constraint stiffening": "as some sectonal areas approach zero", "the ratio $|F_i|/\sigma_{al}$ increases[2], repelling the design away from the configuration where a member is to be dropped from the assembly".
- Example of the feasible region for such problems is also given in the thesis (Fig. 2), implying a full explanation of singular topologies and the computational difficulties caused by them.
- The following modified constraint is proposed, resulting in "constraint softening":

$$g_i' = \left(\frac{|\sigma_i|}{\sigma_{al}} - 1 \right) \frac{A_i}{A_{ref}} \leq 0 \,, \tag{1}$$

where σ_i is the stress in the member i, σ_{al} is the allowable stress, A_i is the cross-sectional area of member i and A_{ref} is "5 to 10 times the starting design value".

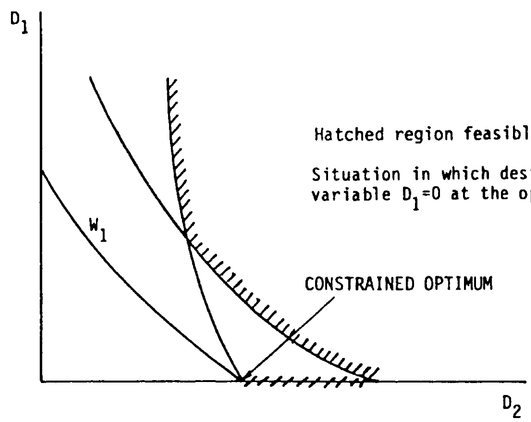

Fig. 2 Illustration in Hajela's (1982) thesis showing the correct type of feasible region for singular topologies. The original subtitle of the diagram "Optimum located in ditch. Problems arise in direct search procedures"

The modified constraint in (1) is satisfied as an equality if either $|\sigma_i| = \sigma_{al}$ or $A_i = 0$. It therefore admits *both* the main body of the (shaded) feasible region and the (shaded) line segment along $D_1 = 0$ in Fig. 2. However, it does not make the latter part ("ditch" or "spoke") wider and therefore it is different from the smooth envelope methods discussed in Sects. 2.10 and 2.11. However, it is possible that some of the approximations and "cumulative constraints" used by Hajela had a similar end effect as smooth envelope functions.

Hajela's pioneering contributions remained largely unnoticed until this year, as can be seen from the next two subsections.

2.4 Introduction of the term "singular topology" and its justification (Kirsch 1990)

The term "singular topology" was introduced by Kirsch (1990) who also made other important contributions to this topic. Kirsch

- illustrated this phenomenon lucidly with a *two-parameter example* (Fig. 3a, note variable linking for A_2), which became a popular benchmark problem later;
- pointed out that the "optimal topology might correspond to a *singular point* in the design space" (without an exact definition of "singular");

[1] Prabhat Hajela sent only a copy of a few pages of his Ph.D. Thesis to the author and the above review had to be inferred from these excerpts

[2] F_i is the force in member i and σ_{al} is the allowable stress in Hajela's thesis

- demonstrated that a "statically determinate" approximation (in which compatibility conditions are ignored, as in "plastic design") may generate the correct optimal topology even in the case of singularity.

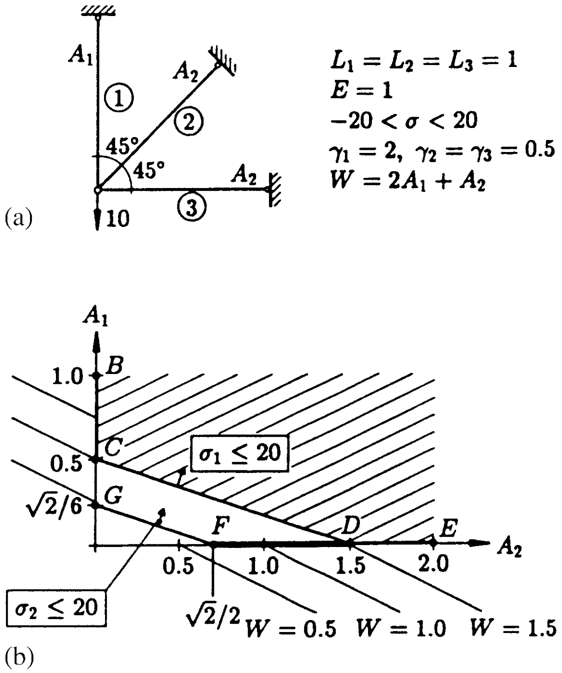

Fig. 3 Kirsch's (1990) example of singular optimal topology. (a) Problem statement; (b) design space

