Struct Multidisc Optim 19, 130–139

Optimum shakedown design under residual displacement constraints*

F. Tin-Loi

Abstract We consider the minimum weight design of suitably discretized elastoplastic structures subjected to variable repeated loads and constraints on the amount of residual (or permanent) deflections. The optimization problem is formulated on the basis of the classical lower bound theorem of shakedown, supplemented by appropriate constraints on deflections obtained from existing bounding results. The primary purpose of the paper is to show that, even for large size structures, this important and challenging problem can be modeled and solved directly as a mathematical programming problem within the industry standard modeling framework GAMS. Examples concerning truss-like structures are presented for illustrative purposes.

1 Introduction

Nowadays, it is widely accepted that the application of shakedown theory for the safety assessment of elastoplastic structures under variable repeated loads is an important, often mandatory, requirement. In this context, "shakedown" — a term coined by Prager — means that the structure responds in a purely elastic manner to the applied load histories, after some initial plastic deformation has occurred. The obverse condition, leading eventually to a structural crisis, is commonly referred to as "inadaptation". The structure then fails by one or both of two modes, namely, alternating plasticity and incremental collapse. The former involves repeated plastic straining in opposite sense (without accumulation of plastic strains) thereby producing a low cycle fatigue phenomenon. The latter involves accumulation of plastic strain increments over each cycle of loading, thus causing a loss of serviceability.

Shakedown analysis belongs to the general class of so-called "simplified" methods which avoid the necessity of computationally intensive evolutive or marching solution methods. It also represents a meaningful generalization of limit analysis theorems and, in particular, is centered on the Melan-Bleich (static) and Koiter (kinematic) theorems. Since these seminal works, a large number of developments and extensions have been carried out, attesting to the fact that this branch of plasticity is by now a mature area. As for these extensions suffice it to mention the following representative achievements: inclusion of thermal loading (König 1969; Gross-Weege and Weichert 1992; Xue *et al.* 1997); incorporation of geometric effects (Weichert 1986; Stumpf 1993; Polizzotto and Borino 1996); extension to dynamics (Maier and Novati 1990; Comi and Corigliano 1991); accommodation of realistic material models (Maier 1989; Pycko and Maier 1995; Polizzotto *et al.* 1996); and consideration of unilateral contact conditions (Polizzotto 1997). Numerous other references can be found in these works; for developments till the late eighties, the interested reader is referred to the comprehensive (and still useful) surveys of König and Maier (1981), Polizzotto (1982) and König (1987).

Parallel with the above-mentioned extensions to the classical shakedown theorems, an enormous amount of research effort has been directed towards supplementing the shakedown limit with bounds on appropriate quantities (e.g. residual displacements) at designated points on the structure. Such bounding techniques represent a useful, often mandatory (as in the case of a theoretically unbounded shakedown limit) safety check, and currently constitute an important branch of shakedown theory. Again the theoretical results for achieving this aim are well-developed. A recent and useful paper in this area (Corigliano *et al.* 1995) summarizes the development of bounding theorems and, more importantly, derives a result of wide applicability. In particular, the so-called "fundamental inequality", a notion attributed to Polizzotto (1982), is extended to provide a general bounding theorem which can elegantly and transparently supply theoretical bounding results on any quantity of interest, such as displacements, for a wide

Received January 18, 1999

F. Tin-Loi

School of Civil and Environmental Engineering, University of New South Wales, Sydney, NSW 2052, Australia
e-mail: `f.tinloi@unsw.edu.au`

* Dedicated to Professor Paul Grundy on the occasion of his 65-th birthday

range of conditions, involving, if required, dynamic loading, nonassociative flow rules and moderate geometric effects. The search for the best bound necessarily involves an optimization problem. The practical usefulness of various bounding techniques has been assessed in a recent, authoritative survey by Maier *et al.* (1996).

Rather than carrying out such checks "independently" of the shakedown analysis, methods of incorporating them directly as necessary constraints in the shakedown theorems have been achieved (e.g. Rizzo 1983; Rizzo and Giambanco 1984; Polizzotto 1984). This elegant solution, unfortunately, appears to be less popular than providing separate bounds.

Whilst it is clear, even from the above necessarily brief survey, that shakedown analysis is a mature area, albeit still awaiting full application to practical engineering situations, the "inverse" problem of optimal shakedown design is far less developed. Research in this area has been sporadic and limited since Heyman's initial work (Heyman 1958) on the minimum weight design of frames under classical shakedown criteria. Heyman's iterative procedure was later used by several others (e.g. Cohn and Parimi 1973; König 1975; Borkauskas and Atkociunas 1975; Save and Prager 1985; Nguyen Dang Hung and Morelle 1990). Numerical schemes based on "optimality criteria" (Rozvany 1989) have also been developed for specific structural types and applied (Giambanco *et al.* 1994a,b; Giambanco and Palizzolo 1995).

This paper considers the elegant and potentially useful approach of combining shakedown with limited ductility constraints in the minimum weight design problem. The formulation, in view of developments in shakedown analysis and bounding procedures, is straightforward. The challenge lies in the numerical solution of practical size (often large-scale) structures. To our knowledge, there has only been a limited attempt (Polizzotto 1984) to solve the most general instance of this problem, in which elastic and work hardening are both design dependent. The difficulty in solving this optimization problem is due to its nonlinearity and nonconvexity. The iterative approach used by Polizzotto (1984) keeps all design-dependent variables constant during every iteration and then adjusts them before the next iteration. This scheme appears to perform reasonably well on the very small examples tested but, as indicated by the author, still needs to be improved.

Motivated by the recent tremendous advances in mathematical programming (MP) computational technology and the development of modeling frameworks for expressing and solving MP problems, including optimization, we investigate the direct solution of the minimum weight problem by means of an industry standard nonlinear programming (NLP) solver named CONOPT (Drud 1994). CONOPT, a sophisticated implementation of a feasible generalized reduced gradient algorithm, is called from within the GAMS (an acronym for General Algebraic Modeling System) modeling system (Brooke *et al.* 1992) which is used to express our problem in point. The merit of this approach is two-fold: first, simplicity of problem formulation and description, especially if carried out via powerful modeling languages such GAMS; and second, we need only a standard and hence readily available NLP solver. It is hoped that both of these advantages will promote the use of MP by practising engineers to solve real engineering problems.

