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A gradient based heuristic algorithm and its application to discrete optimization of bar structures

G. Jivotovski

Abstract In this paper a nonlinear constrained integer optimization problem with a monotonic objective function is considered. Customarily, the optimum is located near the feasible region boundary for this category of problems. A two-stage heuristic algorithm is developed which utilizes this peculiarity. Within the algorithm coordinate descents are computed to move within the feasible region towards the region boundary. Motions along the boundary are performed using discrete antigradients based on linear approximations of the objective function and constraints at the last feasible point. Auxiliary relative vectors are established to find a better point within a polyhedron formed by hyperplanes tangent to the objective function as well as the violated constraint surfaces. In particular, a model for the optimum design of bar structures is presented. It is demonstrated that both the algorithm and the model have been successfully applied to discrete optimization of ten-bar and two hundred-bar trusses and a single-span two-storey frame.

Key words Discrete value optimization, trusses, rigid frames, heuristic algorithms, sequential quadratic programming (SQP)

1 Introduction

At present, engineering design of real structures is more and more carried out by combining Finite Element analysis and mathematical programming methods. The theory and software are well developed to solve problems which can be modelled by differentiable functions and continuous variables (VMA Engineering 1993). Practical structural design, however, often deals with discrete variables which must be selected from a predetermined set of discrete values, e.g. standardized profile sizes, standard sheet thicknesses, material category, number of elements, etc. In these cases, the structural design converts to a discrete nonlinear constrained problem.

Although the first papers on discrete optimization were already published in the 60's, see e.g. Korbut and Finkelshtein (1969), discrete optimization methods are still rarely developed and form an important field for research. Several methods have been proposed to solve discrete problems. Arora *et al.* (1994) classify all methods into the following six categories: branch and bound, simulated annealing, sequential linearization, penalty functions, Lagrangian relaxation, and other methods (among them heuristic techniques). No ideal method exists to solve complex discrete problems. Probabilistic methods including simulated annealing, genetic algorithm and evolution strategy can find the global minimum without using any gradient information, therefore being computationally expensive. Thus, these methods are well elaborated only for small and medium-scale problems. The same holds for branch and bound algorithms. Deterministic approaches do not guarantee that the final point found is the global minimum. A comparative study of five different optimization methods with respect to truss structures was performed by Bouzy and Abel (1995) who suggested a two-step procedure. First, a sequential quadratic programming method (SQP) is applied to solve an equivalent continuous problem. Second, the optimum continuous solution is utilized as a starting point for a local discrete search using a genetic algorithm.

To improve the situation it seems reasonable to develop new heuristic approaches to solve discrete design problems. One of these approaches is to utilize discrete gradients. Amir and Hasegawa (1994) reduced constrained mixed-discrete optimization problems to discrete unconstrained problems using penalty functions and small values of increments for a continuous value as a representation of discrete variables. A gradient based steepest descent technique has been applied to solve discrete unconstrained problems. Chai and Sun (1996) suggested a relative difference quotient algorithm. Here, the procedure starts from the minimum point outside the feasible

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region and advances along the direction of the minimum increment of the objective function and the maximum decrements of the constraints. Then, discrete gradients are used to define a search direction.

In the following, similarly to Jivotovski and Perelman (1997), a new heuristic algorithm is proposed. This approach is developed to solve integer constrained nonlinear problems with monotonic objective functions. Thus, the optimum point is located near the feasible region boundary. This type of problem covers a broad class of real structural design problems. The actual algorithm is divided into two stages. At first, the motion to the feasible region boundary using coordinate descent is determined. Second, discrete antigradients based on linear approximations of the objective function and violated constraints at the last feasible point are computed to cut the polyhedron in the design space. Then, an integer point within the polyhedron is searched. Finally, the values of the objective function in a new point and the last one are compared to each other and a new point is verified with respect to feasibility. The search procedure for the second stage is improved in comparison with the one used by Jivotovski and Perelman (1997). Two auxiliary relative vectors located within a polyhedron with the maximum absolute value of projection equal to 1 and 2, respectively, are established to find a better point.

2
Definition of the optimization problem

Beams, plane and spatial trusses, frames, grillages and arches are considered as different categories of structures under study. Structural topology and geometry as well as kinematic boundary conditions are given. Also, quasi-static loads acting on the structure are assumed, where N_{LC} is the number of loading cases.

Structures consist of N_{bar} prismatic bars. Bars are fabricated from standardized steel or aluminium alloy profiles or welded from standard sheets. The bars are linked to N_{gr} groups according to the design decision accounting $N^{(i)}_{\mathrm{bar}}$ bars for the i-th group, $i = 1, \ldots, N_{\mathrm{gr}}$. The bars in each group are made from the same material and have equivalent cross-sections. The bar cross-sectional shape V_i (e.g. $V_i = 1$ is a flat bar, $V_i = 2$ is a double angle, etc.) and fabrication type U_i ($U_i = 1$ is standardized, $U_i = 2$ is welded profile) for the i-th group are given, $i = 1, \ldots, N_{\mathrm{gr}}$. Alteration of the bar cross-sectional shape V_i and/or fabrication type U_i for the i-th group is possible as well as change of quantity $N^{(i)}_{\mathrm{bar}}$, $i = 1, \ldots, N_{\mathrm{gr}}$. A temporary data base is created for real problems. This data base contains the required material data, e.g. allowable stress, Young's modulus, cross-sectional area, moment of inertia, etc.

The unknown design parameters are ordinal numbers of the elements defined in the sets of standardized profile areas as well as standard sheet thicknesses and widths for welded profiles. The optimum structure has minimum mass or fabrication cost and meets the strength, displacement and stability requirements of the given codes of practice. The problem is formulated as an integer nonlinear constrained mathematical programming problem.

2.1
Design variables

The set of design variables $\{x\}$ consists of a number of subsets

$$\{x\} = \left\{\{x^{(1)}\}, \ldots, \{x^{(i)}\}, \ldots, \{x^{(N_{\mathrm{gr}})}\}\right\}, \tag{1}$$

where each

$$\{x^{(i)}\} = \begin{cases} \{x^{(i)}_1\} & \text{if } U_i = 1\,, \\ \{x^{(i)}_1, \ldots, x^{(i)}_j, \ldots, x^{(i)}_{K_i}, & \\ x^{(i)}_{K_i+1}, \ldots, x^{(i)}_{K_i+j}, \ldots, x^{(i)}_{2K_i}\} & \text{if } U_i = 2\,, \end{cases} \tag{2}$$

is a subset of design variables for the i-th group; $x^{(i)}_1 = n$ is ordinal number of the element from the set $\{A^{(v)}\}$, which defines the cross-sectional area $A^{(v)}_n$ for the bars from the i-th group fabricated from standardized profiles with cross-sectional shape $V_i = v$, K_i is the number of bands which form a welded profile with cross-sectional shape V_i; $x^{(i)}_j$ and $x^{(i)}_{K_i+j}$ are ordinal numbers of the elements from the sets $\{t\}$ and $\{B\}$, which define the j-th band thickness $t_{x^{(i)}_j}$ and width $B_{x^{(i)}_j}$ for welded bars from the i-th group.

