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RESEARCH PAPER

A two-step methodology to apply low-discrepancy sequences in reliability assessment of structural dynamic systems

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Abstract This study introduces various low-discrepancy sequences and then develops a new methodology for reliability assessment for structural dynamic systems. In this methodology, a two-step algorithm is first proposed, in which the most uniformly scattered point set among the low-discrepancy sequences is selected according to the centered L2-discrepancy (CL2 discrepancy) and then rearranged to minimize the generalized F-discrepancy (GF discrepancy). After that, the developed point set is incorporated into the maximum entropy method to capture the fractional moments for deriving the extreme value distribution for reliability assessment of structural dynamic systems. Numerical examples are investigated, where the results are compared with those obtained from Monte Carlo simulations, demonstrating the accuracy and efficiency of the proposed methodology.

Keywords Low-discrepancy sequence · Reliability · Structural dynamic systems · CL2 discrepancy · GF discrepancy · Extreme value distribution · Fractional moments

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1 Introduction

Reliability of structural dynamic systems is defined as the probability that the dynamic response of structures stays below a prescribed threshold over a given time interval (Goller et al. 2013). The definition of the above time-dependent probability, known as the first-passage reliability, has been a persistent challenge in the field of stochastic dynamics of structures. In the past decades, much progress has been made in first-passage reliability analysis methodologies. Among existing methods, the most popular method to evaluate such probability is the upcrossing method based on the Rice formula (Rice 1944), which has the advantage of efficiency, however, its accuracy may be poor (Madsen and Krenk 1984) for structural dynamic systems for the following reasons: (1) the Rice formula assumes that all the upcrossing rates are independent, which may produce large error (Hu and Du 2013), (2) the joint probability density function (PDF) of the target response is usually unavailable in practical problems. Though some empirical modifications have been made to remedy the drawbacks (Vanmarcke 1975; Preumont 1985), these modifications are limited to special problems. In contrast, Monte Carlo simulation (MCS) and its related methods, e.g. importance sampling, subset simulation (Au and Beck 2003) are the most widely-used tools for general uncertainty quantification of structural dynamic systems. The MCS and its related methods can evaluate the first passage reliability of structures accurately, but the required computational cost may not be affordable, especially when large-scale structural dynamic systems are considered. Further, several techniques based on the path integral (Iourtchenko et al. 2006; Naess et al. 2011; Kougioumtzoglou and Spanos 2012) have been developed to solve the stochastic differential equation to derive the first passage reliability straightforwardly. However, the computational efforts related to these techniques could be still demanding for practical structural dynamic systems (Spanos and Kougioumtzoglou 2014). Thus, the first passage reliability problem is still an on-going research topic (Hu and Du 2015a). Examples of the recent developments include the total probability theorem based method (Mourelatos et al. 2015), the nested extreme response surface method (Wang and Wang 2012), the importance sampling approach (Singh et al. 2011), the Markov chain method (Tont et al. 2010) and the Gamma process method (van Noortwijk et al. 2007), etc.

It is well-known that low-discrepancy sequences, also known as quasi-Monte Carlo simulations, have enjoyed increasing popularity in uncertainty quantification in recent years (Dai and Wang 2009). The particular interest in application of low-discrepancy sequences to structural reliability analysis, specifically to static reliability assessment, has arisen very recently (Dai and Wang 2009; Nie and Ellingwood 2004; Zhang et al. 2013). The implementation is often as simple as replacing the pseudo-random point set in MCS by the deterministic low-discrepancy point set in structural reliability analysis. It has been found that the low-discrepancy sequences always have performance superior to the pseudo-random number generators. The main reason behind this situation is that the low-discrepancy sequences are deterministic sequences, which have much better uniformity properties than the pseudo-random point set. In this regard, they may achieve a given accuracy with far fewer samples. In Ref. (Dai and Wang 2009), the authors combine low-discrepancy sequences with importance sampling to investigate four benchmark reliability problems. In their investigation, several thousands of points are required to simulate an event with a failure probability around 1×10^{-4}. This computational time could be called "efficient" in static scenario, however, it is indeed computationally inefficient in structural dynamic systems when the reliability assessment involves time consuming time-domain analysis of engineering structures. As a result, direct use of low-discrepancy sequences to assess the reliability of structural dynamic systems is not recommended.

Alternatively, the extreme value of structural dynamic response can be used to equivalently replace the first passage reliability problem (Chen and Li 2007; Li et al. 2007; Xu et al. 2016). The basic idea is to replace the structural dynamic response in the limit state function with its extreme value and convert the time-variant problem into its time-invariant counterpart. If the extreme value distribution (EVD) can be accurately estimated, the accuracy of reliability analysis will be higher than the upcrossing rate method since the independent upcrossing assumption is eliminated (Hu and Du 2015b). Besides, the reliability can be assessed directly by a simple integral once the EVD is available. Therefore, it is of paramount importance to evaluate the EVD for reliability assessment of structural dynamic systems, where the tradeoff of efficiency and accuracy is of great concern in practical engineering.

Since high-dimensional integrals may be encountered in the evaluation of EVD (Xu et al. 2012) the low-discrepancy sequences could be first employed to numerically evaluate the high-dimensional integrals and then the EVD is determined accordingly. In other words, the reliability of structural dynamic systems could be evaluated "indirectly" based on low-discrepancy sequences. This reveals the possibility of using the low-discrepancy sequences in the reliability analysis of structural dynamic systems. On the other hand, it is known that the more uniformity the integration points are, the more accuracy the numerical integration will be (Hua and Yuan 1981). The discrepancy, which characterizes the maximum error when replacing the ratio of the volumes of hyper-cubes by the ratio of the number of points contained in the volumes (Li and Chen 2009), is an appropriate index to measure the uniformity of a point set and bounds the error of the numerical integration. It is noted that the discrepancy is actually dependent on the dimension and the number of points. If the points are uniformly scattered, the discrepancy will be small. In this regard, a point set with lower discrepancy is highly desirable for the EVD evaluation, which involves numerical integration. Several low-discrepancy sequences have been developed, e.g. Halton, Hammersley, Sobol sequences, etc. for numerical integration and the sequence with the lowest discrepancy in a given dimension among these low-discrepancy sequences is preferred. Nevertheless, practical challenges of using low-discrepancy sequences for numerical integration still arise. First, it is quite difficult to compute the discrepancy, especially in high dimensions. Second, the discrepancy may be only meaningful for the numerical integration when the number of points is large, which indicates the computational effort could be prohibitively large if the low-discrepancy sequences are straightforwardly applied to evaluate the EVD. Thus, when the tradeoff between efficiency and accuracy is considered, further research on the use of low-discrepancy sequences for reliability assessment of structural dynamic systems is needed. In the present paper, the centered L2-discrepancy (CL2 dispcrepancy) and the generalized F-discrepancy (GF discrepancy) will be employed to overcome the mentioned challenges.

The objective of the present paper is to propose a two-step methodology to apply low-discrepancy sequences to reliability assessment of structural dynamic systems. This paper is arranged according to the following: In Section 2, five low-discrepancy sequences that have been successfully and widely applied in many scientific and engineering areas are introduced. Section 3 is devoted to develop the new methodology, in which a low-discrepancy sequence is selected and modified before performing dynamic reliability analysis. The CL2 and the GF discrepancies are employed for this purpose due to their convenience to be computed even in high dimensions. Then, the modified sequence is incorporated into the EVD evaluation technique to obtain the EVD for reliability analysis of structural dynamic systems in Section 4. Numerical examples to demonstrate the proposed methodology are carried out in Section 5. Concluding remarks are included in Section 6.

2 Brief introduction of low-discrepancy sequences

In this section, five well-studied low-discrepancy sequences will be introduced. Throughout this paper, the uniform random vector is denoted as $\mathbf{\Theta} = [\Theta_1, \Theta_2, \ldots, \Theta_d]^T$, where d is the number of random variables. These sequences generate random numbers between $[0,1]^d$, which can be then transformed to the distribution of interest.

