RESEARCH PAPER

System reliability-based design optimization with interval parameters by sequential moving asymptote method

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Abstract

Reliability-based design optimization (RBDO) offers a powerful tool to deal with the structural design with heterogeneous interval parameters concurrently. However, it is time-consuming in the practical engineering design. Therefore, a novel sequential moving asymptote method (SMAM) is proposed to improve the computational efficiency for convex model in this study, in which the nested double-loop optimization problem is decoupled to a sequence of deterministic suboptimization problems based on the method of moving asymptotes. In addition, the sensitivity of reliability index is derived, so the finite difference for the nested optimization loop can be avoided to tremendously improve the computational efficiency. Then, the accuracy of the SMAM is proved based on the error analysis. Furthermore, the Kreisselmeier-Steinhauser (KS) function is used to assemble the multiple constraints to deal with the parallel and series RBDO problems. One benchmark mathematical example, three numerical examples, and one complex civil engineering example, i.e., tower crane, are tested to demonstrate the efficiency of the proposed method by comparison with other existing methods, and the results indicate that SMAM offers a general and effective tool for non-probabilistic reliability analysis and optimization.

Keywords Reliability-based design optimization · Super parametric convex model · Sequential moving asymptotes method · Sensitivity analysis

Nomenclature

RBDO	Reliability-based design optimization
$\mathbf{d}$	Design variables
SMAM	Sequential moving asymptote method
p	Super parameter
KS	Kreisselmeier-Steinhauser
η	Non-probabilistic reliability index
ANVM	Advanced nominal value method
$\widehat{\eta}$	Approximated reliability index
LSF	Limit state function
$\underline{\eta}$	Target non-probabilistic reliability index
MCP	Most concerned point
η_{series}	Non-probabilistic reliability index of series system
FORM	First-order reliability method
$\eta_{parallel}$	Non-probabilistic reliability index of parallel system
CCSTM	Chaotic conjugate stability transformation method
$f(\cdot)$	Objective function
MMA	Method of moving asymptotes
$g(\cdot)$	Performance function
STM	Stability transformation method
∇	The first-order sensitivity operator
DSTM	Directional stability transformation method
∇^2	The second-order sensitivity operator
NRIA	Non-probabilistic reliability index approach
δ	Perturbation operator
CPA	Concerned performance approach
λ	Lagrange multiplier
FDM	Finite difference method
E	Failure event
$\mathbf{x}$	Uncertain variables in physical space
ε	Convergence precision
$\mathbf{q}$	Uncertain variables in q-space
ξ	Controlling parameter of KS function

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1 Introduction

It is today widespread acknowledged that computational approaches permit the analyses and design with respect to real-world engineering systems, while the inevitable effects of uncertainties have led the scientific community to recognize the importance of the uncertain approach for engineering applications (Hu and Du 2015; Xiao et al. 2020; Youn and Wang 2008; Zhang and Han 2020). As a result, the probability model, non-probabilistic convex model, and fuzzy model are developed to account these uncertain behaviors during optimization design phase (Kang et al. 2016; Kang et al. 2019; Moens and Vandepitte 2006; Wang et al. 2018a). Among them, non-probabilistic convex model shows tremendous potential for handling uncertain problems with limited or poor-quality experimental data (Ni et al. 2018; Qiu and Elishakoff 1998; Wang et al. 2019). An exhaustive comparison between interval analysis method and probabilistic approach was conducted by Qiu et al. (2004), and the results indicated that the variations of dynamical responses yielded by the probabilistic approach were tighter than those produced by non-probabilistic approach. Zhao et al. (2018) approximated the failure shear stress of simulated lunar soil using the convex model. In such case, the application of reliability-based design optimization (RBDO) using convex model becomes a promising way to account the interval parameters in optimization design (Elishakoff 1995; Meng et al. 2020; Zhang et al. 2019).

The precursor works of non-probabilistic RBDO could be traced back to the works of Ben-Haim (1994) and Ben-Haim and Elishakoff (1995), who established the basis of non-probabilistic model. Until now, a series of non-probabilistic convex models were put forward to reasonably measure the experimental samples, in which interval set and ellipsoid models are two distinguished representatives. Majumder and Rao (2009) tried to adopt the interval set model to acquire a reasonably structural safety estimation for the aircraft wings. Kang et al. (2011) and Guo (2014) gave a new mathematical definition for reliability index of convex model, which was inspired by the first-order reliability method (FORM) in probabilistic theory. In the study of Jiang et al. (2011), a correlation analysis technique for ellipsoid convex model was created, and this concept was extended into the interval set model (Jiang et al. 2015). Wang et al. (2008) and Jiang et al. (2013) used the ratio of the volume of the safe region to the total volume for measuring the structural safety of interval and convex models. Moreover, Elishakoff and Bekel (2013) suggested a new super ellipsoid model, which provided more alternatives for non-probabilistic theory from a broader perspective (Elishakoff and Elettro 2014). Furthermore, Meng et al. (2018) gave a new mathematical definition of the super parametric convex model to enclose the experimental samples accurately, where the advanced nominal value method (ANVM) was further created to solve RBDO problems

effectively. Moreover, it was proved that interval and ellipsoid models are two special cases of super parametric convex model, and the superiorities were demonstrated from both theoretical and experimental aspects (Meng and Zhou 2018). In general, it is well known that RBDO is composed by a nested double-loop optimization structure, where the deterministic optimization (outer loop) repeatedly calls the non-probabilistic reliability analysis at each iterative step (Guo et al. 2009; Hamzehkolaei et al. 2018; Wu et al. 2020). Consequently, it needs a large amount of numerical estimation number and is computationally challenging (Jiang et al. 2020; Keshtegar and Chakraborty 2018; Keshtegar et al. 2020).

Pursuing high efficiency is a glorious and vital target to ease the heavy computational burden caused by nested optimization loops of RBDO (Jiang et al. 2019). As Tsompanakis and Papadrakakis (2004) pointed out, the calculation of nested optimization problem is an extremely computationally intensive task, and how to accelerate the iterative process is crucial. To this end, a series of promising exploratory works could be found to circumvent the problem of expensive computationally involvement in reliability analysis and optimization (Papadrakakis and Lagaros 2002). Lombardi and Haftka (1998) integrated the anti-optimization strategy and worst-case scenario approach to alleviate the computational cost of RBDO. Kang and Luo (2010) employed the linearized expansion for the concerned performance function, and the number of function calls of reliability assessment was significantly decreased. Based on interval model, Wang et al. (2018b) suggested an efficient single-loop strategy to circumvent the double loops of RBDO through shifting and updating the constraints, and then it was applied to address the multidisciplinary optimization design issue. Hao et al. (2017) constructed an adaptive loop method by combing the single-loop and double-loop strategies to efficiently compute the optimum of ellipsoid model. Sofi and Romeo (2018) used the response surface method to enhance the efficiency of the finite element method involving the interval model. Meng and Zhou (2018) established a target performance approach based on inverse optimization technique. However, all these existing RBDO methods still cannot avoid solving the nested double-loop structure. For probability model, Yi et al. (2008) developed the sequential approximate method, which was deemed as one of the most efficient strategies in RBDO (Aoues and Chateauneuf 2010). However, as an indispensable part of uncertain optimization, there are rare works on concerning sequential approximation strategy for non-probabilistic RBDO problems. What's more, although a huge improvement has been achieved, a common limitation shared by the aforementioned approaches lies in that most of existing works only focus on solving one type of special RBDO problem (interval model or ellipsoid model) efficiently. Thus, the improvement of generality and efficiency for RBDO approaches is urgently required. Additionally, sensitivity analysis is another

indispensable and crucial task. It not only determines the efficiency of RBDO approach to a large extent, but also reflects the influence degree of non-probabilistic parameters onto the reliability of a given system. Unfortunately, very scarce works focus on this challenging task up to now.

In this work, we propose a generalized sequential moving asymptote method (SMAM) in order to decrease the unbearable computational cost of system RBDO problem through converting the nested double-loop optimization model into a series of deterministic suboptimization models, in which the Kreisselmeier-Steinhauser (KS) function is adopted to handle the parallel and series systems effectively. The sensitivity propagation from inner sensitivity of performance function to outer sensitivity of reliability index constraint is derived, and thus, the number of function calls is greatly reduced. The rest of this study is arranged as follows: Section 2 presents a brief review of RBDO. Then, the proposed SMAM is introduced in Section 3, and the error analysis of approximate reliability index constraints is also analyzed to validate the accuracy of the proposed sensitivity calculation method and reliability index constraint. Some case studies are presented to demonstrate the feasibility and effectiveness of SMAM in Section 4. In Section 5, conclusions are summarized and listed.

2 Reliability-based design optimization for interval parameters

This section reviews the reliability-based design optimization with interval parameters, which includes interval model, ellipsoid model, and super parametric convex model.

2.1 Reliability index based on convex model

Convex model characterizes the domain of uncertain parameters through a bounded convex set rather than giving an accurate probability distribution. Theoretically, different categories of convex sets are used to construct the uncertain domain. The most frequently utilized models include interval model and ellipsoid model, which can be employed to address the RBDO problem effectively.

In convex model, the status of a structure is represented by limit state function (LSF), i.e., $G(\mathbf{x}) = 0$. To conveniently implement the reliability analysis, we should transform the uncertain variables $\mathbf{x}$ into $\mathbf{q}$ in the q-space; the details can be seen in the work of Kang et al. (2011). Then, the LSF $G(\mathbf{x}) = 0$ becomes $g(\mathbf{q}) = 0$. The uncertain domain of structures can be divided into two parts by LSF: failure domain $g(\mathbf{q}) < 0$ and safe domain $g(\mathbf{q}) \geq 0$. For interval model, the geometric figure in two-dimensional q-space is depicted as a square (as shown in Fig. 1a) (Kang et al. 2011), and the mathematical definition is as follows:

$$
\begin{aligned}
&\text{find} \quad \mathbf{q} \\
&\min_{\mathbf{d}} \quad \eta = \text{sgn}(g(\mathbf{0})) \cdot \max\{|q_1|, \ldots, |q_i|, \ldots, |q_n|\} \\
&\text{s.t.} \quad g(\mathbf{q}) = 0
\end{aligned}
\tag{1}
$$

where $\text{sgn}(g(\mathbf{0}))$ is the signum function. η is the non-probabilistic reliability index. The optimum $\mathbf{q}^*$ of (1) is called the most concerned point (MCP). If $\eta = 1$, the square is tangent to the limit surface and the MCP is located on the boundary, and then the critical state is reached. If $\eta > 1$, it means all possible points are located in the reliable domain and the structure is safe; otherwise, if $\eta < 1$, the structure is unsafe.

For ellipsoid model, the reliability index in two-dimensional q-space is represented by Euclidean norm $\|\mathbf{q}\|$, and the corresponding geometric figure is a unit circle, as illustrated in Fig. 1b. The reliability index for ellipsoid model is defined as follows (Kang et al. 2011).

$$
\begin{aligned}
&\text{find} \quad \mathbf{q} \\
&\min_{\mathbf{d}} \quad \eta = \text{sgn}(g(\mathbf{0})) \cdot \|\mathbf{q}\| \\
&\text{s.t.} \quad g(\mathbf{q}) = 0
\end{aligned}
\tag{2}
$$

From (2), the absolute value of reliability index is defined as the shortest Euclidean distance from the origin to LSF in q-space, which is similar to the reliability index of FORM. Therefore, the typical FORM methods, such as Hasofer-Lind Rackwitz-Fiessler algorithm, chaotic conjugate stability transformation method (CCSTM), and hybrid descent mean value method (Keshtegar 2016; Keshtegar and Hao 2018), can be employed to solve (2) without much modifications. However, these FORM methods cannot be utilized to estimate other complex non-probabilistic models, such as interval model, super parametric convex model, and exponential convex model (Zhu et al. 2020).