Elements from the set $\{A^{(v)}\}$ of the cross-sectional areas for standardized profiles with cross-sectional shape v are put in ascending order $A^{(v)}_1 < \ldots < A^{(v)}_n < \ldots < A^{(v)}_{Q_v}$, where Q_v is the number of standardized profiles with shape v. Elements from the sets of standard sheet thicknesses $\{t\}$ and widths $\{B\}$ are also arranged in ascending order $t_1 < \ldots < t_n < \ldots < t_{Q_t}$, $B_1 < \ldots < B_n < \ldots < B_{Q_B}$, where Q_t and Q_B are the numbers of thicknesses and widths for standard sheets, respectively.

Design variables have an integer type and belong to the corresponding sets $\{X^{(v)}_1\}$, $\{X_2\}$ and $\{X_3\}$

$$x^{(i)}_1 \in \{X^{(v)}_1\} = \{1, \ldots, n, \ldots, Q_v\}\,, \quad v = V_i\,,$$

$$\text{if } U_i = 1\,,$$

$$x^{(i)}_j \in \{X_2\} = \{1, \ldots, n, \ldots, Q_t\}\,, \quad \text{if } U_i = 2\,,$$

$$x^{(i)}_{K_i+j} \in \{X_3\} = \{1, \ldots, n, \ldots, Q_B\}\,, \quad \text{if } U_i = 2\,. \tag{3}$$

Design variable set (1) is considered as a vector $\mathbf{x} = (x_1, \ldots, x_m, \ldots, x_N)^T$, where $N = \sum_{i=1}^{N_{\mathrm{gr}}} N_i$ is the total number of design variables.

Further,

$$N_i = \begin{cases} 1 & \text{if } U_i = 1\,, \\ 2K_i & \text{if } U_i = 2\,, \end{cases}$$

is the number of design variables for the i-th group. The two-dimensional matrix $\mathbf{D}$ is established. The element $d_{m,n}$ of the matrix $\mathbf{D}$ denotes the n-th value of the set $\{A^{(v)}\}$, $\{t\}$ or $\{B\}$ correspondingly to the m-th design variable $x_m = n$.

2.2 Objective function

The objective function is the structural mass or fabrication cost. The structural mass is defined as

$$C(\mathbf{x}) = \sum_{k=1}^{N_{\mathrm{bar}}} \varrho_k L_k A_k = \sum_{i=1}^{N_{\mathrm{gr}}} \varrho_i A_i \sum_{k=1}^{N_{\mathrm{bar}}^{(i)}} L_k\,, \tag{4}$$

where L_k denotes the length of the k-th bar; $\varrho_k = \varrho_i$ is the specific weight of the k-th bar from the i-th group. The quantity

$$A_k = A_i = \begin{cases} \left.\begin{matrix} A_n^v \\ v = V_i \\ n = x_1^{(i)} \end{matrix}\right\} & \text{if } U_i = 1\,, \\ \sum\limits_{j=1}^{K_i} t_{x_j^{(i)}} B_{x_{K_i+j}^{(i)}} & \text{if } U_i = 2\,, \end{cases}$$

denotes the cross-sectional area of the k-th bar from the i-th group.

The fabrication cost (without mounting) is given by

$$C_1(\mathbf{x}) = \sum_{k=1}^{N_{\mathrm{bar}}} L_k(s_k + sT_k) = \sum_{i=1}^{N_{\mathrm{gr}}} (s_i + sT_i) \sum_{k=1}^{N_{\mathrm{bar}}^{(i)}} L_k\,, \tag{5}$$

where s is the labour rate including overheads (cost/hour). The material price per metre length for the k-th bar from the i-th group is represented by

$$s_k = s_i = \begin{cases} \left.\begin{matrix} \varrho_i S(A_n^v) A_n^v \\ v = V_i \\ n = x_1^{(i)} \end{matrix}\right\} & \text{if } U_i = 1\,, \\ \varrho_i \sum\limits_{j=1}^{K_i} S(t_{x_j^{(i)}}) t_{x_j^{(i)}} B_{x_{K_i+j}^{(i)}} & \text{if } U_i = 2\,. \end{cases}$$

Herein, $S(A_n^{(v)})$ is the material price per kilogram mass for standardized profile with cross-sectional area $A_n^{(v)}$ and shape $v = V_i$; $S(t_{x_j^{(i)}})$ is the material price per kilogram mass for standard sheet with thickness $t_{x_j^{(i)}}$; furthermore T_k, where

$$T_k = T_i = \begin{cases} 0 & \text{if } U_i = 1\,, \\ T_{\mathrm{weld}}^{(i)}(K_i - 1) & \text{if } U_i = 2\,, \end{cases}$$

is labour hour required per metre welding of K_i bands to form the profile with cross-sectional shape V_i and $T_{\mathrm{weld}}^{(i)}$ is labour hour required per metre welding of two bands from the i-th group depending on the number of welding junctions.

2.3 Design constraints

The constraints and their number N_{DC} depend on the class of structure, material and the associated design codes of practice. The given bar joint connections and the structural topology define the number of structural members N_{mem} and the bar behaviour: whether it behaves as a truss (axial tension-compression) or a plane beam (bending) or a spatial beam (tension-compression, bending, twisting with bending and tension-compression if eccentricity takes place). Functional design constraints may be written as

$$G_j(\mathbf{x}) \le 1\,, \quad j = 1, \dots, N_{\mathrm{DC}}\,, \tag{6}$$

where the terms

$$G_i(\mathbf{x}) = \frac{1}{[\sigma_i]} \max_k \max_\Psi \max_{W_k} \left|\sigma_k^{(\Psi)}(\mathbf{x})\right| \le 1\,,$$

$$G_{N_{\mathrm{gr}}+i}(\mathbf{x}) = \frac{1}{[\tau_i]} \max_k \max_\Psi \max_{W_k} \left|\tau_k^{(\Psi)}(\mathbf{x})\right| \le 1\,,$$

define the normal and shear stress constraints for bars from the i-th group, respectively and $i = 1, \dots, N_{\mathrm{gr}}$, $k = 1, \dots, N_{\mathrm{bar}}^{(i)}$, $\Psi = 1, \dots, N_{\mathrm{LC}}$, the quantity

$$G_{2N_{\mathrm{gr}}+\xi}(\mathbf{x}) = \frac{\max\limits_k \max\limits_\Psi \max\limits_{W_k} \left|P_k^{(\Psi)}(\mathbf{x})\right|}{f \min\left(P_\xi^{(\mathrm{cr})}, P_{x,\xi}^{(\mathrm{cr})}, P_{y,\xi}^{(\mathrm{cr})}\right)} \le 1$$

is the overall buckling constraint for the ξ-th compressed-bending structural member and $\xi = 1, \dots, N_{\mathrm{mem}}$, $k = 1, \dots, K_{\mathrm{bar}}^{(\xi)}$, $\Psi = 1, \dots, N_{\mathrm{LC}}$,

$$G_{2N_{\mathrm{gr}}+N_{\mathrm{mem}}+\xi}(\mathbf{x}) =$$

$$\max_k \max_\Psi \max_{W_k} \sqrt{\left[\frac{\sigma_k^{(\Psi)}(\mathbf{x})}{\sigma_\xi^{(\mathrm{cr})}}\right]^2 + \left[\frac{\tau_k^{(\Psi)}(\mathbf{x})}{\tau_\xi^{(\mathrm{cr})}}\right]^2} \le 1$$

is the local buckling constraint for the ξ-th compressed-bending structural member and $\xi = 1, \dots, N_{\mathrm{mem}}$, $k = 1, \dots, K_{\mathrm{bar}}^{(\xi)}$, $\Psi = 1, \dots, N_{\mathrm{LC}}$,

$$G_{2N_{\mathrm{gr}}+2N_{\mathrm{mem}}+\ell}(\mathbf{x}) = \frac{1}{[\delta_\ell]} \max_\Psi \left|\delta_\ell^\Psi(\mathbf{x})\right| \le 1$$

is the nodal displacement constraint for the ℓ-th degree of freedom and $\ell = 1, \dots, N_{\mathrm{ND}}$, $\Psi = 1, \dots, N_{\mathrm{LC}}$.