Before introducing the low-discrepancy sequences, the concept of discrepancy needs to be defined. Usually, the star discrepancy of a point set $\wp$ is considered, which is defined by (Hua and Wang 2012)

$$D(N, \wp) = \sup_{\mathbf{v}\in C^d} \left[\frac{n(\wp, \mathbf{v})}{N} - V([\mathbf{0}, \mathbf{v}])\right] \tag{1}$$

where $C^d = [0, 1]^d$, $\mathbf{v} = (v_1, v_2, \ldots v_d) \in C^d$, $0 \le v_j \le 1, j = 1, 2, \ldots, d$, $n(\wp, \mathbf{v})$ is the number of the points satisfying $\|\Theta_q\| \le \|\mathbf{v}\|$ ($\|\cdot\|$ is the 2-norm), N is the total number of points in C^d and $V([\mathbf{0}, \mathbf{v}])$ is the volume of the hyper-rectangle $[\mathbf{0}, \mathbf{v}] = \prod_{j=1}^{d} [0, v_j]$ (Li and Chen 2009). A smaller value of $D(N, \wp)$ indicates the point sequence $\wp$ is distributed more uniformly and better represents the uniform distribution. It will be demonstrated in Section 3 that a more accurate solution will be achieved in the numerical integration if a sequence with lower discrepancy is applied.

2.1 Halton sequence

Let p be a prime number, and then any arbitrary integer q can be represented in the form

$$q = \sum_{i=0}^{M} a_i(q) \cdot p^i \tag{2}$$

where $a_i(q)$ s are the coefficients for integer q; and $M = [\log_p q]$, $[\log_p q]$ denotes the smallest integer that is larger or equal to $\log_p q$.

From this decomposition, function $\phi_p(q) \in [0, 1]$ for every q can be defined as

$$\phi_p(q) = \sum_{i=0}^{M} \frac{a_i(q)}{p^{i+1}} \tag{3}$$

Let $p_i (1 \le i \le d)$ be the first d prime numbers, which is the prime number seed. Then, the sequence

$$\wp = \left\{_q = \left(\phi_{p_1}(q), \phi_{p_2}(q), \ldots, \phi_{p_d}(q)\right), \right. \\ \left. \Theta q = 1, 2, \ldots\right\} \tag{4}$$

is called the Halton sequence.

It has been proven that the sequence formed by the first N ($N > \max(p_1, p_2, \ldots, p_d)$) points has the discrepancy (Halton 1960; Wang and Hickernell 2000)

$$D(N, \wp) \le N^{-1} \prod_{i=1}^{d} \frac{p_i \log(p_i N)}{\log p_i} \\ = O\left(N^{-1} (\log N)^d\right) \tag{5}$$

While the performance of standard Halton sequences is very good in low dimensions, problems with correlation have been observed between sequences generated from higher primes (Hess and Polak 2003). This can cause serious problems in the estimation of models with high-dimensional integrals. Various methods have been proposed to deal with this; one of the most prominent solutions is the Halton leaped sequence, which uses qL instead of q in the construction of the standard sequence such that (Kocis and Whiten 1997)

$$\wp = \left\{\Theta_q = \left(\phi_{p_1}(qL), \phi_{p_2}(qL), \ldots, \phi_{p_d}(qL)\right), \right. \\ \left. q = 1, 2, \ldots\right\} \tag{6}$$

where L is the leap, which is a prime number different from $p_i (1 \le i \le d)$. The best leap values were found to be31, 61, 149, 409, and 1949 (Kocis and Whiten 1997; Robinson and Atcitty 1999). In the present paper, the leap value 1949 is adopted. In a Halton sequence, each dimension uses a different base.

2.2 Harmmersley sequence

Hammersley sequence is constructed based on the Halton sequence, and could be considered as a variant of Halton sequence. Let $d \ge 2$ and $p_i (1 \le i \le d-1)$ be $d-1$ distinct prime numbers, the corresponding q -th d -dimensional Hammersley point is

$$\wp = \left\{\Theta_q = \left(\frac{2q-1}{N}, \phi_{p_1}(q), \phi_{p_2}(q), \ldots, \phi_{p_{d-1}}(q)\right), \right. \\ \left. q = 1, 2, \ldots, N\right\} \tag{7}$$

where N is the total number of Hammersley points.

The discrepancy gives

$$D(N, \wp) \le N^{-1} \prod_{i=1}^{d-1} \frac{p_i \log(p_i N)}{\log p_i} \\ = O\left(N^{-1} (\log N)^{d-1}\right) \tag{8}$$

2.3 Sobol sequence

Here, the Sobol sequence in just one dimension is illustrated. With the aim being to generate a sequence of values $\theta_1, \theta_2, \ldots, 0<\theta_q<1$, first a primitive polynomial of degree s in the field Z_2 could be expressed as (Joe and Kuo 2008; Joe and Kuo 2003; Bratley and Fox 1988)

$$\theta^s + a_1\theta^{s-1} + \ldots + a_{s-1}\theta + 1 \tag{9}$$

where each coefficient a_i is 0 or 1.

Then, a set of direction numbers $\nu_1, \nu_2, \ldots$ are defined by

$$\nu_q = m_q/2^q \tag{10}$$

where m_q is an odd integer, $0<m_q<2^q$.

Once a polynomial is chosen, its coefficients can be used to define a recurrence for calculating ν_q, thus

$$\begin{aligned}\nu_q &= a_1\nu_{q-1}\oplus a_2\nu_{q-2}\oplus\ldots\oplus a_{s-1}\nu_{q-s+1}\\ &\quad\oplus\nu_{q-s}\oplus\left[\nu_{q-s}/2^s\right], q>s\end{aligned} \tag{11}$$

where $\oplus$ denotes a bit-by-bit exclusive-or operation. For example, if $s=3$ and $a_1=0$, $a_2=a_3=1$, we have

$$\nu_q = \nu_{q-2}\oplus\nu_{q-3}\oplus\left[\nu_{q-3}/2^3\right], q>3 \tag{12}$$

Then, let $m_1=1, m_2=3, m_3=7$ with the corresponding $\nu_q=m_q/2^q$ i.e. $\nu_1=0.1$, $\nu_2=0.11, \nu_3=0.111$. The recurrence of ν_4, ν_5, ν_6 can be obtained by

$$\begin{aligned}\nu_4 &= \nu_2\oplus\nu_1\oplus\left[\nu_1/2^3\right]\\ &= 0.11\oplus0.1\oplus0.0001\\ &= 0.0101\end{aligned} \tag{13}$$

$$\begin{aligned}\nu_5 &= \nu_3\oplus\nu_2\oplus\left[\nu_2/2^3\right]\\ &= 0.111\oplus0.11\oplus0.00011\\ &= 0.00111\end{aligned} \tag{14}$$

$$\begin{aligned}\nu_6 &= \nu_4\oplus\nu_3\oplus\left[\nu_3/2^3\right]\\ &= 0.0101\oplus0.111\oplus0.000111\\ &= 0.101011\end{aligned} \tag{15}$$

Equivalently, the recurrence in terms of m_q can be expressed as (Joe and Kuo 2003)

$$\begin{aligned}m_q &= 2a_1m_{q-1}\oplus2^2a_2m_{q-2}\\ &\quad\oplus\ldots\oplus2^{s-1}a_{s-1}m_{q-s+1}\\ &\quad\oplus2^sm_{q-s}\oplus m_{q-s}\end{aligned} \tag{16}$$

Using the primitive polynomial of degree s, the values m_1, $m_2, \ldots, m_s$ can be chosen freely provided that each m_q $(q\le s)$ is odd and $m_q<2^q$. Then, subsequent values $m_{s+1}, m_{s+2}, \ldots$ are determined by the recurrence.

Finally, the sequence is generated by

$$\theta_q = b_1\nu_1\oplus b_2\nu_2\oplus\ldots \tag{17}$$

where $\ldots b_3b_2b_1$ refers to the binary representation of q.

Generalizing this procedure to d dimensions, the Sobol sequences can be generated accordingly

$$\begin{aligned}\wp = \Big\{&\boldsymbol{\theta}_q = \left(\theta_q^1, \theta_q^2, \ldots, \theta_q^d\right),\\ &q=1,2,\ldots\Big\}\end{aligned} \tag{18}$$

The detailed algorithm for generating Sobol sequences is elaborately explained in Ref. (Bratley and Fox 1988).