Unlike interval and ellipsoid models, super parametric convex model describes the uncertain parameters from a more generalized perspective, which uses the p-norm, i.e., $\|\mathbf{q}\|_p = \left(\sum_{i=1}^{n} |q_i|^p \right)^{\frac{1}{p}}$, to compute the reliability index. Without loss of generality, the reliability index can be computed by the following formulations:

$$
\begin{aligned}
&\text{find} \quad \mathbf{q} \\
&\min_{\mathbf{d}} \quad \eta = \text{sgn}(g(\mathbf{0})) \cdot \|\mathbf{q}\|_p \\
&\text{s.t.} \quad g(\mathbf{q}) = 0
\end{aligned}
\tag{3}
$$

where the subscript p is super parameter. The two-dimensional geometric figure is schematically illustrated in Fig. 1c. It can be observed that the super parametric model becomes the classical ellipsoid model when the value of p is set to be 2. Moreover, if the super parameter is ∞, the super parameter model degrades into a classical interval model. Evidently, the interval and ellipsoid models are only two special cases of super parametric convex model, and thus, we can
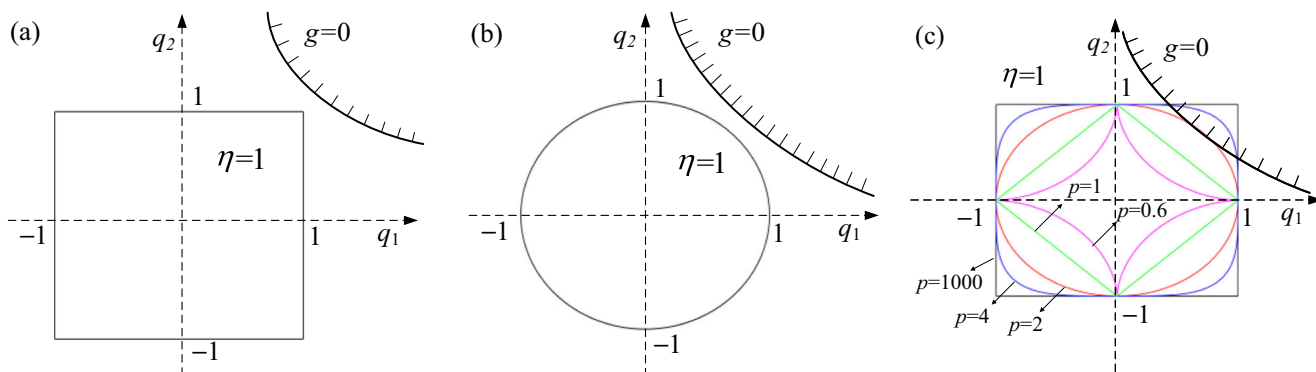
Fig. 1 Different non-probabilistic models. **a** Interval set model. **b** Ellipsoid model. **c** Super parameter model

conclude that super parameter model is more general than the classical non-probabilistic RBDO models. Moreover, its superiorities in terms of theory and experiment are also validated in Meng et al. (2018). So it is adopted as a general non-probabilistic RBDO model in this work.

2.2 General formula of RBDO of convex model

Based on convex model, the typical RBDO formulation is expressed as follows:

$$
\begin{aligned}
&\text{find} && \mathbf{d} \\
&\min && f(\mathbf{d}) \\
&\text{s.t.} && \eta_j(\mathbf{d}, \mathbf{x}) \geq \underline{\eta}_j \qquad j = 1, \cdots, ng \\
& && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U
\end{aligned}
\tag{4}
$$

where $f(\mathbf{d})$ is the objective function. η_j and $\underline{\eta}_j$ are the jth non-probabilistic reliability index and the target non-probabilistic reliability index, respectively. ng is the number of constraints. $\mathbf{d}$ are the design variables with lower bound $\mathbf{d}^L$ and upper bound $\mathbf{d}^U$, which can be selected as deterministic design variables or nominal values of uncertain design variables. $\mathbf{x}$ are the interval parameters. It should be noted that the reliability constraints are different with different convex models. Recently, the authors introduced the ANVM to compute the reliability index of super parametric convex model simply and accurately (Meng et al. 2018), and the iterative formulas are expressed as follows:

$$
\eta_j^k = \frac{\left(g_j^k - \left(\nabla_q g_j^k\right)^{\mathrm{T}} q^k \right) \left(\left\| \nabla_q g_j^k \right\|_{\frac{p}{p-1}} \right)^{\frac{1}{p-1}}}{\left| \nabla_q g_j^k \right|^{\mathrm{T}} \left| \nabla_q g_j^k \right|^{\frac{1}{p-1}}}
$$

$$
q_i^{k+1} = -sign\left(\frac{\partial g_i^k}{\partial q_i^k}\right) \eta_j^k \left(\frac{\left| \frac{\partial g_j^k}{\partial q_i^k} \right|}{\left\| \nabla_q g_j^k \right\|_{\frac{p}{p-1}}} \right)^{\frac{1}{p-1}} \qquad i = 1, \cdots, n
\tag{5}
$$

where $\mathbf{q}$ are the normalized uncertain variables that are transformed from uncertain design variables $\mathbf{d}$ or uncertain

parameters $\mathbf{p}$. n is the number of uncertain variables $\mathbf{q}$. k is the number of iteration. $\nabla_{\mathbf{q}} g_j = \left[\frac{\partial g_j}{\partial q_1}, \frac{\partial g_j}{\partial q_2}, \ldots, \frac{\partial g_j}{\partial q_n} \right]$ is the sensitivity vector of performance function with respect to uncertain variables $\mathbf{q}$. For RBDO, the outer deterministic optimization needs estimating the MCP repeatedly, and the design sensitivities of non-probabilistic constraint should constantly perform finite difference with respect to optimization formula of (3). Therefore, it results in computational cost unbearable.

3 The proposed sequential moving asymptote method for reliability-based design optimization

In this section, a generalized sequential moving asymptote method (SMAM) is established to deal with the uncertainty parameters in Section 3.1, where the method of moving asymptotes (MMA) is employed for updating the design variables efficiently. Then, the generalized sensitivity estimation method is established and the error analysis of SMAM is proved in Section 3.2. The framework and flowchart are given in Section 3.3.

3.1 Sequential moving asymptote method

In SMAM, the non-probabilistic reliability analysis and optimization are both implemented in parallel, and then the inner loop is eliminated. Therefore, high efficiency can be achieved by SMAM. At the mth iterative step, SMAM uses the approximate reliability index to substitute the actual reliability index, and the formulation can be expressed as follows:

$$
\begin{aligned}
&\text{find} && \mathbf{d} \\
&\min && f(\mathbf{d}) \\
&\text{s.t.} && \eta_j(\mathbf{d}^m, \mathbf{x}^m) \geq \underline{\eta}_j \qquad j = 1, \cdots, ng \\
& && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U
\end{aligned}
\tag{6}
$$

where $\eta_j(\mathbf{d}^m, \mathbf{x}^m)$ denotes approximate reliability index at the mth iterative step. Like the sequential approximate

programming in deterministic optimization, we use the linear Taylor expansion of the reliability index $\eta_j(\mathbf{d}^m, \mathbf{x}^m)$ with respect to the deterministic design variables $\mathbf{d}^m$, which is expressed as

$$\eta_j(\mathbf{d}^m, \mathbf{x}^m) = \eta_j(\mathbf{d}^{m-1}, \mathbf{x}^{m-1}) + \left(\nabla_{\mathbf{d}^{m-1}}\eta_j\right)^T (\mathbf{d}-\mathbf{d}^{m-1}) \qquad (7)$$

where reliability index $\eta_j(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})$ requires complete reliability computation loop. In this way, the proposed method avoids solving the nested optimization loops, and this is adverse to the efficiency. To address this issue, we use the following

approximation formulation:

$$\eta_j(\mathbf{d}^m, \mathbf{x}^m) = \widehat{\eta}_j(\mathbf{d}^{m-1}, \mathbf{x}^{m-1}) + \left(\nabla_{\mathbf{d}^{m-1}}\widehat{\eta}_j\right)^T (\mathbf{d}-\mathbf{d}^{m-1}) \qquad (8)$$

In (8), the approximate reliability index $\widehat{\eta}_j$ is used to substitute η_j. Accordingly, the approximate sensitivities $\nabla_{\mathbf{d}^{m-1}}\widehat{\eta}_j$ and $\nabla_{\mathbf{x}^{m-1}}\widehat{\eta}_j$ are adopted to replace the actual sensitivities. Based on ANVM, the approximate reliability index $\widehat{\eta}_j$ can be calculated as follows:

$$\widehat{\eta}_j = \frac{\left(g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1}) - \left(\nabla_q g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})\right)^T \mathbf{q}^{m-1}\right)\left(\left\|\nabla_q g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})\right\|_{\frac{p}{p-1}}\right)^{\frac{1}{p-1}}}{\left|\nabla_q g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})\right|^T \left|\nabla_q g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})\right|^{\frac{1}{p-1}}}$$

$$q_i^m = -sign\left(\frac{\partial g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})}{\partial q_i^{m-1}}\right)\widehat{\eta}_j^{m-1}\left(\frac{\left|\frac{\partial g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})}{\partial q_i^{m-1}}\right|}{\left\|\nabla_q g_j^{m-1}(\mathbf{d}^{m-1}, \mathbf{x}^{m-1})\right\|_{\frac{p}{p-1}}}\right)^{\frac{1}{p-1}} \qquad i = 1, \cdots, n$$

$$(9)$$

where the superscript $m-1$ represents the $(m-1)$th iterative step of outer deterministic optimization. In this way, we perform the approximate performance function instead of actual performance function at the approximated MCP $\mathbf{q}^{m-1}$. Then, we can substitute (9) into (6), and the formulas are described as follows:

$$\begin{aligned}
&\text{find} \quad \mathbf{d} \\
&\min \quad f(d) \\
&\text{s.t.} \quad \widehat{\eta}_j^{m-1} + \left(\nabla_{\mathbf{d}^{m-1}}\widehat{\eta}_j\right)^T (d-d^{m-1}) \geq \eta_{-j} \qquad j = 1, \cdots, ng \\
&\qquad d^L \leq d \leq d^U
\end{aligned} \qquad (10)$$

where $\mathbf{d}^{m-1}$ is the optimum design at the $(m-1)$th iteration. It is well known that the convex optimization approach provides more opportunity to search the optimum, and MMA is one

promising convex optimization method that has been widely used in the structural optimization domain (Svanberg 1987). Thus, we employ the MMA by introducing four intermediate variables: $1/(d_i^U-d_i)$, $1/(d_i-d_i^L)$, $1/(x_i^U-x_i)$, and $1/(x_i-x_i^L)$. At the mth iterative step, we can construct the following suboptimization problem.

