Research papers

Struct Multidisc Optim 21, 128–139

On the trajectories of penalization methods for topology optimization

M. Stolpe, K. Svanberg

Abstract We consider the discretized zero-one minimum compliance topology optimization problem of elastic continuum structures under multiple load conditions. The binary design variables indicate presence or absence of material in the finite elements. A common approach to solve these problems is to relax the binary constraints, i.e. allow the design variables to attain values between zero and one, and penalize intermediate values to obtain a "black and white" (zero-one) design. To avoid convergence to a local minimum, it has been suggested that a continuation method should be used, where the penalized problems are solved with increasing penalization.
In this paper, the trajectories associated with optimal solutions to the penalized problems, for continuously increasing penalization, are studied on some carefully chosen examples. Two different penalization techniques are used. The global trajectory is defined as the path followed by the global optimal solutions to the penalized problems, and we present examples for which the global trajectory is discontinuous even though the original zero-one problem has a unique solution. Furthermore, we present examples where the penalization method combined with a continuation approach fails to produce a black and white design, no matter how large the penalization becomes.

Key words topology optimization

1 Introduction

The worst case minimum compliance topology optimization problem of an elastic structure subject to volume constraints is considered. The infinite dimensional problem consists in finding a subdomain with prescribed volume within a predefined design domain such that the maximum compliance over all load conditions is minimized. The design variable is a density function defined on the design domain with value one on the subdomain and zero elsewhere. In practice, the topology optimization problem is treated by discretizing the design domain into n finite elements and approximating the density function to be constant in each element. It is well-known that the underlying infinite dimensional problem lacks a solution in general. It is possible to construct a design with lower compliance by increasing the number of holes, and hence the set of feasible designs is not closed. When solving the discretized zero-one problem, the non-existence of solutions is evident, since the designs become more complex as the number of finite elements is increased. This phenomenon is commonly referred to as mesh-dependence.

To obtain a well-posed problem, restrictions on the variation of the density may be imposed. One method is to control the perimeter of the design (see Ambrosio and Buttazzo 1993; Haber *et al.* 1996). Another approach is to introduce local gradient constraints on the density variation (Petersson and Sigmund 1998; Zhou *et al.* 1999). Other numerical problems in topology optimization such as checkerboards, where alternating regions of solid material and void are formed can also be eliminated by the restriction methods mentioned above. For an overview of numerical instabilities in topology optimization such as mesh-dependence and checkerboards, see Sigmund and Petersson (1998) and references therein. For an overview of topology optimization of discrete and continuum structures see Bendsøe (1995) and Rozvany *et al.* (1995).

To avoid cumbersome branch and bound techniques when solving the discretized zero-one problem, it has been suggested that the binary constraints should be relaxed, i.e. the design variables should be allowed to attain values between zero and one, and the intermediate values should be penalized to obtain a "black and white" (zero-one) design. Several penalization techniques have been suggested. In the SIMP approach (Solid Isotropic Microstructure with Penalization) (Bendsøe 1989; Rozvany *et al.* (1992), a power-law material model is used,

Recevied January 28, 2000
Revised manuscript received February 28, 2000

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where intermediate densities give very little stiffness in comparison to their weight. Another penalization technique is to add a penalty function that suppresses intermediate densities to the objective function. Numerical examples using this technique are presented by Haber *et al.* (1996), Allaire and Francfort (1993), and Allaire and Kohn (1993).

Since the penalized problems in general are nonconvex, with a possibly large number of local minima, the use of continuation methods has been proposed to increase the possibility to obtain a global optimal solution to the zero-one problem. The idea is to start with no penalization, solve the convex variable thickness sheet problem, increase the amount of penalization, solve the possibly nonconvex penalized problem, using the solution to the previous problem as starting point, and continue this until the penalization is sufficiently large. Continuation methods are used in minimum compliance optimization by Allaire and Kohn (1993), Allaire and Francfort (1993), Sigmund (1994), and Petersson and Sigmund (1998). Proof of existence of solutions and proof of convergence for a class of penalization techniques are given by Petersson (1999), under the assumption that global optimal solutions to the penalized problems can be obtained. Rietz (1999) shows that under certain restrictive assumptions a finite penalization in SIMP is in some sense sufficient.

The purpose of this paper is to study the behaviour of the analytical and computational trajectories, defined as the path followed by the solutions to the penalized problem as the penalization is increased.

The paper is organized as follows. In Sect. 2, we state the original zero-one problem and the penalized problems and define the trajectory of global optimal solutions to the penalized problems. In Sect. 3, we consider a certain two-dimensional truss structure and show that the trajectory of global optimal solutions may be discontinuous. In Sect. 4, we show that similar behaviour may occur also when considering discretized continuum structures. In Sect. 5, we study the behaviour of the SIMP approach when applied to an example with only one load condition.

1.1 Notations and basic relations

The symmetric positive semi-definite stiffness matrix is defined as

$$\mathbf{K}(\mathbf{x}) = \sum_{j=1}^{n} x_j \mathbf{K}_j \in I\!R^{d\times d}\,, \tag{1}$$

where x_j is a binary variable denoting presence ($x_j = 1$) or absence ($x_j = 0$) of material in the j-th element, n is the number of elements, and d is the number of degrees of freedom of the structure; $\mathbf{K}_j$ denotes the symmetric positive semi-definite j-th local stiffness matrix in global coordinates, divided by the density of solid material $\rho_j^{\max}$. The vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m \in I\!R^d$ are the given external loads, and $\mathbf{u}_1, \ldots, \mathbf{u}_m \in I\!R^d$ are the corresponding nodal displacement vectors. The displacement vectors are connected to the external loads through the equilibrium equations

$$\mathbf{K}(\mathbf{x})\mathbf{u}_\ell = \mathbf{f}_\ell, \quad \ell = 1, \ldots, m\,. \tag{2}$$

2 Problem formulation

We consider the discretized worst case minimum compliance topology optimization problem subject to perimeter and simple resource constraints

$$(\mathcal{P}):\ \min_{x,u_\ell}\ \max_{1\le\ell\le m} \{\mathbf{f}_\ell^T \mathbf{u}_\ell\}\,,$$

$$\text{s.t. } \mathbf{K}(\mathbf{x})\mathbf{u}_\ell = \mathbf{f}_\ell\,, \quad \ell = 1, \ldots, m\,,$$

$$P(\mathbf{x}) \le P^{\max}\,, \quad \sum_{j=1}^{n} x_j = M\,,$$

$$x_j \in \{0, 1\}\,, \quad j = 1, \ldots, n\,, \tag{3}$$

where $P(\mathbf{x})$ is the perimeter function; M is assumed to be a given positive integer.

In the SIMP approach the binary constraints are relaxed and intermediate values are penalized by replacing x_j by x_j^p in the equilibrium equations. For a given penalization parameter $p \ge 1$, the problem is formulated as

$$(\mathcal{P}_p):\ \min_{x,u_\ell}\ \max_{1\le\ell\le m} \{\mathbf{f}_\ell^T \mathbf{u}_\ell\}\,,$$

$$\text{s.t. } \mathbf{K}(\mathbf{x}^p)\mathbf{u}_\ell = \mathbf{f}_\ell\,, \quad \ell = 1, \ldots, m\,,$$

$$P(\mathbf{x}) \le P^{\max}\,, \quad \sum_{j=1}^{n} x_j = M\,,$$

$$0 \le x_j \le 1\,, \quad j = 1, \ldots, n\,, \tag{4}$$

where $\mathbf{x}^p = (x_1^p, \ldots, x_n^p)^T$. When $p = 1$, $(\mathcal{P}_p)$ corresponds to the variable thickness sheet problem.