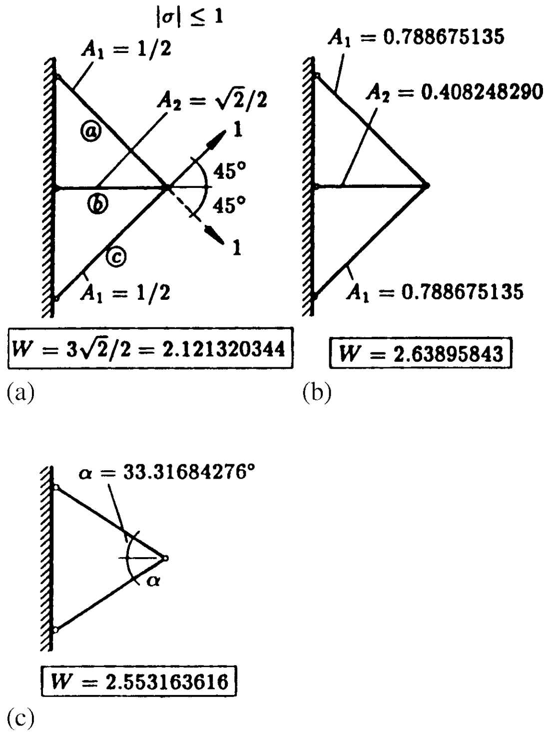

Fig. 4 "Statically determinate" approximation. (a) Problem statement and optimal "plastic" design (neglecting compatibility); (b) elastic design based on optimal "plastic" topology; (c) true optimal elastic design (after Rozvany and Birker 1994)

2.5 Limitations of the "statically determinate" approximation for deriving singular topologies (Rozvany and Birker 1994)

The limitations of the method based on ignoring elastic compatibility were pointed out in the above paper. The "statically determinate" approximation (or "plastic design") indeed does generate the correct topology *if* the resulting layout is statically determinate but can yield an entirely nonoptimal solution if this is not the case.

Based on detailed proofs of *global* optimality by Rozvany and Birker (1994), the optimal plastic design for two alternative loads is shown in Fig. 4a, in which elastic compatibility is not enforced. The best elastic design with the same topology is shown in Fig. 4b and the true elastic optimal design (with two bars) in Fig. 4c.

It will be seen that that the optimal "plastic " design in Fig. 4a is entirely different from the true optimal elastic design in Fig. 4c and the best elastic design based on the topology derived by ignoring elastic compatibility is uneconomical (Fig. 4b).

2.6 "Connectedness" of the feasible set and general properties for singular topologies (Cheng and Jiang 1992; Rozvany and Birker 1994)

It was pointed out by Cheng and Jiang (1992) that for singular topologies the optimal solution is *connected* to the main body of the feasible set by an *infinitesimally narrow* strip of the design space. Since the above authors (and others in the field) were clearly unaware of Hajela's (1982) prior contributions, this (re)discovery represented considerable progress in dealing with computational problems caused by singular topologies.

The *basic features* of singular topologies can be seen in Fig. 3b which shows the design space for the problem in Fig. 3a. The optimal solution is at point F and the main body of the feasible set is BCDE (shaded area). However, the line segment FD is also contained in the feasible set, because the stress constraint for $\sigma_1 \leq 20$ is not valid any more if $A_1 = 0$.

In general, in the case of singular topologies the feasible set in n-dimensional design space consists of

- a "solid" n-dimensional region ("main body" of the feasible set),
- some connected k-dimensional hyperplane segments with $(k < n)$, ("spokes"),

- with the optimal solution located in one of these infinitesimally narrow "spokes".

A similar description of singular topologies was given by Rozvany and Birker (1994).

2.7
Computational difficulties caused by singular topologies

The main difficulty in iterative computational procedures is caused by the narrowness of the "spokes" of the feasible set. In Fig. 3b, for example, the iterative procedure would first have to find point D (which represents a 50% higher weight than point C) and then one would have to move exactly in the direction of the line segment DF. In partially randomized search processes this is practically impossible without modifications of the solution strategy.