$$\begin{aligned}
&\text{find} \quad \mathbf{d} \\
&\min \quad \sum_{i=1}^{nd}\left(\frac{P_{di}^m}{U_{xi}^m-d_i} + \frac{Q_{di}^m}{d_i-L_{di}^m}\right) + r^m \\
&\text{s.t.} \quad \sum_{i=1}^{nd}\left(\frac{P_{dji}^m}{U_{di}^m-d_i} + \frac{Q_{dji}^m}{d_i-L_{di}^m}\right) + r_j^m \geq \eta_{-j} \qquad j = 1, \cdots, ng \\
&\qquad \max\{d_i^L, \alpha_{di}^m\} \leq d_i \leq \min\{d_i^U, \beta_{di}^m\} \qquad i = 1, \cdots, nd
\end{aligned} \qquad (11)$$

where

$$P_{dji}^m = \begin{cases} \left(U_{di}^m \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i}, & \text{if } \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i} > 0 \\[3mm] 0, & \text{if } \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i} \leq 0 \end{cases} \qquad Q_{dji}^m = \begin{cases} 0, & \text{if } \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i} \geq 0 \\[3mm] -\left(d_i^m \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i}, & \text{if } \dfrac{\partial\widehat{\eta}_j^{m-1}}{\partial d_i} < 0 \end{cases}$$

$$r_j^m = \widehat{\eta}_j^{m-1} - \sum_{i=1}^{nd}\left(\frac{P_{dji}^m}{U_{di}^m-d_i^m} + \frac{Q_{dji}^m}{d_i^m-L_{di}^m}\right)$$

$$L_{di}^m < \alpha_{di}^m < d_i^m < \beta_{di}^m < U_{di}^m$$

$$\alpha_{di}^m = 0.9L_{di}^m + 0.1d_i^m, \beta_{di}^m = 0.9U_{di}^m + 0.1d_i^m$$

$$(12)$$

The moving asymptotes exhibit convex characteristic at each iterative step. For $m = 0$ and $m = 1$, the parameters $L_{\mathbf{d}i}^m$ and $U_{\mathbf{d}i}^m$ are as follows:

$$L_{\mathbf{d}i}^m = d_i^m - s\left(d_i^U - d_i^L\right), U_{\mathbf{d}i}^m = d_i^m + s\left(d_i^U - d_i^L\right) \tag{13}$$

where s is a real number and is less than 1.0. It is set to be 0.7 according to the reference (Svanberg 1987). For $m \geq 2$,

$$\begin{cases} L_{di}^m = d_i^m - s\left(d_i^{m-1} - L_{di}^{m-1}\right), U_{di}^m = d_i^m + s\left(U_{di}^{m-1} - d_i^{m-1}\right), \text{if } \mathrm{sgn}\left(d_i^m - d_i^{m-1}\right) \neq \mathrm{sgn}\left(d_i^{m-1} - d_i^{m-2}\right) \\ L_{di}^m = d_i^m - \dfrac{\left(d_i^{m-1} - L_{di}^{m-1}\right)}{s}, U_{di}^m = d_i^m + \dfrac{\left(U_{di}^{m-1} - d_i^{m-1}\right)}{s}, \text{if } \mathrm{sgn}\left(d_i^m - d_i^{m-1}\right) = \mathrm{sgn}\left(d_i^{m-1} - d_i^{m-2}\right) \end{cases} \tag{14}$$

In MMA, $L_{\mathbf{d}i}$ and $U_{\mathbf{d}i}$ are given parameters. Also, we can expand the objective function by the same way. When $L_{\mathbf{d}i} = -\infty$ and $U_{\mathbf{d}i}^m = \infty$, the MMA is degraded into typical sequential approximate programming. From the above analysis, the asymptote values are adaptively modified from current iteration to next, and then the degree of convexity is accordingly adjusted to enhance the convergence. Then, we can obtain all required parameters for SMAM except the sensitivities, which will be given in the next section.

3.2 Sensitivity analysis for SMAM

For RBDO, the sensitivities of non-probabilistic reliability index with respect to design variables have also critical importance. In general, it is easy to compute the design sensitivities of non-probabilistic reliability constraint functions by finite difference method (FDM), but it requires preforming the optimization of reliability analysis repeatedly. For a RBDO problem with n-dimensional design variables, it needs solving the optimization problem of (3) $n + 1$ times. Therefore, it incurs extremely expensive computational cost. To address this issue, an efficient sensitivity computation method of non-probabilistic reliability index with respect to design variables is developed in this paper. Firstly, we can give a small perturbation to the non-probabilistic reliability index at the MCP $\mathbf{q}^*$ for non-probabilistic reliability analysis in (3).

$$\delta\eta = \nabla_{\mathbf{q}}\eta\delta\mathbf{q}^* \tag{15}$$

Besides, the following conditions must be satisfied.

$$\begin{cases} \nabla_{\mathbf{q}}\eta + \lambda\nabla_{\mathbf{q}}g\left(\mathbf{d}, \mathbf{q}^*\right) = 0 \\ g\left(\mathbf{d}, \mathbf{q}^*\right) = 0 \end{cases} \tag{16}$$

where λ is the Lagrange multiplier. For (16), we can also give a perturbation for variables $\mathbf{d}$ and $\mathbf{q}^*$ with respect to the limit state function, as well as the variables $\mathbf{x}$ and $\mathbf{q}^*$.

$$\delta g\left(\mathbf{d}, \mathbf{q}^*\right) = \nabla_{\mathbf{d}}g\left(\mathbf{d}, \mathbf{q}^*\right)\delta\mathbf{d} + \nabla_{\mathbf{q}}g\left(\mathbf{d}, \mathbf{q}^*\right)\delta\mathbf{q}^* = 0 \tag{17}$$

Substituting (16) and (17) into (15), one obtains

$$\delta\eta = \lambda\nabla_{\mathbf{d}}g\left(\mathbf{d}, \mathbf{q}^*\right)\delta\mathbf{d} \tag{18}$$

According to author's previous work (Meng et al. 2018), the Lagrangian multiplier λ is

$$\lambda = \frac{1}{\left\|\nabla_{\mathbf{q}}g\right\|_{\frac{p}{p-1}}} \tag{19}$$

Then, we can compute the design sensitivities of non-probabilistic reliability index by combining (18) and (19).

$$\nabla_{\mathbf{d}}\eta = \frac{\nabla_{\mathbf{d}}g\left(\mathbf{d}, \mathbf{q}^*\right)}{\left\|\nabla_{\mathbf{q}}g\left(\mathbf{d}, \mathbf{q}^*\right)\right\|_{\frac{p}{p-1}}} \tag{20}$$

Especially, if uncertain variables $\mathbf{x}$ are selected as design variables, the design sensitivities $\nabla_{\mathbf{d}}\eta$ can be assessed using chain rule.

$$\nabla_{\mathbf{d}}\eta = \frac{\nabla_{\mathbf{d}}\mathbf{q}\nabla_{\mathbf{x}}g\left(\mathbf{d}, \mathbf{q}\right)}{\left\|\nabla_{\mathbf{q}}g\left(\mathbf{d}, \mathbf{q}\right)\right\|_{\frac{p}{p-1}}} \tag{21}$$

It should be noted that the above formulation is valid at the MCP for LSF, while the SMAM formulation in the previous section is constructed using the approximate design sensitivities. Then, we use the following formulas to provide the sensitivity for SMAM.

$$\nabla_{\mathbf{d}^m}\widehat{\eta} = \frac{\nabla_{\mathbf{d}^m}g\left(\mathbf{d}^m, \mathbf{q}^m\right)}{\left\|\nabla_{\mathbf{q}^m}g\left(\mathbf{d}^m, \mathbf{q}^m\right)\right\|_{\frac{p}{p-1}}} \tag{22}$$

When the nominal values are selected as design variables, the approximate design sensitivities can be computed as follows:

$$\nabla_{\mathbf{d}^m}\widehat{\eta} = \frac{\nabla_{\mathbf{d}^m}\mathbf{q}\nabla_{\mathbf{q}^m}g\left(\mathbf{d}^m, \mathbf{q}^m\right)}{\left\|\nabla_{\mathbf{q}^m}g\left(\mathbf{d}^m, \mathbf{q}^m\right)\right\|_{\frac{p}{p-1}}} \tag{23}$$

It is observed that the design sensitivities can be calculated by the sensitivities of performance function. Compared to the

FDM for the non-probabilistic reliability index, the proposed method can avoid complete reliability iterations, which can be directly obtained according to $\nabla_{\mathbf{d}^m} g$ and $\nabla_{\mathbf{q}^m} g$. In other words, the design sensitivities can be calculated by computing the optimization formula of (3) only once, which means the computational cost of the proposed sensitivity computational method is only about $1/(n+1)$ of the FDM.

Since the SMAM uses the first-order Taylor expansion, it produces some errors for non-linear problems. Thus, the error between actual and approximate reliability index constraint should be analyzed. It should be emphasized that MMA only introduces the intermediate variables to provide the convex property, and this does not affect the accuracy at the current design point. Thus, we perform the error analysis for the approximated performance function in (10) for simplicity. The error of reliability constraint is formulated as follows:

$$\eta_j - \left(\widehat{\eta}_j^m + \left(\nabla_{\mathbf{d}^m} \widehat{\eta}_j \right)^T (\mathbf{d} - \mathbf{d}^m) \right) = O\left(\| (\mathbf{d} - \mathbf{d}^m)^2 \| \right) + O\left(\| (\mathbf{q}^m - \mathbf{q}^*) \| \| \mathbf{d} - \mathbf{d}^m \| \right)$$

(24)

The details of derivative process are given in the Appendix A. From (24), it can be concluded that the difference between (6) and (10) is the second-order small quantities of $\mathbf{d} - \mathbf{d}^m$ and cross product of $\mathbf{q}^m - \mathbf{q}^*$ and $\mathbf{d} - \mathbf{d}^m$. Thus, when the point $(\mathbf{d}, \mathbf{q}^m)$ is located in a small neighborhood of $(\mathbf{d}^m, \mathbf{q}^*)$, the difference between (6) and (10) is of second-order small quantities. Consequently, it can be concluded that in the ε-vicinity of optimum design and corresponding MCP, the difference between the actual MCP and approximate MCP is of higher order of ε. Moreover, with the increase of iteration numbers, the iterative points $\mathbf{q}^m$ and $\mathbf{d}^m$ gradually converge to the optimum $\mathbf{q}^*$ and $\mathbf{d}$, so the relative error in (24) vanishes at the optimum. Therefore, SMAM can provide accurate results for RBDO.

3.3 Procedure and flowchart of the proposed SMAM

In general, the computation procedure of the proposed SMAM is outlined as follows:

Step 1. Define the performance function g, super parameter p, and target non-probabilistic reliability η. Initialize the design variables $\mathbf{d}$ and $\mathbf{x}$. Set iterative step $m = 0$.

Step 2. Transform the interval variables $\mathbf{x}^m$ into normalized variables $\mathbf{q}^m$ by the method in Kang et al. (2011).

Step 3. Compute the approximate reliability index $\widehat{\eta}_j$ and iterative point q_i^m. Estimate the sensitivity of reliability index using (22) and (23). Set $m = m + 1$.

Step 4. Perform the deterministic optimization by MMA in (12) and update the design variables $\mathbf{d}^{m+1}$.

