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RESEARCH PAPER

Reliability analysis for k-out-of-n systems with shared load and dependent components

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Abstract
Many structural systems require a minimal number of components to be operational, and predicting the reliability of such systems is a challenge because surviving components share the original system workload with higher component loads after the failure of some components. The states of all the components are also dependent. Such dependence, however, is generally neglected in many existing methods. In this study, we develop a new reliability method for systems with dependent components that share the system load equally before and after other components have failed. The components are also subjected to other loads, such as a preload. The new method is based on limit-state functions that predict the states of components, and the First Order Reliability Method is used. The advantage of the proposed method is that it can directly link the system reliability with design variables and random parameters because of the use of a physics-based approach. High accuracy is maintained with the consideration of dependent component states. Two examples are used to demonstrate the good accuracy and efficiency of the proposed method.

Keywords Reliability · k-out-of-n system · Limit-state function · Simulation

1 Introduction

In many engineering applications, there are systems that require at least a certain number of components to work successfully to ensure the normal functionality of the entire system. For example, an aircraft with four engines could still work properly if at least two out of its four engines remain functioning. Another example is the bolted flange connection that is commonly used in structural systems, as shown in Fig. 1. There are n bolts, and at least k bolts are required to be operational in order for the connection system to operate. This kind of systems is referred to as k-out-of-n systems, where n is the total number of components in the system, and k is the minimal number of components that have to work successfully.

Many reliability engineering studies assume independence across the components in the system (Kuo and Zuo 2003; Yang 2007). This means that the component states (operational or failed) are statistically independent; in other words, the failure of a component does not affect the failures of other components in the system. In reality, the states of components are dependent because the components share the same environment, such as loads, temperature, and humidity (Cheng et al. 2017; Hu and Du 2017). After a number of components have failed, the remaining components will share the same system load with increased component loads. Take the connection system in Fig. 1 as an example. Initially all the bolts evenly share the system load. If one or more bolts have failed, the system load is redistributed, and the remaining bolts also evenly share the same system workload. Then the load acting on each of the remaining bolts increases, resulting in a higher probability of component failure.

Many studies have been conducted for k-out-of-n systems with load sharing components (Amari and Bergman 2008; Huamin 1998; Kong and Ye 2017; Liu et al. 2016; Taghipour and Kassaei 2015). These studies produce statistics-based reliability methods for estimating the system

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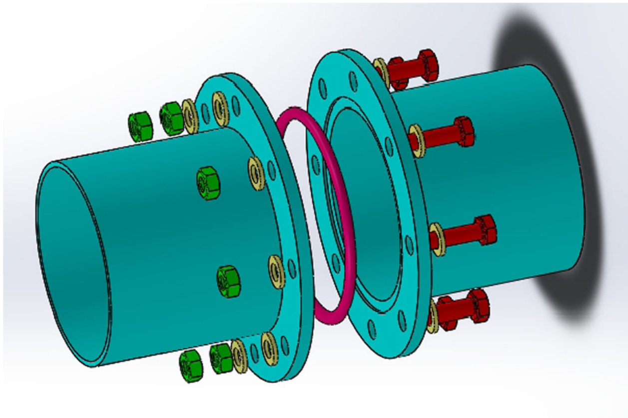

Fig. 1 Bolted flange connection system

reliability, meaning that the life distributions of components are obtained from statistics or assumed. Then the system reliability is analyzed with an assumption that the life distributions or the failure rates of remaining components change after a number of components have failed. In some of the studies, the dependence of component states is neglected for the calculation of the probability of intersection of the event that several components work successfully together. The independence assumption may produce a large error if the component states are strongly dependent. The advantage of the statistics-based methods is that they can easily estimate the system reliability based on statistical data; but they lack a direct link with design variables and parameters, such as component structures, dimensions, material properties, manufacturing imprecisions, and component loads, which do not come from the system load.

This study develops a reliability method for *k*-out-of-*n* systems with physics-based approaches. The scope of this study is for systems with identical components, for example, an aircraft with four identical engines, a power generating system with six identical generators, and a joint connection system (as shown in Fig. 1) with six identical joints. The new method has the following advantages: (1) It can deal with systems whose components are subjected to not only the load that comes from the system load, but also other loads, such as the preload acting on a bolt in the joint connection system in Fig. 1. It therefore overcomes the drawback of the existing methods that deal with component loads directly proportional to the system load. (2) It produces the joint probability density of the states of all components before any component failures and the joint probability density of the states of remaining components after a number of components have failed. Consequentially, component dependence can be accommodated implicitly and automatically. (3) The method is based on the commonly used physics-based method, the First Order Reliability Method (FORM). It has a good balance between efficiency and accuracy. And (4) the new method is a physics-based method that relies on a component limit-state function derived from physics. As a result, the method directly links the system reliability with design variables. When used in a design stage, the method helps identify new design variables for design improvement if the system reliability does not meet the reliability requirement.

The rest of this paper is organized as follows. In Sec. 2, we review the reliability engineering methods for *k*-out-of-*n* systems and FORM. We then discuss the proposed method in Sec. 3, followed by examples in Sec. 4. Sec. 5 concludes the paper.

2 Related work

In this section, we review existing reliability methods for *k*-out-of-*n* systems. Since the proposed method uses a physics-based method that is based on limit-state functions, we also review the First Order Reliability Methods (FORM), which is the most commonly used physics-based method for structural reliability analysis.