2.8
Insights arising from the exact layout theory (Rozvany and Birker 1994)

The so-called "exact layout theory" is a generalization of Michell's (1904) truss theory by Rozvany and Prager (e.g. 1977), for a detailed review see Rozvany, Bendsøe and Kirsch (1995). It uses analytical methods and an infinite number of members in the ground structure. Important features of this theory are as follows (e.g. Prager and Rozvany 1977).

- For each optimization problem, there exists an "adjoint" strain field, which is defined for all points of the structural domain (even along vanishing, i.e. nonoptimal members of zero cross-sectional area).
- Optimality criteria must also be satisfied, in the form of inequalities, for vanishing members (i.e. along any line segment within the structural domain).
- Multiplying the "real" strain for a vanishing member by Young's modulus E, we obtain the equivalent of the "limiting stress", introduced later by Cheng and Jiang (1992).

In the above paper, Rozvany and Birker (1994)

- derived singular topologies for exact truss layout problems (ground structures with an infinite number of elements);
- explored deeper reasons for singularity (see Sect. 3);
- outlined classes of problems which can never have singular optima.

It is a pity that very few readers familiarize themselves with the basic concepts of optimal layout theory which would be necessary (and well worth it) for gaining a deeper insight into singular topologies.

2.9
Modified constraints for the admission of "spokes" of the feasible set (Cheng 1995)

In the above paper, the following modified constraint was suggested (in our notation):

$$(\sigma_i - \sigma_{\mathrm{al}})A_i \leq 0\,, \tag{2}$$

in which σ_{al} is the allowable stress. It will be seen that (2) is an unscaled version of (1), which admits but does not widen the "spokes" of the feasible set. Again, since Hajela's pioneering contributions were not generally known at the time, Cheng's proposal was a major step in the right direction.

2.10
Introduction of smooth envelope functions (SEF's) for stress constraints

The causes of singular topologies and their avoidance is explained in Fig. 5 which, as explained subsequently, orginates from 1994. Singularity is due to the fact that, once a cross-sectional area becomes zero, the permissible stress σ_{al} in the corresponding stress constraints jumps suddenly from a finite value (the permissible stress σ_{al0} for tensile members) to infinity. The above problem is even more severe for compression members with local buckling, because in that case the permissible stress jumps from a near-zero value (very slender member) to infinity (vanishing member). The relation between the permissible stress σ_{al} and cross-sectional area A_i or member force F_i is shown graphically for tension and compression members in Figs. 5a and b. A method for making the feasible set nonsingular involves the use of smooth envelope functions (broken lines in Fig. 5) to replace the discontinuous functions $\sigma_{\mathrm{al}}(A_i)$.

This proposal, together with Fig. 5, was first put forward in an extended abstract submitted to an AIAA/NASA/USAF/ISSMO meeting (Rozvany and Kirsch 1994). The author was a cochairman of that meeting and due to restriction to one presentation by an author the full paper was not presented. However, the same technique was proposed in the author's contribution to a feature article (Rozvany, Bendsøe and Kirsch 1995, February, submitted in 1994, p. 72):

> "This difficulty" (i.e. singularity) "could be solved by using smooth envelope functions (SEF's) for rounding the above mentioned discontinuity (e.g. Rozvany and Sobieszczanski-Sobieski 1992)" ... "increasing the permissible stress as the cross-section becomes very small".

The above quotation makes it clear that the author proposed a KS function type envelope, which was discussed in detail for optimality criteria type methods by Rozvany and Sobieszczanski-Sobieski (1992).

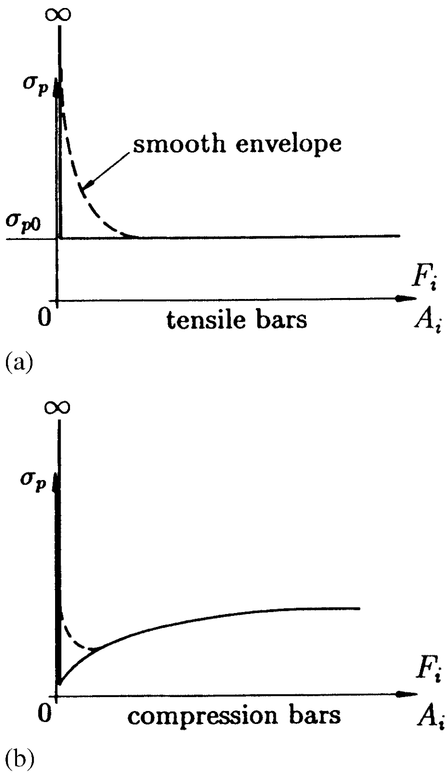

Fig. 5 The relation between allowable stress ($\sigma_{\rm al}$) and cross-sectional (A_i) or member force (F_i) in topology optimization (continuous line) and the corresponding smooth envelope functions (broken line). (a) Tension bars; (b) comparison bars (after Rozvany and Kirsch 1994; Rozvany 1996a)

KS functions were used extensively by the author's research group for stress and local buckling constraints (e.g. Birker 1996).