In this first study, we confine ourselves to discretized truss-like structures, quasistatic conditions, inviscid behavior and small displacements. Classical elastic-plastic constitutive laws which admit, besides perfect plasticity, the well-known kinematic and isotropic hardening models are adopted. Without undue loss of generality, we only consider the case of residual displacement constraints although the bounding results available (Corigliano *et al.* 1995) permit us to consider other quantities such as total deflections and plastic work as well.

The organization of this paper is as follows. In the next section, we first review the shakedown analysis problem, considering residual displacement constraints. Only the static approach is considered. This leads naturally to the formulation of the minimum weight shakedown problem under displacement constraints, which we describe in Sect. 3. We then present, in Sect. 4, a necessarily brief overview of the GAMS modeling environment and in particular stress on its advantages in making the construction and solution of large and complex MP models easy for the nonmathematically oriented user. In the ensuing Sect. 5, we present a number of numerical examples, the last being of reasonable size, before concluding with some general remarks in Sect. 6.

2
Shakedown analysis under displacement constraints

Consider a suitably space-discretized structural system made up of n elements, each of which exhibits an elastic perfectly-plastic or hardening behavior. An m-vector $\mathbf{Q}$ of generalized stresses collects, in a typical finite element fashion, the contribution of each element. Let the structure consist of d degrees of freedom so that the nodal displacements are given by a d-vector $\mathbf{u}$ and the nodal loads by a d-vector $\mathbf{F}$. The actual structure, however, is subjected to a quasistatic load regime $F(t)$ which varies repeatedly with time t.

The classical Melan-Bleich theorem allows us to determine the maximum load amplification factor (also called the "shakedown limit") for which the structure shakes down by responding in a purely elastic manner. This limit can be formally calculated via the following familiar optimization problem (e.g. Maier 1973; Giambanco and Palizzolo 1995):

maximize ξ ,

subject to

$$\mathbf{f}_i \equiv \mathbf{N}^T(\mathbf{Q}_i^e + \overline{\mathbf{Q}}) - \mathbf{H}\mathbf{z} - \mathbf{r} \leq \mathbf{0}\,, \quad \forall i \in \mathcal{I}\,,$$

$$\mathbf{C}^T\overline{\mathbf{Q}} = \mathbf{0}\,,$$

and

$$\mathbf{C}^T\mathbf{Q}_i^e = \xi\mathbf{F}_i, \quad \mathbf{Q}_i^e = \mathbf{S}\mathbf{C}\mathbf{u}_i^e\,, \quad \forall i \in \mathcal{I}\,, \tag{1}$$

where, as is conventional, time t has been removed from the formulation by assuming that $F(t)$ can be identified with any load path that does not exceed a given convex polyhedron whose vertices correspond to a set of ℓ mutually independent nodal load vectors $\xi\mathbf{F}_i$, $i \in \mathcal{I} \equiv \{1, 2, \dots, \ell\}$, where ξ is a scalar proportionally increasing load factor (leading to a homothetic expansion of the convex load polyhedron)

The optimization problem given by (1) simply attempts to maximize the load factor ξ subject to the requirement that there exists a self-equilibrated generalized stress vector $\overline{\mathbf{Q}}$ such that the superposition of purely elastic stresses $\mathbf{Q}_i^e$ and residual stresses $\overline{\mathbf{Q}}$ does not violate the yield condition $\mathbf{f}_i \leq \mathbf{0}$, $\forall i \in \mathcal{I}$.

The particular yield functions adopted clearly postulate a piecewise linearized representation, say with a total of y yield hyperplanes, which can accommodate, through the $y \times y$ matrix $\mathbf{H}$, Maier's remarkable class of hardening models (Maier 1970); $\mathbf{N}$ is an $m \times y$ matrix of outward normals to the yield surface, $\mathbf{r}$ is an obvious y-vector of yield limits and $\mathbf{z}$ is a y-vector of plastic multipliers.

The second constraint set enforces the self-stress nature of $\overline{\mathbf{Q}}$ by setting it to be in equilibrium with a zero load vector, through the $m \times d$ geometric compatibility matrix $\mathbf{C}$ of full column rank (transpose $\mathbf{C}^T$ is the equilibrium matrix).

Instead of calculating $\mathbf{Q}_i^e$ explicitly for $\xi = 1$, as is commonly done, we prefer for ease and transparency of modeling, at the possible expense of a larger system, to express it implicitly in the well-known form given by the last sets of constraints in (1), where $\mathbf{S}$ is an $m \times m$ elastic matrix of unassembled symmetric positive definite element stiffnesses.

Finally, we note that in the above algebraic description, vector and matrix quantities for the whole structure have been assembled via the contributions of corresponding elemental entities, as concatenated vectors and block diagonal matrices, respectively. For example, $\mathbf{Q}^T = (\mathbf{Q}_1^T \dots \mathbf{Q}_n^T)$, $\mathbf{S} = \mathrm{diag}(\mathbf{S}_1, \dots, \mathbf{S}_n)$, etc., where an element index has been denoted by a subscript.

Three points are worthy of note at this stage. Firstly, optimization problem (1) is a standard linear programming (LP) problem (in ξ, $\mathbf{Q}_i^e$, $\overline{\mathbf{Q}}$, $\mathbf{z}$, $\mathbf{u}_i^e$) since the objective function and all constraints are linear. Secondly, as discussed by Tin-Loi and Grundy (1978), the shakedown limit can lose its meaningfulness for certain hardening models. For instance, it is unbounded for isotropic hardening and always leads to an alternating plasticity failure mode for kinematic hardening — hence the motivation for supplementing the shakedown analysis with estimates of bounds on appropriate quantities. Thirdly, both elastic and plastic collapse limits can be obtainedfrom basically the same LP problem (1), by setting $\overline{\mathbf{Q}} = \mathbf{0}$ in the former case and by adopting independent self-stress vectors $\overline{\mathbf{Q}}_i$ for each load condition in the latter.