Step 5. Convergence check. If $\| \mathbf{d}^{m+1} - \mathbf{d}^m \| / \| \mathbf{d}^{m+1} \| \leq \varepsilon_{\mathbf{d}}$, the computational procedure of SMAM is terminated. Here $\varepsilon_{\mathbf{d}}$ is set to be 10^{-3}. Otherwise, transform $\mathbf{q}^m$ to $\mathbf{x}^m$ and go back to Step 2.

The iterative framework of SMAM is given in Fig. 2, which demonstrates the iterative process is simplified as a single deterministic optimization loop. Generally, there are three major different differences between the SMAM and other non-probabilistic RBDO algorithms. Firstly, from the viewpoint of mathematics, FORM and classical non-probabilistic RBDO models are two special cases of the super parametric convex model. Secondly, the single-loop strategy is implemented in SMAM. In classical RBDO problem, an efficient adaptive-loop method and target performance approach still requires solving the nested optimization problem (Hao et al. 2017; Kang et al. 2011; Meng and Zhou 2018). Thirdly, the computational cost of sensitivity analysis of reliability index is greatly reduced. In the traditional RBDO algorithm, even the estimation of sensitivity of reliability index for one uncertain parameter requires running the reliability optimization formula twice. Besides, the sensitivity analysis is also a hot topic, which can measure the influence of input model parameters on the response of reliability index. Consequently, SMAM not only can accelerate the convergence rate to a great extent, but also provide a new non-probabilistic sensitivity analysis tool.
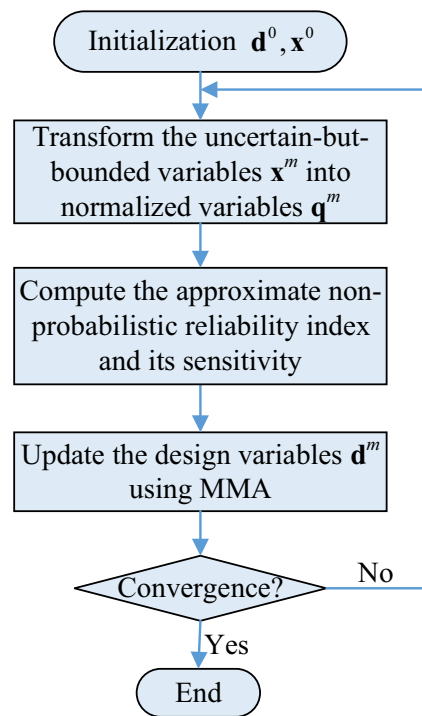

Fig. 2 Iterative framework of SMAM

4 System non-probabilistic reliability-based design optimization using SMAM

For RBDO, there are always involving multiple non-probabilistic constraints. Unfortunately, there are rare researches focusing on system RBDO problem with interval variables. Because the connectivity between different events are complex, the evaluation of system BRDO model is more difficult than that with single non-probabilistic constraint. For physical system with multiple constraints, the series and parallel are two primary assemblies of each component, as illustrated in Fig. 3. Assume that E_j denotes the failure event corresponding to the failure domain $g_j(\mathbf{x}) < 0$. Inspired the reliability concepts in probabilistic theory, the reliability indexes of series and parallel systems are defined in (25) and (26), respectively.

$$\eta_{series} = \eta\left(\bigcup_{j=1}^{p} E_j \right) = \eta\left(\bigcup_{j=1}^{p} g_j(\mathbf{x}) < 0 \right) \tag{25}$$

$$\eta_{parallel} = \eta\left(\bigcap_{j=1}^{p} E_j \right) = \eta\left(\bigcap_{j=1}^{p} g_j(\mathbf{x}) < 0 \right) \tag{26}$$

Besides, it is always possible to convert series arrangement into parallel one and vice versa using De Morgan's laws, $\overline{\cup E_j} = \cap \overline{E_j}$. More details of probabilistic reliability system can be referred in Ditlevsen and Madsen (1996). For both series and parallel systems, the multiple performance functions can combine into a single composite function using the following formulas:

$$\left\{ \mathbf{x} : \bigcup_{j=1}^{p} g_j(\mathbf{x}) < 0 \right\} = \left\{ \mathbf{x} : g_{series}(\mathbf{x}) = \min_j g_j(\mathbf{x}) < 0 \right\} \tag{27}$$

$$\left\{ \mathbf{x} : \bigcap_{j=1}^{p} g_j(\mathbf{x}) < 0 \right\} = \left\{ \mathbf{x} : g_{parallel}(\mathbf{x}) = \max_j g_j(\mathbf{x}) < 0 \right\} \tag{28}$$

Based on (27) and (28), the system reliability/RBDO problem is transformed to a component reliability/RBDO problem, and then it can be solved easily. The solid line in Fig. 3a, b denotes the composite limit state function of series system and parallel system, respectively. For a RBDO problem with same performance functions, when the series system is considered,

the optimum is located in the vicinity of the limit state function $\min_j g_j(\mathbf{x}) = 0$. However, when the parallel system is considered, the optimum is located in the vicinity of the limit state function $\max_j g_j(\mathbf{x}) = 0$.

However, it should be underlined that the composite function may lead to highly non-linearity and non-differential feature, especially at the intersections of different performance functions. To address this issue, we can use the KS function (i.e., aggregate function method) (Kang and Bai 2013; Kreisselmeier and Steinhauser 1983), which can transform multiple constraints into one composite constraint. The KS functions of series and parallel systems can be depicted by (29) and (30), respectively.

$$g_{series}(\mathbf{x}) = -\frac{1}{\xi} \ln\left(\sum_{i=1}^{N} e^{-\xi g_i(\mathbf{x})} \right) \tag{29}$$

$$g_{parallel}(\mathbf{x}) = \frac{1}{\xi} \ln\left(\sum_{i=1}^{N} e^{\xi g_i(\mathbf{x})} \right) \tag{30}$$

where $g_i(\mathbf{x})$ ($i = 1, 2, ..., N$) are the performance functions with N numbers, and ξ denotes the controlling parameter. A one-dimensional example of series system is shown in Fig. 4 with two functions: $g_1 = -(x - 3/2)^2 + 5/2$ and $g_2 = (x - 2)^2 + 1/2$. The composite function $\widetilde{g}$ denotes the envelop function approaches $\min(g_1, g_2)$. As the increase of ξ, the composite function gradually tends to actual series system, but it remains maintaining the differentiability. Thus, the value of ξ should be large enough and it is set to be [50, 100] in this study. Also, we can obtain the sensitivities of composite function as follows:

$$\nabla_{\mathbf{d}} g_{series}(\mathbf{x}) = -\left(\sum_{i=1}^{N} \nabla_{\mathbf{d}} g_i(\mathbf{x}) \right) \tag{31}$$

$$\nabla_{\mathbf{d}} g_{parallel}(\mathbf{x}) = \sum_{i=1}^{N} \nabla_{\mathbf{d}} g_i(\mathbf{x}) \tag{32}$$

Since the traditional RBDO problem is always deemed as series system, (28) and (29) can be avoided. Therefore, we can use the classical non-probabilistic RBDO strategy to handle the series system.

Fig. 3 RBDO for different systems. a Series system. b Parallel system
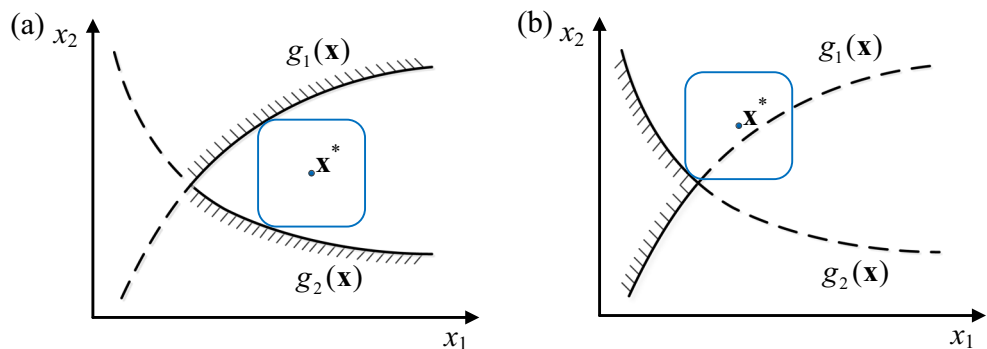
5 Illustrative examples

In this section, five examples are presented to demonstrate the accuracy, robustness, and efficiency of the proposed SMAM. The first and second examples are mathematical RBDO examples, which illustrate the validity and efficiency of the SMAM. In the next three examples, the mechanical structure examples are considered. The last example is related to a complex tower crane in civil engineering. Additionally, we compare the results computed by SMAM with those obtained from other three prevalent FORM algorithms, i.e., stability transformation method (STM) (Yang 2010), CCSTM (Keshtegar 2016), and directional stability transformation method (DSTM) (Meng et al. 2017), and four non-probabilistic RBDO algorithms, i.e., non-probabilistic reliability index approach (NRIA) (Kang et al. 2011), concerned performance approach (CPA) (Kang and Luo 2009), ANVM (Meng and Zhou 2018), and SMAM.

5.1 Non-linear series example

In this case, a RBDO problem is investigated with two uncertain variables, and their nominal values are selected as design variables (Meng and Keshtegar 2019). The interval radiuses of both uncertain variables are considered as unchanged during the optimization process, and their values are set to be 0.5. The RBDO model is expressed as follows:

$$
\begin{aligned}
&\text{find} \quad \mathbf{d} = \left[x_1^C, x_2^C \right]^T \\
&\text{min} \quad x_1^C + x_2^C \\
&\text{s.t.} \quad \eta_j [g_i(\mathbf{x}) \geq 0] \geq \underline{\eta}_j, \qquad j = 1, 2, 3 \\
&\qquad 0 \leq x_1^C \leq 10, \qquad 0 \leq x_1^C \leq 10 \\
&\text{where} \qquad g_1(\mathbf{x}) = \frac{x_1^2 x_2}{20} - 1
\end{aligned}
\tag{33}
$$

$$
g_2(\mathbf{x}) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1
$$

$$
g_3(\mathbf{x}) = \frac{80}{(x_1^2 + 8x_2 + 5)} - 1
$$

$$
\underline{\eta}_1 = \underline{\eta}_2 = \underline{\eta}_3 = 1, \quad \mathbf{d}^0 = [5.0, 5.0]^T
$$

The optimal results are summarized in Table 1, in which six different super parameters $p = 1, 2, 4, 8, 16$, and ∞ are considered. The improved FORM algorithm is applied for NRIA, and CPA and ANVM are served as a comparison group. The number of function evaluations is denoted as F-evaluations in Table 1, in which the number of function evaluations of both objective and constraints is given to demonstrate the efficiency. The iterative
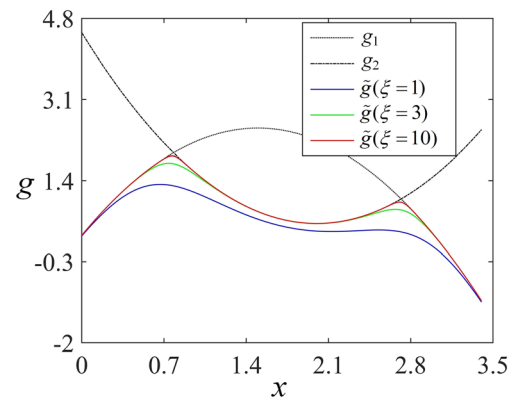

Fig. 4 Envelops of KS function with different ξ for a one-dimensional example

histories of different methods are illustrated in Fig. 5. The non-probabilistic reliability index at the optimum is validated by ANVM to test whether the optimum can provide enough accuracy for different methods.