The proposed smooth envelope function method and Fig. 4 were repeated in a paper by the author (Rozvany 1996a), in which an exponential envelope function (similar to KS functions) of the form

$$\sigma_{\rm al} = \sigma_{\rm al0} \exp(A_0/A_i)\,, \tag{3}$$

was suggested, where A_0 is a small prescribed value.

The effect of using smooth envelope functions for Kirsch's (1990) problem (see Fig. 3 herein) can be seen in Fig. 6, in which the feasible set is clearly nonsingular due to widening of its "spoke" that contains the optimal solution.

In all SEF methods, one starts off with a highly smooth envelope function, and later a shape parameter is changed progressiely to make the corners sharper and sharper. This is necessary

- to try to avoid the wrong local minimum (by using initially highly smooth envelope function), and
- to avoid inaccuracies of the approximation (by using sharp corners at the end, with small deviation from the true constraints).

In the methods discussed here and in Sect. 2.11, this procedure (called also "*continuation method*") is achieved by increasing the parameter ϱ progressively in KS functions, or decreasing the A_0-value in (3) or the ε-value in (4).

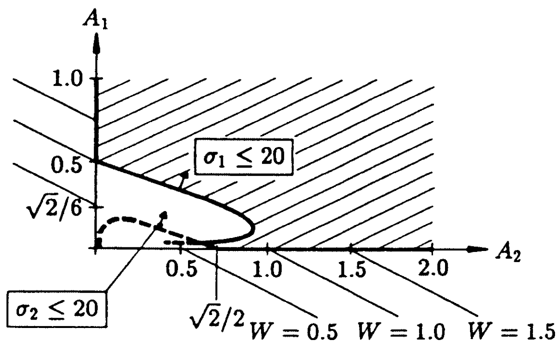

Fig. 6 The effect of smooth envelope functions on the feasible set in Kirsch's (1990) example (after Rozvany 1997, presented in June 1996)

2.11 The "ε-relaxation" method (Cheng and Guo 1997) as a special case of smooth envelope functions (SEF's)

Cheng and Guo (1997) introduced the so-called "ε-relaxation" method, in which the stress constraint in (2) is changed to (in our notation)

$$(\sigma_i - \sigma_{\rm al0})A_i - \varepsilon \leq 0\,. \tag{4}$$

Clearly, (4) implies the *reciprocal* relation for the varying allowable stress value

$$\sigma_{\rm al}(A_i) = \sigma_{\rm al0} + \varepsilon/A_i\,, \tag{5}$$

which is one possible "envelope function" introduced in Fig. 5.

Using $\sigma_{\rm al0} = 1$, $A_0 = 0.4$ in (3) and $\sigma_{\rm al0} = 1$, $\varepsilon = 0.5$ in (5) we obtain the smooth envelope functions shown in Fig. 7. It can be seen that these two curves are almost indentical (and could be brought even closer). It can be concluded that the "ε-relaxation" method represents one of many possible smooth envelope functions, which are very similar. However, Cheng's milestone contributions to the field of singular topologies are of outstanding importance, because he not only developed independently a smooth envelope function method (the ε-relaxation technique) but also studied this method intensively on test examples and introduced additional refinements, such as the "extrapolation approach" (e.g. Guo and Cheng 2000).

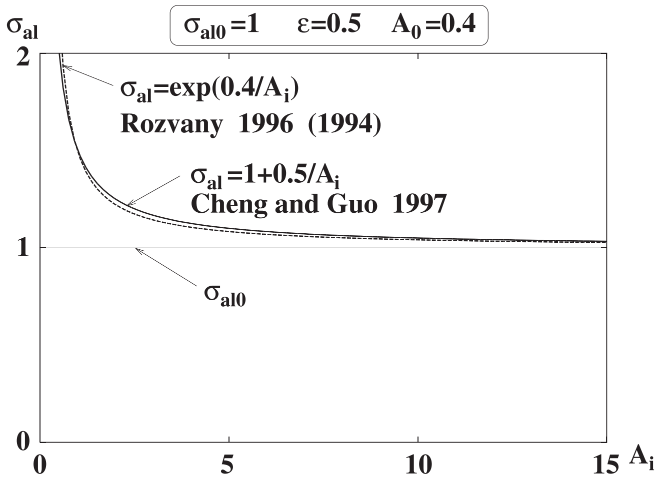

Fig. 7 Comparison of exponential and reciprocal smooth envelope functions for avoiding singularities

