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RESEARCH PAPER

Structural sensitivity reanalysis formulations based on the polynomial-type extrapolation methods

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Abstract
This paper presents new structural sensitivity reanalysis formulations based on the polynomial-type extrapolation methods. In these formulations, the displacement vector of the modified structure is expressed in the form of the vector sequences based on the fixed-point iteration method. By using these vector sequences, the *minimal polynomial extrapolation* (MPE) and the *reduced rank extrapolation* (RRE) methods calculate the approximate displacement vector of the modified structure by solving reduced linear least-square problems. Based on the definitions of the MPE and RRE methods, two sensitivity reanalysis formulations are derived, in which the first- and second-order sensitivities of the modified structure are obtained by solving a set of the over-determined least-square problems with much smaller size than the complete set of equations of the exact sensitivity analyses. The performance of the proposed sensitivity reanalysis formulations is evaluated by using four structural sensitivity reanalysis problems under multiple modifications in their initial designs. The results obtained from the numerical test problems indicate that the proposed sensitivity reanalysis formulations approximate the first- and second-order sensitivities of the modified structure with a high level of accuracy and they are able to converge to the exact solutions.

Keywords Sensitivity · Reanalysis · Polynomial-type extrapolation · Minimal polynomial extrapolation · Reduced rank extrapolation

1 Introduction

Repeated structural and sensitivity analyses of the modified structures are the main parts of today's iterative structural optimization procedures, in which the structures are gradually modified until an optimal design satisfying both of the safety and economical requirements is reached. For each of the modified structures, the derivatives of the structural response with respect to the design variables, which are called sensitivity coefficients, should be calculated by solving a set of modified equations. The sensitivity information is crucial to find search direction during the optimization process, and their calculation in large-scale structures with a high number of design variables is often computationally expensive procedure (Adelman and Haftka 1986). For large-scale structural designs having certain modifications at some components, due to high computational cost of direct analysis, it may not be good choice to perform sensitivity analysis by solving repeatedly complete set of the modified equations. As a result, developing efficient sensitivity analysis techniques with fewer amount of computational effort than the regular sensitivity analysis is one of the active research topics in the field of the structural engineering and performing structural sensitivity analysis more quickly can significantly enhance the performance of structural optimization methods.

In recent years, structural reanalysis methods such as local, global, and combined approximation (CA) methods have been developed to calculate the response of the modified structure without solving complete set of modified equations. Binomial series expansion and the first-order Taylor series expansion about a given initial design are examples of local or single point reanalysis methods. In the local reanalysis methods, the response of the modified structure is calculated by using available information from a single initial design. The local or single point methods have shown good performance in reanalysis problems with smaller changes in the initial design; however, they reported poor accuracy for reanalysis problems with

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larger amount of changes in the design space (Barthelemy and Haftka 1993). Unlike local methods, global or multipoint methods, such as the polynomial fitting or reduced basis methods (Fox and Miura 1971; Haftka et al. 1987; Noor 1994), construct the approximate structural response based on the available analysis information of multiple initial designs. The global methods show great advantage over the local methods in terms of accuracy in the reanalysis problems with larger changes in the design space. However, there are serious concerns about the accuracy and computational effort of the global methods (Zuo et al. 2012; Wu et al. 2003). CA method has been developed by Kirsch (2003) as a unified approach for solving various structural reanalysis problems. CA tries to approximate the response of the modified structure by providing global qualities to the local approximations. In CA, the approximate response is calculated by combining the reduced basis method with the first terms of series expansion. The application results to various reanalysis and sensitivity reanalysis problems show that the obtained solutions are accurate under relatively larger changes in the design space (Amir et al. 2008; Kirsch and Bogomolni 2004; Leu and Huang 2000; Kirsch et al. 2006; Kirsch 2010; Kirsch et al. 2007; Zuo et al. 2017; Sun et al. 2011; Zuo et al. 2011; Xu et al. 2010; Zuo et al. 2019). To increase the efficiency of CA method, Zuo et al. (2016) proposed a hybrid static sensitivity reanalysis method by combining Taylor series expansion and CA method. In comparison to CA method, the hybrid method may largely increase efficiency with small loss of accuracy of the sensitivity analysis (Zuo et al. 2016).

Recently, Hosseinzadeh et al. (2018) applied a new structural reanalysis approach based on the polynomial-type extrapolation methods to approximate the response of the modified structure under multiple types of modifications in the initial design. In this approach, the displacement vector of the modified structure is expressed in the form of the vector sequences based on the fixed-point iteration method. By using these vector sequences, the *minimal polynomial extrapolation* (MPE) and *the reduced rank extrapolation* (RRE) methods calculate the approximate displacement vector of the modified structure. In the MPE and RRE methods, the complete set of analysis equations of the modified structure is reduced to the linear least-square problems with significantly smaller size. Following successful application of the polynomial-type extrapolation methods for structural reanalysis, this paper presents new structural sensitivity reanalysis formulation based on the MPE and RRE methods. To demonstrate the efficiency of the proposed structural sensitivity reanalysis approach, a comprehensive numerical investigation has been carried out by using four sensitivity reanalysis problems with relatively larger changes in their initial designs.

The rest of the paper is organized as follows. The mathematical formulation of the structural sensitivity reanalysis problem is briefly described in Sect. 2. In Sect. 3, a brief review of the CA method for the structural sensitivity reanalysis is presented. In Sect. 4, the structural sensitivity reanalysis based on the MPE and RRE methods is described, and then, the derivation of the proposed formulation for structural sensitivity reanalysis is explained in detail. Section 5 presents the application of the proposed approach on set of four structural sensitivity reanalysis problems. Finally, some concluding remarks are given in Sect. 6.

2 Mathematical formulation of structural sensitivity reanalysis problem

The main aim of the structural sensitivity reanalysis problems is to calculate the sensitivities of a modified structure by using available exact analysis information from an initial design without solving complete set of modified equations. In the present section, since the calculation of sensitivities for a given structure involves structural analysis, at first the problem of structural reanalysis is formulated and, subsequently, the first- and second-order structural sensitivity problems are presented.

2.1 Structural reanalysis

The main objective of a structural reanalysis problem is to calculate the displacement vector of the modified structure without solving complete set of the modified equations. Let us consider a given structure with n_{dof} degrees of freedoms (DOFs), initial stiffness matrix $\boldsymbol{K}_0\in\mathbb{C}^{n_{\mathrm{dof}}\times n_{\mathrm{dof}}}$, and load vector $\boldsymbol{F}_0\in\mathbb{C}^{n_{\mathrm{dof}}}$. In structural reanalysis problems, it is assumed that the displacement vector $\boldsymbol{r}_0\in\mathbb{C}^{n_{\mathrm{dof}}}$ for the initial design is given from the following equation:

$$\boldsymbol{K}_0\boldsymbol{r}_0 = \boldsymbol{F}_0 \tag{1}$$

where the decomposed form of the initial stiffness matrix $\boldsymbol{K}_0$ is given as follows:

$$\boldsymbol{K}_0 = \boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{U}_0 \tag{2}$$

in which $\boldsymbol{U}_0\in\mathbb{C}^{n_{\mathrm{dof}}\times n_{\mathrm{dof}}}$ is a upper triangular matrix.

If structure is subjected to a set of modifications in its initial design, the modified stiffness matrix $\boldsymbol{K}\in\mathbb{C}^{n_{\mathrm{dof}}\times n_{\mathrm{dof}}}$ and the modified load vector $\boldsymbol{F}\in\mathbb{C}^{n_{\mathrm{dof}}}$ can be simply written in the following form:

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}, \tag{3}$$

$$\boldsymbol{F} = \boldsymbol{F}_0 + \Delta\boldsymbol{F}, \tag{4}$$

where $\Delta\boldsymbol{K}\in\mathbb{C}^{n_{\mathrm{dof}}\times n_{\mathrm{dof}}}$ and $\Delta\boldsymbol{F}\in\mathbb{C}^{n_{\mathrm{dof}}}$ represent the changes in the stiffness matrix and load vector, respectively. Usually, matrix $\Delta\boldsymbol{K}$ is related to the changes in the cross-sectional properties, length, and material properties of structural elements. On the other hand, vector $\Delta\boldsymbol{F}$ is related to the changes in the loading conditions as well as geometrical and physical properties of structure (Kirsch 2000).

Now, the load-displacement relation for the modified structure can be written as follows:

$$\boldsymbol{K}\boldsymbol{r} = \boldsymbol{F} \tag{5}$$

where $\boldsymbol{r} \in \mathbb{C}^{n_{\text{dof}}}$ is the displacement vector of the modified structure. The main aim of a structural reanalysis problem is to calculate the displacement vector $\boldsymbol{r}$ without solving complete set of modified equations in (5). After calculating the displacement vector $\boldsymbol{r}$, the stress in the members of structure can be simply obtained accordingly.

2.2 Sensitivity reanalysis

By direct differentiating of (5) with respect to a design variable x_i and rearranging, the first-order derivative of the modified displacement vector $\boldsymbol{r}$ can be obtained as follows:

$$\boldsymbol{K}\frac{\partial \boldsymbol{r}}{\partial x_i} = -\frac{\partial \boldsymbol{K}}{\partial x_i}\boldsymbol{r} \tag{6}$$

where it is assumed that the load vector $\boldsymbol{F}$ is independent of design variables (that is, $\frac{\partial \boldsymbol{F}}{\partial x_i} = 0$) for simplicity. It should be noted that the proposed sensitivity reanalysis approach is also suitable for the cases where the elements of the load vector $\boldsymbol{F}$ are functions of the design variables.

If (6) is differentiated with respect to a design variable x_i, the modified second-order derivative of the modified displacement vector $\boldsymbol{r}$ can be written as follows:

$$\boldsymbol{K}\frac{\partial^2 \boldsymbol{r}}{\partial {x_i}^2} = -\frac{\partial^2 \boldsymbol{K}}{\partial {x_i}^2}\boldsymbol{r} - 2\frac{\partial \boldsymbol{K}}{\partial x_i}\frac{\partial \boldsymbol{r}}{\partial x_i} \tag{7}$$

Both of the (6) and (7) are systems of equations with the size of $n_{\text{dof}} \times n_{\text{dof}}$, where the decomposed form of the modified stiffness matrix $\boldsymbol{K}$ is not available. For the case of multiple design variables, (6) and (7) should be solved for each design variable separately. Therefore, the main aim of the structural sensitivity reanalysis is to calculate the first- and second-order derivatives of the modified displacement vector without direct solving of (6) and (7). After calculating the derivatives of the modified displacement vector, the stress derivatives can also be obtained by explicit differentiation of stress-displacement relations.

3 CA-based sensitivity reanalysis

In this study, the results obtained by the proposed approach will be compared to those yielded by the well-known CA method developed by Kirsch (2000). Therefore, this section provides a brief review of the structural sensitivity reanalysis formulation based on the CA method.

3.1 Structural reanalysis based on the CA method

In the CA method, a linear combination of s basis vectors is used to approximate the displacement vector of the modified structure as follows:

$$\widetilde{\boldsymbol{r}}_s^{\text{CA}} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \ldots + y_s\boldsymbol{r}_s = \boldsymbol{r}_{\text{B}}\boldsymbol{y},\ \boldsymbol{r}_{\text{B}} \in \mathbb{C}^{n_{\text{dof}} \times s}, \boldsymbol{y} \in \mathbb{C}^s \tag{8}$$

where $\widetilde{\boldsymbol{r}}_s^{\text{CA}}$ represents the approximate displacement vector of the modified structure obtained by the CA method with s basis vectors, $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s$ are the linearly independent basis vectors, $\boldsymbol{r}_{\text{B}}$ is the matrix containing the basis vectors, and $\boldsymbol{y}$ is the vector containing constant parameters. The matrix $\boldsymbol{r}_{\text{B}}$ and vector y are in the following forms:

$$\boldsymbol{r}_{\text{B}} = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s], \boldsymbol{y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_s \end{Bmatrix} \tag{9}$$

The basis vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s$ are calculated as follows (Kirsch 2000):

$$\boldsymbol{r}_i = \boldsymbol{r}^{(1)} = \boldsymbol{K}_0^{-1}\boldsymbol{F}, \boldsymbol{r}_2 = -\boldsymbol{B}\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s = -\boldsymbol{B}\boldsymbol{r}_{s-1} \tag{10}$$

where $\boldsymbol{B} = \boldsymbol{K}_0^{-1}\Delta\boldsymbol{K}$.

By premultiplying the load-displacement relation of the modified structure in (5) by $\boldsymbol{r}_{\text{B}}^{\text{T}}$ and expressing the displacement vector by the definition expressed in (8), following system of equations can be obtained:

$$\boldsymbol{K}_{\text{R}}\boldsymbol{y} = \boldsymbol{F}_{\text{R}}, \boldsymbol{K}_{\text{R}} \in \mathbb{C}^{s \times s}, \boldsymbol{F}_{\text{R}} \in \mathbb{C}^s \tag{11}$$

where $\boldsymbol{K}_{\text{R}} = \boldsymbol{r}_{\text{B}}^{\text{T}}\boldsymbol{K}\boldsymbol{r}_{\text{B}}$ represents the reduced order stiffness matrix and $\boldsymbol{F}_{\text{R}} = \boldsymbol{r}_{\text{B}}^{\text{T}}\boldsymbol{F}$ indicates the reduced order load vector. For a very smaller values of s, the vector of unknown coefficients $\boldsymbol{y}$ can be obtained by solving a linear $s \times s$ system of equations in (11), which has much smaller size than the original load-displacement relation of the modified structure. Finally, the displacement vector of the modified structure $\boldsymbol{r}$ can be simply obtained by substituting the vector $\boldsymbol{y}$ in (8).

3.2 First-order sensitivity reanalysis based on the CA method

By differentiation of (8), the first-order sensitivity of the approximate displacement vector provided by the CA method can be obtained as follows:

$$\frac{\partial \widetilde{\boldsymbol{r}}_s^{\text{CA}}}{\partial x_i} = \frac{\partial \boldsymbol{r}_{\text{B}}}{\partial x_i}\boldsymbol{y} + \boldsymbol{r}_{\text{B}}\frac{\partial \boldsymbol{y}}{\partial x_i} \tag{12}$$

where the derivatives $\frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i}$ and $\frac{\partial \boldsymbol{y}}{\partial x_i}$ are unknown and should be obtained.

By taking first-derivative from the (11) and rearranging, the derivatives $\frac{\partial \boldsymbol{y}}{\partial x_i}$ can be obtained by solving following linear system of equations:

$$\boldsymbol{K}_{\mathrm{R}} \frac{\partial \boldsymbol{y}}{\partial x_i} = \left(\frac{\partial \boldsymbol{F}_{\mathrm{R}}}{\partial x_i} - \frac{\partial \boldsymbol{K}_{\mathrm{R}}}{\partial x_i} \boldsymbol{y} \right) \tag{13}$$

where:

$$\frac{\partial \boldsymbol{F}_{\mathrm{R}}}{\partial x_i} = \frac{\partial \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i} \boldsymbol{F} \tag{14}$$

and

$$\frac{\partial \boldsymbol{K}_{\mathrm{R}}}{\partial x_i} = \frac{\partial \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i} \boldsymbol{K} \boldsymbol{r}_{\mathrm{B}} + \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}} \frac{\partial \boldsymbol{K}}{\partial x_i} \boldsymbol{r}_{\mathrm{B}} + \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}} \boldsymbol{K} \frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i} \tag{15}$$

In (14), it is assumed that the load vector $\boldsymbol{F}$ is independent of design variables (that is, $\frac{\partial \boldsymbol{F}}{\partial x_i} = 0$).

Taking the derivative of (10) with respect to the design variable x_i and rearranging yield:

$$\frac{\partial \boldsymbol{r}_1}{\partial x_i} = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{r}_2}{\partial x_i} = -\frac{\partial \boldsymbol{B}}{\partial x_i} \boldsymbol{r}_1, \frac{\partial \boldsymbol{r}_3}{\partial x_i} = -\frac{\partial \boldsymbol{B}}{\partial x_i} \boldsymbol{r}_2 - \boldsymbol{B} \frac{\partial \boldsymbol{r}_2}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{r}_s}{\partial x_i} = -\frac{\partial \boldsymbol{B}}{\partial x_i} \boldsymbol{r}_{s-1} - \boldsymbol{B} \frac{\partial \boldsymbol{r}_{s-1}}{\partial x_i} \tag{16}$$

where $\frac{\partial \boldsymbol{B}}{\partial x_i} = \boldsymbol{K}_0^{-1} \frac{\partial \Delta \boldsymbol{K}}{\partial x_i}$.

Now, the first-derivative of the approximate displacement vector obtained by the CA method can be simply calculated by substituting $\frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i}$ and $\frac{\partial \boldsymbol{y}}{\partial x_i}$ into (12).

3.3 Second-order sensitivity reanalysis based on the CA method

By differentiation of (12), the second-order sensitivity of the approximate displacement vector provided by the CA method can be obtained as follows:

$$\frac{\partial^2 \tilde{\boldsymbol{r}}_s^{\mathrm{CA}}}{\partial x_i^2} = \frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}}{\partial x_i^2} \boldsymbol{y} + 2 \frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i} \frac{\partial \boldsymbol{y}}{\partial x_i} + \boldsymbol{r}_{\mathrm{B}} \frac{\partial^2 \boldsymbol{y}}{\partial x_i^2} \tag{17}$$

where calculating the derivatives $\frac{\partial^2 \boldsymbol{y}}{\partial x_i^2}$ and $\frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}}{\partial x_i^2}$ are necessary for computing second-order sensitivity of the approximate displacement vector.