2.1 Reliability of *k*-out-*n* systems

Suppose that the n components in a k-out-of-n system are identical. Consider no load sharing and assume the states of components are independent. Let the reliability of a component be R and the number of operational components be x. Then the probability of having k components operational is (Yang 2007)

$$\Pr(x=k) = C_n^k R^k (1-R)^{n-k}, \quad k = 1, 2, \ldots, n \tag{1}$$

Since it is required that at least k components have to be operational, the system reliability is calculated by

$$R_s(n,k) = \Pr(x \geq k) = \sum_{i=k}^{n} C_n^i R^i (1-R)^{n-i} \tag{2}$$

When components share the same system load, the load acting on each of remaining components will increase after a number of components have failure. For example, when $k = n$, the n components share the same load equally, and we have a series system. The system reliability is (Yang 2007)

$$R_s(n,n) = R_0^n \tag{3}$$

where R_0 is the component reliability under the initial component load.

When $k = 1$, one component is allowed to fail. The system reliability becomes

$$R_s(n,n-1) = R_s(n,n) + nR_1^{n-1}(1-R_0) \tag{4}$$

where R_1 is the component reliability subjected to an increased component load after one component has failed.

Equation (3) and (4) are derived based on the assumption that component states are independent. This assumption may not hold because components share the same random environment in the system. With statistics-based approaches, the component reliability is calculated with the distribution of the component life. As a result, a direct link between the system reliability and design variables does not exist. In this study, we develop a physics-based approach to establish such a link by using FORM, which is reviewed in Sec. 2.2.

2.2 Physics-based First Order Reliability Method (FORM)

In this subsection, we review how the reliability of a component in a k-out-of-n system is calculated by a physics-based reliability approach. When the state of a component, either operational or failed, can be predicted by a computational model that is derived from physics, the reliability of the component can then be predicted with a physics-based reliability approach. The computational model is called a limit-state function, and the reliability is calculated by

$$R = \Pr\{g(\mathbf{X}) \geq 0\} = \int_{g(\mathbf{X}) \geq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{5}$$

where $g(\mathbf{X})$ is a limit-state function, $\mathbf{X} = [X_1, \ldots, X_m]$ is a vector of random input variables, and $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of $\mathbf{X}$. The associated probability of failure is given by

$$p_f = 1 - R = \Pr\{g(\mathbf{X}) < 0\} = \int_{g(\mathbf{X}) < 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{6}$$

Calculating the integrals in (5) and (6) is difficult and may be impossible for high dimensional random variables. For this reason, approximation methods are usually used, including the First Order Second Moment Method (FOSM) (Dolinski 1982; Lee and Kwak 1987), the First Order Reliability Method (FORM) (Du 2008; Du et al. 2008; Du and Sudjianto 2004; Yang and Gu 2004; Zou and Mahadevan 2006), the Second Order Reliability Method (SORM) (Lim et al. 2014), and other physics-based approaches (das Neves Carneiro and Antonio 2017; Drignei et al. 2016; Ramu et al. 2010; Wang and Wang 2016; Xi et al. 2017; Xie et al. 2017; Youn and Wang 2008; Zhang 2017). Among them, FORM is the most commonly used method because of its good balance between accuracy and efficiency.

With the assumption that all the components of $\mathbf{X}$ are independent, we herein briefly review the procedure of FORM. FORM at first transforms $\mathbf{X}$ into standard normal variables $\mathbf{U}$ by

$$F_{X_i}(X_i) = \Phi(U_i), \quad i = 1, 2, \ldots, m \tag{7}$$

in which $F_{X_i}(\cdot)$ and $\Phi(\cdot)$ represent the cumulative distribution functions (CDF) of X_i and U_i, respectively. FORM then linearizes the limit-state function at the most probable point (MPP), which minimizes the error of the linearization. The MPP $\mathbf{u}^*$ is found by solving

$$\begin{cases} \min & \beta = \sqrt{\mathbf{U}\mathbf{U}^{\mathrm{T}}} \\ s.t. & g(\mathbf{X}) = g(T(\mathbf{U})) = 0 \end{cases} \tag{8}$$

where $T(\cdot)$ stands for the U-to-X transformation defined in (7).

The magnitude of $\mathbf{u}^*$ is the reliability index and is calculated by

$$\beta = \sqrt{(u_1^*)^2 + \ldots + (u_m^*)^2} \tag{9}$$

When the limit-state function is linearized at $\mathbf{u}^*$, the event that the component is in an operational state is defined by

$$Y' = \beta + \sum_{i=1}^{m} \alpha_i U_i > 0 \tag{10}$$

where α_i is the component of the unit vector of the gradient of the limit-state function at the MPP.

And the component reliability is approximated by

$$R = \Phi(\beta) \tag{11}$$

3 The proposed method

We now discuss the proposed reliability method for k-out-of-n systems whose components equally share the system load before and after some components have failed. The components of the system are identical, and they are also subjected to other component loads, such as a preload. As a result, the component load is not directly proportional to the system load as some existing methods assume (Huamin 1998; Kuo and Zuo 2003). Let the system load be L. Since L is a random variable shared by all the components, the states of the components are therefore dependent. In this study, we consider static reliability, and this means that the reliability of a component or the system does not change over time. It is the situation where the limit-state function of a component is time independent. We at first provide necessary equations of the component reliability in Sec. 3.1 and then discuss the details of the proposed system reliability analysis in Sec. 3.2.