As a preliminary to formulating the shakedown analysis under residual displacement constraints, we now briefly state, without any derivation, a key result concerning the calculation of residual displacement bounds. For details of the fundamental so-called "weak" bounding theorem on which it is based, the reader is referred especially to Polizzotto (1982).

Let $\hat{\mathbf{F}}$ be a (dummy) load vector which is in equilibrium with generalized stresses $\hat{\mathbf{Q}}$ and which is suitable for extracting, in usual virtual work fashion, the appropriate (at present single) residual displacement value u^r at the desired degree of freedom. In our description, following the same formalism as (1), an upper bound on u^r for a prescribed ξ is given by the following:

$$u^r \leq B \equiv \frac{1}{2\omega}\left(\mathbf{z}^T\mathbf{H}\mathbf{z} + \overline{\mathbf{Q}}\mathbf{S}^{-1}\overline{\mathbf{Q}}\right)\,,$$

if

$$\mathbf{N}^T(\mathbf{Q}_i^e + \overline{\mathbf{Q}}) - \mathbf{H}\mathbf{z} - \mathbf{r} - \omega\mathbf{N}^T\hat{\mathbf{Q}} \leq \mathbf{0}\,, \quad i \in \mathcal{I}\,,$$

$$\mathbf{C}^T\overline{\mathbf{Q}} = \mathbf{0}\,,$$

$$\mathbf{C}^T\mathbf{Q}_i^e = \xi\mathbf{F}_i, \quad \mathbf{Q}_i^e = \mathbf{S}\mathbf{C}\mathbf{u}_i^e\,, \quad \forall i \in \mathcal{I}\,,$$

$$\mathbf{C}^T\hat{\mathbf{Q}} = \hat{\mathbf{F}}, \quad \hat{\mathbf{Q}} = \mathbf{S}\mathbf{C}\hat{\mathbf{u}}\,,$$

and

$$\mathbf{z} \geq \mathbf{0}, \quad \omega > 0\,. \tag{2}$$

It should be noted that (2) provides an, not necessarily the best, upper bound on u^r. The tightest bound can be obtained (Polizzotto 1982) by solving an optimization problem involving minimization of the quadratic term B on the RHS of the first expression in (2), subject to the indicated constraints. This is still an open problem; some of the computational difficulties were discussed more that a decade ago by Kaneko (1982) for a specific instance of this problem involving discrete truss-like structures. As will soon become clear, we will not attempt such a task in the shakedown analysis under residual displacement constraints, but will merely apply the constraint $B \leq \hat{u}^r$, where $\hat{u}^r$ is a specified serviceability limit. Further, in general, we can set s such serviceability constraints and consequently need to check B_j, $j \in \mathcal{J} \equiv \{1, 2, \dots, s\}$ deflection components.

We are now in a position to formulate the shakedown analysis problem under residual deflections constraints. From (1) and (2), this takes the form of the following NLP problem:

$$\text{maximize } \xi ,$$

subject to

$$\mathbf{N}^T(\mathbf{Q}_i^e + \overline{\mathbf{Q}}_j) - \mathbf{H}\mathbf{z}_j - \mathbf{r} - \omega_j \mathbf{N}^T \hat{\mathbf{Q}}_j \leq \mathbf{0} ,$$

$$\forall i \in \mathcal{I} , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{C}^T \overline{\mathbf{Q}}_j = \mathbf{0} , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{C}^T \mathbf{Q}_i^e = \xi \mathbf{F}_i , \quad \mathbf{Q}_i^e = \mathbf{S}\mathbf{C}\mathbf{u}_i^e , \quad \forall i \in \mathcal{I} ,$$

$$\mathbf{C}^T \hat{\mathbf{Q}}_j = \hat{\mathbf{F}}_j , \quad \hat{\mathbf{Q}}_j = \mathbf{S}\mathbf{C}\hat{\mathbf{u}}_j , \quad \forall j \in \mathcal{J} ,$$

$$B_j \equiv \frac{1}{2\omega_j} \left(\mathbf{z}_j^T \mathbf{H} \mathbf{z}_j + \overline{\mathbf{Q}}_j \mathbf{S}^{-1} \overline{\mathbf{Q}}_j \right) \leq \hat{u}_j^r , \quad \forall j \in \mathcal{J} ,$$

and

$$\mathbf{z}_j \geq \mathbf{0} , \quad \omega_j \geq 0 , \quad \forall j \in \mathcal{J} , \tag{3}$$

where we have replaced the strict positivity condition on ω with a nonnegativity constraint. In practice, a small positive lower bound for ω may be necessary to present numerical difficulties. In the form shown, the NLP can be easily modeled within GAMS and solution attempted by invoking a standard NLP solver such as GAMS/CONOPT.

3 Shakedown design under displacement constraints

The structural optimization problem we are concerned with is one of minimum weight (or volume since the density is assumed to be constant). Under the assumptions of a fixed topology and a bound on the load parameter ξ, we therefore wish to minimize the volume of the structure under the traditional shakedown constraints as well as attempting to limit the magnitudes of residual displacements at certain specified freedoms of the discretized structure. In effect, the problem is the design counterpart of the state problem (3).