By direct differentiating of (13) with respect to a design variable x_i and rearranging, a new linear system of equations is obtained as follows:

$$\boldsymbol{K}_{\mathrm{R}} \frac{\partial^2 \boldsymbol{y}}{\partial x_i^2} = \left(\frac{\partial^2 \boldsymbol{F}_{\mathrm{R}}}{\partial x_i^2} - 2 \frac{\partial \boldsymbol{K}_{\mathrm{R}}}{\partial x_i} \frac{\partial \boldsymbol{y}}{\partial x_i} - \frac{\partial^2 \boldsymbol{K}_{\mathrm{R}}}{\partial x_i^2} \boldsymbol{y} \right) \tag{18}$$

where:

$$\frac{\partial^2 \boldsymbol{F}_{\mathrm{R}}}{\partial x_i^2} = \frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i^2} \boldsymbol{F} \tag{19}$$

and

$$\frac{\partial^2 \boldsymbol{K}_{\mathrm{R}}}{\partial x_i^2} = \frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i^2} \boldsymbol{K} \boldsymbol{r}_{\mathrm{B}} + 2 \frac{\partial \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i} \frac{\partial \boldsymbol{K}}{\partial x_i} \boldsymbol{r}_{\mathrm{B}} + 2 \frac{\partial \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}}}{\partial x_i} \boldsymbol{K} \frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i} + 2 \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}} \frac{\partial \boldsymbol{K}}{\partial x_i} \frac{\partial \boldsymbol{r}_{\mathrm{B}}}{\partial x_i} + \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}} \frac{\partial^2 \boldsymbol{K}}{\partial x_i^2} \boldsymbol{r}_{\mathrm{B}} + \boldsymbol{r}_{\mathrm{B}}^{\mathrm{T}} \boldsymbol{K} \frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}}{\partial x_i^2} \tag{20}$$

As it can be seen from (18), the derivatives $\frac{\partial^2 \boldsymbol{y}}{\partial x_i^2}$ can be obtained by solving a linear system of equations with the size of $s \times s$.

By taking the derivative of (16) with respect to the design variable x_i, the second derivatives of basis vaectors $\frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}}{\partial x_i^2}$ can be written as follows:

$$\frac{\partial^2 \boldsymbol{r}_1}{\partial x_i^2} = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{r}_2}{\partial x_i} = -\frac{\partial^2 \boldsymbol{B}}{\partial x_i^2} \boldsymbol{r}_1, \frac{\partial^2 \boldsymbol{r}_3}{\partial x_i^2} = -\frac{\partial^2 \boldsymbol{B}}{\partial x_i^2} \boldsymbol{r}_2 - 2 \frac{\partial \boldsymbol{B}}{\partial x_i} \frac{\partial \boldsymbol{r}_2}{\partial x_i} - \boldsymbol{B} \frac{\partial^2 \boldsymbol{r}_2}{\partial x_i^2}, \ldots, \frac{\partial^2 \boldsymbol{r}_s}{\partial x_i^2} = -\frac{\partial^2 \boldsymbol{B}}{\partial x_i^2} \boldsymbol{r}_{s-1} - 2 \frac{\partial \boldsymbol{B}}{\partial x_i} \frac{\partial \boldsymbol{r}_{s-1}}{\partial x_i} - \boldsymbol{B} \frac{\partial^2 \boldsymbol{r}_{s-1}}{\partial x_i^2} \tag{21}$$

where $\frac{\partial^2 \boldsymbol{B}}{\partial x_i^2} = \boldsymbol{K}_0^{-1} \frac{\partial^2 \Delta \boldsymbol{K}}{\partial x_i^2}$.

Finally, the second-order derivative of the approximate displacement vector obtained by the CA method $\frac{\partial^2 \tilde{\boldsymbol{r}}_s^{\mathrm{CA}}}{\partial x_i^2}$ can be simply calculated by substituting $\frac{\partial^2 \boldsymbol{r}_{\mathrm{B}}}{\partial x_i^2}$ and $\frac{\partial^2 \boldsymbol{y}}{\partial x_i^2}$ into (17).

4 Proposed sensitivity reanalysis approach

Nowadays, the solutions of many engineering problems can be approximated by a series expansion or a sequence converging to the exact solution. However, approximating the limits of such sequences is not an easy task. In many problems of practical interest, either the convergence of these sequences to their limits is very slow or even divergences are observed, which makes their direct use to approximate their limits computationally expensive or impossible. In mathematical science, one practical way of tackling this problem effectively is by applying to such sequences some convergence *extrapolation methods* (or equivalently *convergence acceleration methods*), which are especially suitable when the dimension of the vector sequences is very large. Usually, an extrapolation method takes a finite or hopefully small number of given sequence and produces another sequence that converges to the former's limit more quickly when this limit exists. In some cases, if the limit of original sequence does not exist, the new sequence produced by the extrapolation methods converges to some meaningful quantities or diverge more slowly than the original sequence (Sidi 2003). In this paper, we use this idea to propose a new structural sensitivity reanalysis approach based on the polynomial extrapolation methods.

Minimal polynomial extrapolation (MPE) method introduced by Cabay and Jackson (1976) and *reduced rank extrapolation* (RRE) method proposed by Kaniel and Stein (1974), Eddy (1979), and Mešina (1977) belong to the category of the polynomial-type vector extrapolation methods. Until now, MPE and RRE methods have been applied successfully as efficient convergence accelerators in various areas of science and engineering (Bertelle et al. 2011; Duminil et al. 2014; Duminil et al. 2015; Loisel and Takane 2011). The convergence and stability analysis of MPE and RRE methods was discussed by Sidi (1986, 1994) and some reviews about these methods are available in Refs. (Sidi 2012; Sidi et al. 1986; Smith et al. 1987). In this section, we show how MPE and RRE methods can be modeled to develop an efficient structural sensitivity reanalysis approach. We only use those equations which will be used directly in the proposed approach. For more information about the derivation and related mathematical proofs, the interested reader may refer to (Sidi 2012).

Back to the load-displacement relation of the modified structure in (5), this equation can be rewritten in terms of the change in the stiffness matrix as follows:

$$(\boldsymbol{K}_0 + \Delta\boldsymbol{K})\boldsymbol{r} = \boldsymbol{F} \tag{22}$$

which can be rearranged to obtain following recurrence formula:

$$\boldsymbol{r}_{n+1} = \boldsymbol{T}\boldsymbol{r}_n + \boldsymbol{b}, \quad \boldsymbol{b}, \boldsymbol{r}_n \in \mathbb{C}^{n_{\mathrm{dof}}}, \boldsymbol{T} \in \mathbb{C}^{n_{\mathrm{dof}} \times n_{\mathrm{dof}}} \tag{23}$$

where

$$\boldsymbol{T} = -\boldsymbol{K}_0^{-1}\Delta\boldsymbol{K}, \ \boldsymbol{b} = \boldsymbol{K}_0^{-1}\boldsymbol{F} \tag{24}$$

In (23), $\boldsymbol{r}_{n+1}$ and $\boldsymbol{r}_n$ indicate the displacement vector of the modified structure at the $(n+1)$th and nth iterations, respectively. Since the decomposed form of the initial stiffness matrix $\boldsymbol{K}_0$ is available, calculating vectors $\boldsymbol{r}_n$ requires only forward and backward substitutions. If (23) is written in the form of $(\boldsymbol{I}-\boldsymbol{T})\boldsymbol{r}=\boldsymbol{b}$, it turns out that the uniqueness of the solution is guaranteed for any nonsingular matrix $\boldsymbol{I}-\boldsymbol{T}$, in which $\boldsymbol{T}$ does not have 1 as its eigenvalue. Let us to assume the unique solution of (23) as $\boldsymbol{r}_{\mathrm{exact}} = \lim_{n\to\infty}\boldsymbol{r}_n$. Now, for any initial vector $\boldsymbol{r}_0$ sufficiently close to $\boldsymbol{r}_{\mathrm{exact}}$ satisfied $\rho\left(\boldsymbol{K}_0^{-1}\Delta\boldsymbol{K}\right) < 1$, (23) converges to the exact displacement vector of the modified structure $\boldsymbol{r}_{\mathrm{exact}}$, where $\rho(\boldsymbol{A})$ is the spectral radius of the square matrix $\boldsymbol{A}$ (Süli and Mayers 2003). If we choose the initial displacement vector $\boldsymbol{r}_0 \in \mathbb{C}^{n_{\mathrm{dof}}}$ as an initial solution vector to (23), the vector sequence $\{\boldsymbol{r}_n\}$ can be generated as follows:

$$\boldsymbol{r}_{n+1} = \boldsymbol{T}\boldsymbol{r}_n + \boldsymbol{b}, \qquad n = 0, 1, \ldots \tag{25}$$

Let us also define

$$\boldsymbol{u}_n = \boldsymbol{r}_{n+1} - \boldsymbol{r}_n, \qquad n = 0, 1, \ldots, \quad \boldsymbol{u}_n \in \mathbb{C}^{n_{\mathrm{dof}}} \tag{26}$$

$$\boldsymbol{w}_n = \boldsymbol{u}_{n+1} - \boldsymbol{u}_n, \qquad n = 0, 1, \ldots, \quad \boldsymbol{w}_n \in \mathbb{C}^{n_{\mathrm{dof}}} \tag{27}$$

In the following subsections, a new sensitivity reanalysis formulation is derived based on the MPE and RRE methods, separately.

4.1 Sensitivity reanalysis based on the minimal polynomial extrapolation (MPE)

4.1.1 Approximate modified displacement vector

Consider vector sequence $\{\boldsymbol{r}_n\}$ in $\mathbb{C}^{n_{\mathrm{dof}}}$ and let us choose k to be an arbitrary positive integer that is usually much smaller than the total number of DOFs of the structure ($k \ll n_{\mathrm{dof}}$). Then, form the matrix $\boldsymbol{U}_{k-1}$ as follows:

$$\boldsymbol{U}_{k-1} = [\boldsymbol{u}_0 | \boldsymbol{u}_1 | \cdots | \boldsymbol{u}_{k-1}] \in \mathbb{C}^{n_{\mathrm{dof}} \times k} \tag{28}$$

where $\boldsymbol{u}_n$ is defined in (26). Let us now imagine that $\boldsymbol{c}' = [c_0, c_1, \cdots, c_{k-1}]^{\mathrm{T}}$ represents the least-square solution of the following overdetermined linear system:

$$\boldsymbol{U}_{k-1}\boldsymbol{c}' = -\boldsymbol{u}_k; \qquad \boldsymbol{c}' \in \mathbb{C}^k \tag{29}$$

where $\boldsymbol{c}'$ can be defined as a solution of the following optimization problem:

$$\min_{c_0, c_1, \cdots, c_{k-1}} \left\| \sum_{j=0}^{k-1} c_j \boldsymbol{u}_j + \boldsymbol{u}_k \right\| \tag{30}$$

By setting $c_k = 1$, $\gamma_0^{\mathrm{MPE}}, \gamma_1^{\mathrm{MPE}}, \cdots, \gamma_k^{\mathrm{MPE}}$ can be calculated as follows:

$$\gamma_j^{\mathrm{MPE}} = \frac{c_j}{\sum_{l=0}^{k} c_l}, \quad j = 0, 1, \cdots, k \tag{31}$$

It should be noted that $\sum_{l=0}^{k} c_l \neq 0$ (Sidi 2012). Finally, the MPE approximation to the displacement vector of the modified structure $\widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}$ is calculated as follows:

$$\widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}} = \sum_{j=0}^{k} \gamma_j^{\mathrm{MPE}} \boldsymbol{r}_j \tag{32}$$

where $\boldsymbol{r}_j$ is the displacement vector of the modified structure at the jth iteration yielded by (25).

4.1.2 First-order sensitivity of the modified displacement vector based on the MPE method

By differentiation of (32), the first-order sensitivity of the approximate displacement vector provided by the MPE method can be obtained as follows:

$$\frac{\partial \tilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial x_i} = \sum_{j=0}^{k} \left(\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i} \boldsymbol{r}_j + \gamma_j^{\mathrm{MPE}} \frac{\partial \boldsymbol{r}_j}{\partial x_i} \right) \tag{33}$$

As it is observed from (33), computation of derivatives $\frac{\partial \boldsymbol{r}_j}{\partial x_i}$ and $\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$ is necessary for calculating the first-order derivatives of the approximate displacement vector. In the following subsections, we obtain these derivatives.

First-order derivatives of vector sequences $\boldsymbol{r}_n$ If it is assumed that the load vector $\boldsymbol{F}$ is independent of design variables (that is, $\frac{\partial \boldsymbol{F}}{\partial x_i} = 0$), the first-order derivatives of vector sequences $\{\boldsymbol{r}_n\}$ in $\mathbb{C}^{n_{dof}}$ are obtained as follows:

$$\frac{\partial \boldsymbol{r}_{n+1}}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i} \boldsymbol{r}_n + \boldsymbol{T} \frac{\partial \boldsymbol{r}_n}{\partial x_i}, \qquad n = 0, 1, \tag{34}$$

where

$$\frac{\partial \boldsymbol{T}}{\partial x_i} = -\boldsymbol{K}_0^{-1} \frac{\partial \Delta \boldsymbol{K}}{\partial x_i} = -\boldsymbol{K}_0^{-1} \frac{\partial \boldsymbol{K}}{\partial x_i} \tag{35}$$

In (35), it is assumed that the first-order derivatives of initial displacement vector $\boldsymbol{r}_0$ and stiffness matrice $\boldsymbol{K}_0$ are equal to zero. So, we can write:

$$\frac{\partial \boldsymbol{r}_0}{\partial x_i} = 0, \quad \frac{\partial \boldsymbol{r}_1}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i} \boldsymbol{r}_0, \quad \frac{\partial \boldsymbol{r}_2}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i} \boldsymbol{r}_1 + \boldsymbol{T} \frac{\partial \boldsymbol{r}_1}{\partial x_i}, \quad \frac{\partial \boldsymbol{r}_3}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i} \boldsymbol{r}_2 + \boldsymbol{T} \frac{\partial \boldsymbol{r}_2}{\partial x_i}, \cdots, \frac{\partial \boldsymbol{r}_k}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i} \boldsymbol{r}_{k-1} + \boldsymbol{T} \frac{\partial \boldsymbol{r}_{k-1}}{\partial x_i} \tag{36}$$

Calculating the derivatives $\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$ By taking first derivative from the (31), the derivatives $\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$ can be obtained as follows:

$$\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i} = \frac{\frac{\partial c_j}{\partial x_i} \left(\sum_{l=0}^{k} c_l\right) - \left(\sum_{l=0}^{k} \frac{\partial c_l}{\partial x_i}\right) c_j}{\left(\sum_{l=0}^{k} c_l\right)^2}, \quad j = 0, 1, \cdots, k \tag{37}$$

where only the derivatives $\frac{\partial \boldsymbol{c}}{\partial x_i}$ are unknown.

By differentiating the overdetermined linear system in (29) and rearranging, a new overdetermined linear system is obtained as follows:

$$\boldsymbol{U}_{k-1} \frac{\partial \boldsymbol{c}'}{\partial x_i} = -\left(\frac{\partial \boldsymbol{u}_k}{\partial x_i} + \frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i} \boldsymbol{c}' \right); \quad \frac{\partial \boldsymbol{c}'}{\partial x_i} = \left[\frac{\partial c_0}{\partial x_i}, \frac{\partial c_1}{\partial x_i}, \cdots, \frac{\partial c_{k-1}}{\partial x_i} \right]^T \tag{38}$$

where $\frac{\partial \boldsymbol{u}_k}{\partial x_i} \in \mathbb{C}^{n_{\mathrm{dof}}}$ and $\frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i} \in \mathbb{C}^{n_{\mathrm{dof}} \times k}$ are given from:

$$\frac{\partial \boldsymbol{u}_k}{\partial x_i} = \frac{\partial \boldsymbol{r}_{k+1}}{\partial x_i} - \frac{\partial \boldsymbol{r}_k}{\partial x_i} \tag{39}$$

$$\frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i} = \left[\frac{\partial \boldsymbol{u}_0}{\partial x_i} \middle| \frac{\partial \boldsymbol{u}_1}{\partial x_i} \middle| \cdots \middle| \frac{\partial \boldsymbol{u}_{k-1}}{\partial x_i} \right] \in \mathbb{C}^{n_{\mathrm{dof}} \times k} \tag{40}$$

From (38), it can be seen that the derivatives $\frac{\partial \boldsymbol{c}'}{\partial x_i} \in \mathbb{C}^k$ are the least-square solution of an overdetermined linear system. After calculating $\frac{\partial \boldsymbol{c}'}{\partial x_i}$, the derivatives $\frac{\partial \boldsymbol{c}}{\partial x_i} \in \mathbb{C}^{k+1}$ can be written as

$$\frac{\partial \boldsymbol{c}}{\partial x_i} = \left[\frac{\partial \boldsymbol{c}'}{\partial x_i} \ 0 \right] \tag{41}$$

Now, the derivatives $\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$ can be easily calculated by substituting (41) in (37).

4.1.3 Second-order sensitivity of the modified displacement vector based on the MPE method

By differentiation of (33), the second-order sensitivity of the approximate displacement vector provided by the MPE method can be obtained as follows:

$$\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial x_i^2} = \sum_{j=0}^{k} \left(\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2} \boldsymbol{r}_j + 2 \frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i} \frac{\partial \boldsymbol{r}_j}{\partial x_i} + \gamma_j^{\mathrm{MPE}} \frac{\partial^2 \boldsymbol{r}_j}{\partial x_i^2} \right) \tag{42}$$

As it is observed from (42), computation of derivatives $\frac{\partial^2 \boldsymbol{r}_j}{\partial x_i^2}$ and $\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2}$ is necessary for calculating the second-order derivatives of the approximate displacement vector. In the following subsections, we obtain these derivatives.