In the quadratic penalty approach a concave quadratic penalty function is added to the objective function in order to force the variables to become either zero or one. For a given penalization factor $q \ge 0$, the problem statement is

$$(\mathcal{P}_q):\ \min_{x,u_\ell}\ \max_{1\le\ell\le m} \{\mathbf{f}_\ell^T \mathbf{u}_\ell\} + q\sum_{j=1}^{n} x_j(1 - x_j)\,,$$

$$\text{s.t. } \mathbf{K}(\mathbf{x})\mathbf{u}_\ell = \mathbf{f}_\ell\,, \quad \ell = 1, \ldots, m\,,$$

$$P(\mathbf{x}) \le P^{\max}\,, \quad \sum_{j=1}^{n} x_j = M\,,$$

$$0 \le x_j \le 1\,, \quad j = 1, \ldots, n\,. \tag{5}$$

When $q = 0$, $(\mathcal{P}_q)$ corresponds to the variable thickness sheet problem.

We assume that $\mathbf{K}(\mathbf{x})$, and $\mathbf{K}(\mathbf{x}^p)$ are positive definite for all vectors $\mathbf{x}$ such that $0 \le x_j \le 1$ for $j = 1, \ldots, n$, and $\sum_{j=1}^{n} x_j = M$. This is in practice often guaranteed by assigning a small lower bound > 0 on the variables x_j, but that is not necessary in our examples below. Then a nested problem in the variables $\mathbf{x}$ only can be obtained by defining the function $\mathbf{u}_\ell(\mathbf{x})$ by $\mathbf{u}_\ell(\mathbf{x}) = \mathrm{K}^{-1}(\mathbf{x})\mathbf{f}_\ell$ for $\ell = 1, \ldots, m$. The compliance under load condition ℓ is denoted by $c_\ell(\mathbf{x}) = \mathbf{f}_\ell^T \mathbf{u}_\ell(\mathbf{x})$. The penalized compliances are denoted by

$$c_{\ell,p}(\mathbf{x}) = \mathbf{f}_\ell^T \mathbf{u}_\ell(\mathbf{x}^p)\,, \tag{6}$$

and

$$c_{\ell,q}(\mathbf{x}) = \mathbf{f}_\ell^T \mathbf{u}_\ell(\mathbf{x}) + q \sum_{j=1}^{n} x_j(1 - x_j)\,. \tag{7}$$

The objective functions for $(\mathcal{P}_p)$ and $(\mathcal{P}_q)$ are denoted by $c_p(\mathbf{x})$ and $c_q(\mathbf{x})$ and defined as

$$c_p(\mathbf{x}) = \max_{1 \le \ell \le m} \{c_{\ell,p}(\mathbf{x})\}\,, \quad c_q(\mathbf{x}) = \max_{1 \le \ell \le m} \{c_{\ell,q}(\mathbf{x})\}\,. \tag{8}$$

The nonempty set of global optima to the nested problem $(\mathcal{P}_p)$ is denoted by $\overline{\mathcal{X}}_p$ and we let $\overline{\mathbf{x}}(p)$ denote a trajectory of global optimal solutions to $(\mathcal{P}_p)$ generated when p is increased continuously, i.e. $\overline{\mathbf{x}}(p) \in \overline{\mathcal{X}}_p$ for all $p \ge 1$. Similarly, the nonempty set of global optima to the nested problem $(\mathcal{P}_q)$ is denoted by $\overline{\mathcal{X}}_q$ and a trajectory of global optimal solutions to $(\mathcal{P}_q)$ generated when q is increased continuously is denoted by $\overline{\mathbf{x}}(q)$. Throughout, these trajectories will be called the global trajectories.

3
A truss example

The first example under study is the six-bar truss structure shown in Fig. 1. The bars in the truss are divided into two groups, such that x_j denotes the common design variable for the bars in the j-th group.

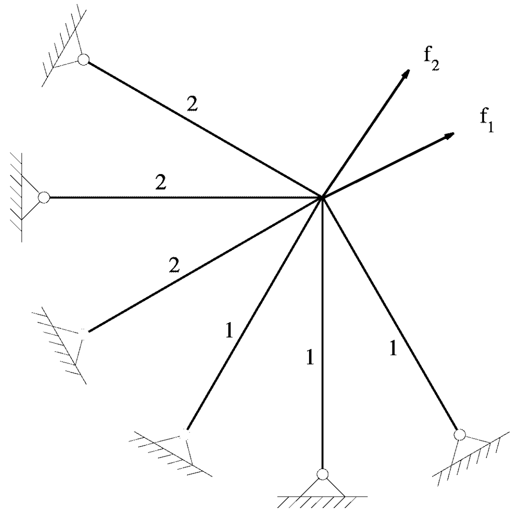

Fig. 1 Six-bar truss

There are several benefits of using variable linking in this six-bar example:

- the number of variables is reduced so that we can illustrate the results more easily;
- the stiffness matrix becomes diagonal, which allows us to obtain analytical expressions for the trajectories and local optima; and
- the local stiffness matrices are no longer rank one, so that the situation resembles topology optimization of continuum structures.

Throughout, the following resource limits are used

$$P^{\max} = 1 \text{ and } M = 1\,. \tag{9}$$

Assuming unit length of the bars, unit cross-sectional areas, and unit modulus of elasticity, the stiffness matrix for the six-bar truss is given by

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} x_1 + 5x_2 & 0 \\ 0 & 5x_1 + x_2 \end{pmatrix}\,. \tag{10}$$

The example consists of a family of problems parameterized by the scalar α. The parameter enables us to control the number, as well as the positions, of the global optimal solutions. The load conditions for this example are defined as

$$\mathbf{f}_1 = \sqrt{\alpha}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\,, \ \mathbf{f}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\,, \tag{11}$$

where α is assumed to satisfy $2.3 \le \alpha \le 2.9$.

The perimeter constraint becomes $|x_1 - x_2| \le 1$ since the interface length is equal to one. The constraint is redundant since $|x_1 - x_2| \le 1$ for all $0 \le x_1, x_2 \le 1$, and it is therefore removed. If $P^{\max} < 1$ it is not possible to obtain a black and white solution.

The nested example problem is given by

$$(\mathcal{P}^\alpha): \ \min \ \max\{c_1(\mathbf{x}), c_2(\mathbf{x})\}\,,$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad x_1, x_2 \in \{0,1\}\,, \tag{12}$$

where the compliances are given by

$$c_1(\mathbf{x}) = \frac{8\alpha}{x_1 + 5x_2} + \frac{2\alpha}{5x_1 + x_2}\,, \tag{13}$$

and

$$c_2(\mathbf{x}) = \frac{8}{x_1 + 5x_2} + \frac{18}{5x_1 + x_2}\,. \tag{14}$$

The global optimal solutions to $(\mathcal{P}^\alpha)$ are given by

$$\overline{\mathbf{x}} = \begin{cases} (1,0)^T \text{ and } (0,1)^T & \text{if } \alpha = 7/3, \\ (0,1)^T & \text{if } \alpha > 7/3, \\ (1,0)^T & \text{if } \alpha < 7/3\,. \end{cases} \tag{15}$$

3.1 The SIMP approach

The example problem penalized using the SIMP approach can be formulated as

$$(\mathcal{P}_p^\alpha): \ \min \ \max\{c_{1,p}(\mathbf{x})\,,\ c_{2,p}(\mathbf{x})\}$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad 0 \le x_1, x_2 \le 1\,, \tag{16}$$

where the penalized compliances are given by

$$c_{1,p}(\mathbf{x}) = \frac{8\alpha}{x_1^p + 5x_2^p} + \frac{2\alpha}{5x_1^p + x_2^p}\,, \tag{17}$$

and

$$c_{2,p}(\mathbf{x}) = \frac{8}{x_1^p + 5x_2^p} + \frac{18}{5x_1^p + x_2^p}\,. \tag{18}$$

Since $x_2 = 1 - x_1$ at all feasible solutions to $(\mathcal{P}_p^\alpha)$, the variable x_2 can be eliminated, and the compliances become univariate functions of the single variable x_1. These compliances, again denoted by $c_{1,p}$ and $c_{2,p}$, are given by

$$c_{1,p}(x_1) = \frac{8\alpha}{x_1^p + 5(1-x_1)^p} + \frac{2\alpha}{5x_1^p + (1-x_1)^p}\,, \tag{19}$$

and

$$c_{2,p}(x_1) = \frac{8}{x_1^p + 5(1-x_1)^p} + \frac{18}{5x_1^p + (1-x_1)^p}\,. \tag{20}$$

In Figs. 2 and 3, $c_{1,p}(x_1)$ and $c_{2,p}(x_1)$ are shown for various values on p and α. As indicated in these figures there exists a point $x_1^*(p)$ such that $c_{1,p}(x_1^*(p)) = c_{2,p}(x_1^*(p))$. This point is given by

$$x_1^*(p) = \frac{(98-18\alpha)^{1/p}}{(42\alpha-58)^{1/p} + (98-18\alpha)^{1/p}}\,. \tag{21}$$

For all $p \ge 1$ it holds that

$$x_1^*(p) = \begin{cases} < 0.5 & \text{if } \alpha > 2.6, \\ > 0.5 & \text{if } \alpha < 2.6, \\ = 0.5 & \text{if } \alpha = 2.6\,. \end{cases} \tag{22}$$

Further, for each $\alpha \in [2.3, 2.9]$, $x_1^*(p) \to 0.5$ as $p \to \infty$.