2.12 Limitations of smooth envelope function ($=\varepsilon$-relaxation) methods

It has been shown by Svanberg and Stolpe [2000, see also Stolpe and Svanberg (2001)] that a smooth envelope function (SEF) method may not converge to the correct global optimal solution. Some deeper reasons for the occasional failure of envelope methods and possible new ways of handling singular topologies are discussed in Sect. 3.

3 Deeper reasons for the difficulties caused by singular topologies – simply explained

3.1 Advantages of exact analytical methods and solutions in explaining and removing computational difficulties

Exact analytical methods and solutions are useful owing to the following features.

- They provide a proof of global optimality of a topology – out of all possible configurations. This can be used for checking the validity and convergence of numerical methods.
- They are based on ground structures consisting of an infinite number of potential members – finite ground structures often modify and thereby obscure the true nature of the exact optimal solution.
- Discretized methods often generate a nonoptimal solution due to various computational pitfalls and difficulties.

The primary aim of this research field, however, is the development of reliable and robust numerical methods.

Since few readers are familiar with the rather intricate concepts of optimal layout theory, we will try to explain the fundamental causes of singularity through a simple example.

3.2 A useful benchmark example of an exact optimal topology (Rozvany and Birker 1994)

The example discussed in this subsection can be used for

- understanding the fundamental reasons for singularity and
- checking on various methods for locating singular topologies.

Most other benchmark examples in this field have some artificial features, such as member-dependent permissible stresses (e.g. Sved and Ginos 1968) or cost functions not proportional to the given member lengths (Kirsch 1990). In the problem in Fig. 8a, we have a vertical support, a single vertical point load and the stress constraint

$$-1/3 < \sigma_i < 1\,. \tag{6}$$

The truss topology is to be optimized for minimum weight.

This problem is more natural in the sense that the allowable stresses are *not* member-dependent and the cost (weight) *is* proportional to the member lengths. Moreover, the smaller allowable stress value in compression represents some allowance for local buckling.

It was shown analytically on the basis of optimal layout theory (Rozvany and Birker 1994) that the optimal topology for this problem consists of two bars at 30° and 60° to the vertical (Fig. 8a). This global optimal solution has the member areas and total weight:

$$A_1 = \sqrt{3}/2 = 0.8660\,,$$

$$A_3 = 1.5\,, \quad W = 2\sqrt{3} = 3.4641\,. \tag{7}$$

Moreover, it was proved by Rozvany and Birker (1994) that any member radiating from the point load within the shaded cone in Fig. 8a would have a greater strain than 1 [the upper limit in (6)]. This means that for any ground structure containing such member(s) the optimal topology must be singular.

A simple example of such singular problem has one additional member (2) at 30° to the horizontal (Fig. 8b). It can be easily checked that in the optimal topology with $A_2 \to 0$, the strain in member 2 is $\varepsilon = 4/3 = 1.3333$, which corresponds to a greater stress than the allowable one in (6). This means that any stress ratio type (zero-order) method would produce large increases in the size of member 2 if it had a small size in the previous iteration. This explains Hajela's (1982) observation about