We use the following notations for our discussions.

- L: Total system workload
- $R_s(n, i)$: Reliability of an i-out-of-n system
- L_{n-i}: Component load with the system load shared by $n-i$ remaining components
- $Y(L_{n-i})$: Component state variable when the component load is L_{n-i}
- Y_i: The state variable of the i-th component

3.1 Component reliability analysis

Let the state of the i-th component be Y_i, $i = 1, 2, \ldots, n$. It is a function of the component load, which is from the system load

L, and other basic random variables $\mathbf{X}$, such as other component loads, material properties and component dimensions. For brevity, when we discuss a general component in the system, we drop the subscript of Y_i. To emphasize the component load, we also separate it from other basic random variables. Thus the limit-state function of a general component is written as

$$Y\left(L_{n-j}\right) = g\left(L_{n-j}, \mathbf{X}\right) \tag{12}$$

where L_{n-i} is the component load distributed from L, or the portion of the system load L shared by the component after j components have failed. L_{n-j} is given by

$$L_{n-j} = \frac{L}{n-j} \tag{13}$$

We assume that L and the components of $\mathbf{X}$ are independent. Recall that if $Y(L_{n-j}) > 0$, the component is operational; otherwise, the component fails. Then the component reliability is given by

$$R = \Pr\left\{Y\left(L_{n-j}\right) = g\left(L_{n-j}, \mathbf{X}\right) > 0\right\}$$
$$= \int_{g\left(L_{n-j}, \mathbf{X}\right)>0} f_L(l) f_{\mathbf{X}}(\mathbf{x}) dl d\mathbf{x} \tag{14}$$

where $f_L(l)$ is the PDF of L, and $f_{\mathbf{X}}(\mathbf{x})$ is the joint PDF of $\mathbf{X}$.

When FORM is used, the component reliability is computed by

$$R = \Pr\left\{Y'\left(L_{n-j}\right) = \beta\left(L_{n-j}\right) + \sum_{i=1}^{m} \alpha_i\left(L_{n-j}\right) U_i > 0\right\}$$
$$= \Phi\left[\beta\left(L_{n-j}\right)\right] \tag{15}$$

3.2 System reliability analysis

Before discussing the details of the proposed method, we summarize the assumptions we use in this study.

1) The system consists of n identical components. It is required that at least k components be operational for the system to work properly.
2) The system load is initially shared by all the components equally.
3) The system load is shared by surviving components equally after other components have failed. This results in an increased component load.
4) The system load and other basic random variables are independent.
5) In addition to the shared system load, components are also subjected to other component loads. Hence component loads are not directly proportional to the system load.

With the above assumes, the limit-state function of component i is given by

$$Y_i\left(L_{n-j}\right) = g\left(L_{n-j}, \mathbf{X}_i\right) \tag{16}$$

where $\mathbf{X}_i$, $i = 1, 2, \ldots, n$, are independent and identically distributed random vectors. In other words, $X_{\cdot j}$, $j = 1, 2, \ldots, m$, are independently and identically distributed. The PDF of $\mathbf{X}_i$, $i = 1, 2, \ldots, n$, is $f_{\mathbf{X}}(\mathbf{x})$ as shown in (14). Since all the components are identical, they have the same limit-state function $g(\cdot)$, which has the common random variable L. Then all the component states Y_i, $i = 1, 2, \ldots, n$, are dependent.

As will be discussed later, for dependent components that equally share the system load, more than one component can fail simultaneously; there are also a number of possible paths along which the system may fail. For these reasons, it is difficult to derive general equations that lead to automated or recursive algorithms. We hence discuss the system reliability analysis with various cases.

$k = n$

For this case, no failure is allowed. This case represents a series system. All components are required to be functional for the system to work properly; in other words, $\bigcap_{i=1}^{n} Y_i(L_n) > 0$. Then the system reliability is given by

$$R_s(n, n) = \Pr\left\{\bigcap_{i=1}^{n} Y_i(L_n) > 0\right\} \tag{17}$$

Recall that Y_i is the state variable of component i and is given by $Y_i(L_n) = g(L_n, \mathbf{X}_i)$. As discussed previously, all the component state variables $Y_i(L_n)$, $i = (1, 2, \ldots, n)$, are dependent. The joint PDF of $Y_i(L_n)$, $i = (1, 2, \ldots, n)$, is therefore required.

In Sec. 3.3, we will discuss how to obtain the joint PDF in order to calculate the system reliability with the use of FORM.

$k = n-1$

One component is allowed to fail. To better understand all the possible paths to the success of the system, we use a tree plot to show all the combinations of events that lead to the proper function of the system. The following notations are used in a tree plot.

- SG: The system in a good (operational) state.
- iF: i components fail.