Here the following notations are used:

$\max\limits_{W_k} \left|\sigma_k^{(\Psi)}(\mathbf{x})\right|$, $\max\limits_{W_k} \left|\tau_k^{(\Psi)}(\mathbf{x})\right|$: maximal absolute values of normal and shear stresses in the volume W_k of the k-th bar from the i-th group during the Ψ-th loading case,

$[\sigma_i]$, $[\tau_i]$: allowable normal and shear stresses for material of bars from the i-th group,

$\max\limits_{W_k} \left|P_k^{(\Psi)}(\mathbf{x})\right|$: maximal absolute value of axial compression force in the volume W_k of the k-th bar from the ξ-th structural member during the Ψ-th loading case,

$\sigma_k^{(\Psi)}(\mathbf{x})$, $\tau_k^{(\Psi)}(\mathbf{x})$: compression and mean shear stresses for the flange and web of the k-th bar from the ξ-th structural member during the Ψ-th loading case,

f: factor of safety,

$P_\xi^{(\mathrm{cr})}$, $P_{x,\xi}^{(\mathrm{cr})}$, $P_{y,\xi}^{(\mathrm{cr})}$: collapsing forces attached to nonaxial compression, longitudinal bending with regard to the x and y axes for the ξ-th structural member including the k-th bar

$\sigma_\xi^{(\mathrm{cr})}$, $\tau_\xi^{(\mathrm{cr})}$: critical values of compression and shear stresses for the flange and web for the ξ-th structural member including the k-th bar,

$\delta_\ell^{(\Psi)}(\mathbf{x})$: calculated displacement along the ℓ-th degree of freedom during the Ψ-th loading case,

$[\delta_\ell]$: allowable nodal displacement limit along the ℓ-th degree of freedom,

$K_{\mathrm{bar}}^{(\xi)}$: number of bars that compose the ξ-th structural member,

N_{ND}: number of limited nodal directions.

2.4 Problem formulation

An optimum design problem is formulated as a nonlinear constrained integer mathematical programming problem. The problem is to find the optimum point $\mathbf{x}^*$ that minimizes the objective function $C(\mathbf{x})$ (4) or (5) for a possible solution set of the vector $\mathbf{x}$ and subjected to the constraints (3) and (6). Thus, we have

$$C(\mathbf{x}^*) = \min_{\mathbf{x}\in D} C(\mathbf{x})\,,$$

$$D = \left\{ \mathbf{x} : G_j(\mathbf{x}) \le 1\,,\ x_m \in \{X_1^{(v)}\}, \{X_2\} \text{ or } \{X_3\}\,, \right.$$

$$\left. m = 1, \ldots, N\,,\ j = 1, \ldots, N_{\mathrm{DC}} \right\}. \qquad (7)$$

3 Structural analysis

Stress-strain analysis adopted for structural optimization is carried out primarily on the basis of the finite element method (FEM) with displacements as unknowns. A single mesh generation is executed before optimization since the structural topology remains invariable. Pre-processing data preparation is fully automatic. Separate subroutines modify the finite element stiffness matrices according to the optimization output data. With respect to efficiency numerous types of finite elements are available for each structural class. For truss and beam problems finite elements based on simple Bernoulli-Euler theory can be used. The stiffness matrix for finite elements with 2 and 3 nodal degrees of freedom for plane and 3 and 6 nodal degrees of freedom for spatial problems provide respective standard matrices for tension-compression, bending in two planes, tension-bending and twisting (e.g. Jivotovski 1991). Assemblage of the global stiffness matrix and solution of FEM equations are accomplished by using frontal technique of Irons (1970). Nodal displacements, actual integral forces and moments in bars and corresponding maximal normal and shear stresses are defined for each finite element. In the case of multiple loading cases a corresponding number of right-hand sides of the equation problem is introduced, but assemblage and triangularization of the matrix coefficients is performed only once.

Collapsing forces $P_\xi^{(\mathrm{cr})}$, $P_{x,\xi}^{(\mathrm{cr})}$, $P_{y,\xi}^{(\mathrm{cr})}$ and critical values of local stresses $\sigma_\xi^{(\mathrm{cr})}$, $\tau_\xi^{(\mathrm{cr})}$ for the ξ-th structural member are defined by analytical expressions and can be found in the paper by Jivotovski (1991).

4 Solution to the formulated problem

An optimum solution according to relationship (7) with monotonic objective function is located near the feasible region boundary. Thus, the optimization process is separated into two stages. The first stage is to step forward to the feasible region boundary. The second stage is to move along this boundary to a local optimum.

The iterative optimization procedure begins with the verification of the feasibility at the initial point $\mathbf{x}^{(0)}$. Existing engineering experience or a given continuous solution to the equivalent problem may be applied as an initial point. It is recommended to take unit values for all design variables if it is difficult to choose a better starting point: $x_m^{(0)} = 1$, $m = 1, \ldots, N$.

Motion to the feasible region is performed if the point $\mathbf{x}^{(0)}$ is infeasible. The point $\mathbf{x}^{(\ell+1)}$ is defined during the ℓ-th iteration by

$$x_m^{(\ell+1)} = \begin{cases} x_m^{(\ell)} + 1 & \text{if } x_m^{(\ell)} \ne x_m^{(u)}\,, \\ x_m^{(u)} & \text{if } x_m^{(\ell)} = x_m^{(u)}\,, \end{cases} \qquad m = 1, \ldots, N\,, \qquad (8)$$

where $x_m^{(u)}$ is the maximum value of the m-th design variable equal to Q_v, Q_t or Q_B.

The procedure is terminated if all design variables have their maximum values

$$x_m^{(\ell)} = x_m^{(u)}\,, \quad m = 1, \ldots, N\,. \qquad (9)$$

If the point $\mathbf{x}^{(u)}$ is infeasible, then as a rule the formulated problem has no solution.

The point $\mathbf{x}^{(\ell+1)}$ is taken as an initial point for the next $(\ell+1)$-th iteration if it is infeasible. Then, Step (8) is repeated.

If the initial point $\mathbf{x}^{(0)}$ is feasible or Step (8) leads to the feasible point $\mathbf{x}^{(\ell+1)}$, the algorithm switches to the simplified Gauss-Seidel descent strategy. The ℓ-th iteration begins with the verification of the convergence criterion at the feasible point $\mathbf{x}^{(\ell)}$. The procedure is terminated and the point $\mathbf{x}^{(\ell)}$ is the global minimum if all variables are equal to unity

$$x_m^{(\ell)} = 1\,, \quad m = 1, \ldots, N\,. \tag{10}$$

The explorative discrete step per m-th coordinate direction is performed from the last feasible point $\mathbf{x}^{(\ell)}$ to a neighbour point $\mathbf{x}^{(\ell,m)}$ if criterion (10) is not satisfied and the m-th design variable is not equal to unity

$$\mathbf{x}^{(\ell,m)} = \mathbf{x}^{(\ell)} - \mathbf{e}^{(m)}\,, \tag{11}$$

where ℓ is the iteration index, m is the coordinate direction index and $\mathbf{e}^{(m)}$ is the unit vector, whose components $e_j^{(m)}$ have the value zero for all $j \neq m$ and unity for $j = m$.