Let $\mathbf{x}^*(p) = (x_1^*(p), x_2^*(p))^T = (x_1^*(p), 1 - x_1^*(p))^T$. Then, $\mathbf{x}^*(p) \to (0.5, 0.5)^T$ as $p \to \infty$.

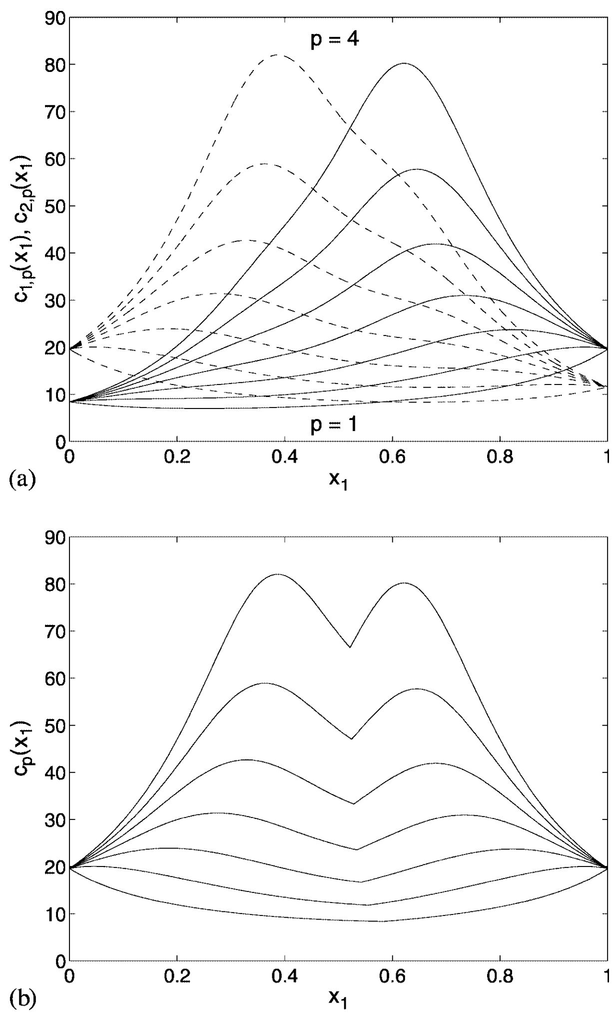

Fig. 2 The penalized compliances for $\alpha = 7/3$. (a) $c_{1,p}(x_1)$ (solid) and $c_{2,p}(x_1)$ (dashed) for $p = 1.0, 1.5, 2.0, \ldots, 4.0$; and (b) $c_p(x_1) = \max\{c_{1,p}(x_1), c_{2,p}(x_1)\}$

Since $c_{1,p}(x_1) < c_{2,p}(x_1)$ if $x_1 < x_1^*(p)$ and $c_{1,p}(x_1) > c_{2,p}(x_1)$ if $x_1 > x_1^*(p)$, the objective function to $(\mathcal{P}_p^\alpha)$ is

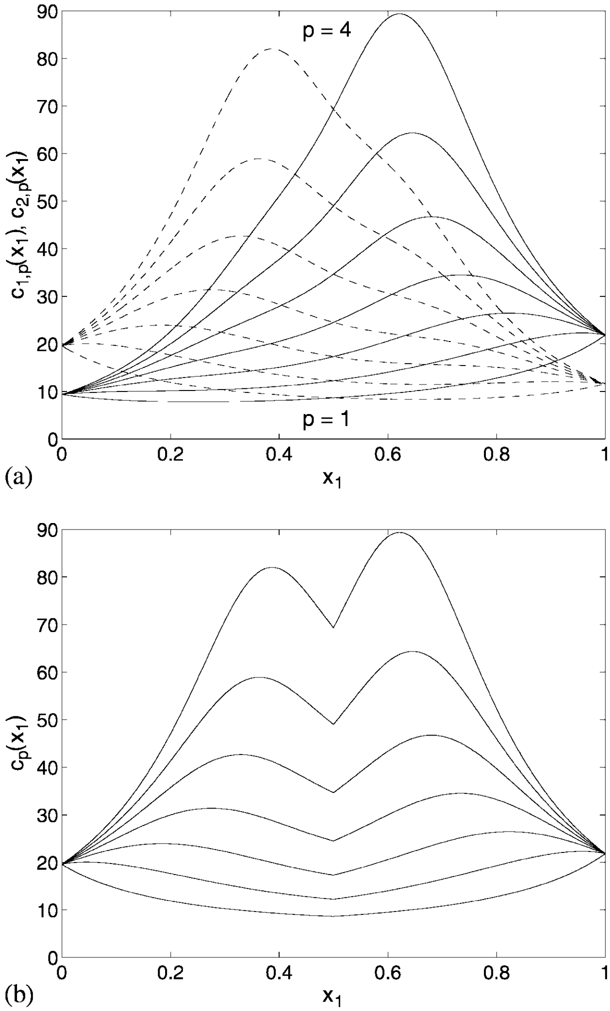

Fig. 3 The penalized compliances for $\alpha = 2.6$. (a) $c_{1,p}(x_1)$ (solid) and $c_{2,p}(x_1)$ (dashed) for $p = 1.0, 1.5, 2.0, \ldots, 4.0$; and (b) $c_p(x_1) = \max\{c_{1,p}(x_1), c_{2,p}(x_1)\}$

given by

$$c_p(x_1) = \begin{cases} c_{2,p}(x_1) & \text{if } x_1 \leq x_1^*(p), \\ c_{1,p}(x_1) & \text{if } x_1 \geq x_1^*(p)\,. \end{cases} \tag{23}$$

Differentiating $c_{1,p}(x_1)$ and $c_{2,p}(x_1)$ gives

$$c'_{1,p}(x_1) = \frac{40\alpha p(1-x_1)^{p-1} - 8\alpha p x_1^{p-1}}{(x_1^p + 5(1-x_1)^p)^2} + \frac{2\alpha p(1-x_1)^{p-1} - 10\alpha p x_1^{p-1}}{(5x_1^p + (1-x_1)^p)^2}\,, \tag{24}$$

and

$$c'_{2,p}(x_1) = \frac{40p(1-x_1)^{p-1} - 8p x_1^{p-1}}{(x_1^p + 5(1-x_1)^p)^2} + \frac{18p(1-x_1)^{p-1} - 90p x_1^{p-1}}{(5x_1^p + (1-x_1)^p)^2}\,. \tag{25}$$

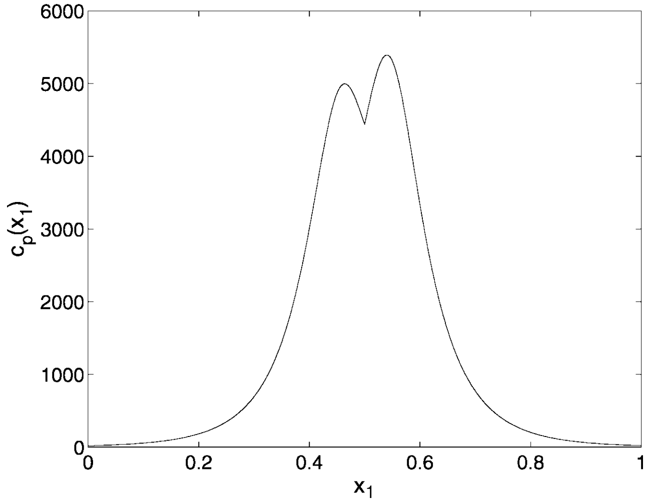

Fig. 4 $c_p(x_1)$ for $p = 10$ and $\alpha = 2.6$

For $p \geq 1$ it follows from (21), (24), and (25) that

$$c'_{1,p}(x_1^*(p)) > 0, \quad \text{and } c'_{2,p}(x_1^*(p)) < 0\,. \tag{26}$$

To illustrate that this holds even for large values on p, $c_p(x_1)$ is depicted in Fig. 4 for $p = 10$ and $\alpha = 2.6$.