Second-order derivatives of vector sequences $\boldsymbol{r}_n$ By differentiation of (34), the second-order derivatives of vector sequences $\{\boldsymbol{r}_n\}$ can be obtained as follows:

$$\frac{\partial^2 \boldsymbol{r}_{n+1}}{\partial x_i^2} = 2 \frac{\partial \boldsymbol{T}}{\partial x_i} \frac{\partial \boldsymbol{r}_n}{\partial x_i} + \frac{\partial^2 \boldsymbol{T}}{\partial x_i^2} \boldsymbol{r}_n + \boldsymbol{T} \frac{\partial^2 \boldsymbol{r}_n}{\partial x_i^2}, \qquad n = 0, 1, \tag{43}$$

where:

$$\frac{\partial^2 \boldsymbol{T}}{\partial x_i^2} = -\boldsymbol{K}_0 \frac{\partial^2 \Delta \boldsymbol{K}}{\partial x_i^2} = -\boldsymbol{K}_0 \frac{\partial^2 \boldsymbol{K}}{\partial x_i^2} \tag{44}$$

So we can write:

$$\frac{\partial^2 \boldsymbol{r}_0}{\partial x_i^2} = 0, \quad \frac{\partial^2 \boldsymbol{r}_1}{\partial x_i^2} = \frac{\partial^2 \boldsymbol{T}}{\partial x_i^2} \boldsymbol{r}_0, \quad \frac{\partial^2 \boldsymbol{r}_2}{\partial x_i^2} = 2 \frac{\partial \boldsymbol{T}}{\partial x_i} \frac{\partial \boldsymbol{r}_1}{\partial x_i} + \frac{\partial^2 \boldsymbol{T}}{\partial x_i^2} \boldsymbol{r}_1 + \boldsymbol{T} \frac{\partial^2 \boldsymbol{r}_1}{\partial x_i^2}, \ldots, \quad \frac{\partial^2 \boldsymbol{r}_k}{\partial x_i^2} = 2 \frac{\partial \boldsymbol{T}}{\partial x_i} \frac{\partial \boldsymbol{r}_{k-1}}{\partial x_i} + \frac{\partial^2 \boldsymbol{T}}{\partial x_i^2} \boldsymbol{r}_{k-1} + \boldsymbol{T} \frac{\partial^2 \boldsymbol{r}_{k-1}}{\partial x_i^2} \tag{45}$$

Calculating the derivatives $\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2}$ By differentiation of (37), the derivatives $\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2}$ can be obtained as follows:

$$\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2} = \frac{\frac{\partial^2 c_j}{\partial x_i^2}\left(\sum_{l=0}^k c_l\right)^2 - \left(\sum_{l=0}^k \frac{\partial^2 c_l}{\partial x_i^2}\right) c_j \left(\sum_{l=0}^k c_l\right) - 2\left(\sum_{l=0}^k c_l\right)\left(\sum_{l=0}^k \frac{\partial c_l}{\partial x_i}\right)\frac{\partial c_j}{\partial x_i} + 2\left(\sum_{l=0}^k \frac{\partial c_l}{\partial x_i}\right)^2 c_j}{\left(\sum_{i=0}^k c_l\right)^3} \tag{46}$$

where the derivatives $\frac{\partial^2 \boldsymbol{c}}{\partial x_i^2}$ are unknown. Differentiating (38) and rearranging gives:

$$\boldsymbol{U}_{k-1}\frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2} = -\left(\frac{\partial^2 \boldsymbol{u}_k}{\partial x_i^2} + \frac{\partial^2 \boldsymbol{U}_{k-1}}{\partial x_i^2}\boldsymbol{c}' + 2\frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i}\frac{\partial \boldsymbol{c}'}{\partial x_i}\right); \quad \frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2} = \left[\frac{\partial^2 c_0}{\partial x_i^2}, \frac{\partial^2 c_1}{\partial x_i^2}, \cdots, \frac{\partial^2 c_{k-1}}{\partial x_i^2}\right]^{\mathrm{T}} \tag{47}$$

where $\frac{\partial \boldsymbol{c}'}{\partial x_i}$ is given from (38). Hence, the $\frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2} \in \mathbb{C}^{n_{\mathrm{dof}} \times k}$ is the least-square solution of the linear system in (47). Then, derivatives $\frac{\partial^2 \boldsymbol{c}}{\partial x_i^2} \in \mathbb{C}^{n_{\mathrm{dof}} \times k+1}$ can be obtained as follows:

$$\frac{\partial^2 \boldsymbol{c}}{\partial x_i^2} = \left[\frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2} \;\; 0\right] \tag{48}$$

Now, the derivatives $\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2}$ can be easily calculated by substituting (48) in (46).

4.2 Sensitivity reanalysis based on the reduced rank extrapolation (RRE)

4.2.1 Approximate modified displacement vector

Again, consider vector sequence $\{\boldsymbol{r}_n\}$ generated from (25) and let us choose k to be an arbitrary positive integer that is usually much smaller than the total number of DOFs of the structure ($k \ll n_{\mathrm{dof}}$). Then, form the matrix $\boldsymbol{U}_k$ as follows:

$$\boldsymbol{U}_k = [\boldsymbol{u}_0|\boldsymbol{u}_1|\cdots|\boldsymbol{u}_k] \in \mathbb{C}^{n_{\mathrm{dof}} \times k+1} \tag{49}$$

where $\boldsymbol{u}_n$ is defined in (26). Let us now imagine that γ^{RRE} represents the least-square solution of the following overdetermined linear system:

$$\boldsymbol{U}_k \boldsymbol{\gamma}^{\mathrm{RRE}} = \boldsymbol{0}; \quad \boldsymbol{\gamma}^{\mathrm{RRE}} = \left[\gamma_0^{\mathrm{RRE}}, \gamma_1^{\mathrm{RRE}}, \cdots, \gamma_k^{\mathrm{RRE}}\right]^{\mathrm{T}} \in \mathbb{C}^{k+1} \tag{50}$$

Besides, $\sum_{j=0}^k \gamma_j^{\mathrm{RRE}} = 1$ is considered as a constraint for (50). $\boldsymbol{\gamma}^{\mathrm{RRE}}$ can also be expressed as a solution of the following constrained optimization problem:

$$\min_{\gamma_0, \gamma_1, \cdots, \gamma_k} \left\| \sum_{j=0}^k \gamma_j^{\mathrm{RRE}} \boldsymbol{u}_j \right\|$$

$$\text{Subject to}: \sum_{j=0}^k \gamma_j^{\mathrm{RRE}} = 1 \tag{51}$$

Finally, the RRE approximation to the displacement vector of the modified structure $\widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}$ is calculated as follows:

$$\widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}} = \sum_{j=0}^k \gamma_j^{\mathrm{RRE}} \boldsymbol{r}_j \tag{52}$$

where $\boldsymbol{r}_j$ is the displacement vector of the modified structure at the jth iteration yielded by (25).

It should be noted that the definition of the RRE presented above is not the only way possible. Another definition of the RRE method is also given in Ref. (Sidi 2012), which is more suitable for computational purposes. According to the definition in Ref. (Sidi 2012), the RRE approximation to $\boldsymbol{r}_{\mathrm{exact}} = \lim_{n\to\infty} \boldsymbol{r}_n$ can also be expressed in the following form:

$$\widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}} = \boldsymbol{r}_0 + \sum_{i=0}^{k-1} \xi_i \boldsymbol{u}_i \tag{53}$$

where there are no any constraint on the ξ_i. In (53), the parameters ξ_i are obtained from the following least-square solution of the overdetermined linear system:

$$\boldsymbol{W}_{k-1}\boldsymbol{\xi} = -\boldsymbol{u}_0; \quad \boldsymbol{\xi} = [\xi_0, \xi_1, \cdots, \xi_{k-1}]^{\mathrm{T}} \in \mathbb{C}^k \tag{54}$$

where

$$\boldsymbol{W}_{k-1} = [\boldsymbol{w}_0|\boldsymbol{w}_1|\cdots|\boldsymbol{w}_{k-1}] \in \mathbb{C}^{n_{\mathrm{dof}} \times k} \tag{55}$$

Here, the vectors $\boldsymbol{w}_n$ are defined in (27).

4.2.2 First-order sensitivity of the modified displacement vector based on the RRE method

By differentiation of (53), the first-order sensitivity of the approximate displacement vector provided by the RRE method can be obtained as follows:

$$\frac{\partial \widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i} = \sum_{i=0}^{k-1}\left(\frac{\partial \xi_i}{\partial x_i}\boldsymbol{u}_i + \xi_i \frac{\partial \boldsymbol{u}_i}{\partial x_i}\right) \tag{56}$$

where

$$\frac{\partial \boldsymbol{u}_i}{\partial x_i} = \frac{\partial \boldsymbol{r}_{i+1}}{\partial x_i} - \frac{\partial \boldsymbol{r}_i}{\partial x_i} \tag{57}$$

In (57), calculation of the derivatives $\frac{\partial \boldsymbol{r}_i}{\partial x_i}$ is quite similar to (36). Hence, only the derivatives $\frac{\partial \xi_i}{\partial x_i}$ are required to calculate the first-order sensitivity of $\frac{\partial \widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i}$.

Taking the derivative of (54) with respect to the design variable x_i and rearranging yield:

$$\boldsymbol{W}_{k-1}\frac{\partial \boldsymbol{\xi}}{\partial x_i} = -\left(\frac{\partial \boldsymbol{u}_0}{\partial x_i} + \frac{\partial \boldsymbol{W}_{k-1}}{\partial x_i}\boldsymbol{\xi}\right) \tag{58}$$

As it can be seen, $\frac{\partial \boldsymbol{\xi}}{\partial x_i}$ is the least-square solution of the linear system in (58).

4.2.3 Second-order sensitivity of the modified displacement vector based on the RRE method

Taking derivative of (56) with respect to the design variable x_i yields the second-order sensitivity of the approximate displacement vector provided by the RRE method as follows:

$$\frac{\partial^2 \widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial {x_i}^2} = \sum_{i=0}^{k-1}\left(\frac{\partial^2 \xi_i}{\partial {x_i}^2}\boldsymbol{u}_i + 2\frac{\partial \xi_i}{\partial x_i}\frac{\partial \boldsymbol{u}_i}{\partial x_i} + \xi_i\frac{\partial^2 \boldsymbol{u}_i}{\partial {x_i}^2}\right) \tag{59}$$

where

$$\frac{\partial^2 \boldsymbol{u}_i}{\partial {x_i}^2} = \frac{\partial^2 \boldsymbol{r}_{i+1}}{\partial {x_i}^2} - \frac{\partial^2 \boldsymbol{r}_i}{\partial {x_i}^2} \tag{60}$$

Table 1 The number of algebraic operations (NAOs) required by the MPE method for the structural sensitivity reanalysis

Structural reanalysis using MPE		
Operation	NAOs	Comments
$\boldsymbol{b} = \boldsymbol{K}_0^{-1}\boldsymbol{F}$	$2{n_{\mathrm{dof}}}^2$	Forward substitution with ${n_{\mathrm{dof}}}^2$ operation and backward substitution with ${n_{\mathrm{dof}}}^2$ operations.
$\boldsymbol{r}_{n+1} = \boldsymbol{T}\boldsymbol{r}_n + \boldsymbol{b}$	$(k+1)(4{n_{\mathrm{dof}}}^2 + 2n_{\mathrm{dof}})$	MPE method requires $k+1$ vector sequences. By using decomposed form of $\boldsymbol{K}_0 = \boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{U}_0$, calculation of each vector sequence requires: • One matrix-vector multiplication with $2{n_{\mathrm{dof}}}^2$ operations. • Forward and backward substitutions with $2{n_{\mathrm{dof}}}^2$ operations. • Negative multiplications with n_{dof} operations. • One vector addition with n_{dof} operations.
$\boldsymbol{U}_{k-1} = [\boldsymbol{u}_0 \| \boldsymbol{u}_1 \| \ldots \| \boldsymbol{u}_{k-1}]$	kn_{dof}	k subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
Solve $\boldsymbol{U}_{k-1}\boldsymbol{c} = -\boldsymbol{u}_k$	$2n_{\mathrm{dof}}k^2$	Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$
$\gamma_j^{\mathrm{MPE}} = \frac{c_j}{\sum_{i=0}^{k} c_i}, j = 0, 1, \ldots, k$	$2(k+1)$	$k+1$ scalar additions and $k+1$ scalar subtractions
$\widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}} = \sum_{j=0}^{k}\gamma_j^{\mathrm{MPE}}\boldsymbol{r}_j$	$2(k+1)n_{\mathrm{dof}}$	$k+1$ vector multiplications and $k+1$ vector additions
First-order structural reanalysis using MPE		
Operation	NAOs	Comments
$\frac{\partial \boldsymbol{r}_{n+1}}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i}\boldsymbol{r}_n + \boldsymbol{T}\frac{\partial \boldsymbol{r}_n}{\partial x_i}$	$(k+1)(8{n_{\mathrm{dof}}}^2 + n_{\mathrm{dof}})$	• Two matrix-vector multiplication with $4{n_{\mathrm{dof}}}^2$ operations. • Two forward and backward substitutions with $4{n_{\mathrm{dof}}}^2$ operations. • One vector addition with n_{dof} operations.
$\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$	$5(k+1)$	$4(k+1)$ scalar additions and $k+1$ scalar subtractions
$\frac{\partial \boldsymbol{u}_0}{\partial x_i}, \frac{\partial \boldsymbol{u}_1}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{u}_k}{\partial x_i}$	$(k+1)n_{\mathrm{dof}}$	$k+1$ subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\boldsymbol{U}_{k-1}\frac{\partial \boldsymbol{c}'}{\partial x_i} = -\left(\frac{\partial \boldsymbol{u}_k}{\partial x_i} + \frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i}\boldsymbol{c}'\right)$	$2n_{\mathrm{dof}}(k^2 + k + 1)$	• One matrix-vector multiplication with $2n_{\mathrm{dof}}k$ operation • One vector addition with n_{dof} operations. • Negative multiplications with n_{dof} operations. • Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$
$\frac{\partial \widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial x_i} = \sum_{j=0}^{k}\left(\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}\boldsymbol{r}_j + \gamma_j^{\mathrm{MPE}}\frac{\partial \boldsymbol{r}_j}{\partial x_i}\right)$	$3(k+1)n_{\mathrm{dof}}$	$2(k+1)$vector multiplications and $(k+1)$ vector additions
Second-order structural reanalysis using MPE		
Operation	NAOs	Comments
$\frac{\partial^2 \boldsymbol{r}_{n+1}}{\partial {x_i}^2} = 2\frac{\partial \boldsymbol{T}}{\partial x_i}\frac{\partial \boldsymbol{r}_n}{\partial x_i} + \frac{\partial^2 \boldsymbol{T}}{\partial {x_i}^2}\boldsymbol{r}_n + \boldsymbol{T}\frac{\partial^2 \boldsymbol{r}_n}{\partial {x_i}^2}$	$(k+1)(12{n_{\mathrm{dof}}}^2 + 3n_{\mathrm{dof}})$	• Three matrix-vector multiplication with $6{n_{\mathrm{dof}}}^2$ operations. • Three forward and backward substitutions with $6{n_{\mathrm{dof}}}^2$ operations. • Two vector addition with $2n_{\mathrm{dof}}$ operations. • One multiplication with n_{dof} operations.
$\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial {x_i}^2}$	$18(k+1)$	$18(k+1)$ scalar additions, multiplications, and subtractions.
$\frac{\partial^2 \boldsymbol{u}_0}{\partial {x_i}^2}, \frac{\partial^2 \boldsymbol{u}_1}{\partial {x_i}^2}, \ldots, \frac{\partial^2 \boldsymbol{u}_k}{\partial {x_i}^2}$	$(k+1)n_{\mathrm{dof}}$	$k+1$ subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\boldsymbol{U}_{k-1}\frac{\partial^2 \boldsymbol{c}'}{\partial {x_i}^2} = -\left(\frac{\partial^2 \boldsymbol{u}_k}{\partial {x_i}^2} + \frac{\partial^2 \boldsymbol{U}_{k-1}}{\partial {x_i}^2}\boldsymbol{c}' + 2\frac{\partial \boldsymbol{U}_{k-1}}{\partial x_i}\frac{\partial \boldsymbol{c}'}{\partial x_i}\right)$	$n_{\mathrm{dof}}(2k^2 + 4k + 3)$	• Two matrix-vector multiplications with $4n_{\mathrm{dof}}k$ operation • Two vector additions with $2n_{\mathrm{dof}}$ operations. • Negative multiplications with n_{dof} operations. • Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$.
$\frac{\partial^2 \widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial {x_i}^2} = \sum_{j=0}^{k}\left(\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial {x_i}^2}\boldsymbol{r}_j + 2\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}\frac{\partial \boldsymbol{r}_j}{\partial x_i} + \gamma_j^{\mathrm{MPE}}\frac{\partial^2 \boldsymbol{r}_j}{\partial {x_i}^2}\right)$	$5(k+1)n_{\mathrm{dof}}$	$3(k+1)$vector multiplications and $2(k+1)$ vector additions.
Total:$(24k+26){n_{\mathrm{dof}}}^2$+$(6k^2+25k+21)n_{\mathrm{dof}} + 25(k+1)$		

Table 2 The number of algebraic operations (NAOs) required by the RRE method for the structural sensitivity reanalysis