For Halton and Hammersley sequences, large prime numbers may be involved in high-dimensional problems. Besides, the seeds $p_i (1\le i\le d)$ are different from each other. In contrast, the Sobol sequence uses the same prime number, which is 2, in all dimensions as the seed to generate points

$$q = \sum_{i=0}^{M} a_i(q)\cdot2^i \tag{19}$$

It has been proven that Sobol sequence achieves the following discrepancy (Bratley and Fox 1988)

$$\begin{aligned}D(N,\wp)&\le C^dN^{-1}(\log N)^d\\ &\quad+O\left(N^{-1}(\log N)^{d+1}\right)\end{aligned} \tag{20}$$

where $C^d=2^d/(d!\times(\log2)^d)$.

2.4 Faure sequence

Faure sequence is a permutation of Halton sequence, but it uses the same base p for each dimension (Faure 1992). The base p can be chosen as the smallest odd prime integer such that $p\ge d$. It should be noted that the j th dimension of a d dimensional Faure sequence is different from that of a d' $(d\ne d')$ Faure sequence because the base p is different. First, the following transformation T is performed

$$T\left[\phi_p(q)\right] = \sum_{i=0}^{M}\frac{b_i(q)}{p^{i+1}} \tag{21}$$

where $b_i(q)=\sum_{l=i}^{M} C_l^i a_l \bmod p$ and C_l^i denote binomial coefficients and coefficients $b_i(q)$ s are actually a permutation of the $a_i(q)$.

The Faure sequence is defined by using successive transformations:

$$\begin{aligned}\wp = \Big\{&\boldsymbol{\theta}_q = \left(\phi_p(q), T\left[\phi_p(q)\right], \ldots, T^{d-1}\left[\phi_p(q)\right]\right),\\ &q=1,2,\ldots\Big\}\end{aligned} \tag{22}$$

The discrepancy of the sequence satisfies

$$D(N,\wp)\le C^dN^{-1}(\log N)^d \tag{23}$$

where C^d is a constant dependent on d and p (Brandimarte 2014)

$$C^d = \frac{1}{d!} \cdot \frac{(p-1)^d}{2\log(p)} \tag{24}$$

2.5 Niederreiter sequence

Let us define an elementary interval in base p as an interval of the form

$$E = \prod_{j=1}^{d} \left[\frac{A_j}{p^{D_j}}, \frac{A_j+1}{p^{D_j}} \right) \tag{25}$$

where A_j and D_j are integers satisfying $D_j \geq 1$ and $0 \leq A_j \leq D_j - 1$ (Tuffin 1996).

Let $0 \leq t \leq m$ be integers. A (t, s) sequence in base p is a sequence θ_q if, for all integers $k \geq 0$, such that for every interval in base p of volume $1/p^{m-t}$, there are exactly p^t points among the p^m with $kp^m \leq q \leq (k+1)p^m$.

The Niederreiter sequence is a d dimensional (t, s) sequence in base p defined as

$$\wp = \left\{ \theta_q = \left(\sum_{i=1}^{M} \frac{y_{q,i}^{(1)}}{p^{i+1}}, \ldots, \sum_{i=1}^{M} \frac{y_{q,i}^{(d)}}{p^{i+1}} \right), \right. \\ \left. q = 1, 2, \ldots, \right\} \tag{26}$$

with

$$y_{q,i}^{(j)} = \sum_{r=0}^{M} c_{i,r}^{j} a_r(q) \tag{27}$$

where $c_{i,r}^j$ are pre-calculated elements (see Ref. (Harald 1992)).

The discrepancy of the sequence satisfies

$$D(N, \wp) \leq N^{-1} (\log N)^d \tag{28}$$

The general differences among these low-discrepancy sequences are discussed and can be found in Ref. (Tuffin 1996). Figure 1 shows the two-dimensional scatter plots with the same sample size 500 for these low-discrepancy sequences, the star discrepancies of which are 0.0056, 0.0024, 0.0076, 0.0080 and 0.0056, respectively. In this regard, the Harmmersley sequence, which possesses the lowest discrepancy, is regarded as the most uniformly distributed sequence in this case. However, when the dimension becomes large, it is always very difficult to compute the star discrepancies. Therefore, efficient methods for computing the discrepancies need to be investigated.

3 The proposed two-step methodology using low-discrepancy sequences

The low-discrepancy sequences above have been widely applied in many fields of numerical integration analysis. Since there are available subroutines written in MATLAB ® to generate these low-discrepancy sequences (Burkardt 2015), they are quite easy to implement in a computer-aided analysis. Another advantage of using these low-discrepancy sequences is that their power degrades little with increasing dimension. The low-discrepancy sequences, which are deterministic sequences, combine the advantage of a random sequence where points can be added incrementally, with the advantage of a lattice where there are no regions of points with higher density than others. Notwithstanding, some practical difficulties still emerge:

(1) The first challenge is that how to decide which sequence should be adopted to achieve the best result when the total number of points N is given.
(2) Another challenge is that the theoretical bound of the discrepancy may only have meaning at large values of N for a relatively large dimension d.

To overcome these challenges, a two-step methodology is developed.

3.1 Step 1

As is mentioned, the discrepancy is a measure of the sequence $\wp$. If a smaller value of $D(N, \wp)$ is considered, more accurate solution may be achieved with the same number of points (Dai and Wang 2009), which is guaranteed by the well-known Koksma-Hlawka inequality (Hua and Wang 2012)

$$\left| \int_{C^d} f(\theta) d\theta - \frac{n}{N} \sum_{q=1}^{N} f(\theta_q) \right| \\ \leq TV(f) \cdot D(N, \wp) \leq TV(f) \cdot D_B \tag{29}$$

where $TV(f)$ is the total variation of the function f and D_B is the theoretical bound of the discrepancy.

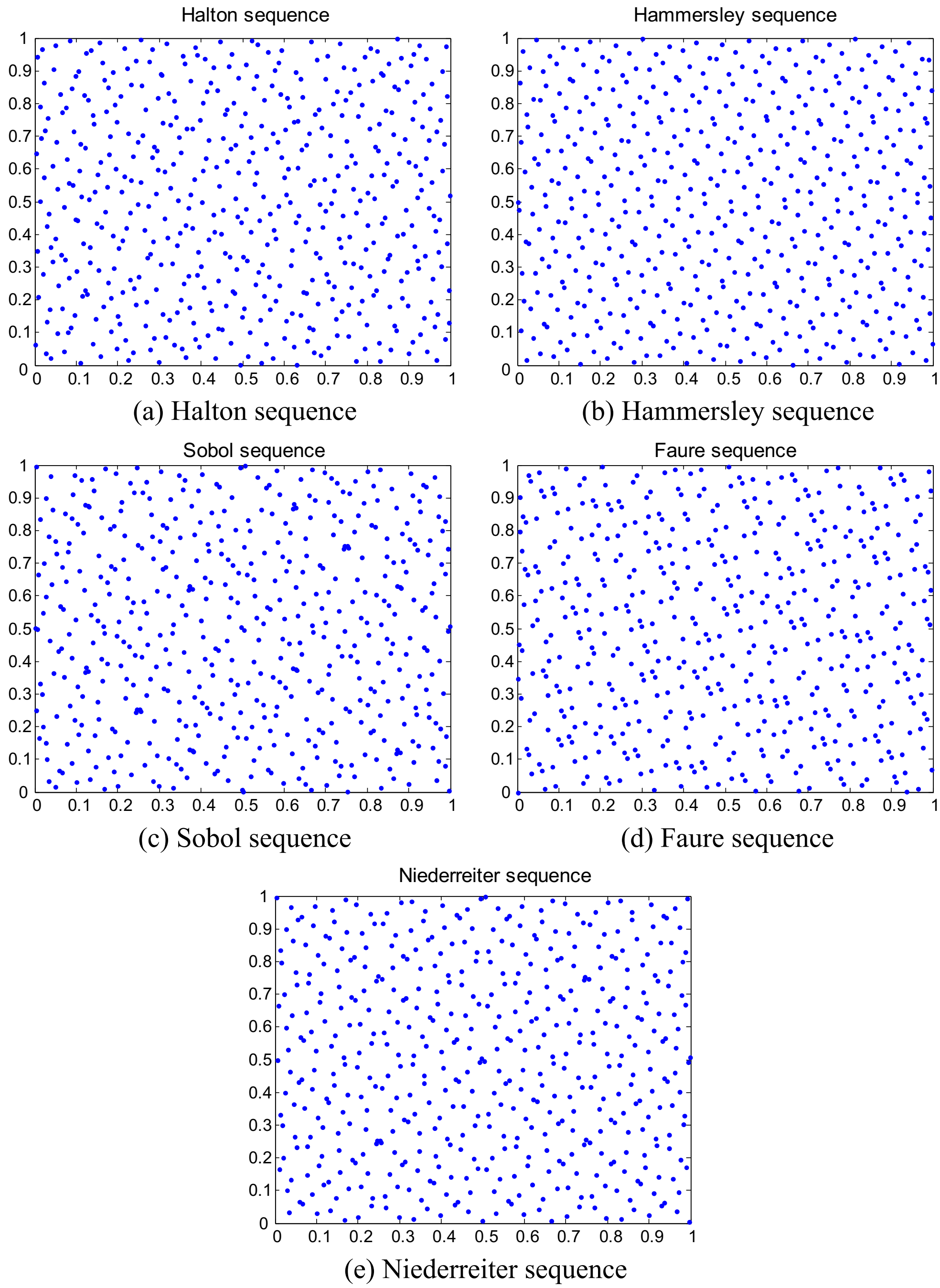
Since a non-uniform random vector, denoted as $\tilde{\boldsymbol{\Theta}} = [\tilde{\Theta}_1, \tilde{\Theta}_2, \ldots, \tilde{\Theta}_d]^T$ can be transformed to be uniform one, we also have