For $p > 1$ it further follows from (24) and (25) that

$$c'_p(1) = c'_{1,p}(1) < 0, \quad \text{and } c'_p(0) = c'_{2,p}(0) > 0\,. \tag{27}$$

Since this is not apparent from Figs. 2 and 3, $c_{2,p}(x_1)$ is shown in magnification in Fig. 5 for x_1 close to zero.

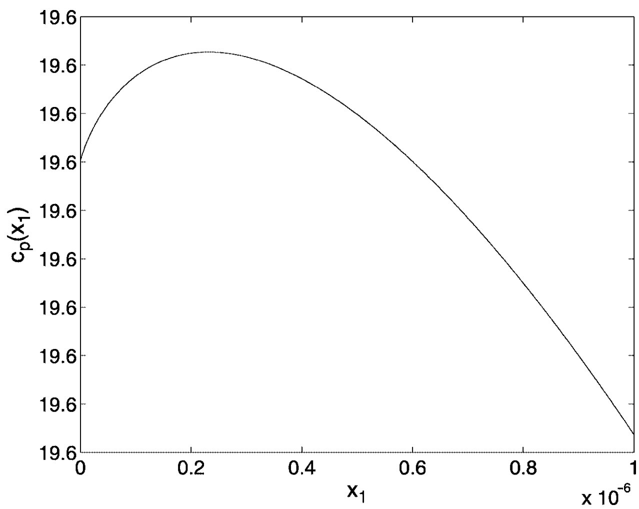

Fig. 5 $c_{2,p}(x_1)$ for $p = 1.1$ and $\alpha = 7/3$

We conclude that, for $p = 1$, $(\mathcal{P}_p^\alpha)$ is a convex problem with one unique optimal solution, namely $\overline{\mathbf{x}}(p) = \mathrm{x}^*(p)$. For $p > 1$, $(\mathcal{P}_p^\alpha)$ has three local minima, the interior point $\mathbf{x} = \mathbf{x}^*(p)$, and the integer points $\mathbf{x} = (1,0)^T$ and $\mathbf{x} = (0,1)^T$. Figures 2 and 3 indicate that there exists a number $p^*(\alpha)$ such that for all $p \geq p^*(\alpha)$ the global optimum of $(\mathcal{P}_p^\alpha)$ is an integer point. $p^*(\alpha)$ is given as the solution to the equation $c_{1,p}(x_1^*(p)) = \min\{c_p(0), c_p(1)\}$.

The global trajectory for $\alpha = 7/3$ is given by

$$\overline{\mathbf{x}}(p) = \begin{cases} \mathbf{x}^*(p) & \text{if } p \le p^*(\alpha), \\ (1,0)^T \text{ or } (0,1)^T & \text{if } p > p^*(\alpha)\,, \end{cases} \tag{28}$$

where $p^*(\alpha) \approx 2.23$.

The global trajectory for $\alpha \neq 7/3$ is given by

$$\overline{\mathbf{x}}(p) = \begin{cases} \mathbf{x}^*(p) & \text{if } p \le p^*(\alpha), \\ (0,1)^T & \text{if } p > p^*(\alpha) \text{ and } \alpha > 7/3, \\ (1,0)^T & \text{if } p > p^*(\alpha) \text{ and } \alpha < 7/3\,. \end{cases} \tag{29}$$

If $\alpha = 2.6$ then $\mathbf{x}^*(p) = (0.5, 0.5)^T$ for all $p \ge 1$, and the global trajectory is given by

$$\overline{\mathbf{x}}(p) = \begin{cases} (0.5,0.5)^T & \text{if } p \le p^*(\alpha), \\ (0,1)^T & \text{if } p > p^*(\alpha)\,, \end{cases} \tag{30}$$

where $p^*(\alpha) \approx 2.18$.

The global trajectories for $\alpha = 7/3$ and $\alpha = 2.6$ are depicted in Fig. 6.

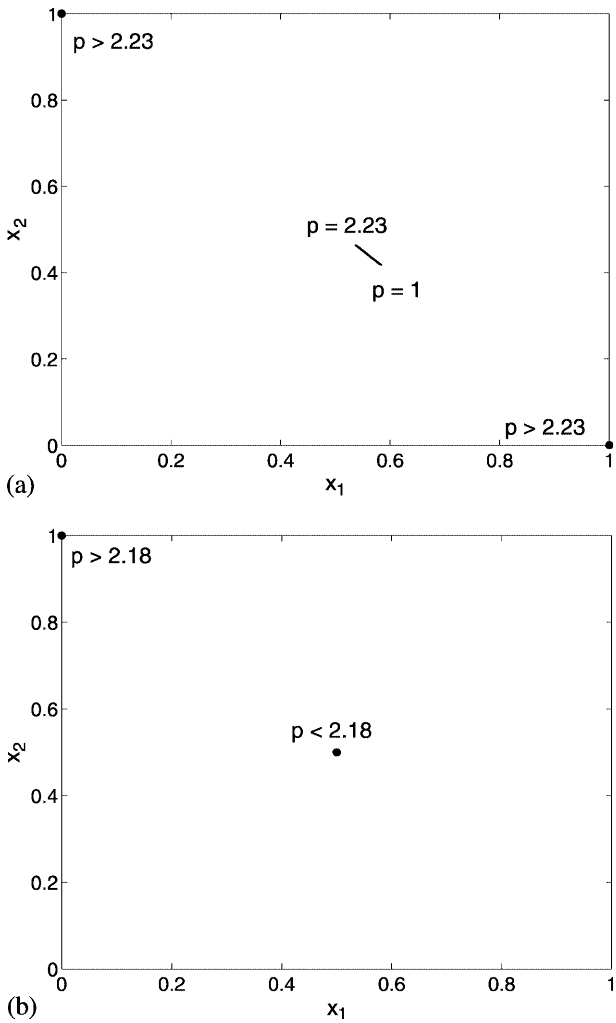

Fig. 6 Global trajectories for the SIMP approach. (a) $\alpha = 7/3$, (b) $\alpha = 2.6$

3.2 The quadratic penalty approach for $\alpha = 2.6$

The example problem with a quadratic penalty function is formulated as

$$(\mathcal{P}_q^\alpha):\ \min\ \max\{c_{1,q}(\mathbf{x})\,,\, c_{2,q}(\mathbf{x})\}\,,$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad 0 \le x_1, x_2 \le 1\,, \tag{31}$$

where the penalized compliances are given by

$$c_{1,q}(\mathbf{x}) = \frac{8\alpha}{x_1 + 5x_2} + \frac{2\alpha}{5x_1 + x_2} + q\sum_{j=1}^{2} x_j(1 - x_j)\,, \tag{32}$$

and

$$c_{2,q}(\mathbf{x}) = \frac{8}{x_1 + 5x_2} + \frac{18}{5x_1 + x_2} + \sum_{j=1}^{2} x_j(1 - x_j)\,. \tag{33}$$

Again, since $x_2 = 1 - x_1$ at all feasible solutions to $(\mathcal{P}_q^\alpha)$, the variable x_2 can be eliminated, and the compliances become univariate functions of the single variable x_1. These penalized compliances, again denoted by $c_{1,q}$ and $c_{2,q}$, are given by

$$c_{1,q}(x_1) = \frac{8\alpha}{5 - 4x_1} + \frac{2\alpha}{4x_1 + 1} + 2qx_1(1 - x_1)\,, \tag{34}$$

and

$$c_{2,q}(x_1) = \frac{8}{5 - 4x_1} + \frac{18}{4x_1 + 1} + 2qx_1(1 - x_1)\,. \tag{35}$$

In Fig. 7, $c_{1,q}(x_1)$ and $c_{2,q}(x_1)$ are shown for $\alpha = 2.6$ and various values on q.