The results in Table 1 indicate classical RBDO methods, i.e., NRIA and CPA, only can solve the classical interval and ellipsoid models. STM, CCSTM, and DSTM have a narrower application range compared to NRIA and CPA, because it only can deal with the ellipsoid model. ANVM and SMAM can solve RBDO problem with different super parameters, so it has broader application range than NRIA, STM, CCSTM, DSTM, and CPA. However, since NRIA consists of nested double-loop structure, it results in inefficiency. Similarly, ANVM requires a large amount of function evaluations. Although the number of objective function evaluations of SMAM is equivalent to that of ANVM, the number of non-probabilistic constraint function evaluations is significantly decreased owing to converting the optimization loop into a series of deterministic constraints. For different super parametric convex models, the number of function evaluations of constraints of SMAM is about 20 times less that of ANVM. Thus, it can be concluded that the computational cost of SMAM is remarkably reduced.

5.2 Mathematical example with series and parallel systems

This RBDO problem also has two uncertain-but-bound variables. The nominal values are considered as design variables, and the initial point is [0, 0]. The interval radiuses of both uncertain variables are 1. The super parameter is selected as 6. Both the series and parallel systems are considered. The formulas are expressed as follows:

Table 1 Optimal results for case 1

| Methods	Objective	Design variables	F-evaluations		η_{min}^{ANVM}
			Objective	Constraint	
$p = 1$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	5.904	(3.240, 2.664)	21	1044	1
SMAM	5.904	(3.240, 2.664)	21	75	1
$p = 2$					
NRIA	6.018	(3.254, 2.764)	18	2700	1
STM	6.018	(3.254, 2.764)	18	3555	1
CCSTM	6.018	(3.254, 2.764)	18	2277	1
DSTM	6.018	(3.254, 2.764)	18	2700	1
CPA	6.018	(3.254, 2.764)	21	801	1
ANVM	6.018	(3.254, 2.764)	18	1602	1
SMAM	6.018	(3.254, 2.764)	21	75	1
$p = 4$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	6.153	(3.290, 2.863)	21	2511	1
SMAM	6.153	(3.290, 2.863)	21	75	1
$p = 8$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	6.232	(3.314, 2.918)	21	2835	1
SMAM	6.232	(3.314, 2.918)	21	75	1
$p = 16$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	6.274	(3.328, 2.947)	21	2772	1
SMAM	6.274	(3.328, 2.947)	21	75	1
$p = \infty$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	6.318	(3.343, 2.975)	21	852	1
ANVM	6.318	(3.343, 2.975)	24	2790	1
SMAM	6.318	(3.343, 2.975)	21	75	1
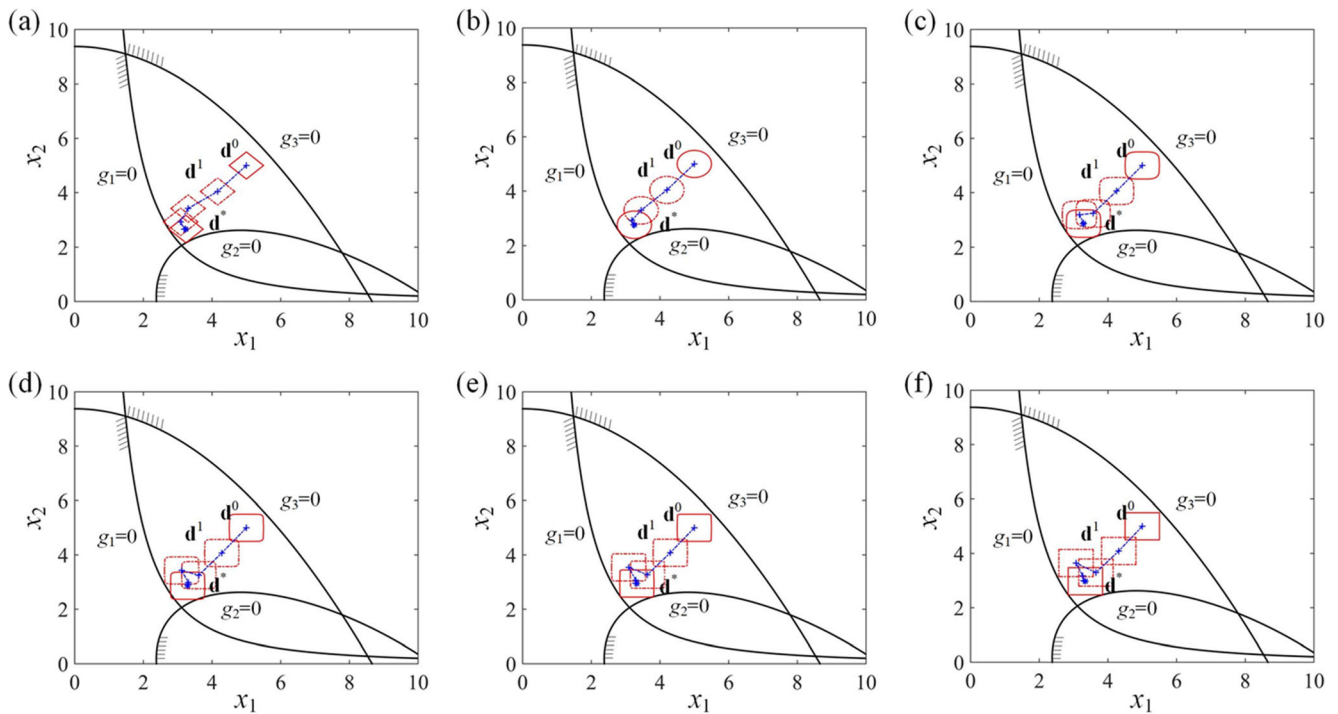
Fig. 5 Iterative histories of different methods for non-linear series example with super parameters **a** $p = 1$, **b** $p = 2$, **c** $p = 4$, **d** $p = 8$, **e** $p = 16$, and **f** $p = \infty$

find $\quad \mathbf{d} = \left[x_1^C, x_2^C \right]^T$

min $\quad x_1^C + x_2^C$ (34)

s.t. $\quad \eta_{series} \geq \underline{\eta}$, or $\quad \eta_{parallel} \geq \underline{\eta}_j$,

$\quad -3 \leq x_1^C < 3, \quad -3 \leq x_2^C \leq 3$

where $\quad g_1(\mathbf{x}) = -2 + 0.1 x_1^2 + x_2$

$\quad g_2(\mathbf{x}) = -2.1 + 0.1(x_1 \cos 60 + x_2 \sin 60)^2$
$\quad \quad \quad + x_2 \cos 60 + x_1 \sin 60$

$\quad \underline{\eta}_{series} = 1 \text{ or } = \underline{\eta}_{parallel} = 1$

The optimal results of series and parallel systems are summarized in Tables 2 and 3, respectively, and the corresponding results are plotted in Fig. 6. KS function is used in all methods to search the optimum of parallel system. Although both series

and parallel systems use the same performance functions, there are significant differences between these two different systems. For series system, only the minimum parts of the performance functions $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ work, and the iterative point converges to an optimum $(-0.407, 0.898)$. For parallel system, the maximum parts of the performance functions $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ work, and the iterative point converges to an optimum $(-1.680, 0.434)$. Comparing with parallel system, series system results in a more conservative result with a larger feasible region. From Tables 2 and 3, it is found that only ANVM and SMAM can find correct optima. However, the SMAM method is the most effective method for this system RBDO problem, and the convergence speed is about three times faster than that of ANVM.

Table 2 Optimal results of series system for case 2

Methods	Objective	Design variables	F-evaluations		$\underline{\eta}_{series}$
			Objective	Constraint	
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	− 2.202	(− 0.407, 0.898)	5	213	1
SMAM	− 2.202	(− 0.407, 0.898)	8	81	1

Table 3 Optimal results of parallel system for case 2

Methods	Objective	Design variables	F-evaluations		$\eta_{parallel}$
			Objective	Constraint	
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	−2.547	(−1.680, 0.434)	13	237	1
SMAM	−2.547	(−1.699, 0.424)	14	90	1

5.3 Stepped cantilever beam

The volume of a stepped cantilever beam with square cross-section (Gandomi et al. 2013), as shown in Fig. 7, is minimized. The left side of the stepped cantilever beam is fixed, while a vertical force is applied at the free end of the cantilever. The design variables are related to the heights/widths of different beam elements, while the thickness t is considered as a fixed value 2/3. All design variables are considered as interval variables with nominal values 5, and the reference coefficients of variations are assumed as 10%. The problem is expressed as follows:

$$
\begin{aligned}
&\text{find} && \mathbf{d} = \left[x_1^C, x_2^C, x_3^C, x_4^C, x_5^C\right]^T \\
&\text{min} && 0.0624\left(x_1^C, x_2^C, x_3^C, x_4^C, x_5^C\right) \\
&\text{s.t.} && \eta[g(\mathbf{x}) \geq 0] \geq \underline{\eta} \\
&&& 0.01 \leq x_i^C \leq 10, \qquad i = 1, \ldots, 5
\end{aligned} \tag{35}
$$

$$
\text{where} \quad g(\mathbf{x}) = 1 - \frac{61}{x_1^3} - \frac{37}{x_2^3} - \frac{19}{x_3^3} - \frac{7}{x_4^3} - \frac{1}{x_5^3}
$$

$$\mathbf{d}^0 = [5.0, 5.0, 5.0, 5.0, 5.0]^T, \quad \underline{\eta} = 2$$

Table 4 presents a comparison of optimal results obtained by the aforementioned RBDO methods, in which
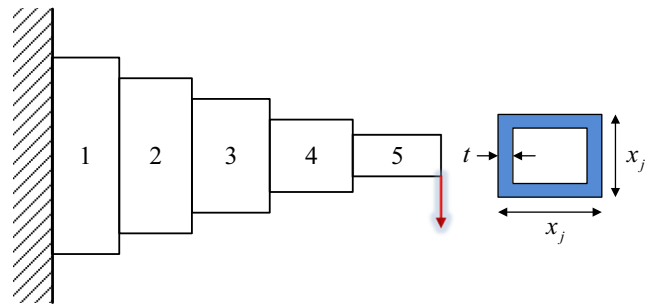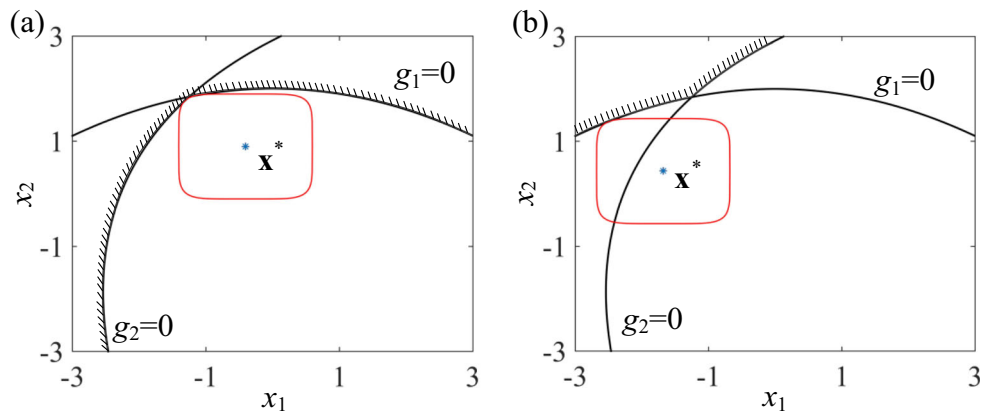

Fig. 7 Stepped cantilever beam

six different super parameters, i.e., super parameters $p = 1.5, 2, 4, 8, 16,$ and ∞, are selected. The comparison results indicate that only ANVM and SMAM can solve super parametric convex models, while other methods, including NRIA, STM, CCSTM, DSTM, and CPA, lack ability for handling this complex RBDO problem. However, SMAM is about 35 times and 43 times faster than ANVM with $p = 1.5$ and $p = 2$, and it is about 50 times faster than ANVM with $p = 4, 8, 16,$ and ∞. So, it can be concluded that SMAM is more general and efficient than NRIA, STM, CCSTM, DSTM, and CPA, and it is far more efficient than ANVM.