Let us assume that the yield limits $\mathbf{r}$, stiffnesses $\mathbf{S}$ and hardening parameters $\mathbf{H}$ of the constituent members of the structure are all regarded as unknown but are (continuous) functions of the cross-sectional areas of all n elements. For simplicity of exposition, we adopt a truss-like structure of n bars, for which the unknown element areas are collected in an n-vector $\mathbf{a}$ and the known element lengths in vector $\mathbf{L}$ of same size n. Then, the yield limits $\mathbf{r}$ can immediately be expressed in terms of $\mathbf{a}$, and if we further assume (as is in fact the case for trusses) that explicit expressions for member stiffnesses $\mathbf{S}(\mathbf{a})$ and hardening matrices $\mathbf{H}(\mathbf{a})$, in terms of $\mathbf{a}$, are available, the minimum volume problem can then be formally stated as the following constrained optimization problem in variables $\mathbf{a}$, ξ, $\mathbf{z}_j$, $\mathbf{Q}_i^e$, $\overline{\mathbf{Q}}_j$, $\hat{\mathbf{Q}}_j$, $\mathbf{u}_i^e$, $\hat{\mathbf{u}}_j$, ω_j:

$$\text{maximize } \mathbf{L}^T \mathbf{a} ,$$

subject to

$$\mathbf{N}^T(\mathbf{Q}_i^e + \overline{\mathbf{Q}}_j) - \mathbf{H}\mathbf{z}_j - \mathbf{r} - \omega_j \mathbf{N}^T \hat{\mathbf{Q}}_j \leq \mathbf{0} ,$$

$$\forall i \in \mathcal{I} , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{C}^T \overline{\mathbf{Q}}_j = \mathbf{0} , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{C}^T \mathbf{Q}_i^e = \xi \mathbf{F}_i , \quad \mathbf{Q}_i^e = \mathbf{S}\mathbf{C}\mathbf{u}_i^e , \quad \forall i \in \mathcal{I} ,$$

$$\mathbf{C}^T \hat{\mathbf{Q}}_j = \hat{\mathbf{F}}_j , \quad \hat{\mathbf{Q}}_j = \mathbf{S}\mathbf{C}\hat{\mathbf{u}}_j , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{z}_j^T \mathbf{H} \mathbf{z}_j + \overline{\mathbf{Q}}_j \mathbf{S}^{-1} \overline{\mathbf{Q}}_j \leq 2\omega_j \hat{u}_j^r , \quad \forall j \in \mathcal{J} ,$$

$$\mathbf{z}_j \geq \mathbf{0} , \quad \omega_j \geq 0 , \quad \forall j \in \mathcal{J} ,$$

$$\xi \geq \hat{\xi} ,$$

and

$$\mathbf{a}_l \leq \mathbf{a} \leq \mathbf{a}_u , \quad \mathbf{T}\mathbf{a} = \mathbf{0} , \tag{4}$$

where $\hat{\xi}$ is a prescribed lower bound on the load parameter; vectors $\mathbf{a}_l$ and $\mathbf{a}_u$ are, respectively, the usual lower and upper bounds on the cross-sectional areas; and $\mathbf{T}$ is a $t \times n$ technological matrix imposing t constraints on the design variables (areas). Such technological constraints represent some desired relationships between the areas, such as the requirement that certain groups of members need to be of the same size. The ω_j variables which appear in the displacement bounds have been relocated in the numerator to obviously reduce numerical instabilities for cases when these variables tend to zero.

This optimization problem is both nonlinear and nonconvex and hence can be difficult to solve for reasonably sized structures. There is also no guarantee that the solution obtained is a global optimum. As mentioned earlier, Polizzotto (1984) considered only trivially small examples with his suggested iterative scheme. In spite of nonconvexity, it may be possible to derive optimality conditions for problem (4). Our approach, however, is to attempt direct solution as an NLP problem within the sophisticated and powerful framework provided by the modeling system GAMS. The next section gives a brief overview of the GAMS environment.

4
GAMS modeling environment

Welch (1987) rightly points out that data manipulation requirements limit MP applications more than optimization requirements. The typical end-user is generally more concerned with model formulation, representation and solution than with the details of the MP techniques involved. There is thus a strong case for making the solution phase as simple as possible while at the same time allowing for easy construction of large and complex models. This aim provided the impetus for the development of modeling languages of which GAMS is one.

GAMS (Brooke *et al.* 1992) had its origin at the Development Research Center of the World Bank in Washington. It is a high level declarative language for formulating small to very large MP models using simple and concise algebraic statements which mirror the actual mathematical constructs involved. A GAMS model is transparent to both human and computer, is easily modified and moved across different computing platforms from notebooks to mainframes, and is independent of the solution algorithm of the MP solvers. It not only frees the model builder from the burdens imposed by the MP solution phase but also takes over the steps required for generation of the model. In addition to providing simplicity and compactness of model construction, it possesses important capabilities such as an internal efficient sparse data representation and automatic differentiation.

A number of MP problems types can be solved via GAMS. In addition to the LP and NLP models, solution procedures are available for MIP (mixed integer programming), RMIP (relaxed mixed integer programming), MINLP (mixed integer nonlinear programming), RMINLP (relaxed mixed integer nonlinear programming), CNS (constrained nonlinear systems) and MCP (mixed complementarity problems). The last problem type is particularly useful for solving plasticity and contact-like problems involving systems of orthogonal (complementarity) vectors. GAMS is continually evolving and adapted as new algorithms and problem classes have been explored. The most widely used solvers for the LP and NLP models — problem classes which concern the present work — are the industry standard large-scale CPLEX and CONOPT codes. In our case, we used the GAMS/CONOPT2 solver for both LP and NLP problems.

A GAMS input model, written as a text-based file, consists of the following basic structure, with GAMS reserved key words given in upper case for clarity:

```
SETS
   Declaration, assignment of members
Data (e.g. PARAMETERS, TABLES)
   Declaration, assignment of values
VARIABLES
   Declaration, assignment of type
   Bounds and/or initial values (optional)
EQUATIONS
   Declaration, definition
MODEL and SOLVE statements
DISPLAY statements (optional)
```

As a very simple illustration, we develop in the following a model appropriate for the limit analysis of the propped cantilever shown in Fig. 1. Through simple statics, the dimensionless bending moment m_i at a generic node i is given as $m_i = rx - 0.5\mu x^2$, where the following nondimensional variables have been defined: $m = M/M_p$, $r = RL/M_p$, $x = X/L$, $\mu = wL^2/M_p$ with M_p denoting the full plastic moment capacity of the cross-section of the uniform beam. The yield conformity condition requires that $-1 \leq m_i \leq 1$. The GAMS model, founded on the static theorem of limit analysis, is then as follows:

```
set i /1*100/;
parameter x(i);
x(i) = (ord(i)−1)/(card(i)−1);
variables mu, m(i), r;
equation bm(i);
bm(i) .. m(i) =e= r*x(i)−0.5*mu*x(i)*x(i);
mu.lo = 0; m.lo(i) = −1; m.up(i) = 1;
model prop /all/;
solve prop using lp maximizing mu;
display mu.l;
```