The tree plot is given in Fig. 2, which shows two paths. The first one represents the case where all components are operational. This is the case of $k = n$ we have just discussed. The second branch indicates that initially all components share the system load with their portion L_n, and then one component fails. After this, the system load is redistributed, and the

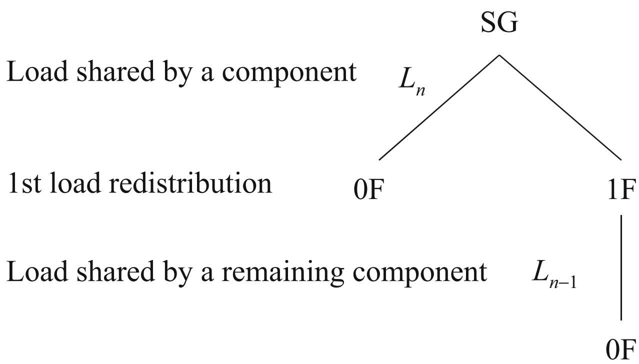

Fig. 2 Tree plot for $k = n - 1$

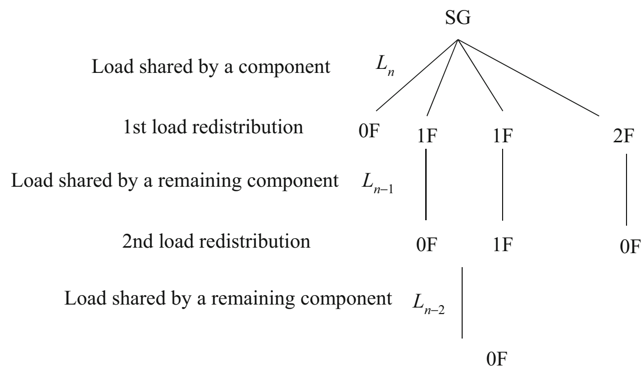

Fig. 3 Tree plot for k=n-2

component load increases to L_{n-1}, under which no further component failures occur.

For the second branch, we consider one of the possible combinations: the n-th component fails under load L_n. After the first load distribution, the component load becomes L_{n-1}, and the first $n - 1$ components do not fail. There are $C_n^{n-1} = n$ such combinations. Then the system reliability is given by

$$R_s(n, n{-}1) = \Pr\left\{\bigcap_{i=1}^{n} Y_i(L_n) > 0\right\} + n\Pr\left\{\left[\bigcap_{i=1}^{n-1} Y_i(L_n) > 0 \cap Y_n(L_n) < 0\right] \bigcap_{i=1}^{n-1} Y_i(L_{n-1}) > 0\right\} \tag{18}$$

The equation can be rearranged as

$$R_s(n, n{-}1) = R_s(n, n) + n\Pr\left\{\bigcap_{i=1}^{n-1} Y_i(L_{n-1}) > 0 \cap Y_n(L_n) < 0\right\} \tag{19}$$

In the rearrangement of the equation, we use $\Pr\{Y_i(L_n) > 0 \cap Y_i(L_{n-1}) > 0\} = \Pr\{Y_i(L_{n-1}) > 0\}$ because $L_{n-1} > L_n$. $Y_i(L_{n-1}) > 0$ automatically leads to $Y_i(L_n) > 0$.

As indicated in (19), the joint PDF of $Y_i(L_{n-1})$, $i = 1, 2, \ldots, n - 1$, and $Y_n(L_n)$ are required.

$k = n{-}2$

As shown in Fig. 3, at most, two components are allowed to fail. The first two branches are the same as the case of $k = n - 1$. The third branch represents two component failures. One component fails with load L_n, and one more component fails with load L_{n-1} after the first load redistribution. No further components fail after the second load redistribution when the load is L_{n-2}. In the equation below, for the third branch, the n-th component fails first followed by the failure of the (n-1)-th component. There are $n(n - 1)$ such combinations. The fourth branch indicates that two components fail before any load redistribution, and the n-th and (n-1)-th components are assumed to fail in the equation below. No more component failures occur after the first load redistribution. There are C_n^{n-2} such combinations. The equation is then given by

$$\begin{aligned} R_s(n, n{-}2) = {} & R_s(n, n{-}1) + \\ & + n(n{-}1)\Pr\left\{\left[\bigcap_{i=1}^{n-1} Y_i(L_n) > 0 \cap Y_n(L_n) < 0\right]\left[\bigcap_{i=1}^{n-2} Y_i(L_{n-1}) > 0 \cap Y_{n-1}(L_{n-1}) < 0\right] \bigcap_{i=1}^{n-2} Y_i(L_{n-2}) > 0\right\} \\ & + C_n^{n-2}\Pr\left\{\left[\bigcap_{i=1}^{n-2} Y_i(L_n) > 0 \cap Y_{n-1}(L_n) < 0 \cap Y_n(L_n) < 0\right]\left[\bigcap_{i=1}^{n-2} Y_i(L_{n-2}) > 0\right]\right\} \end{aligned} \tag{20}$$

The equation is rewritten as

$$\begin{aligned} R_s(n, n{-}2) = {} & R_s(n, n{-}1) + \\ & + n(n{-}1)\Pr\left\{\bigcap_{i=1}^{n-2} Y_i(L_{n-2}) > 0 \cap Y_{n-1}(L_n) > 0 \cap Y_{n-1}(L_{n-1}) < 0 \cap Y_n(L_n) < 0\right\} \\ & + C_n^{n-2}\Pr\left\{\bigcap_{i=1}^{n-2} Y_i(L_{n-2}) > 0 \cap Y_{n-1}(L_n) < 0 \cap Y_n(L_n) < 0\right\} \end{aligned} \tag{21}$$

This process can go on and on until k=1.