$$\begin{aligned} &\left| \int_{\Omega_{\tilde{\boldsymbol{\Theta}}}} f(\tilde{\theta}) d\tilde{\theta} - \frac{1}{N} \sum_{q=1}^{N} f(\tilde{\theta}_q) \right| \\ &= \left| \int_{C^d} f(R^{-1}[\theta]) d\theta - \frac{1}{N} \sum_{q=1}^{N} f(R^{-1}[\theta_q]) \right| \\ &= \left| \int_{C^d} G(\theta) d\theta - \frac{1}{N} \sum_{q=1}^{N} G(\theta_q) \right| \\ &\leq TV(G) \cdot D(N, \wp) \end{aligned} \tag{30}$$

where $G(\theta) = f(R^{-1}[\theta])$ and R^{-1} is the transformation.

The equations above indicate that the discrepancy $D(N, \wp)$ bounds the error of the multi-integral. However, the effort to compute the discrepancies, except for some analytically elegant results for special cases, is usually prohibitively large due to the so-called "curse of dimensionality".

Fig. 1 Two dimensional scatter plots of different sequences



Alternatively, in the present paper, the centered L2 –discrepancy (Hickernell and Wang 2002), which is denoted as CL2 discrepancy herein, is suggested as the index to determine which sequence poses the most uniform property.

The formula of CL2 discrepancy reads

$$\begin{aligned}\mathrm{CL}_2(N,\wp)^2 &= \left(\frac{13}{12}\right)^d - \frac{2}{N}\\ &\sum_{k=1}^{N}\prod_{j=1}^{d}\left(1+\frac{1}{2}\left|\theta_{j,k}-0.5\right|-\frac{1}{2}\left|\theta_{j,k}-0.5\right|^2\right)\\ &+\frac{1}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\prod_{i=1}^{d}\left[1+\frac{1}{2}\left|\theta_{i,k}-0.5\right|+\frac{1}{2}\left|\theta_{i,j}-0.5\right|-\frac{1}{2}\left|\theta_{i,k}-\theta_{i,j}\right|\right]\end{aligned} \tag{31}$$

The advantages of using CL2 discrepancy are

(a) invariant to the rotation of the coordinate system,
(b) convenient for computation.

Similar to (29) and (30), there exists (Dick and Pillichshammer 2010; Song and Chen 2015)

$$\begin{aligned}&\left|\int_{C^d} f(\theta)d\theta - \frac{1}{N}\sum_{q=1}^{N} f\left(\theta_q\right)\right|\\ &\leq \|f\|_2 \cdot \mathrm{CL}_2(N,\wp)\end{aligned} \tag{32}$$

Table 1 CL2 discrepancy (d = 5)

Number of points Sequence	100	200	300	400
Halton	0.0426	0.0252	0.0178	0.0142
Harmmersley	0.0371	0.0210	0.0149	0.0116
Sobol	0.0485	0.0271	0.0182	0.0143
Faure	0.0419	0.0270	0.0198	0.0146
Niederreiter	0.0496	0.0282	0.0189	0.0146

$$\left|\int_{\Omega_{\tilde{\Theta}}} f\left(\tilde{\theta}\right) d\tilde{\theta} - \frac{1}{N}\sum_{q=1}^{N} f\left(\tilde{\theta}_q\right)\right| \le \|G\|_2 \cdot \mathrm{CL}_2(N, \wp) \quad (33)$$

where $\|f\|_2$ and $\|G\|_2$ are the quantities similar to the total variation of the function.

Therefore, the CL2 discrepancy can be utilized as the index to select "the most uniform" point set among the low-discrepancy sequences above as the basic point set, which is denoted as $\wp_S = \{\theta_q = (\theta_{q,1}, \theta_{q,2}, \ldots, \theta_{q,d}), q = 1, 2, \ldots, N\}$. Then, the basic point set can be transformed to be the point set in the interested distribution domain, which is denoted as $\tilde{\wp}_S = \left\{\tilde{\theta}_q = \left(\tilde{\theta}_{q,1}, \tilde{\theta}_{q,2}, \ldots, \tilde{\theta}_{q,d}\right), q = 1, 2, \ldots, N\right\}$.

The CL2 discrepancy of each point set when d = 5 with different numbers of samples (N = 100, 200, 300 and 400) are listed in Table 1. It can be seen that the CL2 discrepancy generally decreases when the number of points increases, which indicates "more uniformity" can be reached by increasing the number of points. Table 2 shows the CL2 discrepancy of each point set with a fixed number of points (N = 200) but increasing dimension (d = 2,5,10 and 13). It is noted that the CL2 discrepancy increases with the dimension, indicating the uniformity of low-discrepancy sequences with a fixed number of points may get worse as the dimension becomes large.

3.2 Step 2

The bound in (29) gives no meaningful information until a very large number of points is used. Considering a realistic engineering problem in which a multiple degree of freedom

Table 2 CL2 discrepancy (N = 200)

Dimension Sequence	2	5	10	13
Halton	0.0071	0.0252	0.1058	0.1910
Harmmersley	0.0045	0.0210	0.0999	0.1862
Sobol	0.0073	0.0271	0.1040	0.1905
Faure	0.0094	0.0270	0.3362	0.5851
Niederreiter	0.0062	0.0282	0.1324	0.2632

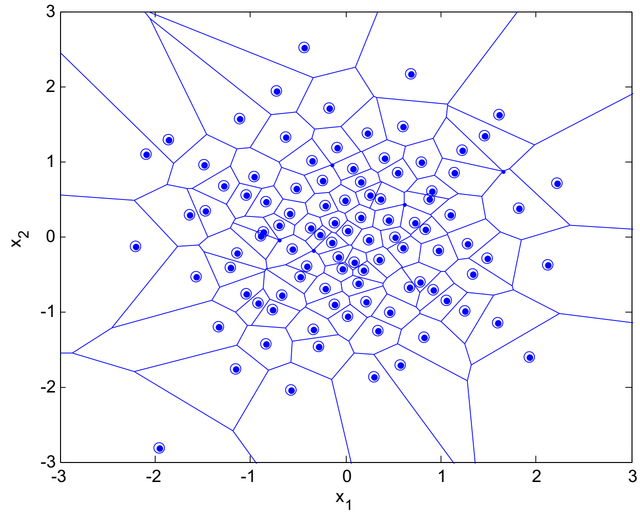


Fig. 2 Voronoi cells in two dimensional standardized normal random-variate space

(MDOF) system under dynamic excitations with possibly strong nonlinearity and several random variables are involved, the computational cost of such stochastic dynamic systems may be intractable.