Operation	NAOs	Comments
Structural reanalysis using RRE		
$\boldsymbol{b} = \boldsymbol{K}_0^{-1}\boldsymbol{F}$	$2n_{\mathrm{dof}}^2$	Forward substitution with n_{dof}^2 operations and backward substitution with n_{dof}^2 operations.
$\boldsymbol{r}_{n+1} = \boldsymbol{T}\boldsymbol{r}_n + \boldsymbol{b}$	$(k+1)(4n_{\mathrm{dof}}^2 + 2n_{\mathrm{dof}})$	RRE method requires $k+1$ vector sequences. By using decomposed form of $\boldsymbol{K}_0 = \boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{U}_0$, calculation of each vector sequence requires: • One matrix-vector multiplication with $2n_{\mathrm{dof}}^2$ operations. • Forward and backward substitutions with $2n_{\mathrm{dof}}^2$ operations. • Negative multiplications with n_{dof} operations. • One vector addition with n_{dof} operations.
$\boldsymbol{U}_k = [\boldsymbol{u}_0 \vert \boldsymbol{u}_1 \vert \ldots \vert \boldsymbol{u}_k]$	$(k+1)n_{\mathrm{dof}}$	$k+1$ subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\boldsymbol{W}_{k-1} = [\boldsymbol{w}_0 \vert \boldsymbol{w}_1 \vert \cdots \vert \boldsymbol{w}_{k-1}]$	kn_{dof}	k subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
Solve $\boldsymbol{W}_{k-1}\boldsymbol{\xi} = -\boldsymbol{u}_0$	$2n_{\mathrm{dof}}k^2 + n_{\mathrm{dof}}$	• Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$ • Negative multiplications with n_{dof} operations.
$\tilde{\boldsymbol{r}}_k^{\mathrm{RRE}} = \boldsymbol{r}_0 + \sum_{i=0}^{k-1} \xi_i \boldsymbol{u}_i$	$(2k+1)n_{\mathrm{dof}}$	• Scalar-vector multiplications with kn_{dof} operations. • Vector addition with $(k+1)n_{\mathrm{dof}}$ operations.
First-order structural reanalysis using RRE		
$\frac{\partial \boldsymbol{r}_{n+1}}{\partial x_i} = \frac{\partial \boldsymbol{T}}{\partial x_i}\boldsymbol{r}_n + \boldsymbol{T}\frac{\partial \boldsymbol{r}_n}{\partial x_i}$	$(k+1)(8n_{\mathrm{dof}}^2 + n_{\mathrm{dof}})$	• Two matrix-vector multiplication with $4n_{\mathrm{dof}}^2$ operations. • Two forward and backward substitutions with $4n_{\mathrm{dof}}^2$ operations. • One vector addition with n_{dof} operations.
$\frac{\partial \boldsymbol{u}_0}{\partial x_i}, \frac{\partial \boldsymbol{u}_1}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{u}_k}{\partial x_i}$	$(k+1)n_{\mathrm{dof}}$	$k+1$ subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\frac{\partial \boldsymbol{W}_{k-1}}{\partial x_i} = \left[\frac{\partial \boldsymbol{w}_0}{\partial x_i} \middle\vert \frac{\partial \boldsymbol{w}_1}{\partial x_i} \middle\vert \cdots \middle\vert \frac{\partial \boldsymbol{w}_{k-1}}{\partial x_i}\right]$	kn_{dof}	k subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\boldsymbol{W}_{k-1}\frac{\partial \boldsymbol{\xi}}{\partial x_i} = -\left(\frac{\partial \boldsymbol{u}_0}{\partial x_i} + \frac{\partial \boldsymbol{W}_{k-1}}{\partial x_i}\boldsymbol{\xi}\right)$	$2n_{\mathrm{dof}}k^2 + 2n_{\mathrm{dof}}(k+1)$	• Matrix-vector multiplication with $2n_{\mathrm{dof}}k$ operations. • One vector addition with n_{dof} operations. • Negative multiplications with n_{dof} operations. • Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$
$\frac{\partial \tilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i} = \sum_{i=0}^{k-1}\left(\frac{\partial \xi_i}{\partial x_i}\boldsymbol{u}_i + \xi_i\frac{\partial \boldsymbol{u}_i}{\partial x_i}\right)$	$3kn_{\mathrm{dof}}$	• Scalar-vector multiplication with $2n_{\mathrm{dof}}k$ operations. • Vector addition with kn_{dof} operations.
Second-order structural reanalysis using RRE		
$\frac{\partial^2 \boldsymbol{r}_{n+1}}{\partial x_i^2} = 2\frac{\partial \boldsymbol{T}}{\partial x_i}\frac{\partial \boldsymbol{r}_n}{\partial x_i} + \frac{\partial^2 \boldsymbol{T}}{\partial x_i^2}\boldsymbol{r}_n + \boldsymbol{T}\frac{\partial^2 \boldsymbol{r}_n}{\partial x_i^2}$	$(k+1)(12n_{\mathrm{dof}}^2 + 3n_{\mathrm{dof}})$	• Three matrix-vector multiplication with $6n_{\mathrm{dof}}^2$ operations. • Three forward and backward substitutions with $6n_{\mathrm{dof}}^2$ operations. • Two vector addition with $2n_{\mathrm{dof}}$ operations. • One multiplication with n_{dof} operations.
$\frac{\partial^2 \boldsymbol{u}_0}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{u}_1}{\partial x_i^2}, \ldots, \frac{\partial^2 \boldsymbol{u}_k}{\partial x_i^2}$	$(k+1)n_{\mathrm{dof}}$	$k+1$ subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\frac{\partial^2 \boldsymbol{W}_{k-1}}{\partial x_i^2} = \left[\frac{\partial^2 \boldsymbol{w}_0}{\partial x_i^2} \middle\vert \frac{\partial^2 \boldsymbol{w}_1}{\partial x_i^2} \middle\vert \cdots \middle\vert \frac{\partial^2 \boldsymbol{w}_{k-1}}{\partial x_i^2}\right]$	kn_{dof}	k subtractions of vectors with the size of $(n_{\mathrm{dof}} \times 1)$
$\boldsymbol{W}_{k-1}\frac{\partial^2 \boldsymbol{\xi}}{\partial x_i^2} = -\left[\frac{\partial^2 \boldsymbol{u}_0}{\partial x_i^2} + 2\frac{\partial \boldsymbol{W}_{k-1}}{\partial x_i}\frac{\partial \boldsymbol{\xi}}{\partial x_i} + \frac{\partial^2 \boldsymbol{W}_{k-1}}{\partial x_i^2}\boldsymbol{\xi}\right]$	$2k^2n_{\mathrm{dof}} + 4(k+1)n_{\mathrm{dof}}$	• Two matrix-vector multiplication with $4n_{\mathrm{dof}}k$ operations. • Two vector addition with $2n_{\mathrm{dof}}$ operations. • One scalar-vector multiplications with n_{dof} operations. • Negative multiplications with n_{dof} operations. • Least square solution of the overdetermined linear system with size of $(n_{\mathrm{dof}} \times k)$
$\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i^2} = \sum_{i=0}^{k-1}\left(\frac{\partial^2 \xi_i}{\partial x_i^2}\boldsymbol{u}_i + 2\frac{\partial \xi_i}{\partial x_i}\frac{\partial \boldsymbol{u}_i}{\partial x_i} + \xi_i\frac{\partial^2 \boldsymbol{u}_i}{\partial x_i^2}\right)$	$6kn_{\mathrm{dof}}$	$3(k+1)$vector multiplications and $2(k+1)$ vector additions.
Total: $24(k+1)n_{\mathrm{dof}}^2 + (6k^2 + 29k + 17)n_{\mathrm{dof}}$		

It should be noted that the calculation of the derivatives of $\frac{\partial^2 \boldsymbol{r}_i}{\partial x_i^2}$ is quite similar to (45). Hence, only the derivatives $\frac{\partial^2 \xi_i}{\partial x_i^2}$ are required to calculate the second-order sensitivity of $\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i^2}$.

Taking the derivative of (58) with respect to the design variable x_i and rearranging yield:

$$\boldsymbol{W}_{k-1}\frac{\partial^2 \boldsymbol{\xi}}{\partial x_i^2} = -\left[\frac{\partial^2 \boldsymbol{u}_0}{\partial x_i^2} + 2\frac{\partial \boldsymbol{W}_{k-1}}{\partial x_i}\frac{\partial \boldsymbol{\xi}}{\partial x_i} + \frac{\partial^2 \boldsymbol{W}_{k-1}}{\partial x_i^2}\boldsymbol{\xi}\right] \qquad (61)$$

In fact, the derivatives $\frac{\partial^2 \boldsymbol{\xi}}{\partial x_i^2}$ are the least-square solutions of the overdetermined linear system in (61).

4.3 Main steps of the proposed sensitivity reanalysis approach

Now, we are ready to summarize the main steps of the proposed reanalysis approach based on the MPE and RRE methods. For a given structural sensitivity reanalysis problem

Table 3 The number of algebraic operations (NAOs) required by the CA method for the structural sensitivity reanalysis

Structural reanalysis using CA		
Operation	NAOs	Comments
$\boldsymbol{r}_s = -\boldsymbol{B}\boldsymbol{r}_{s-1}$ $\boldsymbol{r}_\mathrm{B} = [\boldsymbol{r}_1 \vert \boldsymbol{r}_2 \vert \ldots \vert \boldsymbol{r}_s]$	$s(4n_\mathrm{dof}^2 + n_\mathrm{dof})$	By using decomposed form of $\boldsymbol{K}_0 = \boldsymbol{U}_0^\mathrm{T}\boldsymbol{U}_0$, calculation of each basis vector requires: • Forward and backward substitutions with $2n_\mathrm{dof}^2$ operations. • One matrix-vector multiplication with $2n_\mathrm{dof}^2$ operations. • Negative multiplication with n_dof operations.
$\boldsymbol{K}_\mathrm{R} = \boldsymbol{r}_\mathrm{B}^\mathrm{T}\boldsymbol{K}\boldsymbol{r}_\mathrm{B}$	$2sn_\mathrm{dof}^2 + 2s^2 n_\mathrm{dof}$	• One matrix-matrix multiplication between two matrices with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times n_\mathrm{dof})$. • One matrix-matrix multiplication between two matrices with the sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times s)$, respectively.
$\boldsymbol{F}_\mathrm{R} = \boldsymbol{r}_\mathrm{B}^\mathrm{T}\boldsymbol{F}$	$2sn_\mathrm{dof}$	One matrix-vector multiplication between a matrix and a vector with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times 1)$, respectively.
Solve $\boldsymbol{K}_\mathrm{R}\boldsymbol{y} = \boldsymbol{F}_\mathrm{R}$	$\frac{1}{3}s^3 + 2s^2$	Solving $(s \times s)$ system of equations by Cholesky decomposition method
$\boldsymbol{r} = \boldsymbol{r}_\mathrm{B}\boldsymbol{y}$	$2sn_\mathrm{dof}$	One matrix-vector multiplication between a matrix and a vector with the sizes of $(n_\mathrm{dof} \times s)$ and $(s \times 1)$, respectively.
First-order structural reanalysis using CA		
Operation	NAOs	Comments
$\frac{\partial \boldsymbol{r}_s}{\partial x_i} = -\left(\frac{\partial \boldsymbol{B}}{\partial x_i}\boldsymbol{r}_{s-1} + \boldsymbol{B}\frac{\partial \boldsymbol{r}_{s-1}}{\partial x_i}\right)$ $\frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i} = \left[\frac{\partial \boldsymbol{r}_1}{\partial x_i}\middle\vert\frac{\partial \boldsymbol{r}_2}{\partial x_i}\middle\vert\ldots\middle\vert\frac{\partial \boldsymbol{r}_s}{\partial x_i}\right]$	$s(8n_\mathrm{dof}^2 + 2n_\mathrm{dof})$	• Two matrix-vector multiplications with $4n_\mathrm{dof}^2$ operations. • Two forward and backward substitutions with $4n_\mathrm{dof}^2$ operations. • One vector addition with n_dof operations. • One negative multiplication with n_dof operations
$\frac{\partial \boldsymbol{K}_\mathrm{R}}{\partial x_i} = \frac{\partial \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial x_i}\boldsymbol{K}\boldsymbol{r}_\mathrm{B} + \boldsymbol{r}_\mathrm{B}^\mathrm{T}\frac{\partial \boldsymbol{K}}{\partial x_i}\boldsymbol{r}_\mathrm{B} + \boldsymbol{r}_\mathrm{B}^\mathrm{T}\boldsymbol{K}\frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i}$	$6sn_\mathrm{dof}^2 + 6s^2 n_\mathrm{dof} + 2s^2$	• Three matrix-matrix multiplications between two matrices with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times n_\mathrm{dof})$. • Three matrix-matrix multiplications between two matrices with the sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times s)$, respectively. • Two matrix-matrix additions with the size of $(s \times s)$.
$\frac{\partial \boldsymbol{F}_\mathrm{R}}{\partial x_i} = \frac{\partial \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial x_i}\boldsymbol{F}$	$2sn_\mathrm{dof}$	One matrix-vector multiplication between a matrix and a vector with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times 1)$, respectively.
$\boldsymbol{K}_\mathrm{R}\frac{\partial \boldsymbol{y}}{\partial x_i} = \frac{\partial \boldsymbol{F}_\mathrm{R}}{\partial x_i} - \frac{\partial \boldsymbol{K}_\mathrm{R}}{\partial x_i}\boldsymbol{y}$	$\frac{1}{3}s^3 + 4s^2 + s$	• One matrix-vector multiplication with $2s^2$ operations. • One vector subtraction with s operations. • Solving $(s \times s)$ system of equations by Cholesky decomposition method.
$\frac{\partial \boldsymbol{r}_\mathrm{CA}}{\partial x_i} = \frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i}\boldsymbol{y} + \boldsymbol{r}_\mathrm{B}\frac{\partial \boldsymbol{y}}{\partial x_i}$	$(4s+1)n_\mathrm{dof}$	• Two matrix-vector multiplications between a matrix and a vector with the sizes of $(n_\mathrm{dof} \times s)$ and $(s \times 1)$, respectively. • One vector addition with n_dof operations.
Second-order structural reanalysis using CA		
Operation	NAOs	Comments
$\frac{\partial^2 \boldsymbol{r}_s}{\partial {x_i}^2} = 2\frac{\partial \boldsymbol{B}}{\partial x_i}\frac{\partial \boldsymbol{r}_{s-1}}{\partial x_i} + \frac{\partial^2 \boldsymbol{B}}{\partial {x_i}^2}\boldsymbol{r}_{s-1} + \boldsymbol{B}\frac{\partial^2 \boldsymbol{r}_{s-1}}{\partial {x_i}^2}$ $\frac{\partial^2 \boldsymbol{r}_\mathrm{B}}{\partial {x_i}^2} = \left[\frac{\partial^2 \boldsymbol{r}_1}{\partial {x_i}^2}\middle\vert\frac{\partial^2 \boldsymbol{r}_2}{\partial {x_i}^2}\middle\vert\ldots\middle\vert\frac{\partial^2 \boldsymbol{r}_s}{\partial {x_i}^2}\right]$	$12sn_\mathrm{dof}^2 + 3sn_\mathrm{dof}$	• Three matrix-vector multiplications with $6n_\mathrm{dof}^2$ operations. • Three forward and backward substitutions with $6n_\mathrm{dof}^2$ operations. • Two vector additions with $2n_\mathrm{dof}$ operations. • One multiplication with n_dof operations.
$\frac{\partial^2 \boldsymbol{K}_\mathrm{R}}{\partial {x_i}^2} = 2\frac{\partial \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial x_i}\boldsymbol{K}\frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i} + 2\frac{\partial \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial x_i}\frac{\partial \boldsymbol{K}}{\partial x_i}\boldsymbol{r}_\mathrm{B} + 2\boldsymbol{r}_\mathrm{B}^\mathrm{T}\frac{\partial \boldsymbol{K}}{\partial x_i}\frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i} + \frac{\partial^2 \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial {x_i}^2}\boldsymbol{K}\boldsymbol{r}_\mathrm{B} + \boldsymbol{r}_\mathrm{B}^\mathrm{T}\boldsymbol{K}\frac{\partial^2 \boldsymbol{r}_\mathrm{B}}{\partial {x_i}^2}$	$10sn_\mathrm{dof}^2 + 10s^2 n_\mathrm{dof} + 7s^2$	• Five matrix-matrix multiplications between two matrices with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times n_\mathrm{dof})$. • Five matrix-matrix multiplications between two matrices with the sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times s)$, respectively. • Four matrix-matrix additions with the sizes of $(s \times s)$.
$\frac{\partial^2 \boldsymbol{F}_\mathrm{R}}{\partial {x_i}^2} = \frac{\partial^2 \boldsymbol{r}_\mathrm{B}^\mathrm{T}}{\partial {x_i}^2}\boldsymbol{F}$	$2sn_\mathrm{dof}$	One matrix-vector multiplication between a matrix and a vector with sizes of $(s \times n_\mathrm{dof})$ and $(n_\mathrm{dof} \times 1)$, respectively.
$\boldsymbol{K}_\mathrm{R}\frac{\partial^2 \boldsymbol{y}}{\partial {x_i}^2} = -\left(\frac{\partial^2 \boldsymbol{K}_\mathrm{R}}{\partial {x_i}^2}\boldsymbol{y} + 2\frac{\partial \boldsymbol{K}_\mathrm{R}}{\partial x_i}\frac{\partial \boldsymbol{y}}{\partial x_i} - \frac{\partial^2 \boldsymbol{F}_\mathrm{R}}{\partial {x_i}^2}\right)$	$\frac{1}{3}s^3 + 4s^2 + 4s$	• Two matrix-vector multiplications with $4s^2$ operations.• One vector-scalar multiplication with s operations. • One vector addition with s operations.One vector subtraction with s operations. • One negative multiplication with s operations. • Solving $(s \times s)$ system of equations by Cholesky decomposition method.
$\frac{\partial^2 \boldsymbol{r}_\mathrm{CA}}{\partial {x_i}^2} = \boldsymbol{r}_\mathrm{B}\frac{\partial^2 \boldsymbol{y}}{\partial {x_i}^2} + 2\frac{\partial \boldsymbol{r}_\mathrm{B}}{\partial x_i}\frac{\partial \boldsymbol{y}}{\partial x_i} + \frac{\partial^2 \boldsymbol{r}_\mathrm{B}}{\partial {x_i}^2}\boldsymbol{y}$	$(6s+3)n_\mathrm{dof}$	• Three matrix-vector multiplications between a matrix and a vector with the sizes of $(n_\mathrm{dof} \times s)$ and $(s \times 1)$, respectively. • One scalar multiplication with n_dof operations. • Two vector addition with $2n_\mathrm{dof}$ operations.
Total: $42sn_\mathrm{dof}^2 + (18s^2 + 24s + 4)n_\mathrm{dof} + s^3 + 17s^2 + 5s$		

with the initial displacement vector $\boldsymbol{r}_0$ and initial stiffness matrix $\boldsymbol{K}_0$, we summarize the main steps of the proposed sensitivity reanalysis approach as follows:

- **MPE: Structural reanalysis**
- **Step 1:** Choose k as an arbitrary positive integer that is usually much smaller than the total number of DOFs of the

structure ($k \ll n_{\mathrm{dof}}$) and construct the vectors $\boldsymbol{r}_0, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{k+1}$ by (25).