Since $c_{1,q}(x_1) < c_{2,q}(x_1)$ if $x_1 < 0.5$ and $c_{1,q}(x_1) > c_{2,q}(x_1)$ if $x_1 > 0.5$, the objective function to $(\mathcal{P}_q^\alpha)$ is given by

$$c_q(x_1) = \begin{cases} c_{2,q}(x_1) & \text{if } x_1 \le 0.5 \\ c_{1,q}(x_1) & \text{if } x_1 \ge 0.5\,. \end{cases} \tag{36}$$

Differentiating $c_{1,q}(x_1)$ and $c_{2,q}(x_1)$ gives

$$c'_{1,q}(x_1) = \frac{32\alpha}{(5 - 4x_1)^2} - \frac{8\alpha}{(4x_1 + 1)^2} + 2q(1 - 2x_1)\,, \tag{37}$$

and

$$c'_{2,q}(x_1) = \frac{32}{(5 - 4x_1)^2} - \frac{72}{(4x_1 + 1)^2} + 2q(1 - 2x_1)\,. \tag{38}$$

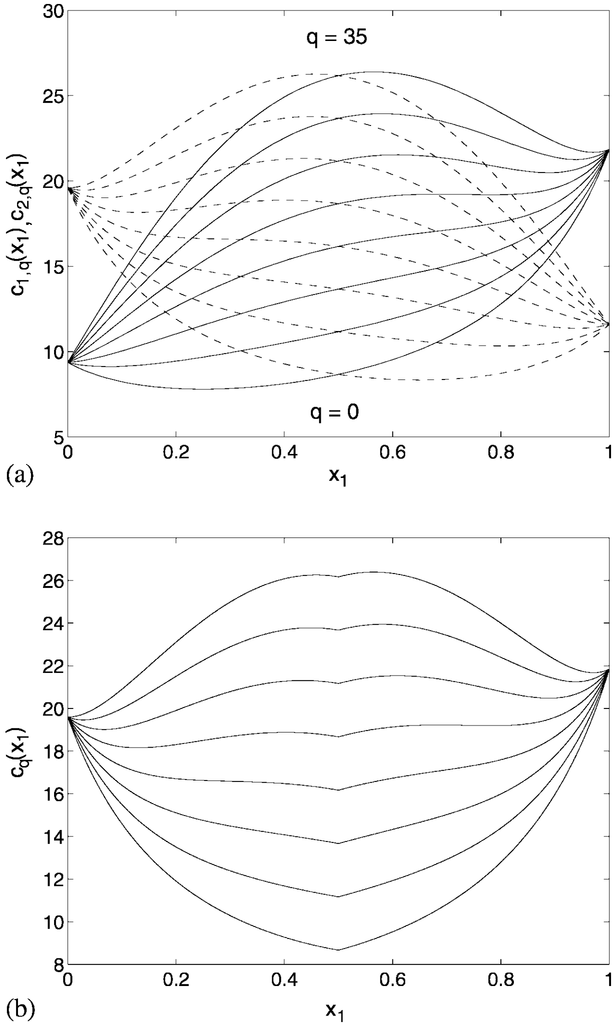

Fig. 7 The penalized compliances for $\alpha = 2.6$. (a) $c_{1,q}(x_1)$ (solid) and $c_{2,q}(x_1)$ (dashed) for $q = 0, 5, 10, \dots, 35$, (b) $c_q(x_1) = \max\{c_{1,q}(x_1), c_{2,q}(x_1)\}$

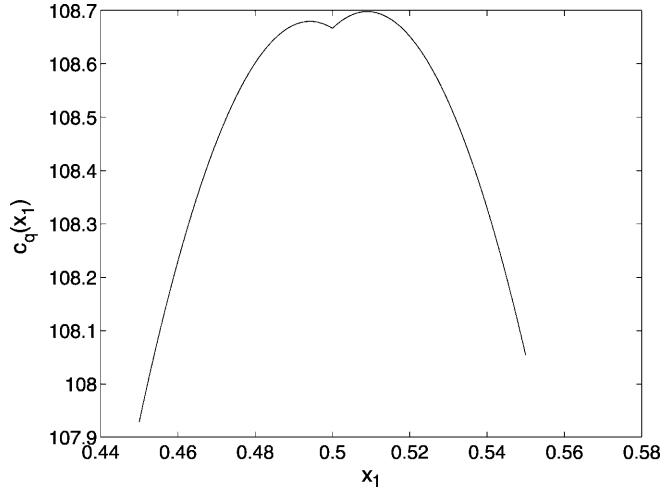

Fig. 8 $c_q(x_1)$ for $q = 200$ and $\alpha = 2.6$

Since

$$c'_{1,q}(0.5) = \frac{8\alpha}{3} > 0, \quad \text{and } c'_{2,q}(0.5) = -\frac{40}{9} < 0\,, \tag{39}$$

independently of q, we conclude that the interior point $\mathbf{x} = (0.5, 0.5)^T$ is a local minimum for all q. The objective function $c_q(x_1)$ is depicted in Fig. 8 for $q = 200$.

Since

$$c'_q(0) = c'_{2,q}(0) > 0 \quad \text{if } q > 884/25\ (= 35.36)\,, \tag{40}$$

and

$$c'_q(1) = c_{1,q}(1) < 0 \quad \text{if } q > 396\alpha/25\ (= 41.184)\,, \tag{41}$$

the point $\mathbf{x} = (0,1)^T$ is a local minimum for all $q > 35.36$, and $\mathbf{x} = (1,0)^T$ is a local minimum for all $q > 41.184$.

Figure 7 indicates that for each q in a certain interval $[q_1, q_2]$ there is a point $0 < \hat{x}_1(q) < 0.2$, such that $\hat{x}_1(q)$ is a local minimum; $\hat{x}_1(q)$ is the unique solution to $c'_{2,q}(x_1) = 0$, with $x_1 < 0.2$. Let $\hat{\mathbf{x}}(q)$ denote the point $\hat{\mathbf{x}}(q) = (\hat{x}_1(q), 1 - \hat{x}_1(q))^T$. Then $\hat{\mathrm{x}}(q) \to (0,1)^T$ as $q \to 35.36$. Furthermore, Fig. 7 shows that there exists a $\hat{q}(\alpha)$ such that the global optimum of $(\mathcal{P}_q^\alpha)$ is given by $\hat{\mathbf{x}}(q)$ for $q \in (\hat{q}(\alpha), 35.36)$. $\hat{q}(\alpha)$ is the solution to the equation $c_{2,q}(\hat{x}_1(q)) = c_{2,q}(0.5)$.

The global trajectory for $\alpha = 2.6$ is given by

$$\overline{\mathbf{x}}(q) = \begin{cases} (0.5, 0.5)^T & \text{if } q \leq \hat{q}(\alpha), \\ \hat{\mathbf{x}}(q) & \text{if } \hat{q}(\alpha) < q < 35.36, \\ (0,1)^T & \text{if } q \geq 35.36\,, \end{cases} \tag{42}$$

where $\hat{q}(2.6) \approx 17.90$.

The global trajectory for $\alpha = 2.6$ is depicted in Fig. 9.

3.3 A continuation method applied to the example problem

The motivation for using continuation methods in topology optimization is to hopefully avoid convergence to a local minimum. The method starts by solving the convex problem $(\mathcal{P}_p)$ or $(\mathcal{P}_q)$ with $p = 1$ or $q = 0$. Using the solution to the previous problem as starting point, $(\mathcal{P}_p)$ or $(\mathcal{P}_q)$ are repeatedly solved by some gradient-based optimization method for a sequence of gradually increasing penalization factors. The following is a reasonable implementation of a continuation method.