In the original low-discrepancy sequences, all the generated points are deemed equally important for propagation/transformation to the domain of interest. When transformed to the domain of interest, however, the original sequences actually no longer have the characteristics of low-discrepancy. This could be the reason for the prohibitive computational cost of using these sequences for structural dynamic problems with a relatively large dimension d. To overcome the difficulty, the selected sequence $\wp_S$ (or $\tilde{\wp}_S$) together with their equal weights, say $1/N$, according to the CL2 discrepancy needs to be rearranged. One of the possible ways of improving the accuracy and efficiency is constructing non-equal weights for the sequence (Chen and Zhang 2013).

Further, the points in the selected sequence are usually not adequately uniformly scattered in the transformed space. To characterize the non-uniformity, the non-equal weight for each point needs to be determined. To this end, the space $\Omega_{\tilde{\Theta}}$ is partitioned by adopting the selected sequence $\tilde{\wp}_S = \{\tilde{\theta}_q = \left(\tilde{\theta}_{q,1}, \tilde{\theta}_{q,2}, \ldots, \tilde{\theta}_{q,d}\right), q = 1, 2, \ldots, N\}$ as the nucleus of the Voronoi cells (Conway and Sloane 2013), which are defined as

$$\Omega_q = \left\{\tilde{\theta} \in \tilde{\Theta} \middle| s\left(\tilde{\theta}, \tilde{\theta}_q\right) \le s\left(\tilde{\theta}, \tilde{\theta}_j\right) \textit{ for all } j \ne q\right\} \quad (34)$$

where $s\left(\tilde{\theta}, \tilde{\theta}_q\right)$ denotes the distance between the points $\tilde{\theta}$ and $\tilde{\theta}_q$.

Figure 2 illustrates the Voronoi cells in the two-dimensional standardized normal random-variate space.

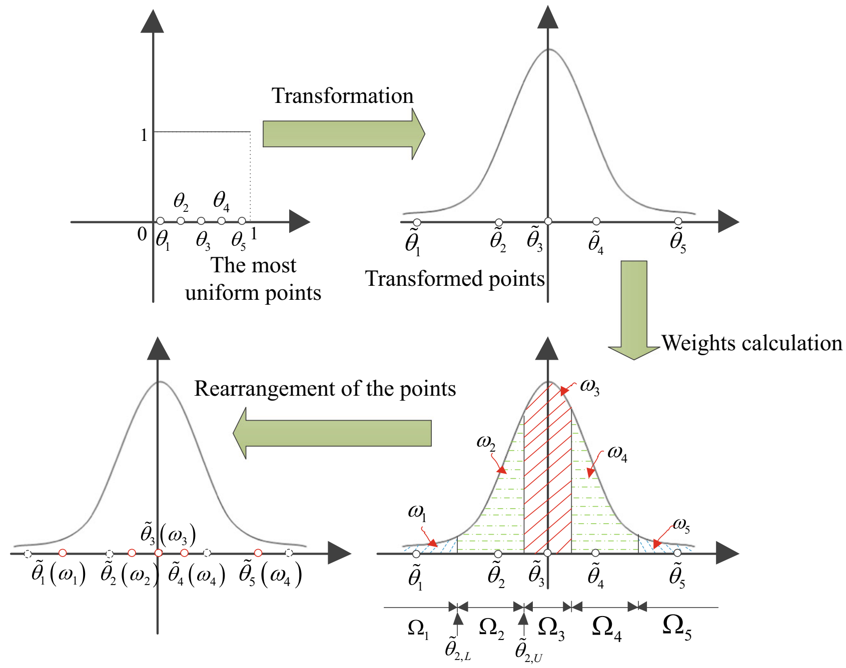


Fig. 3 One-dimensional case for illustration

Clearly, $\Omega_q, q = 1, 2, \ldots, N$ is considered as a partition of the distribution domain $\Omega_{\tilde{\Theta}}$, namely

$$\cup_{q=1}^{N}\Omega_q = \Omega_{\tilde{\Theta}}, \quad \Omega_q \cap \Omega_k = \varnothing, \quad for\ q \neq k \tag{35}$$

which means the partitions are mutually exclusive and exhaustive of the entire domain.

Then, the positive weight, which is the assigned probability, for each point is (Chen et al. 2009)

$$\omega_q = \int_{\Omega_q} p_{\Omega_{\tilde{\Theta}}}\left(\tilde{\theta}\right) d\tilde{\theta} \tag{36}$$

A numerical algorithm to calculate the integral for the weight (36) can be found in Ref. (Chen et al. 2009).

Thereby, the extended Koksma-Hlavka inequality (29) can be represented as (Chen and Zhang 2013)

$$\begin{aligned}&\left|\int_{\Omega_{\tilde{\Theta}}} f\left(\tilde{\theta}\right) d\tilde{\theta} - \sum_{q=1}^{N} \omega_q f\left(\tilde{\theta}_q\right)\right| \\ &\leq TV(G)\cdot D_{GF}\left(N, \tilde{\wp}_S\right)\end{aligned} \tag{37}$$

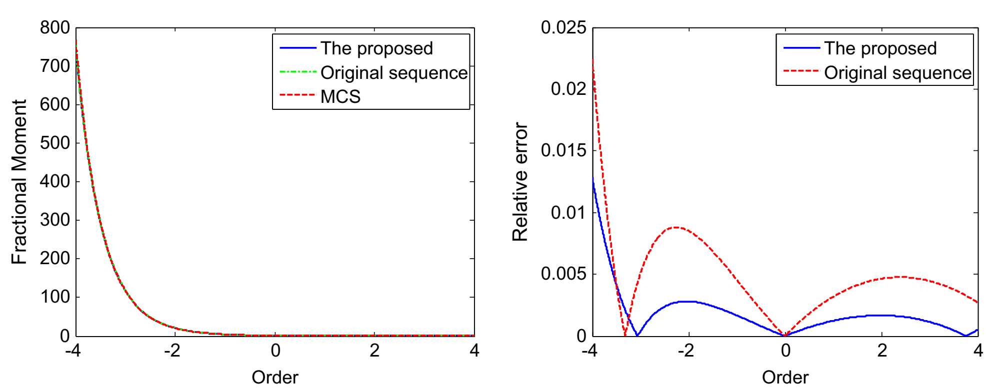
where $D_{GF}(N, \wp)$ denotes the generalized F-discrepancy (GF discrepancy), which is defined as

$$D_{GF}\left(N, \tilde{\wp}_S\right) = \max_{1\leq j\leq d}\left(D_{F,j}\left(N, \tilde{\wp}_S\right)\right) \tag{38}$$

where

$$D_{F,j}\left(N, \tilde{\wp}_S\right) = \sup_{\tilde{\theta}\in\mathbb{R}}\left|F_{N,j}\left(\tilde{\theta}\right) - F_j\left(\tilde{\theta}\right)\right| \tag{39}$$