3.3 The use of FORM

As discussed in Sec. 3.2, we need the joint PDF of the component state variables under varying component loads. For example, to calculate the probability $R_s(n,n) = \Pr\left\{\bigcap_{i=1}^{n} Y_i(L_n) > 0\right\}$ in (17), we must know the joint CDF of $Y_i(L_n) = g(L_n, \mathbf{X}_i) = g(L/n, \mathbf{X}_i)$, $i = 1, 2, \ldots, n$, where $\mathbf{X}_i = [X_{i1}, X_{i2}, \ldots, X_{im}]$.

We now discuss how to use FORM to obtain the joint PDF of component states and then how to use the joint PDF to calculate the system reliability. As indicated in (6), the event that the i-th component works successfully after j components have failed is

$$Y'_i(L_{n-j}) = \beta_i(L_{n-j}) + \alpha_L(L_{n-j})U_L + \sum_{l=1}^{m} \alpha_{il}(L_{n-j})U_{il} > 0 \tag{22}$$

The component state now becomes $Y'_i(L_{n-j})$. We can then replace the original state variables $Y_i(L_{n-j})$with $Y'_i(L_{n-j})$. Then (17) is rewritten as

$$R_s(n,n) = \Pr\left\{\bigcap_{i=1}^{n} Y'_i(L_n) > 0\right\} = P_1 \tag{23}$$

Equation (19) is rewritten as

$$R_s(n,n{-}1) = R_s(n,n) + P_2 \tag{24}$$

where

$$P_2 = \Pr\left\{\bigcap_{i=1}^{n-1} Y'_i(L_{n-1}) > 0 \cap Y'_n(L_n) < 0\right\} \tag{25}$$

Equation (21) is rewritten as

$$R_s(n,n{-}2) = R_s(n,n{-}1) + n(n{-}1)P_3 + C_n^{n-2}P_4 \tag{26}$$

where

$$P_3 = \left\{\bigcap_{i=1}^{n-2} Y'_i(L_{n-2}) > 0 \cap Y'_{n-1}(L_n) > 0 \cap Y'_{n-1}(L_{n-1}) < 0 \cap Y'_n(L_n) < 0\right\} \tag{27}$$

and

$$P_4 = \Pr\left\{\bigcap_{i=1}^{n-2} Y'_i(L_{n-2}) > 0 \cap Y'_{n-1}(L_n) < 0 \cap Y'_n(L_n) < 0\right\} \tag{28}$$

Equation (22) shows that each individual component state variable Y' is the combination of standard normal random variables. Then any combination of component state variables follows a multivariate normal distribution. We now discuss how to obtain the multivariate PDFs to calculate the individual probabilities P_1 through P_4.

P_1

$Y'_i(L_n)$, $i = 1, 2, \ldots, n$, are needed to calculate P_1. We require $Y'_i(L_n) > 0$, $i = 1, 2, \ldots, n$. To use the CDF of a multivariate normal distribution easily, we define a new set of random variables with $\mathbf{Z} = [Z_i]_{i=1,n} = \left[-Y'_i(L_n)\right]_{i=1,n}$, $i = 1, 2, \ldots, n$. From (22), we have the mean vector of $\mathbf{Z}$

$$\boldsymbol{\mu}_1 = [\mu_{1i}]_{i=1,n} = [-\beta(L_n)]_{i=1,n} \tag{29}$$

And the covariance of $\mathbf{Z}$ is

$$\boldsymbol{\Sigma}_2 = \left[\rho_{2ij}\right]_{i,j=1,n} \tag{30}$$

where

$$\rho_{2ij} = \begin{cases} 1 & \text{if } i = j \\ \alpha^2(L_n) & \text{if } i \neq j \end{cases} \tag{31}$$

Then P_1 is given by the following CDF:

$$P_1 = \Phi(\mathbf{0}, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_2) \tag{32}$$

where $\Phi(\cdot)$ is the CDF of the multivariate normal distribution defined by $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

P_2

The calculation of P_2 requires the joint PDF of $Y'_i(L_{n-1})$, $i = 1, 2, \ldots, n-1$, and $Y'_n(L_n)$. Since we require $Y'_i(L_{n-1}) > 0$, $i = 1, 2, \ldots, n-1$, and $Y'_n(L_n) < 0$, we define a new set of random variables with $\mathbf{Z} = [Z_i]_{i=1,n}$ as

$$Z_i = \begin{cases} -Y'_i(L_{n-1}) & i = 1, 2, \ldots, n{-}1 \\ Y'_n(L_n) & i = n \end{cases} \tag{33}$$

Then the mean vector $\boldsymbol{\mu}_2 = [\mu_{2i}]_{i=1,n}$ of $\mathbf{Z}$ is

$$\mu_{2i} = \begin{cases} -\beta(L_{n-1}) & i = 1, 2, \ldots, n{-}1 \\ \beta(L_n) & i = n \end{cases} \tag{34}$$

And the covariance $\boldsymbol{\Sigma}_2 = [\rho_{2ij}]_{i,j=1,n}$ of $\mathbf{Z}$ is

$$\rho_{2ij} = \begin{cases} 1 & \text{if } i = j \\ \alpha^2(L_{n-1}) & \text{if } i \neq j, i, j = 1, 2, \ldots, n{-}1 \\ -\alpha(L_n)\alpha(L_{n-1}) & \text{if } i = 1, 2, \ldots, n{-}1, j = n; i = n, j = 1, 2, \ldots, n{-}1 \end{cases} \tag{35}$$