Algorithm 1. Continuation approach.

Start with an initial topology $\mathbf{x}^0$ and $p_0 = 1$ $(q_0 = 0)$. Set k = 0.

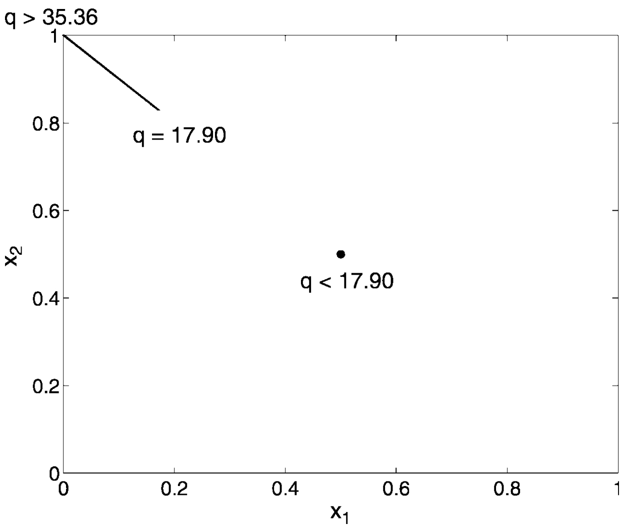

Fig. 9 Global trajectory for $\alpha = 2.6$

while $p_k < p^{max}$ ($q_k < q^{max}$) **do**
Solve ($\mathcal{P}_{p_k}$) or ($\mathcal{P}_{q_k}$) using some gradient-based method with $\mathbf{x}^k$ as starting point.
Denote the solution by $\mathbf{x}^{k+1}$.
Let $p_{k+1} = p_k + \Delta p_k$ ($q_{k+1} = q_k + \Delta q_k$), $k \leftarrow k+1$.
end

This continuation method will in fact not produce a black and white design when applied to the example problem with $\alpha = 2.6$. In both the SIMP and the quadratic penalty approach, the solution to the variable thickness sheet problem, i.e. $\mathbf{x} = (0.5, 0.5)^T$, will also be a local minimum for all $p > 1$ and $q > 0$, respectively. Using this point as a starting point for any penalized problem will give the same point $\mathbf{x} = (0.5, 0.5)^T$ as a solution, at least if the optimization method verifies the optimality conditions prior to performing any iterations.

4
A continuum example

The second example under study is the two-dimensional continuum structure of unit thickness shown in Fig. 10. It is divided into two linear triangles (finite elements).

The following material properties and resource limits are used:

$$\nu = 0.3\,, \quad E = 1\,, \quad P^{\max} = \sqrt{2}\,, \quad \text{and } M = 1\,, \tag{43}$$

where ν is Poisson's ratio and E is the modulus of elasticity of the solid material.

The perimeter constraint becomes $|x_1 - x_2| \le 1$ since the interface length is equal to $\sqrt{2}$. This constraint is redundant since $|x_1 - x_2| \le 1$ for all $0 \le x_1, x_2 \le 1$, and it is therefore removed. If $P^{\max} < \sqrt{2}$ it is not possible to obtain a black and white solution.

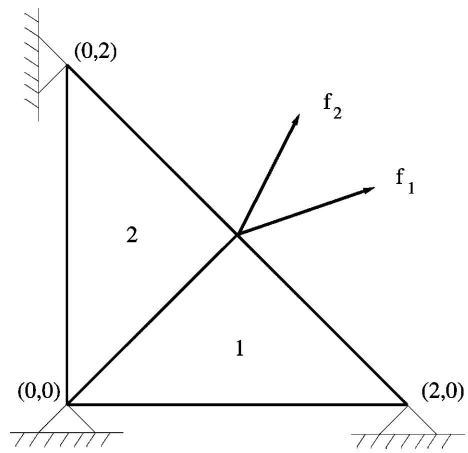

Fig. 10 A continuum example

The stiffness matrix in plane stress is given by

$$\mathbf{K}(\mathbf{x}) = \frac{1}{\beta}\begin{pmatrix} (1-\nu)x_1 + 2x_2 & 0 \\ 0 & 2x_1 + (1-\nu)x_2 \end{pmatrix}, \tag{44}$$

where $\beta = 2(1-\nu^2)/E = 1.82$. All conclusions in this section hold also if, instead, plane strain is used.

The load conditions for this example are defined as

$$\mathbf{f}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \ \mathbf{f}_2 = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}. \tag{45}$$

The nested example problem is given by

$$(\mathcal{P}^c):\ \min\ \max\{c_1(\mathbf{x}),\, c_2(\mathbf{x})\}\,,$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad x_1, x_2 \in \{0, 1\}\,, \tag{46}$$

where the compliances are given by

$$c_1(\mathbf{x}) = \frac{9\beta}{(1-\nu)x_1 + 2x_2} + \frac{\beta}{2x_1 + (1-\nu)x_2}\,, \tag{47}$$

and

$$c_2(\mathbf{x}) = \frac{2\beta}{(1-\nu)x_1 + 2x_2} + \frac{8\beta}{2x_1 + (1-\nu)x_2}\,. \tag{48}$$

The unique global optimal solution to $(\mathcal{P}^c)$ is given by $\overline{\mathbf{x}} = (0, 1)^T$.

The penalized problems are

$$(\mathcal{P}^c_p):\ \min\ \max\{c_{1,p}(\mathbf{x}),\, c_{2,p}(\mathrm{x})\}\,,$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad 0 \le x_1, x_2 \le 1\,, \tag{49}$$

and

$$(\mathcal{P}^c_q):\min\ \max\{c_{1,q}(\mathbf{x}),\, c_{2,q}(\mathbf{x})\}\,,$$

$$\text{s.t. } x_1 + x_2 = 1\,, \quad 0 \le x_1, x_2 \le 1\,. \tag{50}$$

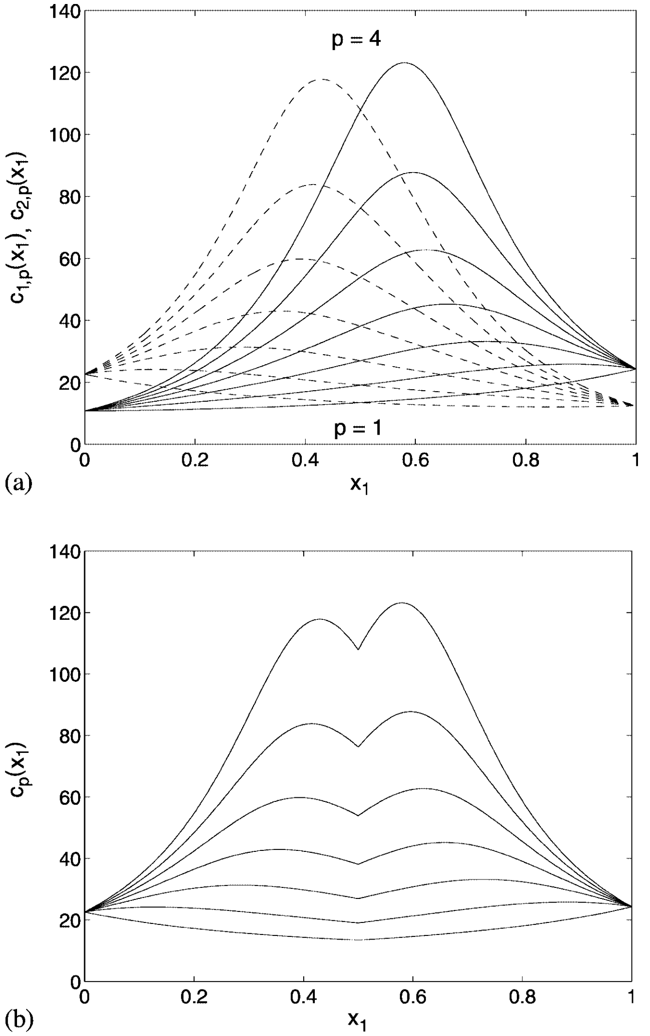

Fig. 11 The compliances penalized with SIMP. (a) $c_{1,p}(x_1)$ (solid) and $c_{2,p}(x_1)$ (dashed) for $p = 1.0, 1.5, 2.0, \ldots, 4.0$, (b) $c_p(x_1) = \max\{c_{1,p}(x_1), c_{2,p}(x_1)\}$