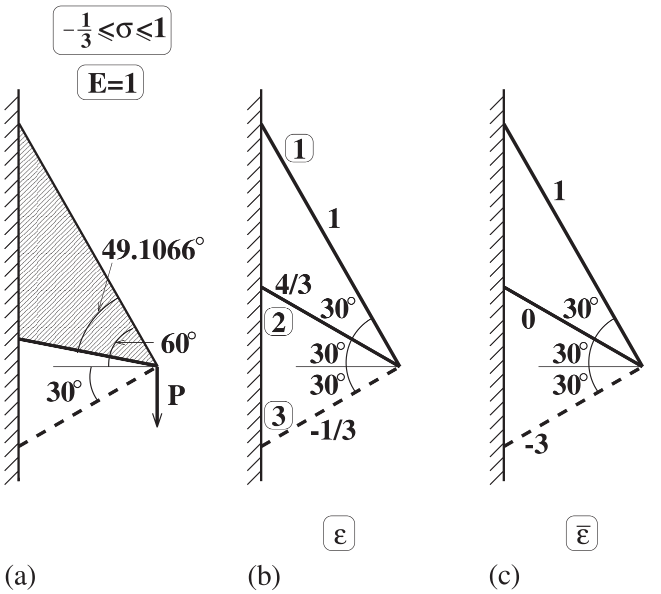

Fig. 8 A benchmark example for singular topologies. (a) Problem statement and exact optimal topology with shaded area showing potential singular members; (b) three-member ground structure with real strains ε; (c) adjoint strains $\bar{\varepsilon}$

"constraint stiffening" or "repelling the design away from the configuration where a member is to be dropped from the assembly".

The solution in Fig. 8a was first derived by Rozvany and Gollub (1990). It was also used as an illustration of an error in Michell's theory (Rozvany 1996b). The same solution was also confirmed by Duysinx (1999), who used a perforated plate formulation.

3.3 Reasons for singularity on the basis of optimal layout theory

Simply stated, optimal layout theory (e.g. Rozvany 1989; Rozvany, Bendsøe and Kirsch 1995) uses optimality criteria in which some "layout function"

- must take on given limiting values along optimal members, and
- must not exceed the above values for vanishing members.

For the elementary stress-based problem in Fig. 8, the layout function is the "adjoint" strain $\bar{\varepsilon}$ and the limiting values are (corresponding to $1/\sigma_{\mathrm{al}}$)

$$\text{in tension:} \quad \bar{\varepsilon} \leq 1.0\,, \quad \text{in compression:} \quad \bar{\varepsilon} \geq -3.0\,. \quad (8)$$

The adjoint strain field, which must be "kinematically admissible"[3], is shown in Fig. 8c. It can be seen that adjoint strains in the optimal members (1 and 3) take on the above limiting values and the adjoint strain in the vanishing member 2 can be easily calculated as $\bar{\varepsilon} = 0$. This means that the above (sufficient) condition for layout optimality is satisfied.

We can draw *two important observations* from the above example.

First, if our redesign formula is based on real stresses/strains (as in the stress ratio method), then we are "barking up the wrong tree" because in this problem the *adjoint strain* values determine which members are optimal. An improved numerical technique could utilize the latter quantities in locating the optimal topology.

Second, in optimal layout theory and optimality criteria methods *adjoint strains* are defined also *for vanishing members* and must be smaller than or equal to their limiting values. It follows, for example, that for so-called self-adjoint problems we never have singular topologies in the optimal solution (Rozvany and Birker 1994). Moreover, limits on the the adjoint strains are not affected by the fact that a member vanishes from our design(!).

4 Other computational difficulties in topology optimization

It was pointed out by Zhou (1996) that so-called *"compressive chains"* cause severe computational difficulties in truss topology optimization for stress and *local buckling constraints*. Similar problems occur if *global buckling constraints* are added to the formulation (Rozvany 1996b). It was suggested by Rozvany and Zhou (1996, p. 1126) already five years ago that *ground structures with overlapping members* could be used for overcoming such difficulties. ("Optimal hinge cancellations could ... also be achieved by an *iterative* procedure with overlapping short and long bars along the same line in the ground structure".) However, numerical experiments have shown that such an iterative procedure tends to move away from the correct solution.

The suggestion of overlapping bars in ground structures (Rozvany and Zhou 1996) was followed up by some outstanding research in this area (e.g. Achtziger 2000; Cheng *et al.* 2000).

5 Concluding remarks

The history and fundamental features of conditional constraints and singular topologies were reviewed.

The causes of singular topologies can be explained at two levels.

- One can observe that singularities are caused by a *discontinuity* in the allowable stress σ_{al}/member area A_i

[3] satisfying kinematic boundary and continuity conditions

relation. The singularity can be removed by employing smooth envelope functions (e.g. KS functions, exponential functions or ε-relaxation = reciprocal functions) which may not always help in finding the global optimum.

- The above discontinuity in the allowable stress function does not necessarily cause a singularity. For delimiting classes of problems with singularity, one must examine *deeper reasons* for singular topologies, namely the correct layout optimality criteria and the possible differences between the actual strains and the so-called "adjoint" strains in the members of the structure.

In view of the detailed historical study in this text, the author would like to appeal to researchers of the field to acknowledge prior contributions in their publications.

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