Then the system reliability is given by the following CDF:

$$P_2 = \Phi(\mathbf{0}, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) \tag{36}$$

P_3

As indicated in (27), P_3 involves $Y'_i(L_{n-1})$, $i = 1, 2, \ldots, n-2$, $Y'_{n-1}(L_n)$, $Y'_{n-1}(L_{n-1})$, and $Y'_n(L_n)$. We require $Y'_i(L_{n-1}) > 0$, $i = 1, 2, \ldots, n-2$, $Y'_{n-1}(L_n) > 0$,

$Y'_{n-1}(L_{n-1}) < 0$, and $Y'_n(L_n) < 0$, a new set of random variables with $\mathbf{Z} = [Z_i]_{i=1,\ n+1}$ is defined by

$$Z_i = \begin{cases} -Y'_i(L_{n-1}) & i = 1, 2, \ldots, n-2 \\ -Y'_{n-1}(L_n) & i = n-1 \\ Y'_{n-1}(L_{n-1}) & i = n \\ Y'_n(L_n) & i = n+1 \end{cases} \tag{37}$$

Then the mean vector $\boldsymbol{\mu}_3 = [\mu_{3i}]_{i=1,\ n+1}$ of $\mathbf{Z}$ is

$$\mu_{3i} = \begin{cases} -\beta(L_{n-1}) & i = 1, 2, \ldots, n-2 \\ -\beta(L_n) & i = n-1 \\ \beta(L_{n-1}) & i = n \\ \beta(L_n) & i = n+1 \end{cases} \tag{38}$$

And the covariance $\boldsymbol{\Sigma}_3 = [\rho_{3ij}]_{i,\ j=1,\ n+1}$ of $\mathbf{Z}$ is

$$\rho_{3ij} = \begin{cases} 1 & \textit{if } i = j \\ \alpha^2(L_{n-2}) & \textit{if } i \neq j, i, j = 1, 2, \ldots, n-2 \\ \alpha(L_{n-1})\alpha(L_n) & \textit{if } i = 1, 2, \ldots, n-2, j = n-1; j = n-1, i = n-1, , 2, \ldots, n-2 \\ -\alpha^2(L_{n-1}) & \textit{if } i = 1, 2, \ldots, n-2, j = n; i = n, j = i = 1, 2, \ldots, n-2 \\ -\left[\alpha(L_{n-1})\alpha(L_n) + \boldsymbol{\alpha}(L_{n-1})\boldsymbol{\alpha}^T(L_n)\right] & \textit{if } i = n-1, j = n; i = n, j = n-1 \\ \alpha^2(L_{n-1}) & \textit{if } i = n-1, j = n+1; i = n+1, j = n-1 \\ \alpha(L_{n-1})\alpha(L_n) & \textit{if } i = n, j = n+1; i = n+1, j = n \end{cases} \tag{39}$$

A dot product appears in (39), and the two associated vectors are $\boldsymbol{\alpha}(L_n) = [\alpha_j(L_n)]_{j=1,\ m}$ and $\boldsymbol{\alpha}(L_{n-1}) = [\alpha_j(L_{n-1})]_{j=1,\ m}$. The dot product is needed to calculate the covariance between $Y'_{n-1}(L_n) = \beta_{n-1}(L_n) + \alpha_L(L_{n-1}) U_{(n-1)}\ L + \sum_{l=1}^{m} \alpha_{(n-1)}\ l(L_n) U_{(n-1)}\ l$ and $Y'_{n-1}(L_{n-1}) = \beta_{n-1}(L_{n-1}) + \alpha_L(L_{n-1}) U_{(n-1)}\ L + \sum_{l=1}^{m} \alpha_{(n-1)}\ l\left(L_{n-j}\right) U_{(n-1)}\ l$.

Then P_3 is given by the following CDF:

$$P_3 = \Phi(\mathbf{0}, \boldsymbol{\mu}_3, \boldsymbol{\Sigma}_3) \tag{40}$$

P_4

We need the joint PDF of $Y'_i(L_n)$, $i = 1, 2, \ldots, n-2$, $Y'_{n-1}(L_n)$, and $Y'_n(L_n)$ to calculate P_4. Since we require $Y'_i(L_n) > 0$, $i = 1, 2, \ldots, n-2$, $Y'_{n-1}(L_n) < 0$, and $Y'_n(L_n) < 0$, we define a new set of random variables with $\mathbf{Z} = [Z_i]_{i=1,\ n}$ as follows:

$$Z_i = \begin{cases} -Y'_i(L_{n-2}) & i = 1, 2, \ldots, n-2 \\ Y'_{n-1}(L_n) & i = n-1 \\ Y'_n(L_n) & i = n-1 \end{cases} \tag{41}$$

The mean vector $\boldsymbol{\mu}_4 = [\mu_{4i}]_{i=1,\ n}$ of $\mathbf{Z}$ is given by

$$\mu_{4i} = \begin{cases} -\beta(L_{n-2}) & i = 1, 2, \ldots, n-1 \\ \beta(L_n) & i = n-1, n \end{cases} \tag{42}$$