The point $\mathbf{x}^{(\ell,m)}$ is taken as an initial point $\mathbf{x}^{(\ell+1)} = \mathbf{x}^{(\ell,m)}$ for the next $(\ell+1)$-th iteration if it is feasible. Then, Step (11) is repeated. Otherwise, the next $(m+1)$-th coordinate direction is taken for the motion to the feasible region boundary.

The step forward process (11) is illustrated in Fig. 1 for a two variable case using black and white circles for infeasible and feasible points, respectively.

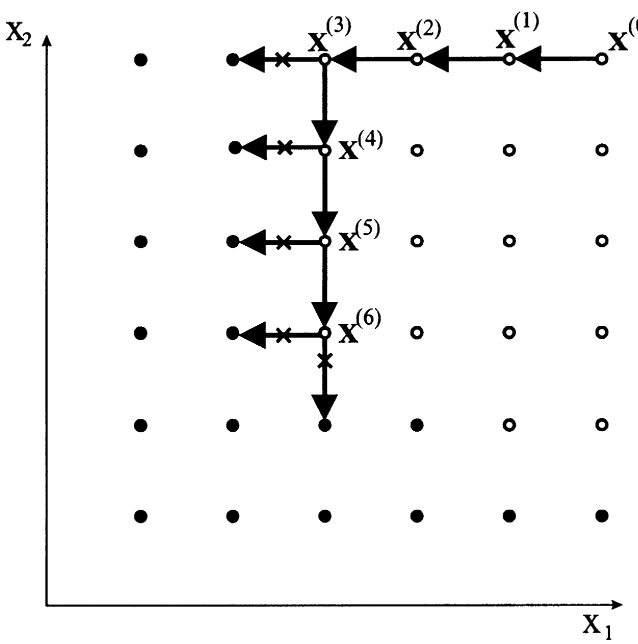

Fig. 1 Graphic representation of the first stage

In the case that the explorative steps in all coordinate directions lead to the infeasible point $\mathbf{x}^{(\ell,m)}$, $m = 1, \ldots, N$ the first discrete derivatives of the objective function and N_{VC} violated constraints are computed at the current design point $\mathbf{x}^{(\ell)}$ by evaluating the functions at neighbour points

$$\frac{\partial C}{\partial x_m^{(\ell)}} = \begin{cases} \frac{C[\mathbf{x}^{(\ell)}] - C[\mathbf{x}^{(\ell,m)}]}{d_{m,n} - d_{m,n-1}} & \text{if } n = x_m^{(\ell)} \neq 1\,, \\ \frac{C[\mathbf{x}^{(\ell)} + \mathbf{e}^{(m)}] - C[\mathbf{x}^{(\ell)}]}{d_{m,2} - d_{m,1}} & \text{if } x_m^{(\ell)} = 1\,, \end{cases}$$

$$m = 1, \ldots, N\,, \tag{12}$$

$$\frac{\partial G_j}{\partial x_m^{(\ell)}} = \begin{cases} \frac{G_j[\mathbf{x}^{(\ell)}] - G_j[\mathbf{x}^{(\ell,m)}]}{d_{m,n} - d_{m,n-1}} & \text{if } n = x_m^{(\ell)} \neq 1\,, \\ \frac{G_j[\mathbf{x}^{(\ell)} + \mathbf{e}^{(m)}] - G_j[\mathbf{x}^{(\ell)}]}{d_{m,2} - d_{m,1}} & \text{if } x_m^{(\ell)} = 1\,, \end{cases}$$

$$m = 1, \ldots, N\,, \quad j = 1, \ldots, N_{\mathrm{VC}}\,. \tag{13}$$

The constraint j^* is selected from the N_{VC} violated constraints by evaluating the expression

$$\cos\beta_{j^*} = \min_j \cos\beta_j\,, \quad j = 1, \ldots, N_{\mathrm{VC}}\,, \tag{14}$$

where

$$\cos\beta_j = \sum_{m=1}^{N} \left[-\frac{\partial C^e}{\partial x_m^{(\ell)}} \right] \left[-\frac{\partial G_j^e}{\partial x_m^{(\ell)}} \right] \tag{15}$$

is the cosine of the angle between the normalized unit vectors of the discrete antigradients of the objective function, $-\nabla C^e$, and the j-th violated constraint, $-\nabla G_j^e$, at the current design point $\mathbf{x}^{(\ell)}$. The projections of the normalized unit vectors of the discrete antigradients are computed by

$$-\frac{\partial C^e}{\partial x_m^{(\ell)}} = -\frac{\partial C}{\partial x_m^{(\ell)}} \left[\sum_{m=1}^{N} \left(\frac{\partial C}{\partial x_m^{(\ell)}} \right)^2 \right]^{-1/2}\,,$$

$$m = 1, \ldots, N\,, \tag{16}$$

$$-\frac{\partial G_j^e}{\partial x_m^{(\ell)}} = -\frac{\partial G_j}{\partial x_m^{(\ell)}} \left[\sum_{m=1}^{N} \left(\frac{\partial G_j}{\partial x_m^{(\ell)}} \right)^2 \right]^{-1/2}\,,$$

$$m = 1, \ldots, N\,, \quad j = 1, \ldots, N_{\mathrm{VC}}\,. \tag{17}$$

The vector $\boldsymbol{\lambda}$ is established

$$\boldsymbol{\lambda} = -\nabla C^e - \nabla G_{j^*}^e\,. \tag{18}$$

This vector $\boldsymbol{\lambda}$ is located within the polyhedron formed by hyperplanes tangent to the objective and violated constraint function surfaces at the current design point $\mathbf{x}^{(\ell)}$. The hyperplanes approximate these surfaces and are perpendicular to the corresponding antigradient vectors $-\nabla C^e$ and $-\nabla G_j^e$ at the current design point $\mathbf{x}^{(\ell)}$, where $j = 1, \ldots, N_{\mathrm{VC}}$.

The vector of design variable changes $\Delta\mathbf{x}$ is attractive for consequent motion if the point $\tilde{\mathbf{x}} = \mathbf{x}^{(\ell)} + \Delta\mathbf{x}$ lies within the polyhedron and the constraints (3) are satisfied. Point $\tilde{\mathbf{x}}$ belonging to the polyhedron implies that the following conditions are met:

$$\sum_{m=1}^{N} -\frac{\partial C^e}{\partial x_m^{(\ell)}} (d_{m,n+p} - d_{m,n}) \geq 0 \,,$$

$$\sum_{m=1}^{N} -\frac{\partial G_j^e}{\partial x_m^{(\ell)}} (d_{m,n+p} - d_{m,n}) \geq 0 \,,$$

$$n = x_m^{(\ell)} \,, \quad p = \Delta x_m \,, \quad j = 1, \ldots, N_{\mathrm{VC}} \,. \tag{19}$$