To demonstrate the performance of different algorithms, we take the super parameters $p = 2$ and 16 as two special cases, as illustrated in Fig. 8. Since super parameter $p = 2$ is the ordinary ellipsoid model, all different algorithms can find the optimum robustly. CPA is more efficient than NRIA, STM, CCSTM, DSTM, and ANVM. However, SMAM is the most efficient method, and the number of function evaluations is only 30 times less than that of CPA. For super parameter $p = 16$, it is a complex RBDO model. As proved in Fig. 8b, only ANVM and SMAM can find the correct optimum, and other methods converge to the incorrect optimum, as well as other super parameters $p \neq 2$ and $p \neq \infty$. Thus, the results indicate that SMAM is the most efficient, robust, and general RBDO method.

Fig. 6 Optimal results for case 2. **a** Series system. **b** Parallel system

Table 4 Optimal results for stepped cantilever beam

Methods	Objective	Design variables	F-evaluations		$\eta_{\min}^{\mathrm{ANVM}}$
			Objective	Constraint	
$p = 1.5$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	1.361	(6.084, 5.377, 4.563, 3.570, 2.221)	167	2310	2
SMAM	1.361	(6.084, 5.377, 4.563, 3.570, 2.221)	66	66	2
$p = 2$					
NRIA	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	153	2580	2
STM	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	146	4206	2
CCSTM	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	153	2682	2
DSTM	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	160	3402	2
CPA	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	134	1800	2
ANVM	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	153	2580	2
SMAM	1.368	(6.106, 5.399, 4.585, 3.591, 2.240)	60	60	2
$p = 4$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	1.382	(6.150, 5.443, 4.628, 3.635, 2.286)	159	2772	2
SMAM	1.382	(6.151, 5.443, 4.627, 3.635, 2.286)	54	54	2
$p = 8$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	1.391	(6.180, 5.473, 4.658, 3.665, 2.316)	137	2388	2
SMAM	1.391	(6.184, 5.474, 4.656, 3.662, 2.317)	48	48	2
$p = 16$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	1.396	(6.197, 5.490, 4.675, 3.682, 2.334)	143	2496	2
SMAM	1.396	(6.200, 5.490, 4.674, 3.681, 2.334)	48	48	2
$p = \infty$					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	1.402	(6.216, 5.509, 4.694, 3.701, 2.353)	120	468	–
ANVM	1.402	(6.216, 5.509, 4.694, 3.701, 2.353)	133	2316	2
SMAM	1.402	(6.219, 5.509, 4.694, 3.700, 2.352)	48	48	2

Fig. 8 Iterative histories of different methods for stepped cantilever beam with super parameters **a** $p = 2$ and **b** $p = 16$
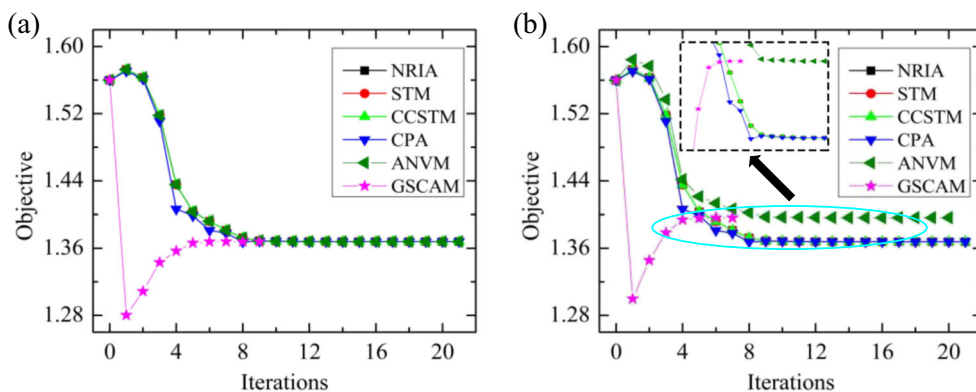
5.4 Tension/compression spring

As illustrated in Fig. 9, the target of this RBDO problem is to minimize the weight of tension/compression spring with four non-linear constraints. There are two design variables: wire diameter x_1 and mean coil diameter x_2, while the reference coefficients of variations are assumed as 5%. The active coil number NP is set to 11. The optimization model is formulated as follows:

$$
\begin{aligned}
\text{find} \quad & \mathbf{d} = \left[x_1^C, x_2^C\right]^T \\
\text{min} \quad & (NP+2)x_2 x_1^2 \\
\text{s.t.} \quad & \eta_j\left[g_j(\mathbf{x}) \geq 0\right] \geq \underline{\eta}_j, \quad j = 1, 2, 3, 4 \\
\text{where} \quad & g_1 = \frac{x_2^3 NP}{71785 x_1^4} - 1 \\
& g_2 = 1 - \frac{4x_2^2 - x_1 x_2}{12566\left(x_2 x_1^3 - x_1^4\right)} + \frac{1}{5108 x_1^2} \\
& g_3 = \frac{140.45 x_1}{x_2^2 NP} - 1 \\
& g_4 = 1 - \frac{x_1 + x_2}{1.5} \\
& 0.05 \leq x_1 \leq 2, \, 0.25 \leq x_2 \leq 1.3 \\
& \mathbf{d}^0 = [0.05, 0.5], \underline{\eta}_j = 1, \, j = 1, 2, 3, 4
\end{aligned}
\tag{36}
$$

The comparison results are tabulated in Table 5, where six different super parameters, including $p = 1, 2, 4, 8, 16$, and ∞, are considered. For comparison purposes, NRIA, STM, CCSTM, DSTM, CPA, and ANVM are also
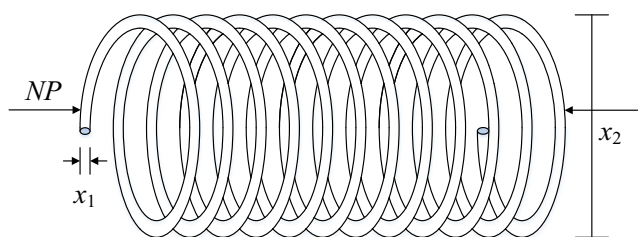

Fig. 9 A tension/compression spring

performed with the same initial point. The optima of different RBDO approaches are verified by ANVM to validate whether the results satisfy the target non-probabilistic reliability requirement. The results of Table 5 indicate that only ANVM and SMAM can solve all different RBDO models. NRIA, STM, CCSTM, and CPA demonstrate the feasibility and validity for conventional ellipsoid model, but it cannot be used to calculate the other complex RBDO problems with $p = 1, 4, 8, 16$, and ∞. DSTM is not robust for this problem. CPA can handle both ellipsoid and interval models, and thus, it has broader application range than NRIA, STM, CCSTM, and DSTM. ANVM exhibits effectiveness for all different RBDO models, but the convergence rate is too slow. For all RBDO models, SMAM can solve them accurately and is approximately ten times faster than ANVM, and thus, it is the most general and efficient method.

5.5 A twenty-five-bar truss

A steel truss with 24 degrees of freedom and 4 uncertain loads is considered here (Ganzerli and Pantelides 2000). All bars have the same elasticity modulus $E = 29,000$ ksi. The height of the truss is 600 in., and the length of the bays is also 600 in. The structure is shown in Fig. 10, and all these loads are assumed as the interval variables. The nominal values of external loads are $P_1 = P_3 = 400$ kip, $P_2 = 500$ kip, and $P_4 = 300$ kip, and the corresponding coefficients are considered as the following values: $\beta_1 = \beta_2 = \beta_3 = 10\%$ and $\beta_4 = 20\%$. The RBDO model is expressed as follows:

$$
\begin{aligned}
\text{find} \quad & \mathbf{d} = \left[A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, \right. \\
& \left. A_{14}, A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20}, A_{21}, A_{22}, A_{23}, A_{24}, A_{25}\right]^T \\
\text{min} \quad & V(\mathbf{d}) \\
\text{s.t.} \quad & \eta_j\left[g_j(\mathbf{d}, P_1, P_2, P_3, P_4) \geq 0\right] \geq \underline{\eta}_j, j = 1, \ldots, 25 \\
\text{where} \quad & g_j = 25 \text{ ksi} - \sigma_j(\mathbf{d}, P_1, P_2, P_3, P_4), j = 1, \ldots, 25 \\
& \underline{\eta}_j = 1, j = 1, \ldots, 25 \\
& 0.1 \leq A_i \leq 25, i = 1, \ldots, 25
\end{aligned}
\tag{37}
$$

Table 5 Optimal results for tension/compression spring

| Methods	Objective	Design variables	F-evaluations		η_{min}^{ANVM}
			Objective	Constraint	
p = 1					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	0.0261	(0.0625, 0.5140)	15	1440	1
SMAM	0.0261	(0.0625, 0.5140)	21	99	1
p = 2					
NRIA	0.0291	(0.0639, 0.5482)	12	810	1
STM	0.0291	(0.0639, 0.5482)	12	2844	1
CCSTM	0.0291	(0.0639, 0.5482)	12	1773	1
DSTM	–	–	–	–	–
CPA	0.0291	(0.0639, 0.5482)	18	489	1
ANVM	0.0291	(0.0639, 0.5482)	12	810	1
SMAM	0.0291	(0.0639, 0.5482)	21	99	1
p = 4					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	0.0319	(0.0653, 0.5757)	12	720	1
SMAM	0.0319	(0.0653, 0.5757)	21	99	1
p = 8					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	0.0336	(0.0661, 0.5920)	12	684	1
SMAM	0.0336	(0.0661, 0.5920)	18	87	1
p = 16					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	0.0346	(0.0665, 0.6008)	15	855	1
SMAM	0.0346	(0.0665, 0.6008)	18	87	1
p = ∞					
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	0.0356	(0.0670, 0.6099)	21	312	1
ANVM	0.0356	(0.0670, 0.6099)	15	837	1
SMAM	0.0356	(0.0670, 0.6099)	15	75	1
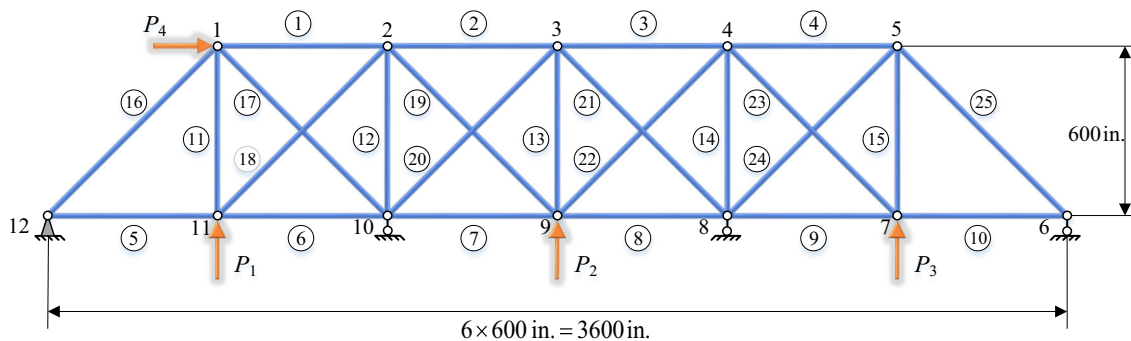
Fig. 10 A twenty-five-bar truss

where the design variables **d** represent the cross-section areas of bars. Additionally, the maximum tension and compression stresses of each bar are 25 ksi, and the finite element method is used to compute the structural response.