The covariance $\boldsymbol{\Sigma}_4 = [\rho_{4ij}]_{i,\ j=1,\ n}$ of $\mathbf{Z}$ is

$$\rho_{4ij} = \begin{cases} 1 & \textit{if } i = j \\ \alpha^2(L_{n-2}) & \textit{if } i \neq j, i, j = 1, 2, \ldots, n-2 \\ -\alpha(L_n)\alpha(L_{n-2}) & \textit{if } i = 1, 2, \ldots, n-2, j = n-1; i = n-1, j = 1, 2, \ldots, n-2 \\ -\alpha(L_n)\alpha(L_{n-2}) & \textit{if } i = 1, 2, \ldots, n-2, j = n; i = n, j = 1, 2, \ldots, n-2 \\ \alpha^2(L_n) & \textit{if } i = n-1, j = n; i = n, j = n-1 \end{cases} \tag{43}$$

Then P_4 is given by the following CDF:

$$P_4 = \Phi(\mathbf{0}, \boldsymbol{\mu}_4, \boldsymbol{\Sigma}_4) \tag{44}$$

3.4 Computational considerations

The proposed method employs FORM and therefore requires the MPP search and the evaluation of a multivariate normal CDF. The MPP search optimization model in (8) can be solved with any constrained nonlinear optimization algorithm, such as the sequential quadratic programming method. MPP search algorithms (Du et al. 2004; Youn et al. 2005; Zhang and Der Kiureghian 1995) can also be used. Since the algorithms are specifically designed for the MPP, their efficiency, measured by the number of calls of a limit-state function, is in general less than a general optimization algorithm.

The other computational aspect is the calculation of the CDF of a multivariate normal distribution. It is a multi-dimensional integral of the PDF of the multivariate normal distribution. When the dimension is low, the integral can be calculated numerically (Drezner 1994; Drezner and Wesolowsky 1990; Genz 2004). When the dimension is high, a sampling method can be used to estimate the integral (Teng et al. 2015).

4 Examples

We use two examples to demonstrate the proposed method. The first example is a simple problem where the component limit-state function is linear with respect to two normally and independently distributed random variables. Example two involves a bolted flange connection system whose components have a nonlinear limit-state function with more random variables, some of which are non-normal variables.

For the comparison study on the accuracy, we also develop a computational procedure for Monte Carlo simulation (MCS). The details are provided in the appendix.

4.1 A 2-out-4 system with a linear limit-state function

The limit-state function of the component of the system is given by

$$Y(L_{n-i}) = g(L, \mathbf{X}) = C - \frac{L}{n-i} \tag{45}$$

where $\mathbf{X} = [C]$, and C is the resistance of the component and is normally distributed. $L/(n-i)$ is the load applied to the component after i components have failed. The system load L also follows a normal distribution. The distributions are given in Table 1. The units of the variables are provided in parentheses, where N standards for newtons.

The results from the proposed method are given in Table 2. To verify the result, we also perform MCS with a sample size of 10^7. As shown in Table 1, the results from both methods are very close. For this problem with a linear limit-state function, the difference comes from only numerical errors. The proposed method produces a very accurate system reliability prediction. The proposed method is also efficient, and it calls the limit-state function 88 times, and the CPU time is only 1.08 s compared with 645.34 s used by MCS. The high efficiency is particularly important for reliability-based design optimization because it will call the system reliability analysis repeatedly.

Table 1 Distributions of random variables

Variable	Mean	Standard deviation	Distribution
L (N)	80	10	Normal
C (N)	75	5	Normal

Table 2 System reliability for example one

	Proposed Method	MCS
R_s	0.9999986	0.9999989
Number of function calls	88	10,000,034
CPU time (seconds)	1.08	645.34

4.2 A 4-out-of 6 bolted flange connection system

Figure 4 shows a bolted flange connection system, which consists of six tension joints. It is required that at least four joints should work to ensure the functionality of the system.

The limit-state function of one bolt after i bolts have failed is given by

$$Y(L_{n-i}) = g(L, \mathbf{X}) = S_p - \frac{4\left(C\dfrac{L}{n-i} + F_p\right)}{\pi D^2} \tag{46}$$

where $\mathbf{X} = [S_p, C, F_p, D]$, in which S_p is the proof strength of the bolt, C is the stiffness constant (the faction of the load carried by a bolt), F_p is the preload acting on the bolt, and D is the diameter of the bolt. L is the system load. The variables are independent and their distributions are provided in Table 3. Some of the random variables are not normally distributed. Two cases are studied. Case 1 involves a lower system load and therefore higher system reliability while Case 2 involves a higher system load and therefore lower system reliability. In Table 3, for a normal or lognormal distribution, parameters 1 and 2 are a mean and standard deviation, respectively; for a Weibull distribution, parameters 1 and 2 are a scale parameter and shape parameter, respectively.