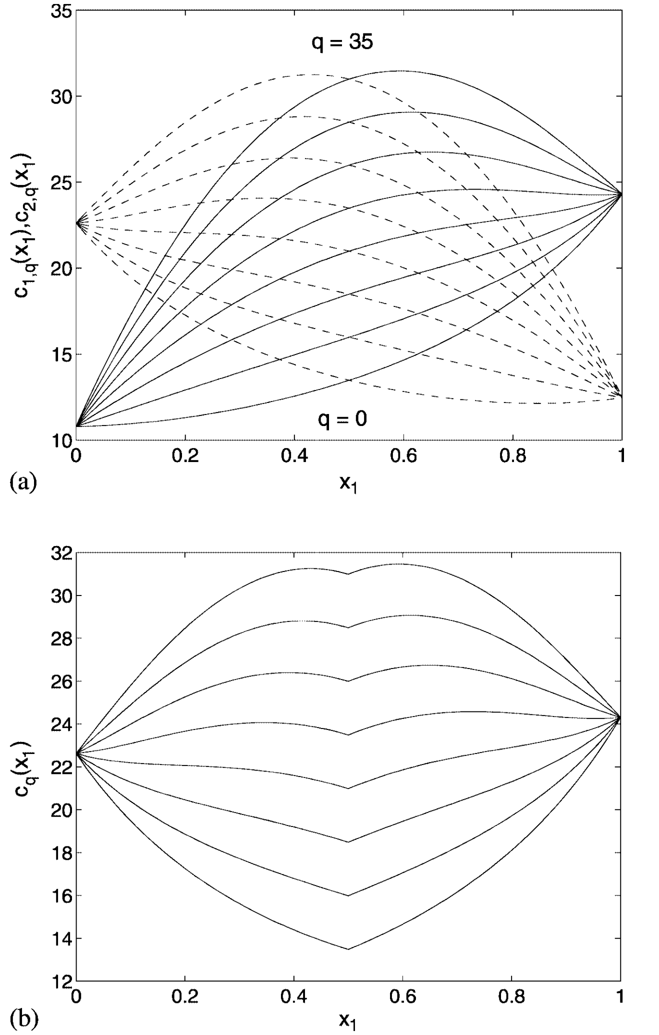

Fig. 12 The compliances with quadratic penalty. (a) $c_{1,q}(x_1)$ (solid) and $c_{2,q}(x_1)$ (dashed) for $q = 0, 5, 10, \ldots, 35$. (b) $c_q(x_1) = \max\{c_{1,q}(x_1), c_{2,q}(x_1)\}$

The penalized compliances are shown in Figs. 11 and 12 for various values on p and q.

The behaviour of this continuum example is very similar to the behaviour of the six-bar truss example with $\alpha = 2.6$. The global trajectories associated with $(\mathcal{P}_p^c)$ and $(\mathcal{P}_q^c)$ are both discontinuous. It also holds that the point $\mathbf{x} = (0.5, 0.5)^T$ is a local minimum to $(\mathcal{P}_p^c)$ and $(\mathcal{P}_q^c)$ for all $p \geq 1$ and $q \geq 0$, respectively. Thus, a continuation approach will get stuck in this local minimum.

5
An example with only one load condition

The discontinuous global trajectories in the previous examples relied heavily on the fact that the objective function was nondifferentiable due to the multiple load conditions. A natural question is then if the global trajectory is better behaved if there is only one single load condition. As our next example will show, the answer is negative, at least for the SIMP approach.

The third example under study is the six-bar truss structure shown in Fig. 13, where the resource limits are given by (9).

Assuming unit length of the bars, unit cross-sectional areas, and unit modulus of elasticity, the stiffness matrix for the six-bar truss is given by

$$\mathbf{K}(\mathbf{x}) = \begin{pmatrix} a\,x_1 + bx_2 & 0 \\ 0 & bx_1 + a\,x_2 \end{pmatrix}, \tag{51}$$

where

$$a = \frac{2 - \sqrt{2}}{2}, \text{ and } b = \frac{4 + \sqrt{2}}{2}. \tag{52}$$

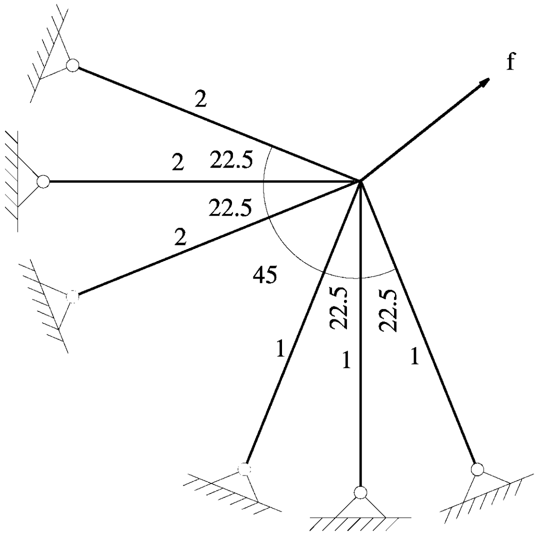

Fig. 13 A six-bar truss under one load condition

The load condition for this example is defined as

$$\mathbf{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} . \tag{53}$$

The nested example problem is given by

$$(\mathcal{P}^1): \min\ c(\mathbf{x}) ,$$

$$\text{s.t. } x_1 + x_2 = 1 , \quad x_1, x_2 \in \{0, 1\} , \tag{54}$$

where the compliance is given by

$$c(\mathbf{x}) = \frac{1}{a\, x_1 + b x_2} + \frac{1}{b x_1 + a\, x_2} . \tag{55}$$

The global optimal solutions to $(\mathcal{P}^1)$ are $\overline{\mathbf{x}} = (1,0)^T$ and $\overline{\mathbf{x}} = (0,1)^T$.

The example problem penalized using the SIMP approach can be formulated as

$$(\mathcal{P}^1_p): \min\ c_p(\mathbf{x}) ,$$

$$\text{s.t. } x_1 + x_2 = 1 , \quad 0 \le x_1, x_2 \le 1 , \tag{56}$$

where the penalized compliance is given by

$$c_p(\mathbf{x}) = \frac{1}{a\, x_1^p + b x_2^p} + \frac{1}{b x_1^p + a\, x_2^p} . \tag{57}$$

Substituting $x_2 = 1 - x_1$, we obtain

$$c_p(x_1) = \frac{1}{a\, x_1^p + b(1-x_1)^p} + \frac{1}{b x_1^p + a\,(1-x_1)^p} . \tag{58}$$

Differentiating $c_p(x_1)$ gives

$$c_p'(x_1) = \frac{bp(1-x_1)^{p-1} - apx_1^{p-1}}{(a\, x_1^p + b(1-x_1)^p)^2} +$$

$$\frac{a\, p(1-x_1)^{p-1} - bpx_1^{p-1}}{(bx_1^p + a\,(1-x_1)^p)^2} . \tag{59}$$

Since $c_p'(0.5) = 0$ for all $p \ge 1$, $\mathbf{x} = (0.5, 0.5)^T$ is a stationary point for all $p \ge 1$.

The second derivative of the penalized compliance evaluated at $x_1 = 0.5$ is

$$c_p''(0.5) = \frac{8p2^p}{(a+b)^3} \left((a+b)^2 + p\left(a^2 + b^2 - 6ab\right) \right) . \tag{60}$$

Since $c_p''(0.5) > 0$ for all $p \ge 1$ if $a^2 + b^2 - 6ab \ge 0$, $\mathbf{x} = (0.5, 0.5)^T$ is a strict local minimum for all $p \ge 1$ if

$$\frac{b}{a} \ge 3 + \sqrt{8} \quad \text{or} \quad \frac{b}{a} \le 3 - \sqrt{8} . \tag{61}$$

Since the first condition in (61) is satisfied for a and b given in (52), we conclude that $\mathbf{x} = (0.5, 0.5)^T$ is a strict local minimum for all $p \ge 1$.