Two auxiliary relative vectors $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$, coplanar to the vector $\boldsymbol{\lambda}$ with the maximum absolute values of projections equal to 1 and 2, respectively are as follows:

$$\boldsymbol{\lambda}^{(1)} = \frac{\boldsymbol{\lambda}}{\lambda_{\max}} \,, \quad \boldsymbol{\lambda}^{(2)} = 2\boldsymbol{\lambda}^{(1)} \,, \tag{20}$$

where $\lambda_{\max} = \max\limits_m |\boldsymbol{\lambda}_m|$, $m = 1, \ldots, N$ is the maximum absolute value of the vector $\boldsymbol{\lambda}$ projection. These vectors are used to define the design variable changes $\Delta\mathbf{x}$

$$\Delta x_m = \begin{cases} 0 & \text{if } |\lambda_m^{(1)}| < \varepsilon_1 \,, \\ \pm 1 & \text{if } |\lambda_m^{(1)}| \geq \varepsilon_1 \,, \end{cases} \quad m = 1, \ldots, N \,, \tag{21}$$

$$\Delta x_m = \begin{cases} 0 & \text{if } |\lambda_m^{(2)}| < \varepsilon_1 \,, \\ \pm 1 & \text{if } \varepsilon_1 \leq |\lambda_m^{(2)}| < \varepsilon_2 \,, \\ \pm 2 & \text{if } |\lambda_m^{(2)}| \geq \varepsilon_2 \,, \end{cases} \quad m = 1, \ldots, N \,. \tag{22}$$

Herein, $0.5 \leq \varepsilon_1 < 1.0$ and $1.5 \leq \varepsilon_2 < 2.0$ are positive values that define the number of design variable changes equal to ± 1 and ± 2, respectively. The sign of design variable change Δx_m depends on the sign of the corresponding projection λ_m.

Design variable changes $\Delta\mathbf{x}$ are taken according to (22) using minimum values of ε_1 and ε_2. Conditions (19) are verified at the point $\tilde{\mathbf{x}}$. The point $\tilde{\mathbf{x}}$ that satisfies conditions (19) is taken as an initial point $\mathbf{x}^{(\ell+1)} = \tilde{\mathbf{x}}$ for the next $(\ell+1)$-th iteration if the value of the objective function in it is less than in the last feasible point $\mathbf{x}^{(\ell)}$

$$C(\tilde{\mathbf{x}}) < C\left(\mathbf{x}^{(\ell)}\right) \,, \tag{23}$$

and it is feasible that

$$G_j(\tilde{\mathbf{x}}) \leq 1 \,, \quad j = 1, \ldots, N_{\mathrm{DC}} \,. \tag{24}$$

The search procedure continues from Step (11) if conditions (23) and (24) are satisfied. If one of the conditions (19), (23) or (24) is violated, the values ε_1 and ε_2 are increased and new vector of design variable changes $\Delta\mathbf{x}$ is constructed according to (22). If ε_1 and ε_2 take their maximum given values, design variable changes $\Delta\mathbf{x}$ are computed by (21), the initial value of ε_1 is equal to 0.5. The search procedure is terminated if there is no point that complies with conditions (19), (23) or (24) for all possible combinations of vector $\Delta\mathbf{x}$. A local minimum is the last feasible point $\mathbf{x}^* = \mathbf{x}^{(\ell)}$.

The second stage is illustrated in Fig. 2 for a two variable case.

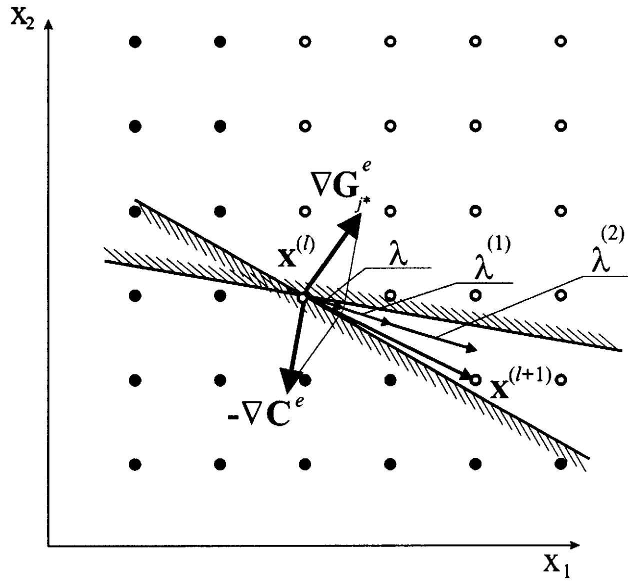

Fig. 2 Graphic representation of the second stage

The iterative steps of the algorithm are given below.

1. Take the iteration index $\ell = 0$. Choose an initial point $\mathbf{x}^{(0)}$.
2. If point $\mathbf{x}^{(\ell)}$ is feasible, go to Step 5. Otherwise, continue.
3. Check Criterion (9). If (9) is satisfied, terminate. Otherwise, construct $\mathbf{x}^{(\ell+1)}$ using (8) and continue.
4. Take $\mathbf{x}^{(\ell)} = \mathbf{x}^{(\ell+1)}$, let $\ell = \ell + 1$ and return to Step 2.
5. Check Criterion (10). If (10) is satisfied, go to Step 13. Otherwise, let $m = 1$ and continue.
6. Construct $\mathbf{x}^{(\ell,m)} = \mathbf{x}^{(\ell)} - \mathbf{e}^{(m)}$.
7. Take $\mathbf{x}^{(\ell+1)} = \mathbf{x}^{(\ell,m)}$ if point $\mathbf{x}^{(\ell,m)}$ is feasible. Let $\ell = \ell + 1$ and return to Step 5. Otherwise, continue.
8. If $m < N$, increase the coordinate direction index $m = m + 1$ and return to Step 6. Otherwise, continue.

9. Compute the normalized unit vectors $-\nabla C^e$ and $-\nabla G_j^e$. Select the constraint j^*. Establish vectors $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$. Take $\varepsilon_1 = 0.5$ and $\varepsilon_2 = 1.5$ and continue.
10. Construct $\Delta\mathbf{x}$ according to (22) or (21) and check conditions (19). If (19) are satisfied, go to Step 12. Otherwise, continue.
11. Increase ε_1 and ε_2. If $\varepsilon_1 < 1.0$ and $\varepsilon_2 < 2.0$, return to Step 10. Otherwise, go to Step 13.
12. Check conditions (23) and (24). If (23) and (24) are satisfied, take $\mathbf{x}^{(\ell+1)} = \tilde{\mathbf{x}}$, let $\ell = \ell + 1$ and go on with Step 5. Otherwise, go to Step 11.
13. Take $\mathbf{x}^* = \mathbf{x}^{(\ell)}$ and terminate.

Table 1 The cross-sectional characteristics of standardized profiles in Example 1

n	A_n (cm^2)	W_n (cm^3)	I_n (cm^4)
1	118.392	1690.16	41623.0
2	144.922	2290.85	62435.0
3	167.342	2842.498	83246.0
4	187.096	3360.34	104058
5	204.594	3852.73	124869
6	221.374	4324.92	145681
7	236.658	4780.51	166492
8	251.019	5222.00	187304
9	264.593	5651.39	208115

5 Numerical examples

In this article the well-known problems of the ten-bar and two hundred-bar trusses and of the frame with two storeys and a single span are presented to show the efficiency of the algorithm.

5.1 Portal frame

The structural geometry of a frame with two storeys and a single span is shown in Fig. 3, where (1), (2) and (3) represent three loading cases, respectively.