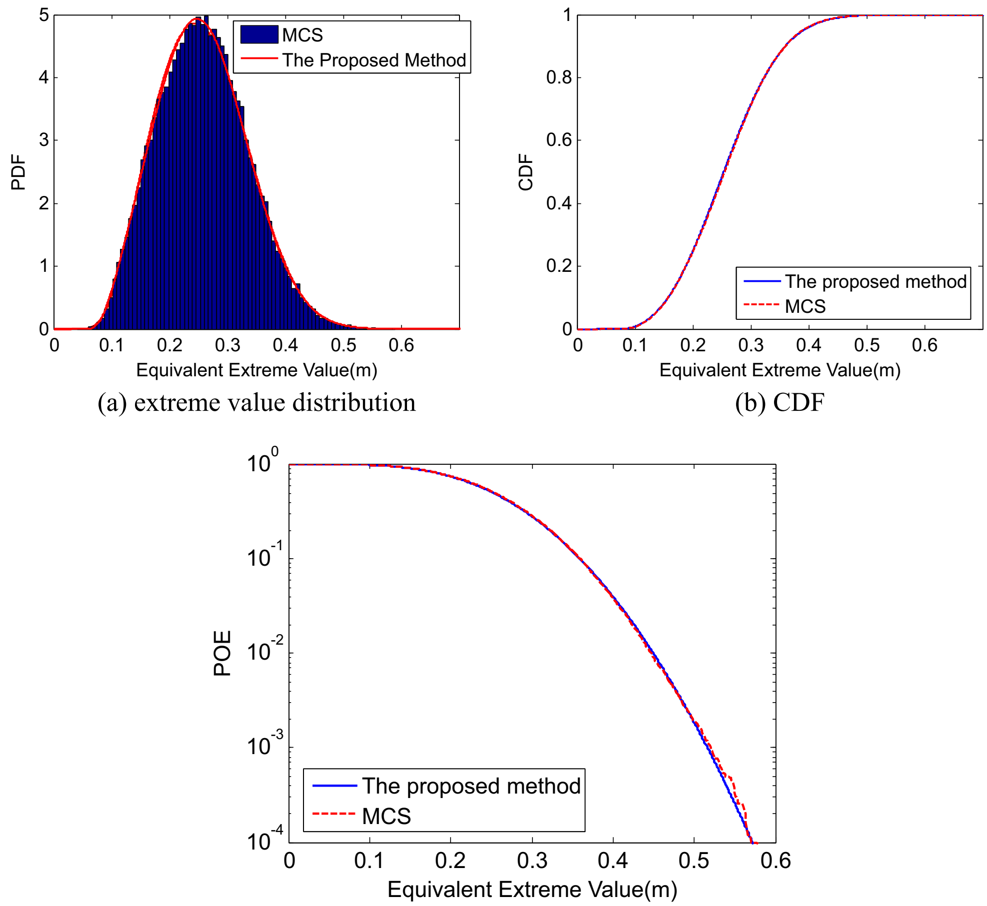
is the marginal F-discrepancies in the direction $\tilde{\theta}_j, j = 1, 2, \ldots, d$. Herein, $F_j\left(\tilde{\theta}\right)$ denotes the marginal



(a) fractional moments (b) relative error

Fig. 4 Fractional moments of extreme value (SDOF)

Fig. 5 Reliability assessment (SDOF)



(a) extreme value distribution (b) CDF

(c) POE

cumulative density function (CDF) of $\tilde{\theta}_j$ and $F_{N,j}(\tilde{\theta})$ is the empirical marginal CDF of $\tilde{\theta}_j$

$$F_{N,j}\left(\tilde{\theta}\right) = \sum_{q=1}^{N} \omega_q \cdot I\left\{\tilde{\theta}_{q,j} \le \tilde{\theta}\right\} \tag{40}$$

where $\tilde{\theta}_{q,j}$ is the j-th component of $\tilde{\Theta}_q$.

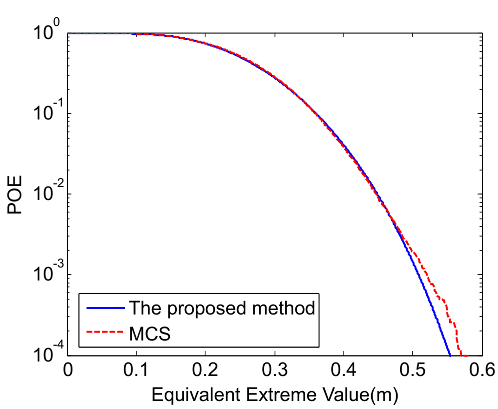


Fig. 6 POE curve without Step 2 (SDOF)

It is noted that the computation of $D_{GF}(N, \tilde{\wp}_S)$ only involves d one-dimensional empirical CDF evaluations, which could be very efficient regardless of the dimension d. In addition, it has been proven that (Chen and Zhang 2013)

$$D_{GF}\left(N, \tilde{\wp}_S\right) \le D\left(N, \tilde{\wp}_S\right) \tag{41}$$

which indicates that the sequence with unequal weights can achieve lower error bound compared to that of the equal-weighted sequence using the same number of points N. In other words, the computational accuracy could be significantly improved by using the sequence with unequal weights when the same efficiency is achieved. In addition, the smaller $D_{GF}(N, \wp)$ is, the better the accuracy of the numerical integration would be. It is also worth pointing out that the sequence with the smaller $D(N, \wp)$ is preferred, which can further reduce $D_{GF}(N, \wp)$ according to (41). This could illustrate the reason why we need to select "the most uniform" sequence in accordance with the CL2 discrepancy in the first step. Therefore, the GF discrepancy is adopted as the objective function in optimally rearranging the sequence $\tilde{\Theta}_q$, $q = 1, 2, \ldots, N$ (the coordinate of each point). The optimization problem is conducted such that

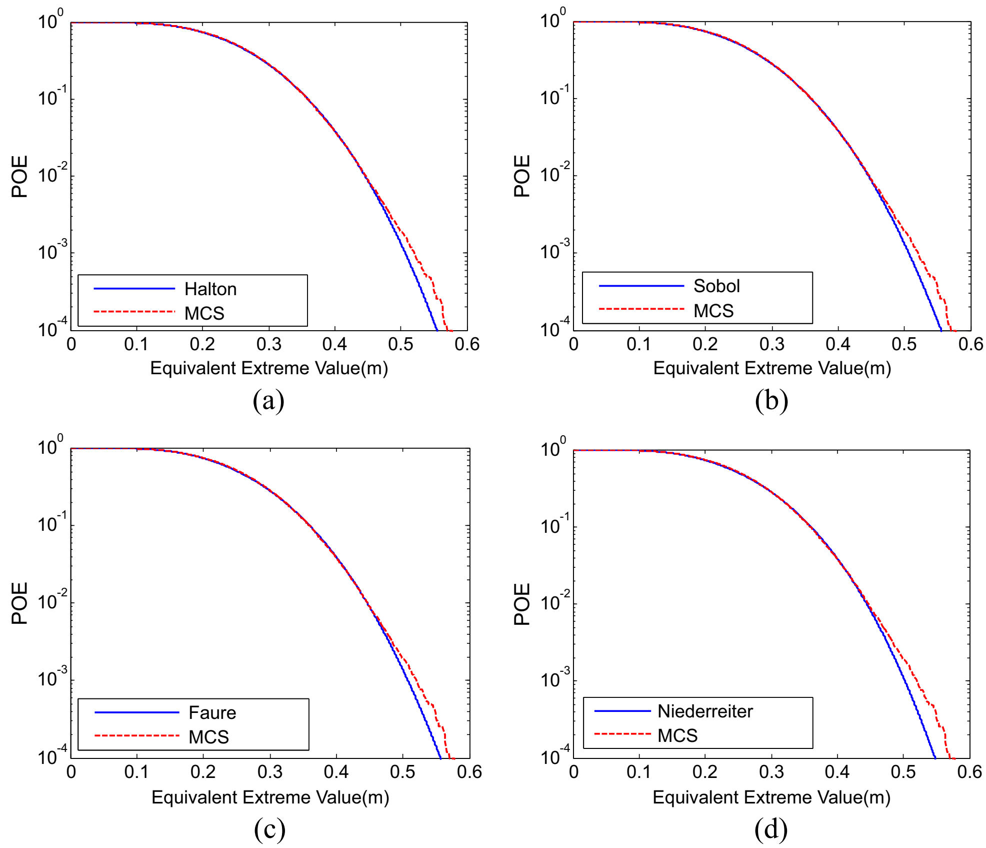


Fig. 7 POE curves by other low-discrepancy sequences (SDOF)