- Step 2: Obtain the vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k-1}$ by (26) and calculate the matrix
  $\boldsymbol{U}_{k-1} = [\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{k-1}]$ with the size of $n_{\mathrm{dof}} \times k$.
- **Step 3:** Solve the overdetermined linear system of (29) with the size of $n_{\mathrm{dof}} \times k$ in the least square sense and calculate $\boldsymbol{c}' = [c_0, c_1, \cdots, c_{k-1}]^{\mathrm{T}}$.
- **Step 4:** Calculate $\gamma_0^{\mathrm{MPE}}, \gamma_1^{\mathrm{MPE}}, \cdots, \gamma_k^{\mathrm{MPE}}$ by (31) with $c_k = 1$.
- **Step 5:** Compute the approximate displacement vector of the modified structure $\widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}$ by (32).
- **MPE: First-order sensitivity reanalysis**
- **Step 1:** Calculate the first-order derivatives of vector sequences $\frac{\partial \boldsymbol{r}_0}{\partial x_i}, \frac{\partial \boldsymbol{r}_1}{\partial x_i}, \frac{\partial \boldsymbol{r}_2}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{r}_k}{\partial x_i}$ by using (36).
- **Step 2:** Calculate the derivatives $\frac{\partial \boldsymbol{c}'}{\partial x_i} = \left[\frac{\partial c_0}{\partial x_i}, \frac{\partial c_1}{\partial x_i}, \cdots, \frac{\partial c_{k-1}}{\partial x_i}\right]^{\mathrm{T}}$ by solving the overdetermined linear system in (38) with the size of $n_{\mathrm{dof}} \times k$ and construct the derivatives $\frac{\partial \boldsymbol{c}}{\partial x_i} = \left[\frac{\partial \boldsymbol{c}'}{\partial x_i}\ 0\right]$.
- **Step 3:** Calculate the derivatives $\frac{\partial \gamma_j^{\mathrm{MPE}}}{\partial x_i}$ by using (37).
- **Step 4:** Calculate the MPE approximation of the first-order sensitivity of the displacement vector of the modified structure $\frac{\partial \widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial x_i}$ by using (33).
- **MPE: Second-order sensitivity reanalysis**
- **Step 1:** Calculate the second-order derivatives of vector sequences $\frac{\partial^2 \boldsymbol{r}_0}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{r}_1}{\partial x_i^2}, \ldots, \frac{\partial^2 \boldsymbol{r}_k}{\partial x_i^2}$ by using (45).
- **Step 2:** Calculate the derivatives $\frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2} = \left[\frac{\partial^2 c_0}{\partial x_i^2}, \frac{\partial^2 c_1}{\partial x_i^2}, \cdots, \frac{\partial^2 c_{k-1}}{\partial x_i^2}\right]^{\mathrm{T}}$ by solving the overdetermined linear system in (47) with the size of $n_{\mathrm{dof}} \times k$ and construct the derivatives $\frac{\partial^2 \boldsymbol{c}}{\partial x_i^2} = \left[\frac{\partial^2 \boldsymbol{c}'}{\partial x_i^2}\ 0\right]$.
- **Step 3:** Calculate the derivatives $\frac{\partial^2 \gamma_j^{\mathrm{MPE}}}{\partial x_i^2}$ by using (46).
- **Step 4:** Calculate the MPE approximation of the second-order sensitivity of displacement vector of the modified structure $\frac{\partial^2 \widetilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial x_i^2}$ by using (42).
- **RRE: Structural reanalysis**
- **Step 1:** Choose k as an arbitrary positive integer that is usually much smaller than the total number of DOFs of the structure ($k \ll n_{\mathrm{dof}}$) and construct the vectors $\boldsymbol{r}_0, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_{k+1}$ by (25).
- **Step 2:** Compute the vectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ and $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1}$ by (26) and (27), respectively, and form the matrix $\boldsymbol{W}_{k-1} = [\boldsymbol{w}_0 | \boldsymbol{w}_1 | \cdots | \boldsymbol{w}_{k-1}]$ with the size of $n_{\mathrm{dof}} \times k$.
- **Step 3:** Solve the overdetermined linear system of (54) with the size of $n_{\mathrm{dof}} \times k$ in the least-square sense and calculate $\boldsymbol{\xi} = [\xi_0, \xi_1, \cdots, \xi_{k-1}]^{\mathrm{T}}$.
- **Step 4:** Compute the approximate displacement vector of the modified structure $\widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}$ by (53).
- **RRE: First-order sensitivity reanalysis**
- **Step 1:** Calculate the first-order derivatives of vector sequences $\frac{\partial \boldsymbol{r}_0}{\partial x_i}, \frac{\partial \boldsymbol{r}_1}{\partial x_i}, \frac{\partial \boldsymbol{r}_2}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{r}_k}{\partial x_i}$ by using (36).
- **Step 2:** Calculate the derivatives $\frac{\partial \boldsymbol{\xi}}{\partial x_i} = \left[\frac{\partial \xi_0}{\partial x_i}, \frac{\partial \xi_1}{\partial x_i}, \cdots, \frac{\partial \xi_{k-1}}{\partial x_i}\right]^{\mathrm{T}}$ by solving the overdetermined linear system in (58) with size of $n_{\mathrm{dof}} \times k$.
- **Step 3:** Calculate the derivatives $\frac{\partial \boldsymbol{u}_0}{\partial x_i}, \frac{\partial \boldsymbol{u}_1}{\partial x_i}, \frac{\partial \boldsymbol{u}_2}{\partial x_i}, \ldots, \frac{\partial \boldsymbol{u}_{k-1}}{\partial x_i}$ by using (57).
- **Step 4:** Calculate the RRE approximation of the first-order sensitivity of displacement vector of the modified structure $\frac{\partial \widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i}$ by using (56).
- **RRE: Second-order sensitivity reanalysis**
- **Step 1:** Calculate the second-order derivatives of vector sequences $\frac{\partial^2 \boldsymbol{r}_0}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{r}_1}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{r}_2}{\partial x_i^2}, \ldots, \frac{\partial^2 \boldsymbol{r}_k}{\partial x_i^2}$ by using (45).
- **Step 2:** Calculate the derivatives $\frac{\partial^2 \boldsymbol{u}_0}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{u}_1}{\partial x_i^2}, \frac{\partial^2 \boldsymbol{u}_2}{\partial x_i^2}, \ldots, \frac{\partial^2 \boldsymbol{u}_{k-1}}{\partial x_i^2}$ by using (60).
- **Step 3:** Calculate the derivatives $\frac{\partial^2 \boldsymbol{\xi}}{\partial x_i^2} = \left[\frac{\partial^2 \xi_0}{\partial x_i^2}, \frac{\partial^2 \xi_1}{\partial x_i^2}, \cdots, \frac{\partial^2 \xi_{k-1}}{\partial x_i^2}\right]^{\mathrm{T}}$ by solving the overdetermined linear system in (61) with the size of $n_{\mathrm{dof}} \times k$.
- **Step 4:** Calculate the RRE approximation of the second-order sensitivity of displacement vector of the modified structure $\frac{\partial^2 \widetilde{\boldsymbol{r}}_k^{\mathrm{RRE}}}{\partial x_i^2}$ by using (59).

According to the sensitivity reanalysis formulation derived based on the MPE and RRE methods, it can be concluded that the proposed sensitivity reanalysis methods calculate the approximate sensitivities of the modified structure by solving the linear least-square problems with sizes of the $n_{\mathrm{dof}} \times k$ and $n_{\mathrm{dof}} \times k+1$, respectively, which are much smaller than the

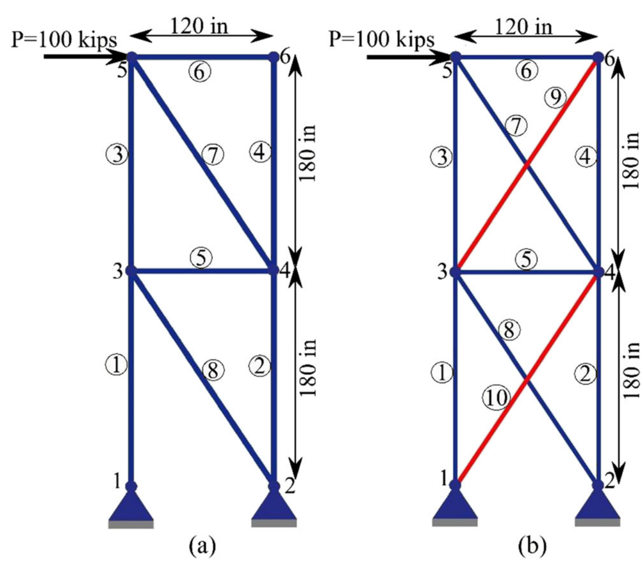

Fig. 1 A simple 10-bar planar truss structure problem: **a** initial structure, **b** modified structure

Table 4 Initial and modified designs for simple 10-bar planar truss structure problem

	Small modifications			Large modifications		
	Initial design (in^2)	Modified design (in^2)	Variations	Initial design (in^2)	Modified design (in^2)	Variations
A_1	5.00	4.50	− 10%	5.00	1.50	− 70%
A_2	5.00	4.50	− 10%	5.00	1.50	− 70%
A_3	5.00	5.50	+ 10%	5.00	8.50	+ 70%
A_4	5.00	5.50	+ 10%	5.00	8.50	+ 70%
A_5	5.00	5.00	0%	5.00	5.00	0%
A_6	5.00	5.50	+ 10%	5.00	8.50	+ 70%
A_7	5.00	5.50	+ 10%	5.00	8.50	+ 70%
A_8	5.00	5.00	0%	5.00	5.00	0%
A_9	–	5.00	Added	–	5.00	Added
A_{10}	–	5.00	Added	–	5.00	Added

complete set of equations of the exact sensitivity analysis with the size of $n_{\mathrm{dof}} \times n_{\mathrm{dof}}$.

5 Numerical tests

5.1 Accuracy

In this section, a set of four structural sensitivity reanalysis problems are presented to examine the accuracy and efficiency of the proposed sensitivity reanalysis methods in calculating the approximate displacement vector as well as its first- and second-order sensitivities. These test examples are a simple 10-bar planar truss structure, a 582-bar tower structure, a 968-bar double layer grid structure, and an 18-bar truss structure. The 10-bar planar truss structure is a simple illustrative test example, which shows how the proposed methods are able to perform structural sensitivity reanalysis. All of the test problems are solved by the CA, MPE, and RRE methods and the results are given in the tables.

Table 5 Approximate displacement and sensitivity vectors obtained by the MPE method with different values of parameter k for 10-bar plane truss structure (small modifications)

k	2	3	4	5	$\boldsymbol{r}_{\mathrm{exact}}$
$\tilde{\boldsymbol{r}}_k^{\mathrm{MPE}}$	0.661747	0.683652	0.682931	0.682932	0.682932
	0.286190	0.300074	0.299616	0.299628	0.299628
	0.665515	0.686557	0.685793	0.685790	0.685790
	− 0.284261	− 0.301129	− 0.300374	− 0.300372	− 0.300372
	1.902144	1.946915	1.946802	1.946811	1.946811
	0.379015	0.387441	0.387585	0.387596	0.387596
	1.867739	1.913913	1.913171	1.913180	1.913180
	− 0.361673	− 0.375384	− 0.376044	− 0.376040	− 0.376040
$\frac{\partial \tilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial A_9}$	− 0.003682	0.001119	0.001074	0.001074	0.001074
	− 0.002769	0.000308	0.000279	0.000279	0.000279
	− 0.005792	− 0.001030	− 0.001074	− 0.001074	− 0.001074
	0.004032	0.000235	0.000279	0.000279	0.000279
	− 0.031386	− 0.020069	− 0.020046	− 0.020045	− 0.020045
	− 0.006519	− 0.004358	− 0.004344	− 0.004343	− 0.004343
	− 0.033980	− 0.022089	− 0.022101	− 0.022100	− 0.022100
	− 0.001804	− 0.004310	− 0.004343	− 0.004343	− 0.004343
$\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\mathrm{MPE}}}{\partial A_9^2}$	0.010533	− 0.000302	− 0.000298	− 0.000298	− 0.000298
	0.006783	− 0.000080	− 0.000077	− 0.000078	− 0.000078
	0.010696	0.000295	0.000298	0.000298	0.000298
	− 0.008454	− 0.000075	− 0.000078	− 0.000078	− 0.000078
	0.027117	0.005573	0.005569	0.005569	0.005569
	0.005215	0.001208	0.001207	0.001207	0.001207
	0.028499	0.006142	0.006140	0.006140	0.006140
	− 0.005345	0.001205	0.001207	0.001207	0.001207

To study the performance of the proposed sensitivity reanalysis procedures, the accuracy of the obtained approximate vectors is evaluated by measuring the relative errors as follows:

$$E_{\text{relative}} = \frac{\left\| \boldsymbol{r}_{\text{exact}} - \widetilde{\boldsymbol{r}} \right\|}{\left\| \boldsymbol{r}_{\text{exact}} \right\|} \tag{62}$$

$$E_{\text{relative}/x_{i,1}} = \frac{\left\| \frac{\partial \boldsymbol{r}_{\text{exact}}}{\partial x_i} - \frac{\partial \widetilde{\boldsymbol{r}}}{\partial x_i} \right\|}{\left\| \frac{\partial \boldsymbol{r}_{\text{exact}}}{\partial x_i} \right\|} \tag{63}$$

$$E_{\text{relative}/x_{i,2}} = \frac{\left\| \frac{\partial^2 \boldsymbol{r}_{\text{exact}}}{\partial {x_i}^2} - \frac{\partial^2 \widetilde{\boldsymbol{r}}}{\partial {x_i}^2} \right\|}{\left\| \frac{\partial^2 \boldsymbol{r}_{\text{exact}}}{\partial {x_i}^2} \right\|} \tag{64}$$

where E_{relative} is the relative error in approximating the displacement vector of the modified structure, $E_{\text{relative}/x_{i,1}}$ indicates the relative error obtained in calculating the first-order sensitivity of the displacement vector, $E_{\text{relative}/x_{i,2}}$ represents the relative error occurred in predicting the second-order sensitivity of the displacement vector, ‖.‖ indicates the L_2 norm, $\boldsymbol{r}_{\text{exact}}$ and $\widetilde{\boldsymbol{r}}$ represent the exact and approximate displacement vectors of the modified structure, respectively, $\frac{\partial \boldsymbol{r}_{\text{exact}}}{\partial x_i}$ and $\frac{\partial \widetilde{\boldsymbol{r}}}{\partial x_i}$ are the exact and approximate first-order sensitivities of the modified structure with respect to the design variable x_i, respectively, $\frac{\partial^2 \boldsymbol{r}_{\text{exact}}}{\partial {x_i}^2}$ and $\frac{\partial^2 \widetilde{\boldsymbol{r}}}{\partial {x_i}^2}$ indicate the exact and approximate second-order sensitivities of the modified structure with respect to the design variable x_i, respectively. It should be noted that the exact displacement vector and its first- and second-order derivative are calculated by solving (5), (6), and (7), respectively, in Matlab software.