The bars are linked to four groups. The first group includes bars 1 and 5, the second bars 2 and 4, the third only bar 3 and the fourth only bar 6. The bars in all groups have the same cross-sectional shape V_i and fabrication type U_i, $i = 1, \ldots, 4$. The material is steel with the following properties: specific weight $\varrho_i = 76\,999.34\,\mathrm{N/m^3}$, Young's modulus $E_i = 206.88\,\mathrm{kN/mm^2}$, allowable normal stress $[\sigma_i] = 163.86\,\mathrm{N/mm^2}$, $i = 1, \ldots, 4$. The objective function of the problem is the weight of the structure. Normal stress constraints are imposed. The displacements of nodes 2, 3, 4 and 5 are limited to $[\delta_\ell] = 25.4\,\mathrm{mm}$ in horizontal direction, $\ell = 1, \ldots, 4$. In this example two different cases are considered.

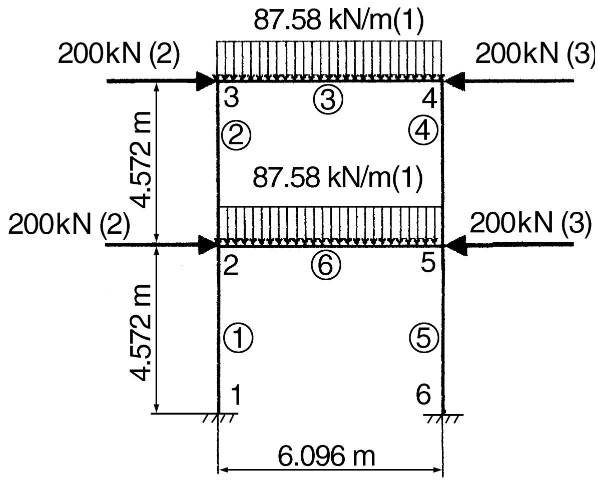

Fig. 3 Frame with two storeys and a single span

5.1.1 Case 1

The frame is fabricated from standardized profiles, $V_i = 1$ and $U_i = 1$, $i = 1, \ldots, 4$. The cross-sectional characteristics are presented in Table 1. Herein, W_n and I_n denote the n-th element from the sets of section modulus $\{W\}$ and moments of inertia $\{I\}$ for standardized profiles with shape $V_i = 1$, respectively.

The results for the first case are given in Table 2. Only 35 constraint computations are necessary to obtain the optimum solution from the unit initial point $x_m^{(0)} = 1$, $m = 1, \ldots, N$. The difference between the optimum weight of Chai and Sun (1996) and the optimization by the present method is 1.05%.

5.1.2 Case 2

Welded profiles with I shape are optimized, $V_i = 2$ and $U_i = 2$, $i = 1, \ldots, 4$. The sets of standard sheet thicknesses $\{t\}$ and widths $\{B\}$ include $Q_t = 26$ and $Q_B = 10$ elements, respectively: $\{t\}$={5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21; 22; 23; 24; 25; 26; 27; 28; 29; 30} in mm and $\{B\}$={100; 200; 300; 400; 500; 600; 700; 800; 900; 1000} in mm.

The results for the second case in comparison with the continuous solution to the equivalent problem are listed in Table 3, where NFE means number of finite element analyses. Initial design variables were taken equal to unity for discrete problem. Lower and upper bounds of continuous variables correspond to $t_1 = 5\,\mathrm{mm}$, $B_1 = 100\,\mathrm{mm}$ and $t_{Q_t} = 30\,\mathrm{mm}$, $B_{Q_B} = 1000\,\mathrm{mm}$, respectively. A sequential linear programming method (SLP) gives the best result in comparison to the feasible directions method and to the sequential quadratic programming method (see VMA Engineering 1993).

Table 2 The results computed for portal frame, Case 1

Method	Weight $C(\mathbf{x}^*)$ (N)	x_1^*	A_1^*	The optimum point, area (cm^2) x_2^*	A_2^*	x_3^*	A_3^*	x_4^*	A_4^*
Method of Chai and Sun (1996)	42979	6	221.374	2	144.922	2	144.922	6	221.374
Present method	42534	5	204.954	2	144.922	2	144.922	7	236.658

Table 3 The results computed for portal frame, Case 2

Group number i	Bar number k	Member	The optimum continuous point. SLP x_j^* (mm)	A_i^* (cm^2)	The optimum discrete point. Present method. x_j^*	t_j^* or B_j^* (mm)	A_i^* (cm^2)
		Lower	5.742		2	6	
		shelf	280.5		3	300	
1	1	Upper	5.858	81.85	1	5	83.0
	5	shelf	269.1		3	300	
		Wall	5.0		1	5	
			999.6		10	1000	
		Lower	6.403		3	7	
		shelf	100.01		1	100	
2	2	Upper	6.403	62.77	2	6	63.0
	4	shelf	100.01		1	100	
		Wall	5.0		1	5	
			999.0		10	1000	
		Lower	6.382		3	7	
		shelf	100.20		1	100	
3	3	Upper	6.382	62.74	4	8	65.0
		shelf	100.20		1	100	
		Wall	5.0		1	5	
			999.0		10	1000	
		Lower	17.118		6	10	
		shelf	127.6		2	200	
4	6	Upper	17.118	93.65	6	10	90.0
		shelf	127.60		2	200	
		Wall	5.0		1	5	
			999.2		10	1000	
NFE			543		1369		
Weight $C(\mathbf{x}^*)$ (N)			17523.1		17555.3		

5.2 Ten-bar truss

The geometry and nodal coordinates of a ten-bar plane truss structure are shown in Fig. 4.

Each group contains only one bar. All bars are fabricated from aluminium alloy standardized profiles with the same cross-sectional shape, $V_i = 1$ and $U_i = 1$, $i = 1, \ldots, 10$. The data are as follows: specific weight $\varrho_i = 27150.68\,\mathrm{N/m^3}$, Young's modulus $E_i = 68.96\,\mathrm{kN/mm^2}$, allowable normal stress $[\sigma_i] = 172.4\,\mathrm{N/mm^2}$, $i = 1, \ldots, 10$, the loads $P_1 = 667\,340\,\mathrm{N}$, and $P_2 = 222\,450\,\mathrm{N}$. There are 10 design variables. The objective function is the weight. The constraints are the member normal stresses and the vertical displacements of nodes 1, 2, 3 and 4. The allowable displacement limit for all nodes is $[\delta_\ell] =$

Table 4 The cross-sectional areas in Example 2

n	A_n (cm^2)	n	A_n (cm^2)	n	A_n (cm^2)
1	0.645	2	3.23	3	6.45
4	12.9	5	19.4	6	25.8
7	32.3	8	38.7	9	41.9
10	45.2	11	48.4	12	51.6
13	54.8	14	58.1	15	61.3
16	64.5	17	70.9	18	77.4
19	83.9	20	90.3	21	96.8
22	103.0	23	110.0	24	116.0
25	123.0	26	129.0	27	135.0
28	142.0	29	148.0	30	155.0
31	161.0	32	168.0	33	174.0
34	181.0	35	187.0	36	194.0
37	200.0	38	206.0	39	213.0

Table 5 The results computed for the ten-bar truss

Group and bar number $i = k$	The optimum continuous point A_i^* (cm^2)	The optimum discrete point, area: Present method x_i^*	A_i^* (cm^2)	Method of Chai and Sun (1996) x_i^*	A_i^* (cm^2)	Evolution strategy x_i^*	A_i^* (cm^2)
1	151.9	31	161.0	26	129.0	30	155.0
2	0.645	1	0.645	1	0.645	1	0.645
3	163.12	30	155.0	26	129.0	30	155.0
4	92.62	20	90.3	22	103.0	21	96.8
5	0.645	1	0.645	1	0.645	1	0.645
6	12.71	4	12.9	4	12.9	4	12.9
7	79.92	19	83.9	15	61.3	19	83.9
8	82.64	19	83.9	29	148.0	19	83.9
9	131.2	26	129.0	28	142.0	26	129.0
10	0.645	1	0.645	1	0.645	1	0.645
Weight $C(\mathbf{x}^*)$ (N)	20808.0	20882.9		21557.0		20985.0	