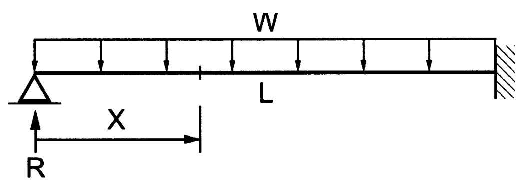

Fig. 1 Propped cantilever

We do not need to describe the model further. Readers familiar with GAMS will immediately understand the “set”, “variable” and “equation” declarations, while those not familiar with GAMS will appreciate the concise yet descriptive style and should recognize that it implements in a transparent algebraic form the limit analysis formulation. On executing this GAMS model, a collapse load factor of 11.657 will be obtained for the adopted 100 nodes discretization (which, as evident, can be easily changed).

For those interested in further details of actual models (from such diverse areas as economics, chemical engineering, trade, etc.), GAMS Development Corporation maintains an extensive library on their website (http://www.gams.com). An example of a GAMS model for the limit analysis of plane frames and grids under various combined stress yield conditions can be found in the paper by Tin-Loi (1995).

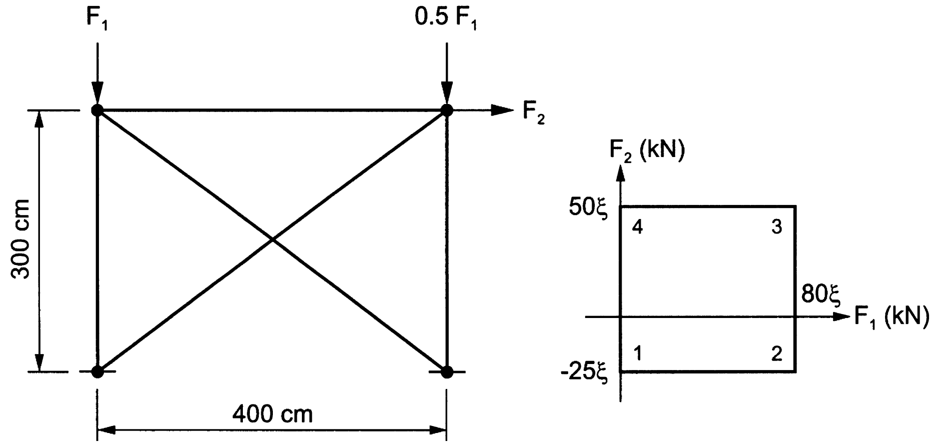

Fig. 2 Example 1: 5-bar truss, geometry and loading

5 Numerical examples

In this section, three examples are presented. The first example concerns a simple academic 5-bar truss similar to the one used by Rizzo (1983); the second is that of a 2-span, 25-member bridge truss; and the third is a 234-member double layer space truss.

As indicated earlier, we adopted the GAMS/CONOPT2 code (version 2.070F) to solve both LP and NLP problems, with all examples being implemented using the same two (analysis and design) generic GAMS models. All runs were carried out on a Win95-based 333 MHz Pentium-II running GAMS 2.50.094. In addition to presenting the optimal solutions obtained, we briefly report on some computational statistics (viz. total execution times and number of major iterations) for the larger problems.

5.1 Example 1

Consider the simple 5-bar truss shown in Fig. 2. For the indicated repeated load regime 1-2-3-4, governed by the load parameter ξ, we carried out in the first instance a series of analyses to investigate the shakedown limit behavior with various prescribed limits on the residual deflection at the top right hand horizontal freedom of the truss. The following data (kN, cm units) were used: Young's modulus $E = 20000$; yield limit σ (tension and compression) $= 40$; cross-sectional areas a (horizontal and diagonal bars) $= 4.5$, cross-sectional areas a (vertical bars) $= 5$. Each bar i was assumed to be governed by a generic 4×4 hardening matrix H_i with diagonal entries $= h_i$ and off-diagonal entries $= \beta h_i$. This was used to simulate perfect-plasticity ($h_i = 0$), Prager's kinematic hardening ($\beta = -1$, $h_i = 0.1Ea_i/L_i$) and isotropic hardening ($\beta = 1$, $h_i = 0.1Ea_i/L_i$).

Without any residual displacement constraints, the following load limits were obtained: ξ (elastic) $= 2.510$, ξ (plastic collapse) $= 3.119$, ξ (shakedown, perfectly plastic) $= 2.986$, and ξ (shakedown, kinematic) $= 4.205$, and ξ (shakedown, isotropic) $= \infty$. It should be noted that in this and the ensuing two examples, both elastic and collapse limits were obtained from the shakedown model (1) with, as mentioned earlier, appropriate constraints on the self-stress field.

Residual deflections constraints were then imposed at the above indicated location in both positive an negative directions. We show in Fig. 3, the variations of the shakedown limits for perfectly plastic constitution (crosses), kinematic hardening (circles) and isotropic hardening (pluses).