As another error measuring criterion, the obtained approximate vectors are also evaluated by the average of errors occurred in each DOF as follows:

$$E_{\text{Av.}} = \frac{\sum_{i=1}^{n_{\text{dof}}} \left(\frac{\boldsymbol{r}_{\text{exact}i} - \widetilde{\boldsymbol{r}}_i}{\boldsymbol{r}_{\text{exact}i}} \right)}{n_{\text{dof}}} \times 100 \tag{65}$$

$$E_{\text{Av.}/x_{i,1}} = \frac{\sum_{i=1}^{n_{\text{dof}}} \left(\frac{\frac{\partial \boldsymbol{r}_{\text{exact}i}}{\partial x_i} - \frac{\partial \widetilde{\boldsymbol{r}}_i}{\partial x_i}}{\frac{\partial \boldsymbol{r}_{\text{exact}i}}{\partial x_i}} \right)}{n_{\text{dof}}} \times 100 \tag{66}$$

Table 6 Approximate displacement and sensitivity vectors obtained by the MPE method with different values of parameter k for 10-bar plane truss structure (large modifications)

k	2	3	4	5	6	$\boldsymbol{r}_{\text{exact}}$
$\widetilde{\boldsymbol{r}}_k^{\text{MPE}}$	0.869582	1.874323	1.573429	1.580613	1.580430	1.580430
	0.463037	1.083837	0.885293	0.898131	0.898277	0.898277
	0.883720	1.887736	1.583853	1.588099	1.588291	1.588291
	− 0.479343	− 1.136202	− 0.901799	− 0.901647	− 0.901723	− 0.901723
	2.727902	5.309613	4.526083	4.549223	4.549508	4.549508
	0.567962	1.106210	0.949034	0.961839	0.961926	0.961926
	2.735470	5.333231	4.507212	4.530467	4.530738	4.530738
	− 0.462316	− 1.083062	− 0.944259	− 0.943849	− 0.943956	− 0.943956
$\frac{\partial \widetilde{\boldsymbol{r}}_k^{\text{MPE}}}{\partial A_9}$	− 0.082195	0.033039	0.001148	0.001247	0.001224	0.001224
	− 0.052577	0.022085	0.000461	0.000518	0.000537	0.000537
	− 0.085923	0.030847	− 0.001243	− 0.001248	− 0.001224	− 0.001224
	0.058466	− 0.024039	0.000529	0.000546	0.000537	0.000537
	− 0.254772	0.066734	− 0.015259	− 0.014944	− 0.014914	− 0.014914
	− 0.053134	0.013971	− 0.002929	− 0.002809	− 0.002799	− 0.002799
	− 0.263374	0.069527	− 0.016764	− 0.016424	− 0.016397	− 0.016397
	0.039112	− 0.017756	− 0.002858	− 0.002784	− 0.002799	− 0.002799
$\frac{\partial^2 \widetilde{\boldsymbol{r}}_k^{\text{MPE}}}{\partial {A_9}^2}$	0.071585	− 0.003326	− 0.000279	− 0.000301	− 0.000296	− 0.000296
	0.045235	− 0.002210	− 0.000112	− 0.000126	− 0.000130	− 0.000130
	0.072259	− 0.002774	0.000301	0.000301	0.000296	0.000296
	− 0.049767	0.002211	− 0.000128	− 0.000132	− 0.000130	− 0.000130
	0.192242	− 0.004275	0.003687	0.003617	0.003610	0.003610
	0.039748	− 0.000960	0.000707	0.000680	0.000677	0.000677
	0.196919	− 0.004328	0.004051	0.003976	0.003969	0.003969
	− 0.039244	0.002091	0.000690	0.000675	0.000677	0.000677

Table 7 Approximate displacement and sensitivity vectors obtained by the RRE method with different values of parameter k for 10-bar plane truss structure (small modifications)

k	2	3	4	5	$\boldsymbol{r}_{\text{exact}}$
$\tilde{\boldsymbol{r}}_k^{\text{RRE}}$	0.662887	0.683611	0.682931	0.682932	0.682932
	0.285963	0.300046	0.299616	0.299628	0.299628
	0.666666	0.686517	0.685793	0.685790	0.685790
	−0.282725	−0.301092	−0.300374	−0.300372	−0.300372
	1.899099	1.946820	1.946802	1.946811	1.946811
	0.378409	0.387423	0.387585	0.387596	0.387596
	1.864283	1.913814	1.913171	1.913180	1.913180
	−0.361060	−0.375356	−0.376044	−0.376040	−0.376040
$\frac{\partial \tilde{\boldsymbol{r}}_k^{\text{RRE}}}{\partial A_9}$	−0.001738	0.001116	0.001074	0.001074	0.001074
	−0.001660	0.000306	0.000279	0.000279	0.000279
	−0.003887	−0.001032	−0.001074	−0.001074	−0.001074
	0.002842	0.000237	0.000279	0.000279	0.000279
	−0.027861	−0.020077	−0.020046	−0.020045	−0.020045
	−0.005856	−0.004360	−0.004344	−0.004343	−0.004343
	−0.030340	−0.022098	−0.022101	−0.022100	−0.022100
	−0.002734	−0.004309	−0.004343	−0.004343	−0.004343
$\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\text{RRE}}}{\partial A_9{}^2}$	0.005689	−0.000301	−0.000298	−0.000298	−0.000298
	0.003987	−0.000079	−0.000077	−0.000078	−0.000078
	0.006043	0.000296	0.000298	0.000298	0.000298
	−0.005399	−0.000075	−0.000078	−0.000078	−0.000078
	0.019155	0.005574	0.005569	0.005569	0.005569
	0.003752	0.001208	0.001207	0.001207	0.001207
	0.020331	0.006143	0.006140	0.006140	0.006140
	−0.002753	0.001205	0.001207	0.001207	0.001207

Table 8 Approximate displacement and sensitivity vectors obtained by the RRE method with different values of parameter k for 10-bar plane truss structure (large modifications)

k	2	3	4	5	6	$\boldsymbol{r}_{\text{exact}}$
$\tilde{\boldsymbol{r}}_k^{\text{RRE}}$	0.835224	1.737231	1.573995	1.580604	1.580430	1.580430
	0.386510	0.991836	0.885661	0.898114	0.898277	0.898277
	0.854152	1.751372	1.584431	1.588094	1.588291	1.588291
	−0.342917	−1.031541	−0.902247	−0.901647	−0.901723	−0.901723
	2.332228	4.916797	4.527432	4.549192	4.549508	4.549508
	0.480061	1.023600	0.949291	0.961821	0.961926	0.961926
	2.315501	4.935092	4.508689	4.530435	4.530738	4.530738
	−0.380553	−0.990378	−0.944419	−0.943850	−0.943956	−0.943956
$\frac{\partial \tilde{\boldsymbol{r}}_k^{\text{RRE}}}{\partial A_9}$	−0.017887	0.015871	0.001154	0.001246	0.001224	0.001224
	−0.012553	0.010502	0.000466	0.000516	0.000537	0.000537
	−0.020881	0.013628	−0.001238	−0.001249	−0.001224	−0.001224
	0.015174	−0.010888	0.000526	0.000546	0.000537	0.000537
	−0.080171	0.017411	−0.015254	−0.014948	−0.014914	−0.014914
	−0.016866	0.003606	−0.002927	−0.002811	−0.002799	−0.002799
	−0.084212	0.018896	−0.016759	−0.016429	−0.016397	−0.016397
	0.006083	−0.007546	−0.002862	−0.002785	−0.002799	−0.002799
$\frac{\partial^2 \tilde{\boldsymbol{r}}_k^{\text{RRE}}}{\partial A_9{}^2}$	0.016816	−0.001763	−0.000280	−0.000301	−0.000296	−0.000296
	0.011343	−0.001166	−0.000113	−0.000126	−0.000130	−0.000130
	0.017409	−0.001221	0.000300	0.000302	0.000296	0.000296
	−0.013180	0.001023	−0.000127	−0.000132	−0.000130	−0.000130
	0.053738	0.000222	0.003686	0.003618	0.003610	0.003610
	0.011232	−0.000014	0.000707	0.000680	0.000677	0.000677
	0.055565	0.000277	0.004049	0.003977	0.003969	0.003969
	−0.009069	0.001146	0.000691	0.000675	0.000677	0.000677

Table 9 The relative errors obtained by the CA, MPE, and RRE methods with different values of parameters k and s for 10-bar plane truss structure under large modifications (derivatives are with respect to A_9)

Displacement						
k or s	2	3	4	5	6	7
$E^{\mathrm{CA}}_{\mathrm{relative}}$	0.24	0.05	5.40×10^{-3}	1.70×10^{-3}	1.52×10^{-12}	3.55×10^{-11}
$E^{\mathrm{MPE}}_{\mathrm{relative}}$	0.41	0.18	5.50×10^{-3}	7.40×10^{-5}	1.77×10^{-13}	4.72×10^{-13}
$E^{\mathrm{RRE}}_{\mathrm{relative}}$	0.49	0.09	5.20×10^{-3}	7.97×10^{-5}	4.63×10^{-14}	1.55×10^{-13}
First-order sensitivity						
$E^{\mathrm{CA}}_{\mathrm{relative}/A_{9,1}}$	18.50	3.35	0.06	0.03	7.02×10^{-9}	9.29×10^{-8}
$E^{\mathrm{MPE}}_{\mathrm{relative}/A_{9,1}}$	16.74	5.88	0.02	2.60×10^{-3}	2.95×10^{-12}	7.98×10^{-11}
$E^{\mathrm{RRE}}_{\mathrm{relative}/A_{9,1}}$	4.48	2.43	0.02	2.80×10^{-3}	1.37×10^{-11}	8.17×10^{-11}
Second-order sensitivity						
$E^{\mathrm{CA}}_{\mathrm{relative}/A_{9,2}}$	10.52	6.16	0.09	0.04	5.83×10^{-8}	7.41×10^{-6}
$E^{\mathrm{MPE}}_{\mathrm{relative}/A_{9,2}}$	55.10	2.34	0.02	2.50×10^{-3}	1.18×10^{-11}	3.39×10^{-10}
$E^{\mathrm{RRE}}_{\mathrm{relative}/A_{9,2}}$	14.48	1.05	0.02	2.70×10^{-3}	6.58×10^{-12}	1.26×10^{-10}

$$E_{\mathrm{Av.}/x_{i,2}} = \frac{\sum_{i=1}^{n_{\mathrm{dof}}} \left(\frac{\frac{\partial^2 \boldsymbol{r}_{\mathrm{exact}\,i}}{\partial x_i^2} - \frac{\partial^2 \widetilde{\boldsymbol{r}}_i}{\partial x_i^2}}{\frac{\partial^2 \boldsymbol{r}_{\mathrm{exact}\,i}}{\partial x_i^2}} \right)}{n_{\mathrm{dof}}} \times 100 \tag{67}$$

where $E_{\mathrm{Av.}}$ represents the average error obtained by a reanalysis method in approximating the displacement vector of the modified structure, $E_{\mathrm{Av.}/x_{i,1}}$ indicates the average error yielded by a reanalysis method in calculating the first-order sensitivity of the displacement vector, $E_{\mathrm{Av.}/x_{i,2}}$ is the average error obtained by a reanalysis method in predicting the second-order sensitivity of the displacement vector, $\boldsymbol{r}_{\mathrm{exact}\,i}$ and $\widetilde{\boldsymbol{r}}_i$ are the exact and approximate displacements of the ith DOF of the structure, respectively, $\frac{\partial \boldsymbol{r}_{\mathrm{exact}\,i}}{\partial x_i}$ and $\frac{\partial \widetilde{\boldsymbol{r}}_i}{\partial x_i}$ are the exact and approximate first-order displacement sensitivities of the ith DOF of the structure, respectively, $\frac{\partial^2 \boldsymbol{r}_{\mathrm{exact}\,i}}{\partial x_i^2}$ and $\frac{\partial^2 \widetilde{\boldsymbol{r}}_i}{\partial x_i^2}$ are the exact and

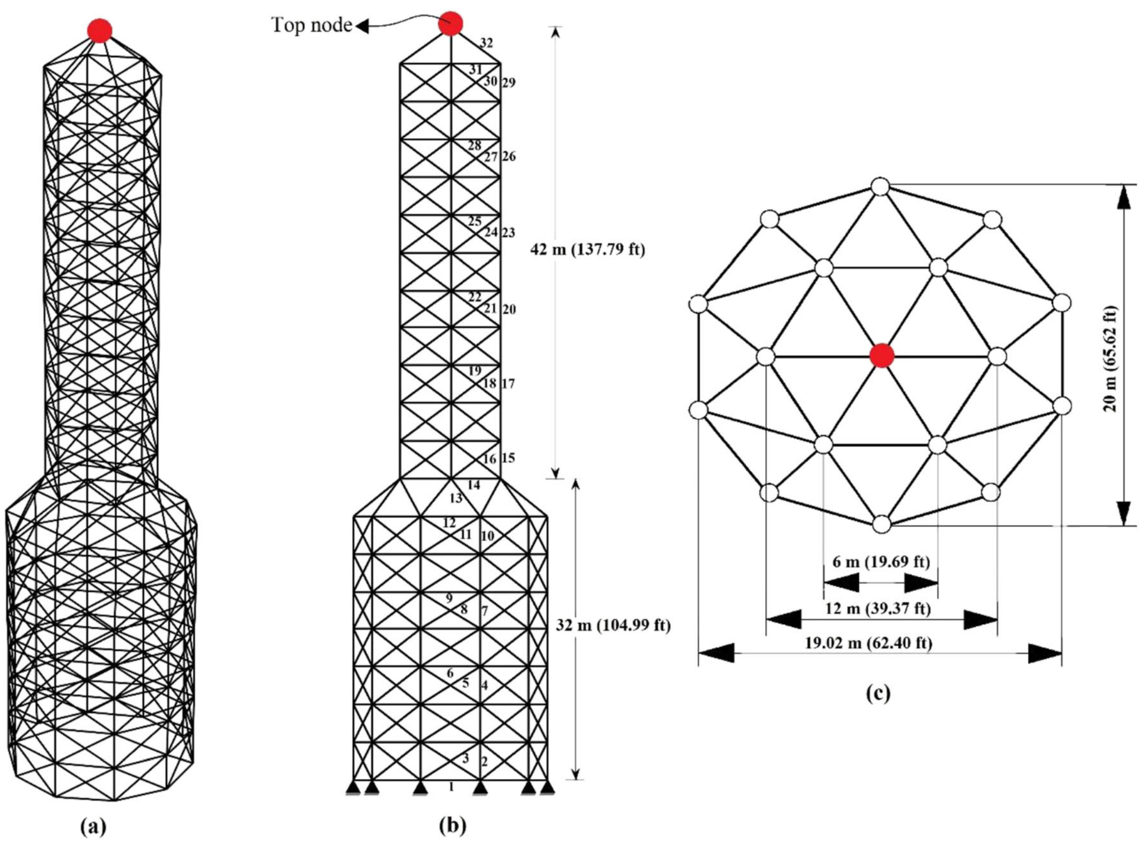

Fig. 2 A 582-bar tower structure: **a** 3D view, **b** side view, **c** top view

approximate second-order displacement sensitivities of the ith DOF of the structure, respectively.

As it is mentioned before, the accuracy of the results obtained by the proposed sensitivity reanalysis procedures depends on the parameter k, which is much smaller than the degree of freedom (n_{dof}) of the structure. In the investigated test problems, the accuracy of the MPE and RRE methods is investigated by setting different values for parameter k.

5.2 Computational effort

To investigate the computational effort required by the proposed methods, the number of algebraic operations (NAOs) required by the MPE and RRE methods for the structural sensitivity reanalysis are presented in Tables 1 and 2. In addition, the required NAOs for the CA-based sensitivity reanalysis method are presented in Table 3. From these tables, it can be seen that the sensitivity reanalysis procedures using the CA, MPE, and RRE methods require $O\left(n_{\text{dof}}^2\right)$ flops, which means that the proposed sensitivity reanalysis methods and CA are computationally equivalent. For the complete second-order sensitivity reanalysis, the required NAOs for both of the MPE and RRE methods are about $24(k+1)n_{\text{dof}}^2$, while it is $42sn_{\text{dof}}^2$ for the CA method. This means that the proposed methods require slightly fewer amount of NAOs than the CA method for any values of the parameters k and s. In the numerical tests, the NAOs required by the proposed methods are compared to those yielded by the CA method.

5.3 A 10-bar planar truss structure problem

A simple 10-bar planar truss structure shown in Fig. 1a is used as the first test problem to show the solution finding process of the proposed sensitivity reanalysis methods. The Young's modulus of truss members is equal to 30,000 ksi. In this test example, two types of modifications in the initial design are considered, including sizing modifications and member additions. It is assumed that the cross-sections of all members in initial structure are equal to 5.0 in^2. In the modified structure, two braced members are added as shown in Fig. 1b and the cross-sections of members are changed as listed in Table 4. As it can be seen from Table 4, small and large modifications are considered to investigate the accuracy of the proposed sensitivity reanalysis methods under different levels of modifications.

Considering small and large modifications in initial structure, Tables 5, 6, 7, and 8 list the obtained structural and sensitivity reanalysis results by the MPE and RRE methods with respect to cross-sectional variable of member 9. From these tables, it is clear that both of the MPE and RRE methods are able to approximate the exact solution with a high accuracy for small and large modifications in initial structure. In addition, for different values of k and s, the relative errors obtained by the MPE and RRE methods are compared to those obtained by the CA method in Table 9. From this table, it can be seen that the proposed methods provide significantly fewer amount of relative errors than CA method. For example, when k and s are equal to 6, the relative errors of the approximate second-order sensitivities provided by the MPE and RRE methods are about 1.18×10^{-11} and 6.58×10^{-12}, respectively, while it is 5.83×10^{-8} for the CA method. Similar conclusion can be made by comparing the reanalysis and first-order sensitivity errors.