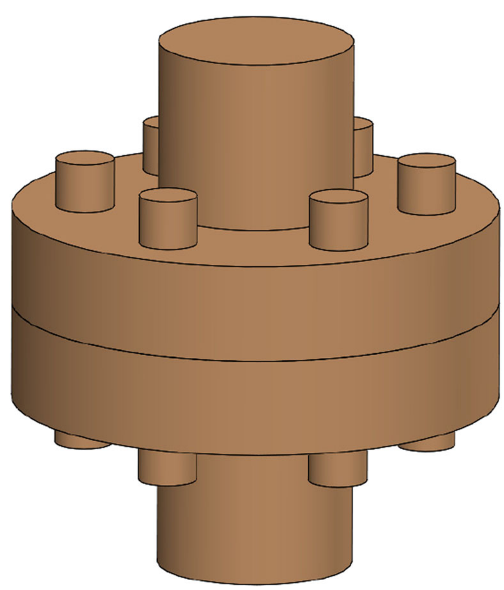

Fig. 4 A bolted flange connection system

Table 3 Distributions of random variables

Variable	Parameter 1	Parameter 2	Distribution
S_p(kpsi)	114	18	Weibull
C	0.38	0.01	Lognormal
L (kip) (Case 1)	75	5	Normal
L (kip) (Case 2)	85	5	Normal
F_p (kip)	15	1	Lognormal
D (in)	0.55	0.01	Normal

To evaluate the accuracy of the proposed method, the result is compared with that from MCS. The results of the two cases are provided in Tables 4 and 5. The MCS solution is obtained from the procedure described in the appendix.

The results show that the proposed method produces accurate solutions because they are close to those from MCS with 10^7 simulation. The accuracy is good for both high and low system reliability cases. The proposed method is also efficient because it uses fewer limit-state function calls and less CPU time.

With the physics-based approach, it is straightforward to change design variables if the system reliability predicted does not meet the reliability requirement. To improve reliability, we may select a different material to change the proof strength S_p, modify the component size to change the stiffness constant C, change the preload F_p, and change the diameter D. We can easily set the system reliability to its target value.

5 Conclusions

This study deals with the reliability prediction for systems that requires its k components out of the total of n identical components so that the system can work properly. Since components equally share the system load before and after some of the components have failed, component states are dependent, and the system load is redistributed with the increase component load.

The proposed method is based on the First Order Reliability Method, which linearizes the component limit-state state function at the most probable point. Then all the component states follow a multivariate normal distribution. This treatment allows for the easy and fast probability integrations for estimating the system reliability. Since there are no general equations for general k-out-of-n systems, equations for specific systems, including $k=n$, $k=n-1$, and $k=n-2$ systems, are derived. The proposed method can also accommodate components, which are subjected to other component loads, such as a preload, which does not come from the system load. It can therefore handle systems whose component loads are not directly proportional to the system load.

This study also develops a Monte Carlo simulation procedure to perform the same system reliability analysis. With a sufficiently large number of samples, the method is accurate. Due to this reason, the accuracy of the proposed method is compared with the Monte Carlo simulation. As indicated by the example, the proposed method is much faster than the Monte Carlo simulation.

The proposed method is based on the First Order Reliability Method, which may not be accurate for the situation where a component limit-state function is highly nonlinear with respect to the system load and other basic input variables. Our future work will be the improvement in accuracy, the investigation of $k \le n-3$ systems, and the extension of the method to time-dependent reliability problems.

Table 4 System reliability for Case 1

	Proposed Method	MCS
R_s	0.999882	0.999855
Number of function calls	267	10,295,808
CPU time (seconds)	2.43	1522.69

Table 5 System Reliability for Case 2

	Proposed Method	MCS
R_s	0.998346	0.997932
Number of function calls	252	10,686,380
CPU time (seconds)	1.77	1562.20

Appendix: Monte Carlo Simulation for *k*-of-of-*n* Systems

The Monte Carlo simulation (MCS) for a k-out-of-n system with $k=n-2$ is discussed herein. The procedure is summarized below.

1) Input n, k, distribution types and parameters of L and $\mathbf{X}$, and number of simulations N_{sim}.
2) Initialize the number of success $N_S=0$.
3) Generate N_{sim} samples of system load L.
4) For $i=1$ to N_{sim} by 1

   (a) Set number of component failures $n_F=0$
   (b) Generate n samples of basic random variables $\mathbf{X}$
   (c) Set component load shared by n components; call $g(\cdot)$ and obtain the state variables of the n components
   (d) Count the number of failures n_F

(e) If $n_F = 0$ (no component failure before the first load redistribution)

$$N_S = N_S + 1$$

End if.

If $n_F = 1$ (one component failure before the first load redistribution).

Delete the failed component.

Set component load shared by the remaining $n-1$ components; call $g(\cdot)$ and obtain the state variables of the $n-1$ components.

Count the number of failures n_{F1}.

If $n_{F1} = 0$ (no failure after the first load redistribution)

$$N_S = N_S + 1$$

EndIf.

If $n_{F1} = 1$ (one component failure after the first load redistribution).

Delete the failed component.

Set component load shared by the $n-2$ remaining components; call $g(\cdot)$ and obtain the state variables of the $n-2$ components.

Count the number of failures n_{F2}.

If $n_{F2} = 0$ (no failure after the first load redistribution)

$$N_S = N_S + 1$$

End if.

End if.

EndIf.

If $n_F = 2$ (two component failures before the first load redistribution).

Delete the two failed components.

Set component load shared by the $n-2$ remaining components; call $g(\cdot)$ and obtain the state variables of the $n-2$ components.

Count the number of failures n_{F1}.

If $n_{F1} = 0$ (no failure after the second load redistribution)

$$N_S = N_S + 1$$

End if.

EndIf.

EndFor

5) System reliability $R_S = N_S/N_{\text{sim}}$

With a sufficiently large value of N_{sim}, the system reliability obtained from MCS can be very accurate. It is the reason we use MCS to evaluate the accuracy of the proposed methods. MCS, however, is extremely computationally expensive because it requires a large number of limit-state function calls for highly reliable systems.

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