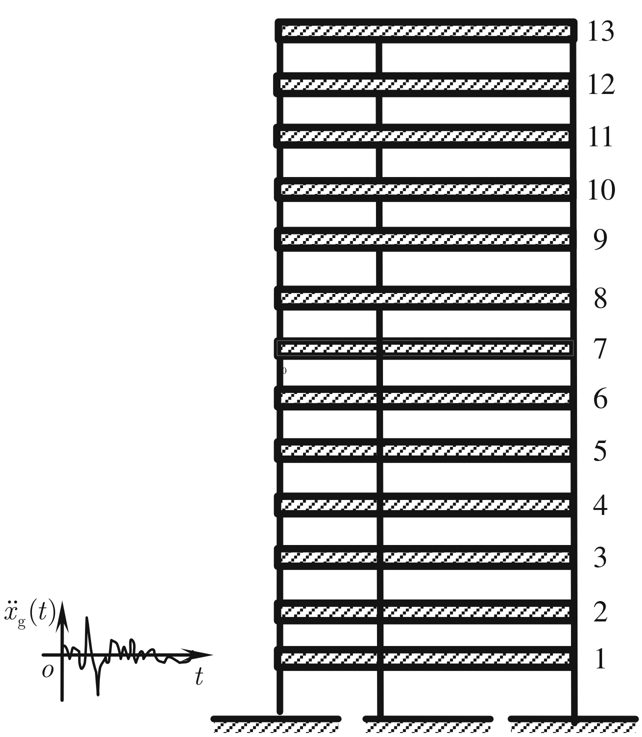


Fig. 8 A 13 storey shear frame structure

$$\begin{array}{ll} \textit{find} & \tilde{\theta}_q \\ \text{minimize} & D_{GF} \\ s.t. & \mu_j^l \le \tilde{\theta}_{q,j} \le \mu_j^u \end{array} \tag{42}$$

where μ_j^l and μ_j^u are the truncated lower and upper bounds for $\tilde{\Theta}_j$.

In practice, some approximation methods can be employed (Chen et al. 2016) for simplifying the optimization problem above. Although the resulted point set is usually sub-optimal, it still performs quite well to minimize the GF discrepancy significantly. Then, the rearranged points $\tilde{\theta}_q$ and their corresponding weights ω_q, $q = 1, 2, \ldots, N$, are formulated based on the low-discrepancy sequences.

To make this point clearer, a one-dimensional case with 5 points are used for the illustration, which is shown in Fig. 3. First, the uniform point set ($\{\theta_1, \theta_2, \ldots, \theta_5\}$) is transformed to be the point set ($\{\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_5\}$) using the probability distribution of interest. Then, the distribution is partitioned via Voronoi cells $\{\Omega_1, \Omega_2, \ldots, \Omega_5\}$ and the weight for each point can be calculated by (36). In the one dimensional case, the Voronoi cell of the point $\tilde{\theta}_q$ is the interval $\Omega_q = [\tilde{\theta}_{q,L}, \tilde{\theta}_{q,U}]$ and the weight for each point is the area over the distribution

Table 3 Structural parameters

Storey number	The lumped mass ($\times 10^5 kg$)	The lateral stiffness ($\times 10^8 N/m$)
1	3.487	0.882
2	3.852	0.875
3	3.225	1.758
4	2.887	1.854
5	2.667	1.662
6	2.558	1.654
7	2.558	1.618
8	2.558	1.618
9	2.558	1.618
10	2.558	1.618
11	2.558	1.618
12	2.558	1.618
13	2.558	1.618

in the interval Ω_q, i.e. $\omega_q = \int_{\Omega_q} p_{\Omega_{\tilde{\Theta}}}\left(\tilde{\theta}\right) d\tilde{\theta} = \int_{\tilde{\theta}_{q,L}}^{\tilde{\theta}_{q,U}} p_{\Omega_{\tilde{\Theta}}}\left(\tilde{\theta}\right) d\tilde{\theta}$. Finally, the positions of these points are rearranged to minimize the GF discrepancy.

To summarize, the two-step methodology is proposed below:

Step 1: Select the most uniformly scattered sequence among the introduced low-discrepancy sequences according to the CL2 discrepancy as the basic point set. Then, transform the basic point set to the point set in the interested distribution domain.
Step 2: Calculate the weight for each point and then conduct an optimization to rearrange the selected point set such that it minimizes the GF discrepancy.

The proposed methodology shows potential for improving the efficiency of uncertainty quantification of structural dynamic systems. In the next section, the application of the proposed algorithm to the reliability assessment of structural dynamic systems will be further explored.

Table 4 CL2 discrepancy (15 random variables)

Number of points Sequence	100	200	300	400	500
Halton	0.4114	0.2788	0.2213	0.1842	0.1603
Harmmersley	0.4006	0.2723	0.2130	0.1774	0.1533
Sobol	0.3740	0.2579	0.2031	0.1658	0.1480
Faure	0.7846	0.7583	0.7483	0.7442	0.7433
Niederreiter	0.5496	0.3817	0.2811	0.2132	0.1632

4 The maximum entropy principle based method for reliability assessment of structural dynamic systems

4.1 The formulation of reliability assessment of structural dynamic systems

Without loss of generality, the state equation of a nonlinear structural dynamic system can be expressed as

$$\dot{\mathbf{X}} = \mathbf{A}\left(\mathbf{X}, \tilde{\mathbf{\Theta}}, t\right) \tag{43}$$

with the initial condition

$$\mathbf{X}(t)|_{t=t_0} = x_0 \tag{44}$$

where $\mathbf{X} = \left(X_1, X_2, \ldots, X_m, \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_m\right)^T$; m is the degree of freedom of the structure; $\mathbf{A} = (A_1, A_2, \ldots, A_{2m})^T$ is the deterministic operator.

For a well-posed structural dynamic system, the solution to (43) exists, is unique and must be a function of $\tilde{\mathbf{\Theta}}$ (Li and Chen 2009). We can assume the solution takes the form

$$\mathbf{X}(t) = \mathbf{H}\left(\tilde{\mathbf{\Theta}}, t\right) \tag{45}$$

and thus the velocity takes the form

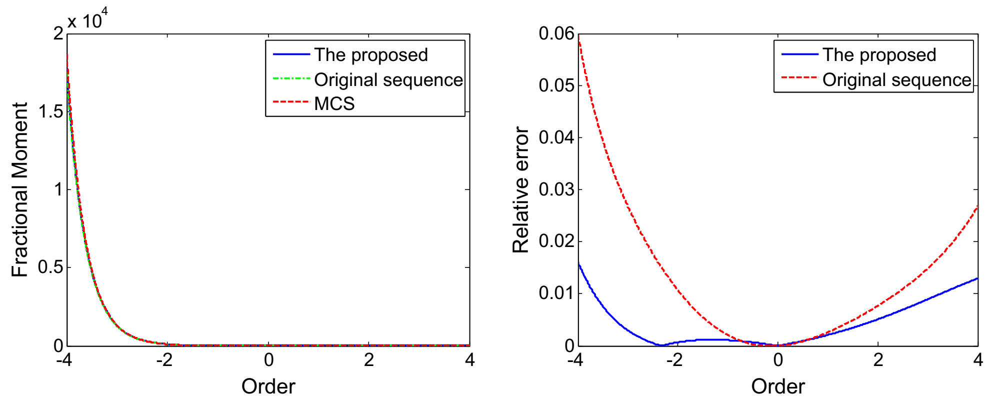
$$\dot{\mathbf{X}}(t) = \mathbf{h}\left(\tilde{\mathbf{\Theta}}, t\right) \tag{46}$$

where $\mathbf{h}\left(\tilde{\mathbf{\Theta}}, t\right) = \partial \mathbf{H}\left(\tilde{\mathbf{\Theta}}, t\right)/\partial t$.

The components of (45) and (46) are

$$X_l(t) = H_l\left(\tilde{\mathbf{\Theta}}, t\right) \tag{47}$$

Fig. 9 Fractional moments of extreme value (MDOF linear structure)



(a) fractiona l moments (b) relative error

$$\dot{X}_l(t) = h_l\left(\tilde{\boldsymbol{\Theta}}, t\right) \tag{48}$$

where the subscript "l" denotes the l-th component. For simplicity, the subscript "l" will be omitted hereafter to remove additional confusion.