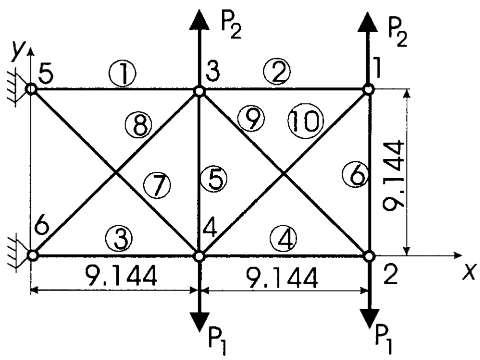

Fig. 4 Ten-bar truss

50.8 mm, $\ell = 1, \ldots, 4$. The set of the cross-sectional areas includes 39 elements and is presented in Table 4.

Initial design variables were taken equal to unity. The results are given in Table 5 in comparison with the solution of Chai and Sun (1996) using a relative difference quotient algorithm, the solution of Grill (1997) using an evolution strategy as well as the continuous solution of Haug and Arora (1979). The present method gives the best result with 1075 constraint computations. The number of finite element analyses using an evolution strategy is 11160.

The solution obtained by Chai and Sun (1996) is infeasible. The reason is that the stress constraints have no monotonic property for statically indeterminate structures. Omitting the stress constraints and increasing the value of the bar cross-sectional area on the second level optimization can violate the stress constraints. In particular, the stress in the fifth bar is $\sigma_S = 280.9\,\mathrm{N/mm}^2 = 1.63[\sigma]$.

A change in material and shape according to DIN2448 (2.81), steel pipes gives the optimum weight $C(\mathbf{x}^*) = 21\,665\,\mathrm{N}$. Accounting for stability requirements using Euler's buckling formula for compressed bars does not change the solution, since only displacement constraints are active.

5.3 Two hundred-bar truss

The geometry and sizes of a plane two hundred-bar truss structure are shown in Fig. 5. The bars are linked to 96 groups. All bars are fabricated from standardized profiles with the same cross-sectional shape, $V_i = 1$ and $U_i = 1$, $i = 1, \ldots, 96$. The material is steel with the following properties: specific weight $\varrho_i = 0.283\,\mathrm{lb/in}^3$ (for references ksi-units are chosen), Young's modulus $E_i = 30\,000\,\mathrm{ksi}$, allowable normal stress $[\sigma_i] = 30\,\mathrm{ksi}$, $i = 1, \ldots, 96$.

There are three loading cases.

1. One kip acting in positive x direction at nodes 1, 6, 15, 20, 29, 34, 43, 48, 57, 62, 71;
2. 10 kips acting in negative y direction at nodes 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 15, 16, 17, 18, 19, 20, 22, 24, ..., 71, 72, 73, 74, 75;
3. Cases 1 and 2 acting together.

The objective function is the weight of the structure. Stress constraints are applied to each group. Displacement constraints are applied on all nodes for both vertical and horizontal directions. The allowable displacement limit is $[\delta_\ell] = 0.5\,\mathrm{in}$, $\ell = 1, \ldots, 150$. The set of the cross-sectional areas includes 30 elements. The areas are taken according to DIN1028, double angle profiles and presented in Table 6.

Initial design variables were taken equal to discrete upper neighbours of the continuous solution of Haug and Arora (1979). The optimum weight of Haug and Arora (1979) is 28 963 lb. Bouzy and Abel (1995) obtain the minimum weight 28 880 lb using the sequential quadratic programming method with 5600 constraint computations. The optimum discrete solution of Cai and Thierauf (1994) using a parallel evolution strategy is 29 737 lb. The number of finite element analyses is 70316. The present method gives the optimum weight 29 168 lb with 81075 constraint computations. The results are given in Table 7.

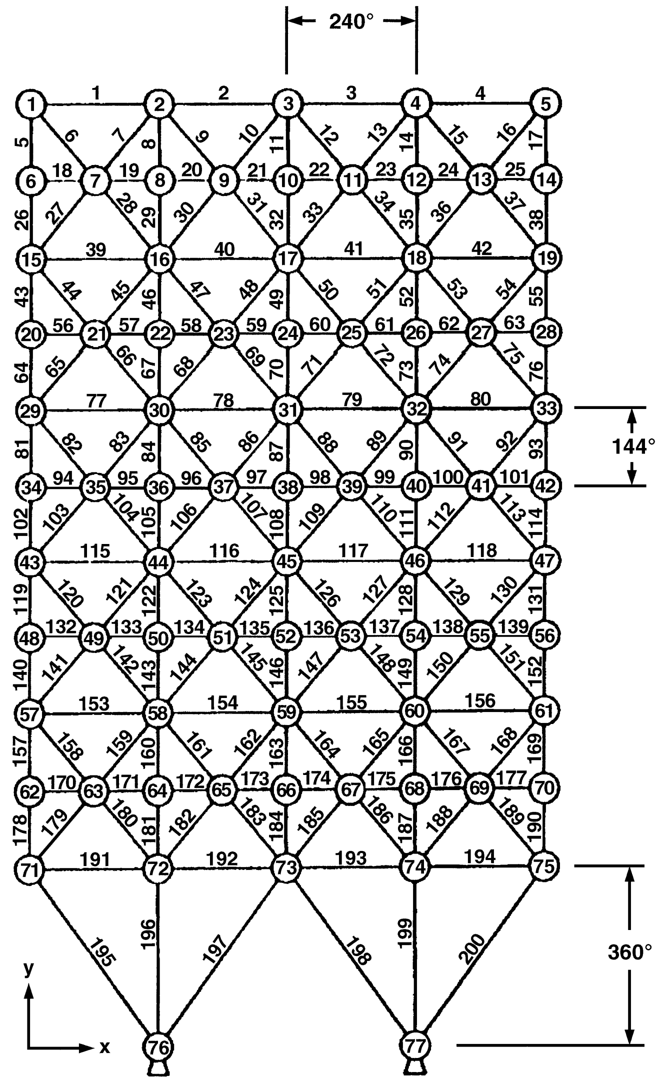

Fig. 5 Two hundred-bar truss

6 Conclusions

In this paper a method for solving integer nonlinear programming is described. The proposed approach is a robust tool for a particular class of problems with monotonic objective function. The algorithm makes use of discrete antigradients to move along the feasible region boundary towards an optimum which is located near this boundary. The algorithm only searches for integer points. It is very simple and natural to engineers. The algorithm, however, terminates on a quasi-optimum and does not guarantee a global solution.

A model of structural optimization for systems of bars is constructed. Design variables are ordinal numbers of the elements from the sets of standardized profile areas as well as standard sheet thicknesses and widths for welded structures.

Test results for the ten-bar and two hundred-bar trusses and a frame with two storeys and a single span indicate a satisfactory convergence. The solutions for all problems considered by the present method are better than results found in the references. The number of finite element analyses performed before convergence is modest for problems with 4, 10 and 24 design variables. The structural analysis effort increases significantly for the problem with 96 design variables. The greatest part of the computations, however, is performed during the last iterations with a small decrement of the objective function.