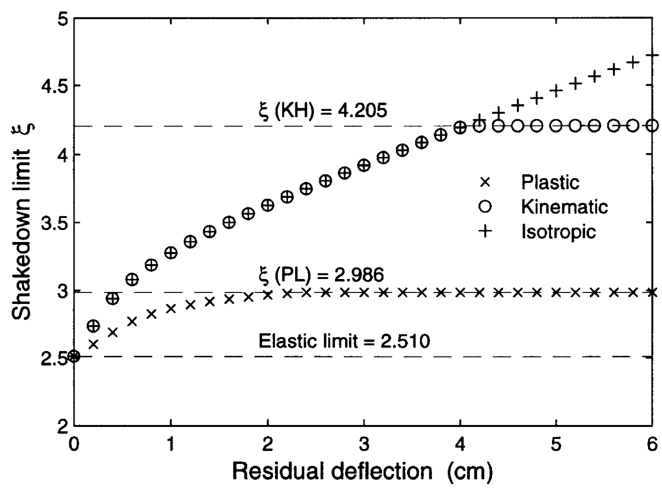

Fig. 3 Example 1: variation of shakedown limit with residual deflection limit

Finally, the minimum weight problem was solved for these three constitutive laws, with $\hat{\xi}$ (lower bound on ξ) set to 4 in all instances. The obtained variation of minimum volume with residual deflection limits (imposed in the same manner as for the analysis runs) are shown in Fig. 4 for two situations, namely when all areas were as-

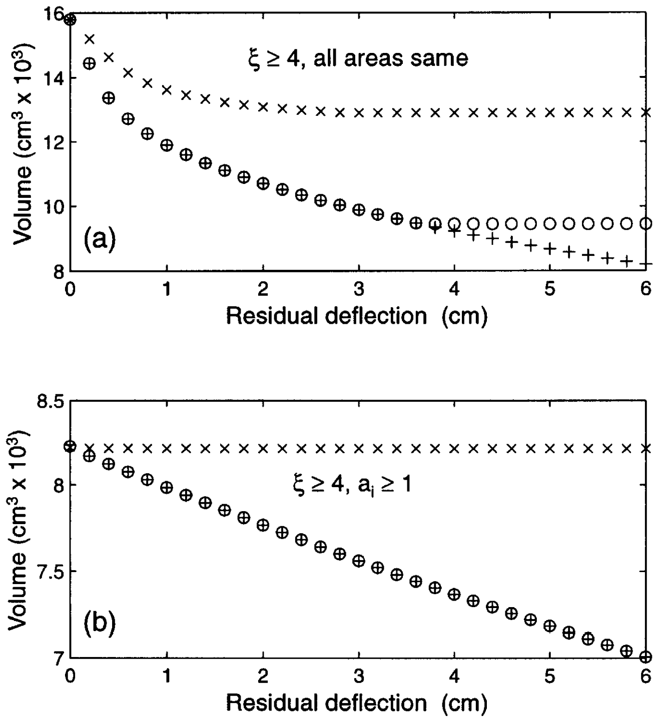

Fig. 4 Example 1: variation of optimal volume with residual deflection limit

sumed to be equal (Fig. 4a) and when a lower bound of 1 was imposed on all areas (Fig. 4b).

5.2 Example 2

This second example concerns the two-span bridge-type truss shown in Fig. 5. The truss consists of 25 members and 20 degrees of freedom. Material properties (kN, cm units) are: Young's modulus $E = 20000$; yield limit σ (tension and compression) $= 30$. Both analysis and minimum weight problems were solved for a repeated load pattern consisting of application and then removal of a vertically downwards 200ξ kN point load at each of the 4 bottom chord freedoms to simulate a moving point load on the structure.

For the analysis runs, we adopted bar areas of $a = 20$ for all members, and when hardening matrix was included we used the same generic matrix as in Example 1. Table 1 summarizes the results obtained; also shown are the number of CONOPT2 iterations and total GAMS solution time. Only the last case involved residual deflection constraints (2 cm vertically down at each of the bottom chord nodes), with isotropic hardening being assumed in that particular instance. Clearly, more computational effort is required to solve the deformation constrained problem.

Three minimum volume cases were then solved. For all of these runs, we assumed $\xi \geq 3.561494$, an isotropic hardening model and residual deflection constraints of 2 cm, as for the last analysis case. Details of these runs are: (a) all bar areas are the same, $a_i \geq 5$, (b) 4 groups (bottom, top, vertical, diagonal) of bars with equal areas for members of the same group, $a_i \geq 5$, and (c) $a_i \geq 5$. Table 2 summarizes the results obtained; for paucity of space, we report on optimal bar areas for the first two cases only. In all runs we initiated the optimization with the arbitrarily chosen starting point: $a_i = 5$, other variables $= 1$. As expected from the corresponding analysis problem, the first case returned the solution $a_i = 20$. Also, surprisingly, the "free" case required less iterations than the other two cases.

Table 1 Analysis results for Example 2

Case	ξ	Iter	Time (sec)
Elastic limit	3.219	4	0.12
Collapse limit	4.660	5	0.11
SD - perfectly plastic	4.616	5	0.11
SD - kinematic	5.161	5	0.11
SD - isotropic ($\hat{u}_j^r = 2$)	3.561	64	8.06

5.3 Example 3

This example concerns the space truss shown in Fig. 6. All members of this double-layer parallel grid are of the same lengths (300 cm). The structure is restrained vertically at each top node along the perimeter and in all directions at the six corner supports. The truss consists of 234 members, 64 nodes and 162 degrees of freedom.

The repeated load program used consisted of a sequential application and removal of single nodal point vertical load of 1.6ξ T to all the interior top nodes, leading to 20 load events (including the no load case) per cycle. Material properties (T, cm units) are: Young's modulus $E = 2000$; yield limit σ (tension and compression) $= 3$. The same type of analyses and minimum weight cases as for Example 2 were considered.

Assumed bar areas for the analyses were $a_i = 15$. The isotropic hardening, deflection constrained case limited the vertical residual deflection at the top centre node to 5 cm. Table 3 gives the results of these analyses. Once again, as expected, more computational effort is required to perform the analysis when deflection constraints are included.