The RBDO with different super parameters $p = 1, 2, 4,$ 8, 16, and ∞ are tested, and the comparison results are tabulated in Table 6. It can be found that NRIA, STM, CCSTM, DSTM, and CPA cannot search the correct optima, while ANVM and SMAM robustly converge to the optimum. Since this RBDO problem contains many design variables and constraints, it requires unbearable computational cost. Taking ellipsoid model as a case, all RBDO approaches can solve this simple RBDO problem, and CPA is more efficient than NRIA, STM, CCSTM, DSTM, and ANVM. However, the number of function calls of proposed SMAM method is about 23 times less than that of CPA. For interval set model, the computational cost of SMAM is about 34 times less than that of CPA and 241 times less than ANVM. Evidently, it can be concluded that SMAM is the most general and efficient method.

The design variable values of SMAM at the optimum are tabulated in Table 7. It can be found that the design variables are drastically changed with the change of super parameters, which means the selection of super parameters is vital for optimization results, and the minimum volume method is suggested to select the reasonable uncertain model (Meng et al. 2018). The larger the super parameter value is, the more conservative of design becomes. It is interesting to mention that the choice of the RBDO models is critical to offer credible design results, and different safety levels can be achieved for SMAM by selecting suitable super parameter (Fig. 11).

5.6 A tower crane

In civil engineering, tower crane is widely applied for lifting materials from one place to another place, which is shown in Fig. 12 (Meng and Zhou 2018). A lightweight design of a

tower crane with 1.52 m × 1.52 m × 87 m is carried out. This tower crane includes 928 bars, in which every 32 bars form a standard component. Sectional dimensions of vertical bars and cross bars are chosen as design variables. Young's modulus E, windward wind load, transverse wind load, and allowable normal stress $\underline{\sigma}$ are assumed as interval variables with reference coefficients of variations 2.5%. The nominal values of these four uncertain variables are [206 GPa, 172.29 kN, 109.16 kN, and 345 MPa]^T. The RBDO model is established as follows:

$$
\begin{aligned}
&\text{find}\quad \mathbf{d} = [b_1, t_1, b_2, t_2]^T \\
&\underset{\mathbf{d}}{\min}\quad C(\mathbf{d})\, s.t.\, \eta(\sigma - \sigma_- \geq 0) \geq \underline{\eta} \\
&\qquad 35 \leq b_1 \leq 200,\, 5 \leq t_1 \leq 15 \\
&\qquad 15 \leq b_2 \leq 100,\, 2 \leq t_2 \leq 6 \\
&\qquad \mathbf{d}^0 = [120, 10, 7, 5]^T
\end{aligned}
\tag{38}
$$

A comparison of the RBDO results obtained by different methods is gathered in Table 8. It is evident that NRIA, STM, CCSTM, and DSTM lack of ability to find the optimum results, while CPA and SMAM converge to the optimum result stably. The weight of initial design is 345,953 kg. After performing RBDO, the weight is decreased to 178,782 kg, which is only about half of the initial design. Thus, it generates huge economic benefits. Besides, we also perform the deterministic optimization, and the corresponding structural weight and design variables are 177,937 kg and [53.954, 5.984, 16, 4], respectively. It is observed that the difference of the structural weight between the deterministic design and RBDO design is very small, but the non-probabilistic reliability index of the deterministic design decreases to 0. It means that the deterministic design is too sensitive to the variations of uncertain parameters and has a risk of failure.

Among all RBDO approaches, NRIA, STM, CCSTM, DSTM, CPA, and ANVM converge to the incorrect optimum, and the non-probabilistic reliability index of ANVM is only

Table 6 Optimal results for twenty-five-bar truss

Methods	Objective	F-evaluations		η_{min}^{ANVM}
		Objective	Constraint	
p = 1				
NRIA	–	–	–	–
STM	–	–	–	–
CCSTM	–	–	–	–
DSTM	–	–	–	–
CPA	–	–	–	–
ANVM	1.2162×10^5	552	828,000	1
SMAM	1.2163×10^5	728	3651	1
p = 2				
NRIA	1.2311×10^5	445	667,500	1
STM	1.2311×10^5	419	5,185,110	1
CCSTM	1.2311×10^5	454	3,119,580	1
DSTM	1.2311×10^5	630	3,738,000	1
CPA	1.2311×10^5	476	64,500	1
ANVM	1.2311×10^5	445	667,500	1
SMAM	1.2311×10^5	468	2401	1
p = 4				
NRIA	–	–	–	–
STM	–	–	–	–
CCSTM	–	–	–	–
DSTM	–	–	–	–
CPA	–	–	–	–
ANVM	1.2535×10^5	764	1,146,000	1
SMAM	1.2537×10^5	988	4901	1
p = 8				
NRIA	–	–	–	–
STM	–	–	–	–
CCSTM	–	–	–	–
DSTM	–	–	–	–
CPA	–	–	–	–
ANVM	1.2690×10^5	988	1,483,860	1
SMAM	1.2691×10^5	1846	9026	1
p = 16				
NRIA	–	–	–	–
STM	–	–	–	–
CCSTM	–	–	–	–
DSTM	–	–	–	–
CPA	–	–	–	–
ANVM	1.2780×10^5	738	1,107,000	1
SMAM	1.2783×10^5	1222	6026	1
p = ∞				
NRIA	–	–	–	–
STM	–	–	–	–
CCSTM	–	–	–	–
DSTM	–	–	–	–
CPA	1.2879×10^5	1208	190,500	1
ANVM	1.2879×10^5	888	1,332,000	1
SMAM	1.2885×10^5	1118	5526	1

Table 7 Design variables at the optimum

Design variables	$p=1$	$p=2$	$p=4$	$p=6$	$p=8$	$p=16$	$p=\infty$
A_1	2.390	2.702	3.502	3.742	3.854	4.015	4.006
A_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1
A_3	0.779	0.726	0.796	0.828	0.853	0.872	0.710
A_4	0.1	0.230	0.374	0.298	0.368	0.373	0.855
A_5	12.764	13.016	13.370	13.527	13.612	13.749	13.850
A_6	5.735	5.667	5.403	5.402	5.418	5.460	5.546
A_7	0.1	0.1	0.1	0.1	0.1	0.1	0.1
A_8	0.322	0.464	0.480	0.507	0.511	0.522	0.343
A_9	0.1	0.1	0.1	0.192	0.140	0.185	0.447
A_{10}	5.916	5.938	5.980	5.994	6.005	6.028	6.217
A_{11}	10.530	10.345	10.036	10.019	10.026	10.054	10.156
A_{12}	12.404	12.792	13.580	13.825	13.944	14.115	14.142
A_{13}	10.968	10.986	11.007	11.019	11.028	11.041	11.071
A_{14}	10.900	11.138	11.273	11.321	11.352	11.398	11.420
A_{15}	11.758	11.726	11.707	11.751	11.715	11.735	11.742
A_{16}	2.313	2.696	3.204	3.425	3.546	3.740	3.905
A_{17}	15	14.820	14.364	14.331	14.336	14.372	14.519
A_{18}	9.999	10.579	11.534	11.786	11.901	12.059	12.081
A_{19}	7.833	7.847	7.913	7.951	7.972	8.004	7.969
A_{20}	7.833	7.846	7.909	7.947	7.967	7.998	7.962
A_{21}	7.692	7.803	7.913	7.957	7.984	8.023	8.067
A_{22}	7.768	7.767	7.809	7.833	7.845	7.867	7.855
A_{23}	8.262	8.307	8.351	8.289	8.312	8.318	8.292
A_{24}	8.262	8.196	8.202	8.253	8.255	8.282	8.282
A_{25}	8.367	8.397	8.457	8.477	8.492	8.526	8.791

0.1904 at the optimum. TPA and SMAM show effectiveness for solving super parametric convex model. However, for this complex engineering example, the convergence speed of SMAM is significantly faster than TPA. It can be concluded that SMAM is the most effective, general, and accurate method.
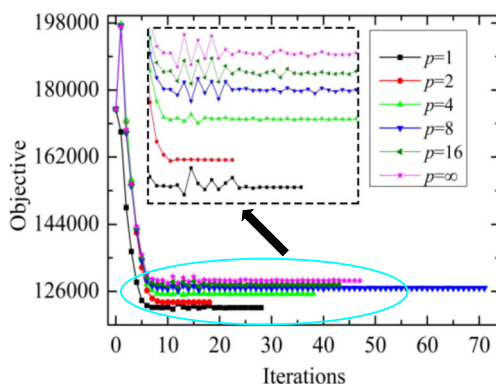

Fig. 11 Iterative histories of SMAM for the twenty-five-bar truss with different super parameters

6 Conclusions

Reliability-based design optimization (RBDO) exhibits unique competitiveness in the situation of inadequate sample data. For system RBDO problem, there is always involving an inefficient and non-universal problem. In this study, considering the reasonability update of random and design variables, a novel sequential moving asymptote method (SMAM) is proposed to simultaneously perform the deterministic optimization and reliability analysis, which leads to a dramatic reduction of computational cost. Furthermore, a general sensitivity computation method is provided, in which the sensitivities of reliability index with respect to design variables can be directly obtained by using the gradient of performance function, and thus, the time-consuming finite difference method can be avoided. To further promote the computational efficiency, the MMA is applied for converting the suboptimization problem into a series of convex optimization models at each iterative point. Furthermore, the KS function is applied to deal with the system reliability analysis effectively.