5.4 A 582-bar tower structure

The second test example is a 582-bar tower structure shown in Fig. 2. This structure has 154 nodes which results 462 DOFs. The structural members are categorized into 32 design groups as displayed in Fig. 2. The loading condition is as follows: 1.12 kips acting in the X and Y directions and −6.74 kips acting in the Z direction at all nodes of the tower. In the initial design, it is assumed that all of the structural members have constant Young's modulus and

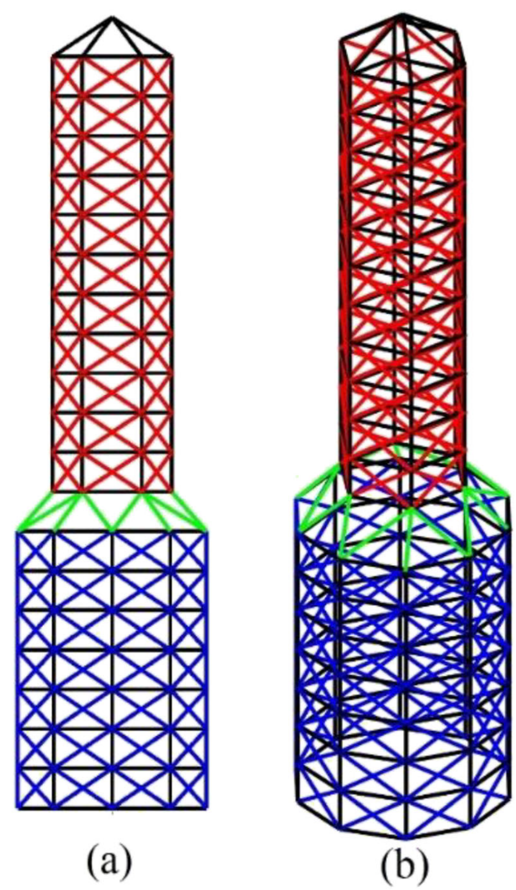

Fig. 3 Modified parts of 582-bar tower structure: **a** side view, **b** 3D view

Table 10 Initial and modified designs for 582-bar tower structure problem

	Cross-sectional areas (in^2)				Young's modulus (ksi)		
	Initial design	Modified design	Variations		Initial design	Modified design	Variations
A_1	10	10	0%	E_1	29,000	29,000	0%
A_2	10	10	0%	E_2	29,000	29,000	0%
A_3	10	13	+ 30%	E_3	29,000	29,000	0%
A_4	10	10	0%	E_4	29,000	29,000	0%
A_5	10	13	+ 30%	E_5	29,000	29,000	0%
A_6	10	10	0%	E_6	29,000	29,000	0%
A_7	10	10	0%	E_7	29,000	29,000	0%
A_8	10	13	+ 30%	E_8	29,000	29,000	0%
A_9	10	10	0%	E_9	29,000	29,000	0%
A_{10}	10	10	0%	E_{10}	29,000	29,000	0%
A_{11}	10	13	+ 30%	E_{11}	29,000	29,000	0%
A_{12}	10	10	0%	E_{12}	29,000	29,000	0%
A_{13}	10	10	0%	E_{13}	29,000	20,300	− 30%
A_{14}	10	10	0%	E_{14}	29,000	29,000	0%
A_{15}	10	10	0%	E_{15}	29,000	29,000	0%
A_{16}	10	7	− 30%	E_{16}	29,000	29,000	0%
A_{17}	10	10	0%	E_{17}	29,000	29,000	0%
A_{18}	10	7	− 30%	E_{18}	29,000	29,000	0%
A_{19}	10	10	0%	E_{19}	29,000	29,000	0%
A_{20}	10	10	0%	E_{20}	29,000	29,000	0%
A_{21}	10	7	− 30%	E_{21}	29,000	29,000	0%
A_{22}	10	10	0%	E_{22}	29,000	29,000	0%
A_{23}	10	10	0%	E_{23}	29,000	29,000	0%
A_{24}	10	7	− 30%	E_{24}	29,000	29,000	0%
A_{25}	10	10	0%	E_{25}	29,000	29,000	0%
A_{26}	10	10	0%	E_{26}	29,000	29,000	0%
A_{27}	10	7	− 30%	E_{27}	29,000	29,000	0%
A_{28}	10	10	0%	E_{28}	29,000	29,000	0%
A_{29}	10	10	0%	E_{29}	29,000	29,000	0%
A_{30}	10	7	− 30%	E_{30}	29,000	29,000	0%
A_{31}	10	10	0%	E_{31}	29,000	29,000	0%
A_{32}	10	10	0%	E_{32}	29,000	29,000	0%

cross-sectional area equal to 29,000 ksi and 10 in^2, respectively. Then, we assume a set of multiple modifications in the initial design as follows: 30% increase in the cross sections of design groups 3, 5, 8, and 11 (braced members); 30% decrease in the cross sections of design groups 16, 18, 21, 24, 27, and 30; and 30% decrease in the Young's

Table 11 Approximate displacement obtained by the CA, MPE, and RRE methods with different values of parameters k and s for the top node of 582-bar tower structure problem

CA									
	$s = 2$	Error (%)	$s = 4$	Error (%)	$s = 6$	Error (%)	$s = 8$	Error (%)	Exact
r_x	5.62298125	0.57	5.65509728	1.20×10^{-3}	5.65502905	3.87×10^{-5}	5.65503123	8.82×10^{-8}	5.65503124
r_y	5.26728087	0.36	5.28626583	6.28×10^{-5}	5.28626781	2.54×10^{-5}	5.28626914	3.09×10^{-7}	5.28626915
r_z	− 0.73904489	0.63	− 0.73442906	4.30×10^{-3}	− 0.73439816	5.73×10^{-5}	− 0.73439774	4.75×10^{-8}	− 0.73439774
MPE									
	$k = 2$	Error (%)	$k = 4$	Error (%)	$k = 6$	Error (%)	$k = 8$	Error (%)	Exact
r_x	5.65557254	9.60×10^{-3}	5.65503623	8.83×10^{-5}	5.65503093	5.51×10^{-6}	5.65503124	1.16×10^{-8}	5.65503124
r_y	5.28623165	7.00×10^{-4}	5.28627844	1.76×10^{-4}	5.28626881	6.46×10^{-6}	5.28626915	3.30×10^{-9}	5.28626915
r_z	− 0.73449172	1.28×10^{-2}	− 0.73439905	1.79×10^{-4}	− 0.73439770	5.21×10^{-6}	− 0.73439774	1.57×10^{-8}	− 0.73439774
RRE									
r_x	5.65557336	9.60×10^{-3}	5.65503551	7.55×10^{-5}	5.65503095	5.12×10^{-6}	5.65503124	4.51×10^{-9}	5.65503124
r_y	5.28623157	7.00×10^{-4}	5.28627782	1.64×10^{-4}	5.28626883	6.09×10^{-6}	5.28626915	1.11×10^{-8}	5.28626915
r_z	− 0.73449181	1.28×10^{-2}	− 0.73439929	2.11×10^{-4}	− 0.73439770	5.05×10^{-6}	− 0.73439774	2.25×10^{-8}	− 0.73439774

Table 12 Approximate first-order sensitivities obtained by the CA, MPE, and RRE methods with different values of parameters k and s for the top node of 582-bar tower structure problem

CA									
	$s=2$	Error (%)	$s=4$	Error (%)	$s=6$	Error (%)	$s=8$	Error (%)	Exact
$\frac{\partial r_x}{\partial A_{13}}$	−0.15128484	5.08	−0.15935592	0.01	−0.15937198	2.80×10^{-3}	−0.15937625	7.86×10^{-5}	−0.15937637
$\frac{\partial r_y}{\partial A_{13}}$	−0.12272230	3.47	−0.12722067	0.07	−0.12713779	3.50×10^{-3}	−0.12713347	1.40×10^{-4}	−0.12713329
$\frac{\partial r_z}{\partial A_{13}}$	0.01170164	9.13	0.01069158	0.29	0.01071973	2.43×10^{-2}	0.01072223	9.73×10^{-4}	0.01072234
MPE									
	$k=2$	Error (%)	$k=4$	Error (%)	$k=6$	Error (%)	$k=8$	Error (%)	Exact
$\frac{\partial r_x}{\partial A_{13}}$	−0.15860953	0.48	−0.15933194	0.03	−0.15937474	1.02×10^{-3}	−0.15937635	1.50×10^{-5}	−0.15937637
$\frac{\partial r_y}{\partial A_{13}}$	−0.12677064	0.29	−0.12711248	0.02	−0.12713214	9.01×10^{-4}	−0.12713328	1.21×10^{-5}	−0.12713329
$\frac{\partial r_z}{\partial A_{13}}$	0.01066696	0.52	0.01072441	0.02	0.01072231	2.76×10^{-4}	0.01072234	6.77×10^{-7}	0.01072234
RRE									
	$k=2$	Error (%)	$k=4$	Error (%)	$k=6$	Error (%)	$k=8$	Error (%)	Exact
$\frac{\partial r_x}{\partial A_{13}}$	−0.15860980	0.48	−0.15933009	0.03	−0.15937459	1.10×10^{-3}	−0.15937635	1.61×10^{-5}	−0.15937637
$\frac{\partial r_y}{\partial A_{13}}$	−0.12677086	0.29	−0.12711126	0.02	−0.12713206	1.00×10^{-3}	−0.12713327	1.30×10^{-5}	−0.12713329
$\frac{\partial r_z}{\partial A_{13}}$	0.01066697	0.52	0.01072458	0.02	0.01072231	2.00×10^{-4}	0.01072234	8.72×10^{-7}	0.01072234

modulus of design group 13. Figure 3 shows the modified parts of 582-bar tower structure and Table 10 lists the initial and modified designs for this test problem.

To investigate the accuracy of the MPE and RRE methods in performing structural and sensitivity reanalysis, the displacement sensitivity of the top node of the tower with respect to the cross-sectional area of member group 13 is investigated. If the displacements of the top node of the tower in x, y, and z directions are indicated by r_x, r_y, and r_z, respectively, Tables 11, 12, and 13 compare the approximate sensitivity results obtained by the CA, MPE, and RRE methods with the corresponding exact values for different values of parameters k and s. For the case of $k=s=2$, it can be seen that the maximum displacement errors yielded by the MPE and RRE methods are about 0.01%, while it is about 0.63% for the CA method. These results indicate that the proposed methods are able to approximate the displacement vector of the modified structure with a very smaller number of parameter k. When $k=s=2$, it can be observed that the

Table 13 Approximate second-order sensitivities obtained by the CA, MPE, and RRE methods with different values of parameters k and s for the top node of 582-bar tower structure problem

CA									
	$s=2$	Error (%)	$s=4$	Error (%)	$s=6$	Error (%)	$s=8$	Error (%)	Exact
$\frac{\partial^2 r_x}{\partial {A_{13}}^2}$	0.02942822	7.66	0.03194360	0.23	0.03191489	0.14	0.03189256	0.07	0.03187047
$\frac{\partial^2 r_y}{\partial {A_{13}}^2}$	0.02359137	7.22	0.02549451	0.27	0.02544798	0.09	0.02544287	0.07	0.02542608
$\frac{\partial^2 r_z}{\partial {A_{13}}^2}$	−0.00216769	1.23	−0.00210996	1.47	−0.00212547	0.74	−0.00213639	0.23	−0.00214137
MPE									
	$k=2$	Error (%)	$k=4$	Error (%)	$k=6$	Error (%)	$k=8$	Error (%)	Exact
$\frac{\partial^2 r_x}{\partial {A_{13}}^2}$	0.03087225	3.13	0.03169684	0.55	0.03187064	5.28×10^{-4}	0.03187075	8.75×10^{-4}	0.03187047
$\frac{\partial^2 r_y}{\partial {A_{13}}^2}$	0.02469799	2.86	0.02532176	0.41	0.02542570	1.50×10^{-3}	0.02542624	6.64×10^{-4}	0.02542608
$\frac{\partial^2 r_z}{\partial {A_{13}}^2}$	−0.00207479	3.11	−0.00214202	0.03	−0.00214119	8.40×10^{-3}	−0.00214136	1.88×10^{-4}	−0.00214137
RRE									
	$k=2$	Error (%)	$k=4$	Error (%)	$k=6$	Error (%)	$k=8$	Error (%)	Exact
$\frac{\partial^2 r_x}{\partial {A_{13}}^2}$	0.03087227	3.13	0.03169192	0.56	0.03187038	3.00×10^{-4}	0.03187075	8.85×10^{-4}	0.03187047
$\frac{\partial^2 r_y}{\partial {A_{13}}^2}$	0.02469800	2.86	0.02531847	0.42	0.02542552	2.20×10^{-3}	0.02542625	6.70×10^{-4}	0.02542608
$\frac{\partial^2 r_z}{\partial {A_{13}}^2}$	−0.00207479	3.11	−0.00214223	0.04	−0.00214118	8.80×10^{-3}	−0.00214136	2.07×10^{-4}	−0.00214137

Table 14 The relative errors obtained by the CA, MPE, and RRE methods with different values of parameters k and s for 582-bar tower structure problem (derivatives are with respect to A_{13})

Displacement						
k or s	2	4	6	8	…	14
$E^{\mathrm{CA}}_{\mathrm{relative}}$	6.00×10^{-3}	2.37×10^{-5}	4.30×10^{-7}	8.15×10^{-9}	…	9.42×10^{-11}
$E^{\mathrm{MPE}}_{\mathrm{relative}}$	1.11×10^{-4}	1.60×10^{-6}	5.76×10^{-8}	2.82×10^{-10}	…	3.51×10^{-13}
$E^{\mathrm{RRE}}_{\mathrm{relative}}$	1.11×10^{-4}	1.50×10^{-6}	5.41×10^{-8}	2.74×10^{-10}	…	3.50×10^{-13}
First-order sensitivity						
$E^{\mathrm{CA}}_{\mathrm{relative}/A_{13,1}}$	0.06	1.81×10^{-3}	1.25×10^{-4}	4.72×10^{-6}	…	3.86×10^{-7}
$E^{\mathrm{MPE}}_{\mathrm{relative}/A_{13,1}}$	4.10×10^{-3}	2.39×10^{-4}	8.38×10^{-6}	1.22×10^{-7}	…	3.33×10^{-12}
$E^{\mathrm{RRE}}_{\mathrm{relative}/A_{13,1}}$	4.10×10^{-3}	2.47×10^{-4}	9.13×10^{-6}	1.40×10^{-7}	…	3.57×10^{-12}
Second-order sensitivity						
$E^{\mathrm{CA}}_{\mathrm{relative}/A_{13,2}}$	0.10	5.00×10^{-3}	2.11×10^{-3}	1.44×10^{-3}	…	6.53×10^{-4}
$E^{\mathrm{MPE}}_{\mathrm{relative}/A_{13,2}}$	0.03	4.29×10^{-3}	4.65×10^{-5}	7.13×10^{-6}	…	1.58×10^{-10}
$E^{\mathrm{RRE}}_{\mathrm{relative}/A_{13,2}}$	0.03	4.42×10^{-3}	4.90×10^{-5}	7.22×10^{-6}	…	1.28×10^{-10}

maximum errors yielded by both of the MPE and RRE methods for the first- and second-order sensitivities are about 0.5% and 3%, respectively, while these values for the CA method are about 9% and 3%, respectively (Tables 12 and 13). By increasing the value of parameter k, the proposed sensitivity reanalysis methods are successfully converged to the exact sensitivity values. Although the performances of the MPE, RRE, and CA methods are relatively same for the structural reanalysis, the proposed methods provide significantly accurate results than the CA method for the first- and second-order sensitivities.

For different values of parameters k and s, Table 14 compares the relative errors obtained by the CA, MPE, and RRE methods in the sensitivity reanalysis of the modified displacement vector with respect to the cross-sectional area of member group 13. Judging from the reported results, it turns out that the convergence speeds of the CA, MPE, and RRE methods in structural reanalysis are faster than sensitivity reanalysis. From Table 14, it can be seen that the proposed methods perform remarkably better than the CA method in terms of the relative errors of the first- and second-order sensitivities. Although the proposed methods converge to the exact solution with $k = 14$, the obtained errors for the case of $k = 8$ are also acceptable from engineering viewpoint. In addition, Table 15 compares the required CPU times and NAOs for each method.

Table 15 The required NAOs and CPU times required by the CA, MPE, and RRE methods for sensitivity reanalysis of 582-bar tower structure problem ($k = s = 14$)

	NAOs	CPU time (s)
CA	109,881,626	0.0245
MPE	67,286,880	0.0187
RRE	66,940,731	0.0179

Figure 4 illustrates the average errors obtained by the MPE and RRE methods in approximating the modified displacements vector and its sensitivities for this test problem. From this figure, it can be clearly seen that the average errors are dramatically reduced by increasing the parameter k. When $k = 6$, the average errors obtained by both of the MPE and RRE methods are smaller than 0.1%, which indicate the efficiency of the proposed approaches in sensitivity reanalysis. For the case of $k = 6$, the average errors obtained by the MPE method for the structural reanalysis, first-order sensitivity, and second-order sensitivity are about

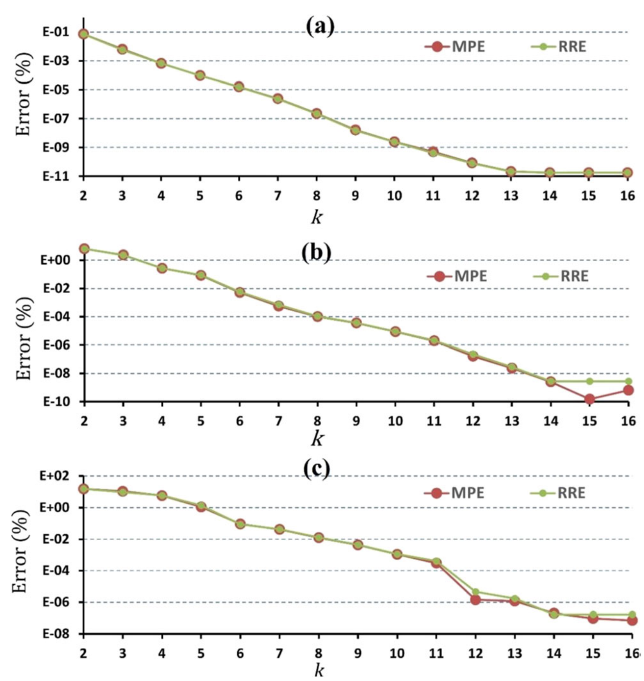

Fig. 4 Average errors obtained by the MPE and RRE methods for different values of parameter k in 582-bar tower structure problem: **a** average reanalysis errors ($E_{\mathrm{Av.}}$), **b** average first-order sensitivity errors ($E_{\mathrm{Av.}/x_{i,1}}$), and **c** average second-order sensitivity errors ($E_{\mathrm{Av.}/x_{i,2}}$)

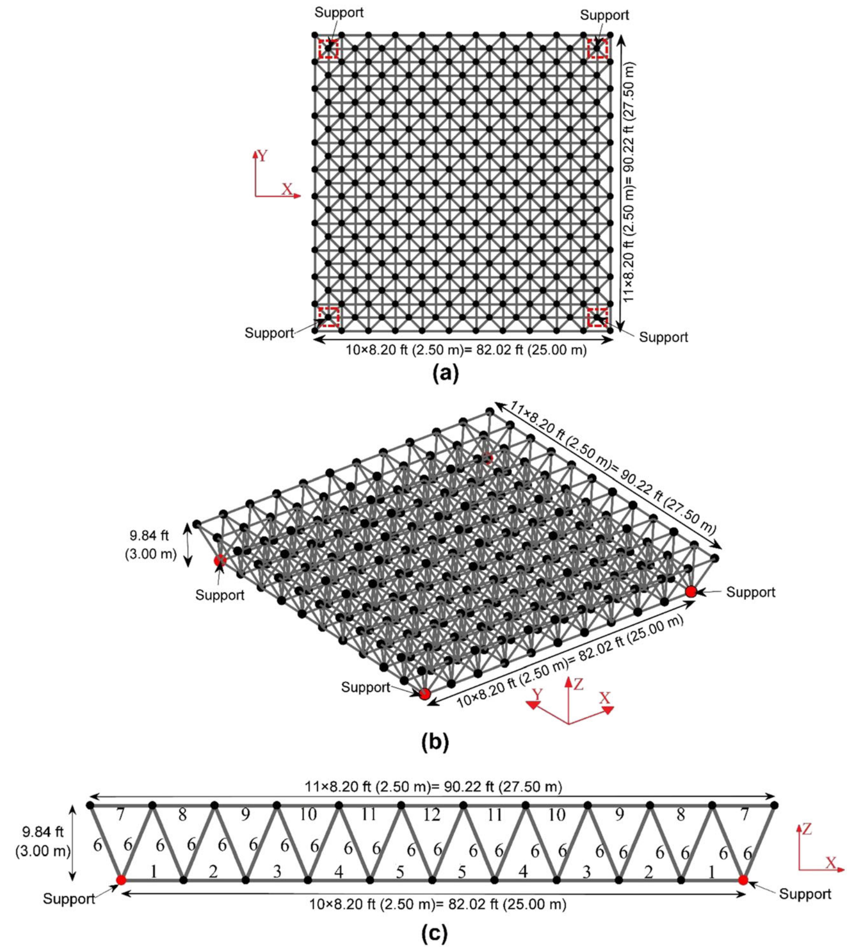

Fig. 5 A 968-bar double layer grid structure: **a** top view, **b** 3D view, **c** side view

$1.59 \times 10^{-5}\%$, $5.10 \times 10^{-3}\%$, and 0.090%, respectively, while the corresponding values for the RRE method are $1.56 \times 10^{-5}\%$, $5.71 \times 10^{-3}\%$, and 0.088%, respectively. These errors can be further reduced by increasing the parameter k.