For the minimum volume runs, we assumed $\xi \geq 73.211892$, an isotropic hardening model and residual deflection constraint of 5 cm, as for the last analysis case. Details of these runs are: (a) all bar areas are the same, $a_i \geq 5$, (b) 3 groups (bottom, top, diagonal) of bars with equal areas for members of the same group, $a_i \geq 5$, and (c) $a_i \geq 5$. Initial attempts to solve the NLP problems directly, as for Examples 1 and 2, were not entirely satisfactory, since CONOPT2 had difficulty in finding an initial feasible starting point. We therefore adopted a simple

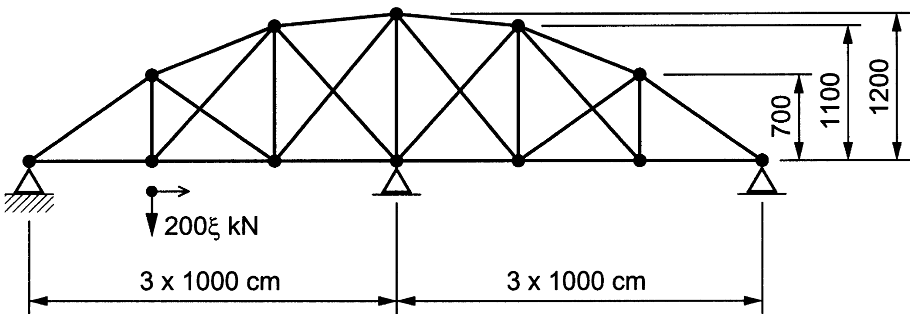

Fig. 5 Example 2: bridge truss

Table 2 Minimum volume results for Example 2

Case	Volume	a	Iter	Time
$a_i \geq 5$	(cm^3)	(bottom, top, vertical, diagonal)		(sec)
Equal areas	5.783438e5	20	1613	197
4 groups	4.589014e5	17.577, 21.514, 14.632, 12.257	1711	275
Free	3.598282e5	–	1214	177

Table 3 Analysis results for Example 3

Case	ξ	Iter	Time (sec)
Elastic limit	56.796	4	5
Collapse limit	68.892	9	96
SD - perfectly plastic	63.075	7	14
SD - kinematic	95.059	8	25
SD - isotropic ($\hat{u}_j^r = 5$)	73.212	355	492

two-phase approach exploiting the "warm start" capability of GAMS. In essence, we fixed all areas (by setting lower bounds to be equal to upper bounds) at a value that would obviously lead to a feasible (stable) structure, carried out a dummy optimization for the total volume, and then used the feasible solution so obtained to warm start the optimization proper. This technique worked well in all cases. We arbitrarily chose $a = 20$ for all bars in all three cases. Table 4 summarizes the results obtained. In this example, the "free" area case, as expected, turned out to be far more computationally demanding. However, a solution was still found.

6 Conclusions

Given the tremendous advances in both computing hardware and optimization technology, especially with the latter being available within a mathematical programming modeling environment, can we solve the important state and design problems concerning shakedown conditions and deflection limits directly as MP problems? From the results obtained, some of which are given in this paper, the answer appears to be "yes".

The use of a modeling framework, GAMS in the present case, provides advantages such as ease of model building, maintenance, solution, large-scale capabilities, etc. which we hope will increase the practical application by engineers of MP techniques in structural mechanics. The end-user can then concentrate primarily on model building, rather than be involved in such specialist tasks as algorithm design and implementation.

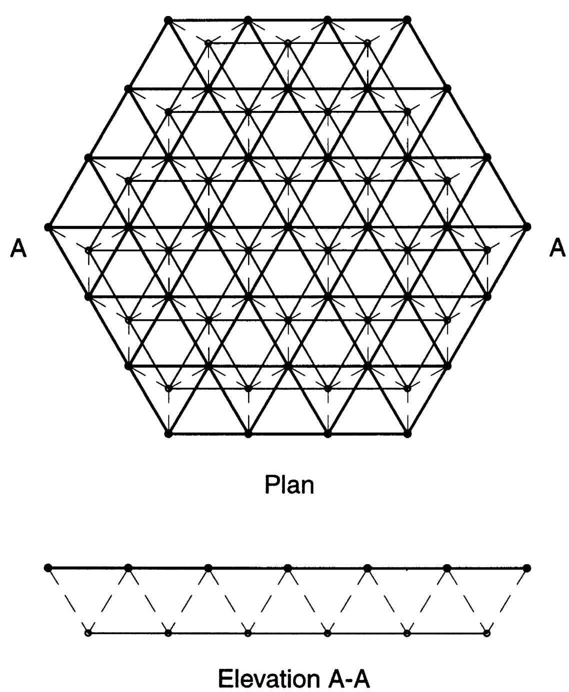

Fig. 6 Example 3: double-layer space truss

Table 4 Minimum volume results for Example 3

Case	Volume	a	Iter	Time
$a_i \geq 5$	(cm^3)	(bottom, top, diagonal)		(sec)
Equal areas	1.053000e6	15	318	1151
3 groups	0.705602e6	9.362, 5.157, 16.026	491	3153
Free	0.618673e6	–	3190	27850

Whilst there is a possibility of using the same approach for topology design, the results, although reasonably encouraging from the runs carried out, are still far from conclusive. At this stage, the optimality criteria approach, if such criteria can be derived, still appears to be the best technique. In this respect, a current development (Ferris and Munson 1998) which may prove useful for modelers is the automatic conversion of an NLP program into the corresponding Karush-Kuhn-Tucker conditions which lead to an MCP. Of course, solving the MCP guarantees only a stationary point for the original problem, if nonconvex.

Clearly, there is still scope for further work in the area covered by this paper. For instance, more extensive computational experiments need to be carried out, generic models for use by engineers need to be developed, and analysis/synthesis of other structural types need to be considered.

Acknowledgements This research was supported by the Australian Research Council. The author would also like to thank Professor Michael Ferris, University of Wisconsin, Madison for advice on GAMS and Professor Giulio Maier, Milan Technical University, for constant inspiration.