Numerical results on mathematical and complex engineering examples verify that (1) SMAM can estimate the
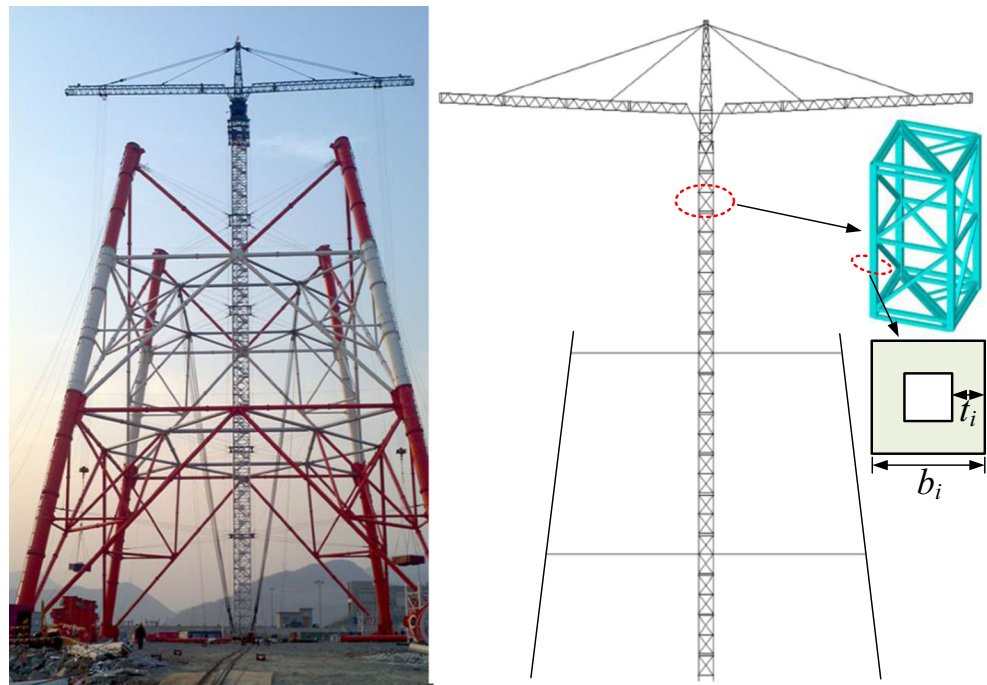
Fig. 12 A tower crane

Table 8 Optimal results for tower crane

Method	Objective	Design variables	F-evaluations		η_{min}^{MVM}
			Objective	Constraint	
NRIA	–	–	–	–	–
STM	–	–	–	–	–
CCSTM	–	–	–	–	–
DSTM	–	–	–	–	–
CPA	–	–	–	–	–
ANVM	178,095 kg	(54.091, 6.00, 16, 4)	156	1404	0.1904
TPA	178,783 kg	(54.877, 6.04, 16, 4)	20	130	1
SMAM	178,782 kg	(54.877, 6.04, 16, 4)	20	50	1

reliability index constraints with high convergence rate, which is more efficient than existing RBDO methods, including ANVM, TPA, CPA, and the first-order reliability methods (NRIA, STM, CCSTM, DSTM, and CPA), and it is more general than FORM algorithms and CPA. By using different non-probabilistic super parameters, different non-probabilistic RBDO tasks can be achieved according to existing experimental data, so it is very flexible. (2) We provide a general sensitivity analysis method for non-probabilistic reliability analysis and optimization, which has been not found until now. In practical engineering application, the proposed SMAM may be of great significance, which can be easily extended to perform the RBDO and reliability analysis for other models and systems.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Replication of results The algorithm provided in this article is part of the software we are developing. As the software cannot be published due to confidential issue of the funded project, detailed explanation about how

the algorithm is implemented in Section 3. Based on the MMA code developed by Svanberg (1987), the proposed algorithm is easy to code. Readers are welcome to contact the authors for details and further explanations.

Appendix

In the proposed SMAM, we develop the approximate sensitivity analysis method to substitute the FDM in order to promote the efficiency, and the corresponding computational error should be performed. By using the Taylor expansion, we can expand the performance function at the MCP, which is formulated as follows:

$$g(\mathbf{d}^m, \mathbf{q}^m) = g(\mathbf{d}^m, \mathbf{q}^*) + \nabla_{\mathbf{q}} g(\mathbf{d}^m, \mathbf{q}^*)^{\mathrm{T}}(\mathbf{q}^m - \mathbf{q}^*) + O\left(\left\| (\mathbf{q}^m - \mathbf{q}^*) \right\|^2\right) \tag{39}$$

$$\nabla_{\mathbf{d}} g(\mathbf{d}^m, \mathbf{q}^m) = \nabla_{\mathbf{d}} g(\mathbf{d}^m, \mathbf{q}^*) + \nabla_{\mathbf{dq}}^2 g(\mathbf{d}^m, \mathbf{q}^*)^{\mathrm{T}}(\mathbf{q}^m - \mathbf{q}^*) + O\left(\left\| (\mathbf{q}^m - \mathbf{q}^*) \right\|^2\right) \tag{40}$$

where $\nabla^2 g$ denotes the second-order derivative of function g. Then, the sensitivity of approximate non-probabilistic reliability index with respect to design variables in SMAM can be written as follows:

$$
\begin{aligned}
\nabla_{d^m} \widehat{\eta} &= \frac{\nabla_{d^m} g(d^m, q^m)}{\left\| \nabla_{q^m} g(d^m, q^m) \right\|_{\frac{p}{p-1}}} \\
&= \left(\frac{\nabla_d g(d^m, q^*) + \nabla_{dq}^2 g(d^m, q^*)^{\mathrm{T}}(q^m - q^*) + O\left(\left\| (q^m - q^*) \right\|^2\right)}{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}} \right) \times \frac{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}}{\left\| \nabla_q g(d^k, q^m) \right\|_{\frac{p}{p-1}}}
\end{aligned}
\tag{41}
$$

In (41), the term $\dfrac{\left\| \nabla_{\mathbf{q}} g(\mathbf{d}^m, \mathbf{q}^*) \right\|_{\frac{p}{p-1}}}{\left\| \nabla_{\mathbf{q}} g(\mathbf{d}^k, \mathbf{q}^m) \right\|_{\frac{p}{p-1}}}$ can be expanded as follows:

$$
\begin{aligned}
\frac{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}}{\left\| \nabla_q g(d^m, q^m) \right\|_{\frac{p}{p-1}}} &= \left(\frac{\sum_{i=1}^{n} \left| \nabla_q g(q_i^*) + \nabla_{qq}^2 g(q_i)^{\mathrm{T}}(q_i^m - q_i^*) + O\left(\left\| (q_i^m - q_i^*) \right\|^2\right) \right|^{\frac{p}{p-1}}}{\sum_{i=1}^{n} \left| \nabla_q g(q_i^*) \right|^{\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\leq \left(1 + \frac{\sum_{i=1}^{nq} \left| \nabla_{qq}^2 g(q_i)^{\mathrm{T}}(q_i^m - q_i^*) + O\left(\left\| (q_i^m - q_i^*) \right\|^2\right) \right|^{\frac{p}{p-1}}}{\sum_{i=1}^{nq} \left| \nabla_q g(q_i^*) \right|^{\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&= 1 + O\left(\left\| (q^m - q^*) \right\|\right)
\end{aligned}
\tag{42}
$$

Hence, $\nabla_{\mathbf{d}^m} \widehat{\eta}$ satisfies:

$$
\begin{aligned}
\nabla_d \eta &= \frac{\nabla_d g(d^m, q^*)}{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}} \\
&= \left(\frac{\nabla_d g(d^m, q^*) + \nabla_{dq}^2 g(d^m, q^*)^{\mathrm{T}}(q^m - q^*) + O\left(\left\| (q^m - q^*) \right\|^2\right)}{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}} \right) \times \left(1 + O\left(\left\| (q^m - q^*) \right\|\right) \right) \\
&= \nabla_d \widehat{\eta} + \frac{\nabla_{dq}^2 g(d^m, q^*)^{\mathrm{T}}(q^m - q^*)}{\left\| \nabla_q g(d^m, q^*) \right\|_{\frac{p}{p-1}}} + O\left(\left\| (q^m - q^*) \right\|^2\right) \\
&= \nabla_d \widehat{\eta} + O\left(q^m - q^*\right)
\end{aligned}
\tag{43}
$$

For reliability index, it can be expanded by Taylor expansion.

$$\eta(\mathbf{d}^m, \mathbf{q}^*) = \widehat{\eta}(\mathbf{d}^m, \mathbf{q}^m) + \left(\nabla_\mathbf{d}\widehat{\eta}(\mathbf{d}^m, \mathbf{q}^m)\right)^T (\mathbf{d} - \mathbf{d}^m) + O\left(\left\|(\mathbf{d} - \mathbf{d}^m)^2\right\|\right) \tag{44}$$

$$\eta(\mathbf{d}^m, \mathbf{q}^*) = \eta(\mathbf{d}^m, \mathbf{q}^m) + (\nabla_\mathbf{d}\eta(\mathbf{d}^m, \mathbf{q}^m))^T (\mathbf{d}^m - \mathbf{d}^*) + O\left(\left\|(\mathbf{d}^m - \mathbf{d}^*)^2\right\|\right) \tag{45}$$

$$\nabla_\mathbf{d}\eta(\mathbf{d}^m, \mathbf{q}^*) = \nabla_\mathbf{d}\widehat{\eta}(\mathbf{d}^m, \mathbf{q}^m) + O\left(\left\|(\mathbf{q}^m - \mathbf{q}^*)\right\|\right) \tag{46}$$

$$\nabla_\mathbf{d}\widehat{\eta}(\mathbf{d}^m, \mathbf{q}^m) = \nabla_\mathbf{d}\eta(\mathbf{d}^m, \mathbf{q}^*) - O\left(\left\|(\mathbf{q}^m - \mathbf{q}^*)\right\|\right) \tag{47}$$

$$\nabla_\mathbf{d}\widehat{\eta}(\mathbf{d}^m, \mathbf{q}^m) = \nabla_\mathbf{d}\eta(\mathbf{d}^m, \mathbf{q}^*) + \left(\nabla_\mathbf{dq}^2\widehat{\eta}(\mathbf{d}^m, \mathbf{q}^*)\right)^T (\mathbf{q}^* - \mathbf{q}^m) + O\left(\left\|(\mathbf{q}^* - \mathbf{q}^*)^2\right\|\right) \tag{48}$$

Considering $\mathbf{q}^*$ is the optimal solution, which satisfies $\nabla_\mathbf{q}\eta(\mathbf{d}^m, \mathbf{q}^*) = 0$, so the error of reliability constraint can be computed as follows:

$$
\begin{aligned}
&\eta_j - \left(\widehat{\eta}_j^m + \left(\nabla_{d^m}\widehat{\eta}_j\right)^T (d - d^m)\right) \\
&= \eta_j - \left(\widehat{\eta}_j^m + \left(\nabla_d\eta(d^m, q^*) - O(q^m - q^*)\right)^T (d - d^m)\right) \\
&= \eta_j - \widehat{\eta}_j^m - \left(\nabla_d\eta(d^m, q^*)(d - d^m) - O\left(\|(q^m - q^*)\|\|d - d^m\|\right)\right) \\
&= \left(\nabla_d\widehat{\eta}(d^m, q^m) - \nabla_d\eta(d^m, q^*)\right)^T (d - d^m) + O\left(\|(d - d^m)^2\|\right) + O\left(\|(q^m - q^*)\|\|d - d^m\|\right) \\
&= O\left(\|(d - d^m)^2\|\right) + O\left(\|(q^m - q^*)\|\|d - d^m\|\right)
\end{aligned}
\tag{49}
$$

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