5.5 968-bar double layer grid structure

A 968-bar double layer grid structure shown in Fig. 5 is the third investigated test problem. The grid structure has 265 nodes which

Table 16 Initial and modified designs for 968-bar double layer grid structure problem

	Cross-sectional areas (in^2)				Young's modulus (ksi)		
	Initial design	Modified design	Variations		Initial design	Modified design	Variations
A_1	10	14	+40%	E_1	29,000	29,000	0%
A_2	10	14	+40%	E_2	29,000	29,000	0%
A_3	10	14	+40%	E_3	29,000	29,000	0%
A_4	10	14	+40%	E_4	29,000	29,000	0%
A_5	10	10	0%	E_5	29,000	29,000	0%
A_6	10	10	0%	E_6	29,000	23,200	−20%
A_7	10	6	−40%	E_7	29,000	29,000	0%
A_8	10	6	−40%	E_8	29,000	29,000	0%
A_9	10	6	−40%	E_9	29,000	29,000	0%
A_{10}	10	6	−40%	E_{10}	29,000	29,000	0%
A_{11}	10	10	0%	E_{11}	29,000	29,000	0%
A_{12}	10	10	0%	E_{12}	29,000	29,000	0%

Table 17 The relative errors obtained by the CA, MPE, and RRE methods with different values of parameters k and s for 968-bar double layer grid structure problem (derivatives are with respect to A_6)

Reanalysis								
k or s	2	4	6	8	…	14	16	18
$E_{\text{relative}}^{\text{CA}}$	0.03	8.53×10^{-4}	5.51×10^{-5}	1.50×10^{-6}	…	8.57×10^{-11}	3.63×10^{-10}	1.95×10^{-9}
$E_{\text{relative}}^{\text{MPE}}$	9.74×10^{-3}	1.81×10^{-4}	7.50×10^{-6}	2.61×10^{-7}	…	2.13×10^{-11}	6.69×10^{-13}	4.41×10^{-14}
$E_{\text{relative}}^{\text{RRE}}$	9.68×10^{-3}	1.80×10^{-4}	7.61×10^{-6}	2.88×10^{-7}	…	2.22×10^{-11}	6.81×10^{-13}	1.28×10^{-13}
First-order sensitivity								
$E_{\text{relative}/A_{6,1}}^{\text{CA}}$	0.06	1.75×10^{-3}	9.33×10^{-5}	4.16×10^{-6}	…	9.70×10^{-10}	1.64×10^{-9}	1.32×10^{-8}
$E_{\text{relative}/A_{6,1}}^{\text{MPE}}$	1.92×10^{-2}	6.01×10^{-4}	2.00×10^{-5}	6.00×10^{-7}	…	7.41×10^{-11}	4.35×10^{-12}	3.40×10^{-13}
$E_{\text{relative}/A_{6,1}}^{\text{RRE}}$	1.93×10^{-2}	6.24×10^{-4}	2.19×10^{-5}	4.97×10^{-7}	…	1.17×10^{-10}	4.52×10^{-12}	6.03×10^{-13}
Second-order sensitivity								
$E_{\text{relative}/A_{6,2}}^{\text{CA}}$	7.15	0.18	0.06	2.00×10^{-3}	…	1.89×10^{-7}	2.03×10^{-6}	1.22×10^{-5}
$E_{\text{relative}/A_{6,2}}^{\text{MPE}}$	4.01	0.05	2.88×10^{-3}	3.13×10^{-4}	…	4.87×10^{-8}	1.41×10^{-9}	1.10×10^{-10}
$E_{\text{relative}/A_{6,2}}^{\text{RRE}}$	3.99	0.44	2.57×10^{-3}	3.73×10^{-4}	…	3.96×10^{-8}	1.38×10^{-9}	4.97×10^{-10}

results 795 DOFs. The members of the structure are categorized into 12 member groups as illustrated in Fig. 5. The displacement of corner nodes at the bottom layer is constrained in x, y, and z directions. All free nodes of the structure are subjected to a vertical load of 5 kips in negative direction of Z-axis. In the initial structure, it is assumed that all of the structural members have constant Young's modulus and cross-sectional area equal to 29,000 ksi and 10 in^2, respectively. In this test problem, a set of multiple changes in the initial design are assumed as follows: 40% increase in the cross-sections of member groups 1 through 4 at the bottom layer, 40% decrease in the cross-sections of member groups 7 through 10 at the top layer, and 20% decrease in the Young's modulus of member group 6 (diagonal members). Table 16 lists the cross-sectional areas and Young's modulus of structural members in the initial and modified designs.

In this test problem, the sensitivities of the modified displacement vector with respect to the cross-sectional areas of diagonal members (A_6) are investigated by considering different values for parameters k and s, and the relative errors obtained by the CA, MPE, and RRE methods are summarized in Table 17. From this table, it can be seen that the MPE and RRE methods provide high-quality solutions for the case of $k = 8$. The relative errors are further decreased by increasing parameter k and the proposed methods converge to the exact solutions when $k = 18$. When comparing the MPE and RRE methods against the CA method, it can be concluded that the proposed methods converge to the exact displacement and sensitivities faster than the CA method. For example, the MPE and RRE methods approximate the second-order sensitivity with relative errors of 1.10×10^{-10} and 4.97×10^{-10}, respectively, while it is 1.22×10^{-5} for the CA method. In addition, Table 18 lists the CPU times and NAOs required by the CA, MPE, and RRE methods for solving this test example.

Table 18 The required NAOs and CPU times required by the CA, MPE, and RRE methods for sensitivity reanalysis of 968-bar double-layer grid structure problem ($k = s = 18$)

	NAOs	CPU time (s)
CA	207,055,028	0.052
MPE	134,127,342	0.043
RRE	132,922,863	0.041

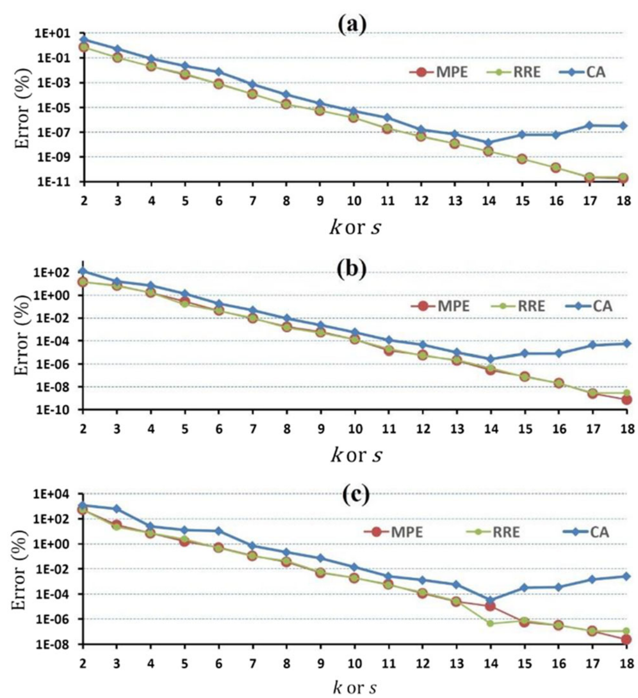

Fig. 6 Average errors obtained by the CA, MPE, and RRE methods for different values of parameter k in 968-bar double-layer grid structure problem: **a** average reanalysis errors ($E_{\text{Av.}}$), **b** average first-order sensitivity errors ($E_{\text{Av.}/x_{i,1}}$), and **c** average second-order sensitivity errors ($E_{\text{Av.}/x_{i,2}}$)

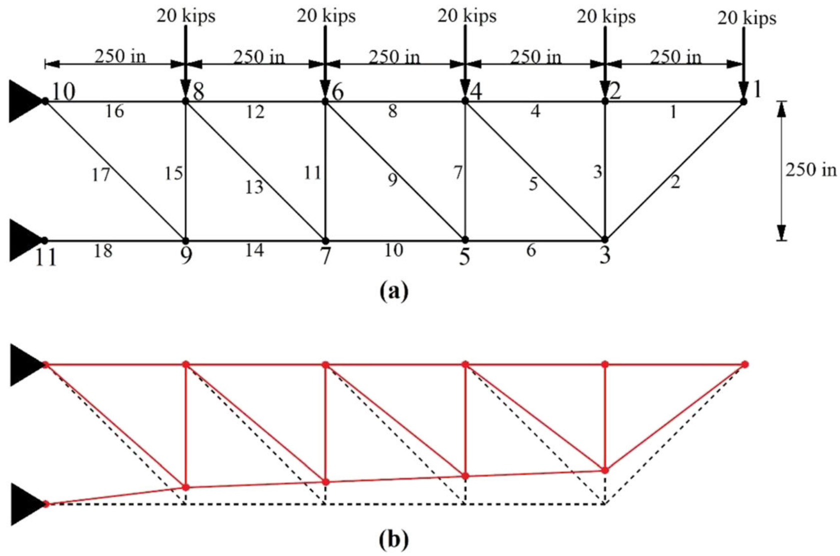

Fig. 7 Sensitivity reanalysis problem of 18-bar truss structure: **a** initial shape, **b** modified shape

For different values of parameters k and s, the average structural and sensitivity reanalysis errors yielded by the CA, MPE, and RRE methods are illustrated in Fig. 6. Once again, it can be seen that the average errors yielded by the proposed methods are significantly reduced by increasing the parameter k. For example, when $k = 7$, the average errors obtained by the MPE method for structural reanalysis, first-order sensitivity, and second-order sensitivity are equal to $1.20 \times 10^{-4}\%$, $9.36 \times 10^{-3}\%$, and 0.11%, respectively. The corresponding average errors yielded by the RRE method are equal to $1.22 \times 10^{-4}\%$, $9.70 \times 10^{-3}\%$, and 0.10%, respectively. When $s = 7$, the average errors yielded by the CA method are about

Table 19 The initial and modified designs for 18-bar truss structure

Size variables (in^2)				Shape variables (in)		
Cross-sections	Initial	Modified	Variations	Coordinates	Initial	Modified
A_1	10	12.025	20.25%	x_1	1250	1250
A_2	10	16.750	67.50%	y_1	250	250
A_3	10	6.175	−38.25%	x_2	1000	1000
A_4	10	12.025	20.25%	y_2	250	250
A_5	10	4.825	−51.75%	x_3	1000	1000
A_6	10	16.750	67.50%	y_3	0	60
A_7	10	6.175	−38.25%	x_4	750	750
A_8	10	12.025	20.25%	y_4	250	250
A_9	10	4.825	−51.75%	x_5	750	750
A_{10}	10	16.750	67.50%	y_5	0	50
A_{11}	10	6.175	−38.25%	x_6	500	500
A_{12}	10	12.025	20.25%	y_6	250	250
A_{13}	10	4.825	−51.75%	x_7	500	500
A_{14}	10	16.750	67.50%	y_7	0	40
A_{15}	10	6.175	−38.25%	x_8	250	250
A_{16}	10	12.025	20.25%	y_8	250	250
A_{17}	10	4.825	−51.75%	x_9	250	250
A_{18}	10	16.750	67.50%	y_9	0	30
				x_{10}	0	0
				y_{10}	250	250
				x_{11}	0	0
				y_{11}	0	0

Table 20 Approximate first-order sensitivities of the 1th node of 118-bar tower with respect to the shape variable of y_3 obtained by the MPE and RRE methods for different values of parameter k for structure problem

MPE									
	$k=2$	Error (%)	$k=4$	Error (%)	$k=6$	Error (%)	$k=8$	Error (%)	Exact
$\frac{\partial r_x^1}{\partial y_3}$	0.0032098	6.74	0.0031377	8.84	0.0034267	0.44	0.0034479	0.17	0.0034419
$\frac{\partial r_y^1}{\partial y_3}$	− 0.0118635	72.49	− 0.0401623	6.86	− 0.0428037	0.73	− 0.0431594	0.10	− 0.0431182
RRE									
$\frac{\partial r_x^1}{\partial y_3}$	0.0002211	93.58	0.0031600	8.19	0.0033410	2.93	0.0034470	0.15	0.0034419
$\frac{\partial r_y^1}{\partial y_3}$	0.0023952	105.55	− 0.0396156	8.12	− 0.0421331	2.28	− 0.0431629	0.10	− 0.0431182

7.00×10^{-4}%, 0.05%, and 0.70%, respectively. The efficiency of the proposed methods is more observable when the values of parameters k and s are increased.

5.6 Shape sensitivity of 18-bar truss structure

In the proposed sensitivity reanalysis methods, the derivatives of the stiffness matrix are required to perform sensitivity reanalysis procedure. However, in the structural shape optimization problems, it is not an easy task to calculate the analytical derivatives of the stiffness matrix with respect to the shape variables. Alternatively, the differential methods can be employed to approximate sensitivity of the stiffness matrix with respect to the shape variables. To illustrate how the accuracy of the proposed methods can be affected by using differential method, a shape sensitivity reanalysis problem is investigated. This test problem is 18-bar planar truss structure shown in Fig. 7a. The structure consists of 11 nodes and 18 members. The upper nodes of the structure are subjected to concentrated loads as shown in Fig. 7a. The Young's modulus of all structural members is equal to 30,000 ksi. In the initial design, the cross-sectional areas of all members are set to 10 in^2. It is assumed that the structure is subjected to the simultaneous size and shape modifications as listed in Table 19. As it can be seen from Table 19, the structure is subjected to the relatively large multi-type modifications in different directions. Figure 7b shows the modified shape of the structure.

In this test problem, the derivatives of the stiffness matrix are obtained by the difference method, i.e., $\frac{\partial \boldsymbol{K}}{\partial x_i} \approx \frac{\Delta \boldsymbol{K}}{\Delta x}$. The first-order sensitivities of the first node of the structure in x and y directions with respect to the shape variable of y_3 are selected to show the performance of the proposed methods. Table 20 compares the approximate sensitivities yielded by the MPE and RRE methods with the exact values. From Table 20, it can be concluded that the proposed methods can provide satisfactory results for the shape sensitivity problems. For the case of $k = 6$, the MPE method calculates the sensitivities with the errors smaller than 1.00%, which are adequate from engineering viewpoint.

6 Concluding remarks

In this paper, new structural sensitivity reanalysis formulations are introduced based on the polynomial-type extrapolation methods. In these formulations, the displacement vector of the modified structure is expressed in the form of the vector sequences based on the fixed-point iteration method. By using these vector sequences, the *minimal polynomial extrapolation* (MPE) and the *reduced rank extrapolation* (RRE) methods calculate the approximate displacement vector of the modified structure. In the structural reanalysis based on the MPE and RRE methods, the complete set of analysis equations of the modified structure is reduced to the linear least-square problems with significantly smaller size. Based on the definitions of the MPE and RRE methods, two sensitivity reanalysis formulations are derived, in which the first- and second-order sensitivities of the structure are obtained by solving a set of the over-determined least-squares problems with much smaller size than the complete set of equations of the exact sensitivity analyses. In the derived sensitivity reanalysis formulations, the approximate sensitivities of the modified structure are calculated by solving the linear least-square problems with sizes of the $n_{\mathrm{dof}} \times k$ and $n_{\mathrm{dof}} \times k + 1$, respectively, in which k is an arbitrary positive integer that is usually much smaller than the total number of DOFs of the structure ($k \ll n_{\mathrm{dof}}$). In order to validate the proposed sensitivity reanalysis formulations, four structural sensitivity reanalysis problems under multiple types of modifications are investigated. The obtained structural and sensitivity reanalysis results indicate that the proposed methods are able to approximate the displacement vector of the modified structure and its sensitivities with a very smaller number of parameter k. In addition, the reanalysis and sensitivity errors are further decreased by increasing parameter k and the proposed methods are also capable to converge to the exact sensitivity vectors of the structure.

7 Replication of results

In the investigated test problems, all of the necessary data are provided to readers and the obtained results can be verified via the presented information.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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