References

Borkauskas, A.; Atkociunas, J. 1975: Optimal design for cyclic loading. In: Sawczuk, A.; Mroz, Z. (eds.) *Optimization in structural design*, pp. 433–440. Berlin, Heidelberg, New York: Springer

Brooke, A.; Kendrick, D.; Meeraus, A. 1992: *GAMS: a user's guide, release 2.25*. Massachusetts: Boyd & Fraser Publishing Company

Cohn, M.Z.; Parimi, S.R. 1973: Optimal shakedown of plastic structures for fixed and shakedown loadings. *J. Appl. Mech.* **40**, 595–599

Comi, C.; Corigliano, A. 1991: Dynamic shakedown in elastoplastic structures with general internal variable constitutive laws. *Int. J. Plasticity* **7**, 679–692

Corigliano, A.; Maier, G.; Pycko, S. 1995: Dynamic shakedown analysis and bounds for elastoplastic structures with nonassociative, internal variable constitutive laws. *Int. J. Solids Struct.* **32**, 3145–3166

Drud, A. 1994: CONOPT – a large-scale GRG code. *ORSA J. Computing* **6**, 207–216

Ferris, M.C.; Munson, T.S. 1998: Complementarity problems in GAMS and the PATH solver. *Technical Report TR98-12*, Computer Sciences Department, University of Wisconsin, Madison

Giambanco, F.; Palizzolo, L. 1995: Optimality conditions for shakedown design of trusses. *Comput. Mech.* **16**, 369–378

Giambanco, F.; Palizzolo, L.; Polizzotto, C. 1994a: Optimal shakedown design of beam structures. *Struct. Optim.* **8**, 156–167

Giambanco, F.; Palizzolo, L.; Polizzotto, C. 1994b: Optimal shakedown design of circular plates. *J. Eng. Mech. Div., ASCE* **120**, 2335–2355

Gross-Weege, J.; Weichert, D. 1992: Elastic-plastic shells under variable mechanical and thermal loads. *Int. J. Mech. Sci.* **34**, 863–880

Heyman, J. 1958: Minimum weight of frames under shakedown loading. *J. Eng. Mech. Div., ASCE* **84**, 1–25

Kaneko, I. 1982: On some recent engineering applications of complementarity problems. *Math. Prog. Study* **17**, 111–125

König, J.A. 1969: A shakedown theorem for temperature dependent elastic moduli. *Bull. Acad. Pol. Sci.* **17**, 161–165

König, J.A. 1975: On optimum shakedown design. In: Sawczuk, A; Mroz, Z. (eds.) *Optimization in structural design*, pp. 402–414. Berlin, Heidelberg, New York: Springer

König, J.A. 1987: *Shakedown of elastic-plastic structures.* Amsterdam: Elsevier

König, J.A.; Maier, G. 1981: Shakedown analysis of elastoplastic structures: a review of recent developments. *Nuclear Eng. Design* **66**, 81–95

Maier, G. 1970: A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes. *Meccanica* **5**, 54–66

Maier, G. 1973: A shakedown matrix theory allowing for workhardening and second-order geometric effects. In: Sawczuk, A. (ed.) *Foundations of plasticity*, pp. 417–433. Leyden: Noordhoff

Maier, G. 1989: A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic-plastic models. *Rendiconti Academia Nazionale dei Lincei* **LXXXI**, 161–174

Maier, G.; Novati, G. 1990: Dynamic shakedown and bounding theory for a class of nonlinear hardening discrete structural models. *Int. J. Plasticity* **6**, 551–572

Maier, G.; Comi, C.; Corigliano, A.; Perego, U.; Hübel, H. 1996: *Bounds and estimates on inelastic deformations: a study of their practical usefulness*. European Commission Report, Nuclear Science and Technology Series, Brussels: European Commission

Nguyen Dang Hung; Morelle, P. 1990: Optimal plastic design and the development of practical software. In: Lloyd Smith, D. (ed.) *Mathematical programming methods in structural plasticity*, pp. 207–229. Berlin, Heidelberg, New York: Springer

Polizzotto, C. 1982: A unified treatment of shakedown theory and related bounding techniques. *SM Arch.* **7**, 19–75

Polizzotto, C. 1984: Shakedown analysis and design in presence of limited ductility behaviour. *Eng. Struct.* **6**, 80–86

Polizzotto, C. 1997: Shakedown of elastic-plastic solids with frictionless unilateral contact boundary conditions. *Int. J. Mech. Sci.* **39**, 819–828

Polizzotto, C.; Borino, G. 1996: Shakedown and steady-state responses of elastic-plastic solids in large displacements. *Int. J. Solids Struct.* **23**, 3415–3437

Polizzotto, C.; Borino, G.; Caddemi, A.; Fuschi, P. 1996: Shakedown problems for material models with internal variables. *Eur. J. Mech. A/Solids* **21**, 621–639

Pycko, S.; Maier, G. 1995: Shakedown theorems for some classes of nonassociative hardening elastic-plastic material models. *Int. J. Plasticity* **11**, 367–395

Rizzo, S. 1983: Shakedown analysis of discrete elastic-work-hardening structures with displacement constraints. *Proc. Int. Conf. on Structural Mechanics in Reactor Technology, SMIRT-7* (held in Chicago), Paper L9-2, pp. 311–318

Rizzo, S.; Giambanco, F. 1984: Shakedown analysis of limited-ductility structures. *Meccanica* **19**, 151–157

Rozvany, G.I.N 1989: *Structural design via optimality criteria.* Dordrecht: Kluwer

Save, M.A.; Prager, W. 1985: *Structural optimization.* New York: Plenum Press

Stumpf, H. 1993: Theoretical and computational aspects in the shakedown analysis of finite elastoplasticity. *Int. J. Plasticity* **9**, 583–602

Tin-Loi, F. 1995: Plastic limit analysis of plane frames and grids using GAMS. *Comput. Struct.* **54**, 15–25

Tin-Loi, F.; Grundy, P. 1978: Deflection stability of work hardening structures. *J. Struct. Mech.* **6**, 331–347

Weichert, D. 1986: On the influence of geometrical nonlinearities on the shakedown of elastic-plastic structures. *Int. J. Plasticity* **2**, 135–148

Welch, J.S.: 1987: PAM - a practitioner's approach to modeling. *Mgmt Sci.* **33**, 610–625

Xue, M.D.; Wang, X.F.; Williams, F.W.; Xu, B.Y. 1997: Lower-bound shakedown analysis of axisymmetric structures subjected to variable mechanical and thermal loads. *Int. J. Mech. Sci.* **39**, 965–976