In Figs. 14 and 15, $c_p(x_1)$ is shown for various values on p.

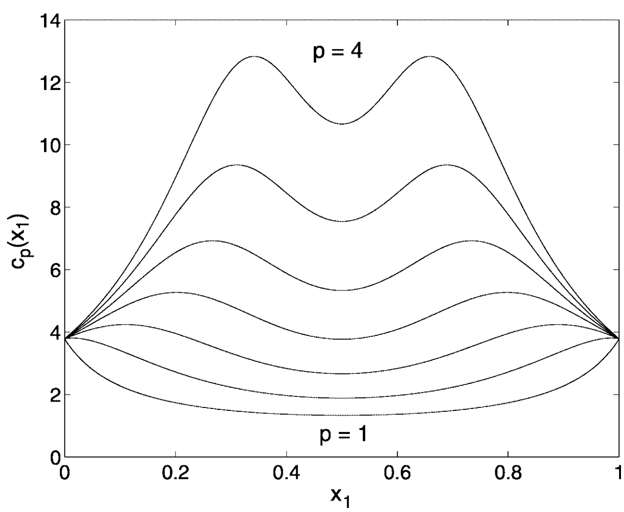

Fig. 14 $c_p(x_1)$ for $p = 1.0, 1.5, 2.0, \dots, 4.0$

Figure 14 indicates that there exists a number p^* such that for all $p > p^*$ the global optimum of $(\mathcal{P}^1_p)$ is an integer point (in fact two integer points).

The global trajectory is given by

$$\overline{\mathbf{x}}(p) = \begin{cases} (0.5, 0.5)^T & \text{if } p \le p^* \\ (1,0)^T \text{ or } (0,1)^T & \text{if } p > p^* , \end{cases} \tag{62}$$

where $p^* \approx 2.50$.

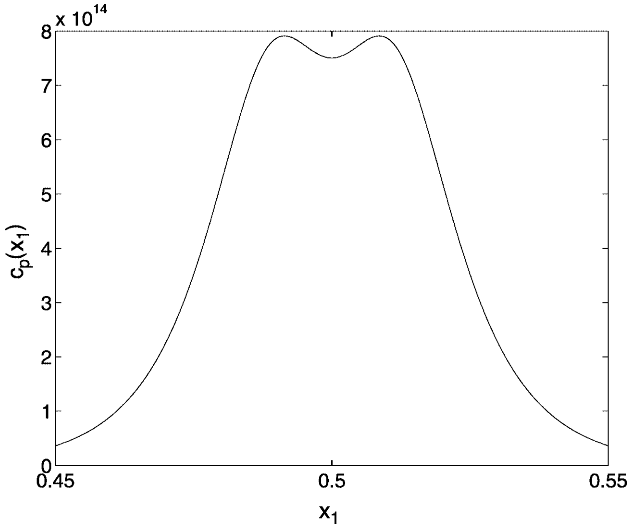

Fig. 15 $c_p(x_1)$ for $p = 50$. Notice the axes

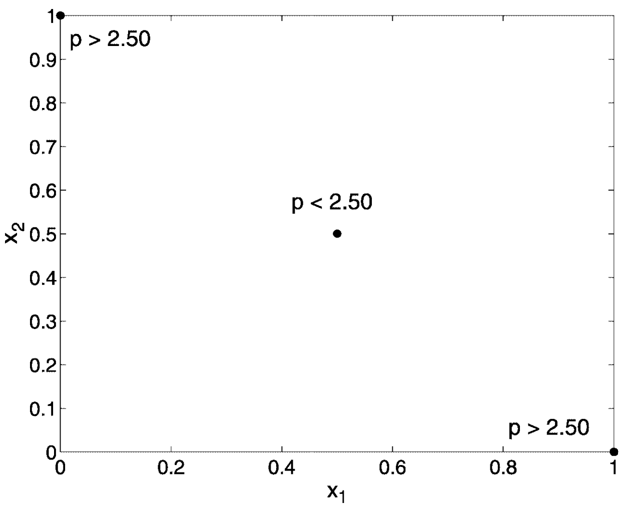

Fig. 16 Global trajectory

The global trajectory is depicted in Fig. 16.

If the load vector is changed slightly to $\mathbf{f} = (1.01, 0.99)^T$, there will be a unique optimal solution of the zero-one problem $(\mathcal{P}^1)$, but there will still be an interior local optimum (close to $\mathbf{x} = (0.5, 0.5)^T$) of the penalized problem $(\mathcal{P}_p^1)$, for all $p \geq 1$.

If the quadratic penalty approach is applied to this single load problem, the penalized compliance $c_q(\mathbf{x})$ becomes strictly concave for sufficiently large values on q (see Figs. 17 and 18). Thus there are no interior local optima of the penalized problem for these large q.

6 Conclusions

We have demonstrated that the global trajectories associated with the SIMP and the quadratic penalty approach may be discontinuous even though the original zero-one problem has a unique optimal solution. This has practical implications for the continuation approach. In general, we cannot expect to be able to follow the global trajectory, no matter how gently p or q are increased.

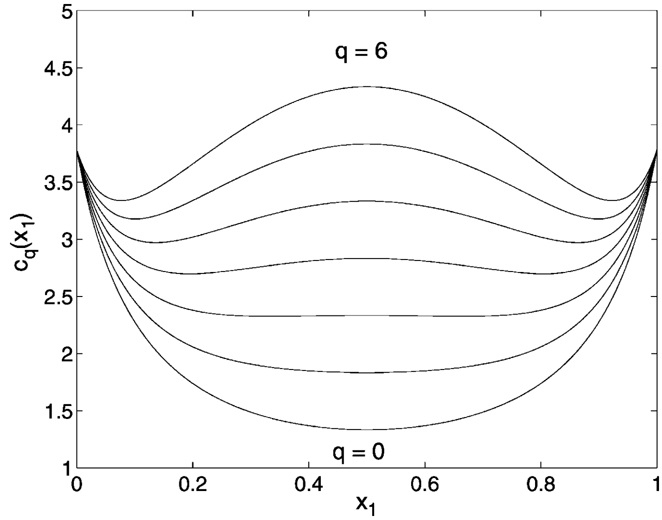

Fig. 17 $c_q(x_1)$ for $q = 0, 1, 2, \ldots, 6$

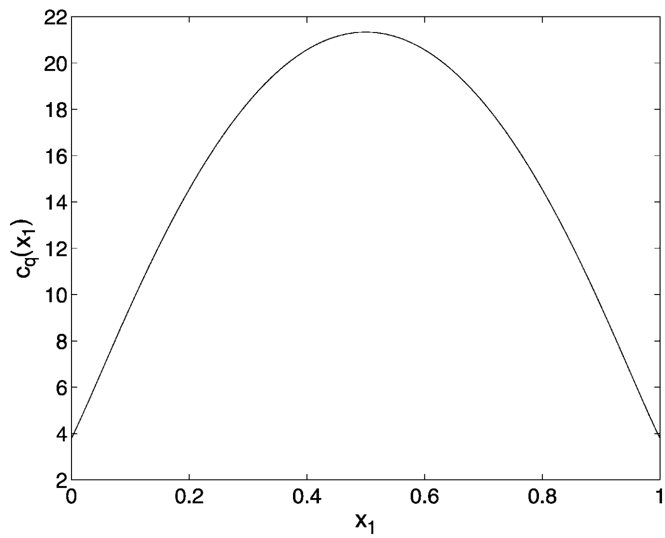

Fig. 18 $c_q(x_1)$ for $q = 40$

Furthermore, we have presented examples where the continuation approach, combined with either SIMP or the quadratic penalty approach, fails to produce a black and white design, no matter how large the penalization parameters p or q becomes.

The main conclusion from these examples is that although the continuation approach combined with some penalization techniques may be a very good heuristic in many cases, it is not possible to prove any convergence results. At least not without some severe assumptions which exclude the examples presented in this paper.

Acknowledgements Research supported by the Swedish Research Council for the Engineering Sciences (TFR).

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