The inherent efficiency of the method suggested renders it potentially suitable for nonlinear optimization with different numbers of design variables and functional constraints.

Table 6 The cross-sectional areas in Example 3

n	A_n (in^2)	n	A_n (in^2)	n	A_n (in^2)
1	0.100	2	0.347	3	0.440
4	0.539	5	0.954	6	1.081
7	1.174	8	1.333	9	1.488
10	1.764	11	2.142	12	2.697
13	2.800	14	3.131	15	3.565
16	3.813	17	4.805	18	5.952
19	6.572	20	7.192	21	8.525
22	9.300	23	10.850	24	13.330
25	14.290	26	17.170	27	19.180
28	23.680	29	28.080	30	33.700

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References

Amir, H.M.; Hasegawa, T. 1994: Shape optimization of skeleton structures using mixed-discrete variables. *Struct. Optim.* **8**, 125–130

Arora, J.S.; Huang, M.W.; Hsieh, C.C. 1994: Methods for optimization of nonlinear problems with discrete variables: a review. *Struct. Optim.* **8**, 69–85

Bouzy, G.; Abel, J.F. 1995: A two-step procedure for discrete minimization of truss weight. *Struct. Optim.* **9**, 128–131

Cai, J.; Thierauf, G. 1994: A parallel evolution strategy for solving discrete structural optimization. In: Topping, B.H.V.; Papadrakakis, M. (eds.) *Advances in parallel and vector processing for structural mechanics*, pp. 239–244. Edinburgh: Civil-Comp Ltd.

Chai, S.; Sun, H.C. 1996: A relative difference quotient algorithm for discrete optimization. *Struct. Optim.* **12**, 45–56

Grill, H. 1997: Learning from nature: structural design using distributed evolution strategies. *3-rd European Cray-SGI Massively Parallel Processing Workshop* (held in Paris)

Table 7 The results computed for the two-hundred-bar truss

Group no. i	Bar no. k	The optimum continuous point A_i^* (in^2)	The optimum discrete point x_i^*	A_i^* (in^2)	Group no. i	Bar no. k	The optimum continuous point A_i^* (in^2)	The optimum discrete point x_i^*	A_i^* (in^2)
1	1, 4	0.1878	11	2.142	48	97, 98	0.1000	2	0.347
2	2, 3	0.1000	1	0.100	49	102, 114	10.4800	9	1.488
3	5, 17	4.7832	2	0.347	50	103, 113	0.1108	1	0.100
4	6, 16	0.1703	12	2.697	51	104, 112	1.0313	14	3.131
5	7, 15	0.1000	1	0.100	52	105, 111	6.8203	25	14.290
6	8, 14	2.3462	13	2.800	53	106, 110	0.5012	9	1.488
7	9, 13	0.1876	12	2.697	54	107, 109	0.3754	3	0.440
8	10, 12	0.1000	8	1.333	55	108	6.4768	3	0.440
9	11	2.8809	1	0.100	56	115, 118	1.9807	9	1.488
10	18, 25, 56, 63,				57	116, 117	1.4784	3	0.440
	94, 101, 132,	0.1000	1	0.100	58	119, 131	9.1546	2	0.347
	139, 170, 177				59	120, 130	3.1979	12	2.697
11	19, 20, 23, 24	0.1000	1	0.100	60	121, 129	0.1000	1	0.100
12	21, 22	0.1000	10	1.764	61	122, 128	9.0271	28	23.680
13	26, 38	6.7767	12	2.697	62	123, 127	0.2074	1	0.100
14	27, 37	0.1000	1	0.100	63	124, 126	0.9717	9	1.488
15	28, 36	0.2361	11	2.142	64	125	6.5338	1	0.100
16	29, 35	3.3133	15	3.565	65	133, 134, 137, 138	0.1000	1	0.100
17	30, 34	0.1732	1	0.100	66	135, 136	0.1219	1	0.100
18	31, 33	0.2227	7	1.174	67	140, 152	9.9624	7	1.174
19	32	4.1473	8	1.333	68	141, 151	0.1341	1	0.100
20	39, 42	0.1000	14	3.131	69	142, 150	3.3000	12	2.697
21	40, 41	0.1000	5	0.954	70	143, 149	9.5771	28	23.680
22	43, 55	8.1292	1	0.100	71	144, 148	0.9814	9	1.488
23	44, 54	0.2476	17	4.805	72	145, 147	0.2269	1	0.100
24	45, 53	0.1000	1	0.100	73	146	7.0561	2	0.347
25	46, 52	4.4206	20	7.192	74	153, 156	2.5500	5	0.954
26	47, 51	0.2802	4	0.539	75	154, 155	0.6074	1	0.100
27	48, 50	0.2673	6	1.081	76	157, 169	7.5376	3	0.440
28	49	4.7929	1	0.100	77	158, 168	4.1216	9	1.488
29	57, 58, 61, 62	0.1000	1	0.100	78	159, 167	0.1000	1	0.100
30	59, 60	0.1002	1	0.100	79	160, 166	13.3290	29	28.080
31	64, 76	9.3889	6	1.081	80	161, 165	1.8691	10	1.764
32	65, 75	0.1000	1	0.100	81	162, 164	0.3045	1	0.100
33	66, 74	0.3362	17	4.805	82	163	7.4246	3	0.440
34	67, 73	5.0733	21	8.525	83	171, 172, 175, 176	0.1000	1	0.100
35	68, 72	0.3008	5	0.954	84	173, 174	0.1000	1	0.100
36	69, 71	0.3096	3	0.440	85	178, 190	8.2183	6	1.081
37	70	5.5744	4	0.539	86	179, 189	0.1000	1	0.100
38	77, 80	0.4967	11	2.142	87	180, 188	4.1916	9	1.488
39	78, 79	0.3865	2	0.347	88	181, 187	13.8330	30	33.700
40	81, 93	9.5196	1	0.100	89	182, 186	0.3354	1	0.100
41	82, 92	0.9366	15	3.565	90	183, 185	1.9082	10	1.764
42	83, 91	0.1000	1	0.100	91	184	7.8840	5	0.954
43	84, 90	6.2617	24	13.330	92	191, 194	5.8649	5	0.954
44	85, 89	0.3508	2	0.347	93	192, 193	3.4248	1	0.100
45	86, 88	0.4835	6	1.081	94	195, 200	10.6560	10	1.764
46	87	5.8679	1	0.100	95	196, 199	17.7770	30	33.700
47	95, 96, 99, 100	0.1000	2	0.347	96	197, 198	7.7140	14	3.131

Haug, E.J.; Arora, J.S. 1979: *Applied optimal design. Mechanical and structural systems*. New York: John Wiley & Sons

Irons, B. 1970: A frontal solution program for finite element analysis. *Int. J. Num. Meth. Eng.* **2**, 5–32

Jivotovski, G.A. 1991: *Optimization of ship pressed panels hull structures using two-level approach*. PhD Thesis, Gorky State University (in Russian)

Jivotovski, G.A.; Perelman, B.S. 1997: Optimization of structures made from standard half-finished products. *Appl. Prob. Strength Plasticity* **55**, 92–102 (in Russian)

Korbut, A.A.; Finkelshtein, J.J. 1969: *Discrete programming*. Moscow: Nauka (in Russian)

VMA Engineering 1993: *DOT. User's manual*