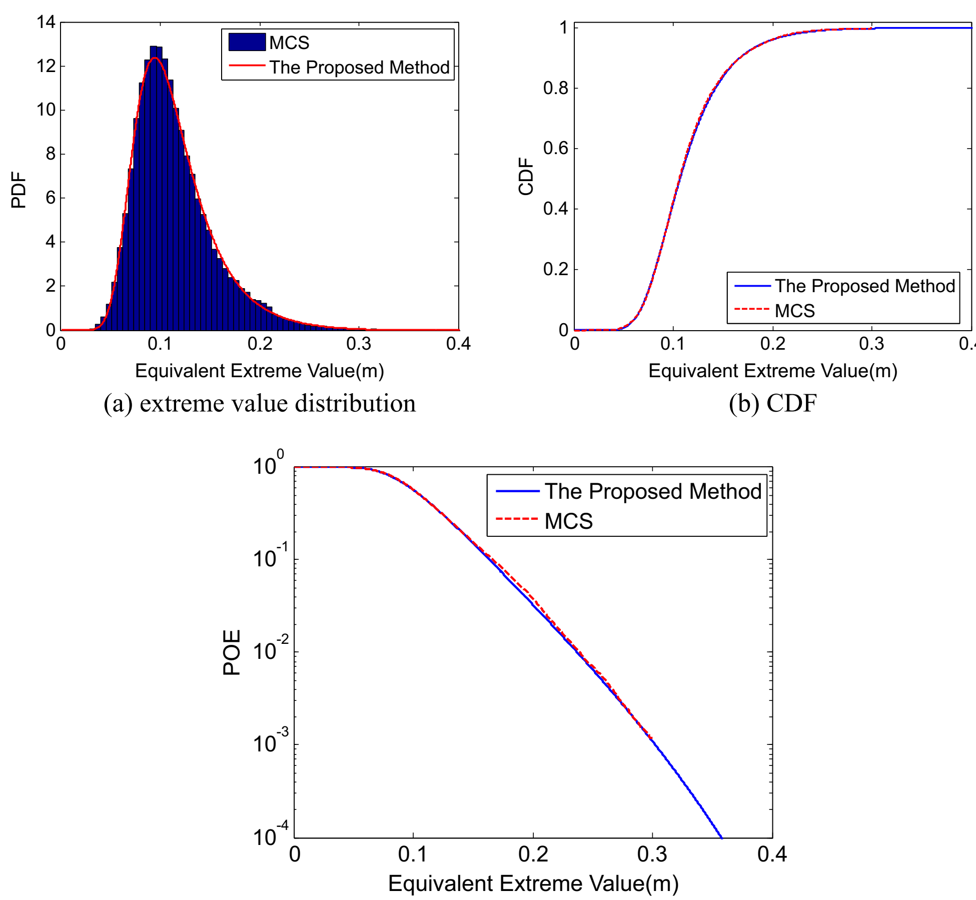
Then, the first passage reliability for structural dynamic systems can be defined as

$$R = \Pr\{X(t) \in \Omega_S, t \in [0, T]\} \tag{49}$$

where Pr denotes the probability and Ω_S is the safe domain. In the present paper, the symmetric double boundary is specifically employed such that

$$R = \Pr\{|X(t)| \le x_B, t \in [0, T]\} \tag{50}$$

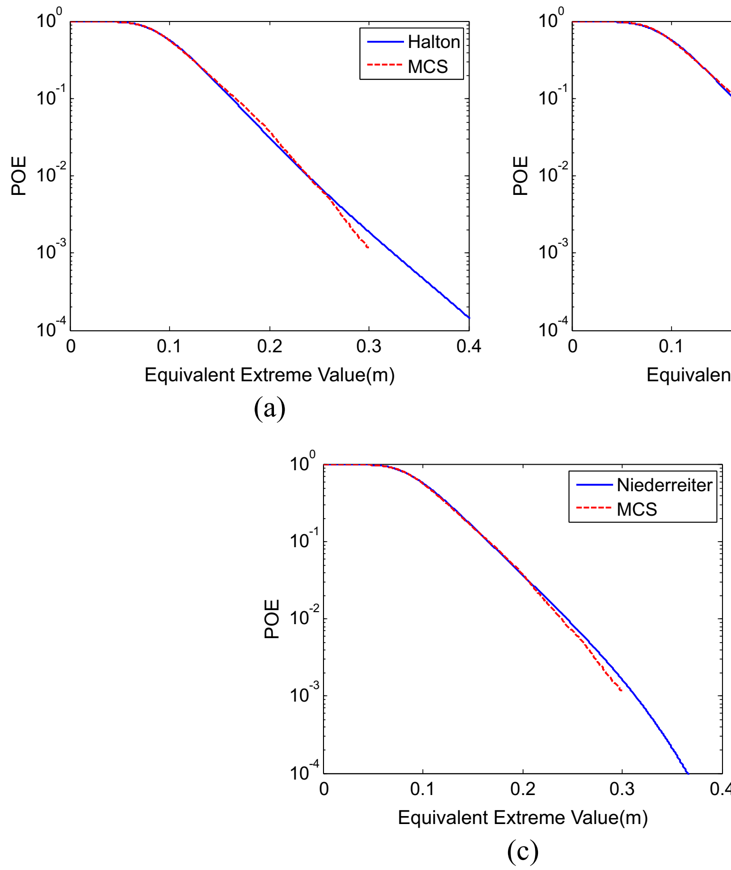
Fig. 10 Reliability assessment (MDOF linear structure)



(a) extreme value distribution (b) CDF

(c) POE

Fig. 11 POE curves by other low-discrepancy sequences (MDOF linear structure)



in which x_B is the threshold.

As is discussed, the dynamic reliability can be evaluated straightforwardly and conveniently from the extreme value distribution (Chen and Li 2007). Thus, the equivalent form of (50) is

$$R = \Pr\{X_{ext} \le x_B,\} \tag{51}$$

where $X_{ext} = \max_{t \in [0,T]} (|X(t)|)$ is the extreme value associate to the failure criterion.

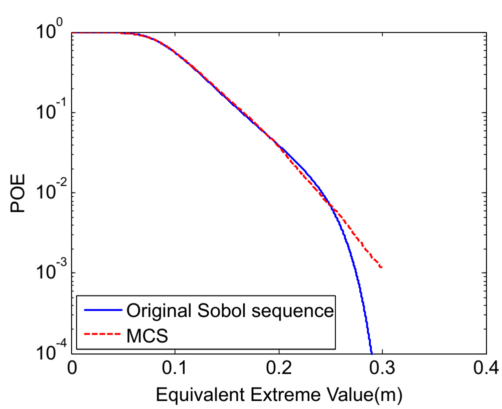


Fig. 12 POE curve without Step 2 (MDOF linear structure)

Once the EVD $p_{X_{ext}}(x)$ is obtained, it is convenient to perform a simple one-dimensional integration to give the dynamic reliability

$$R = \int_0^{x_B} p_{X_{ext}}(x)dx \tag{52}$$

Then, the task is now to evaluate the extreme value distribution $p_{X_{ext}}(x)$. To this end, the maximum entropy method (MEM) will be applied to derive the extreme value distribution $p_{X_{ext}}(x)$.

4.2 Maximum entropy method (MEM) to derive the extreme value distribution

The MEM is based on the concept that the distribution that maximizes the information entropy is the one that is statistically most likely to occur (Kapur and Kesavan 1992). In the context of information theory, the entropy S_{ext} of the EVD $p_{X_{ext}}(x)$ is defined as

$$S_{ext} = -\int p_{X_{ext}}(x) \ln\left(p_{X_{ext}}(x)\right)dx \tag{53}$$

Then, one needs to find $p_{X_{ext}}(x)$ that maximizes the entropy S_{ext} subject to moments constraints

$$u_k = \int x^k p_{X_{ext}}(x)dx; k = 1, 2, \ldots, n \tag{54}$$

Table 5 Parameters in Bouc-Wen model

Parameter	α	A	n	q	p	d_ψ	λ	ψ	β	γ	d_ν	d_η	ξ
Value	0.03	1	1	0.25	1000	5	0.5	0.05	40	20	2000	2000	0.99

where $n+1$ is the number of known integer moments.

The classical solution of this problem is given by

$$p_{X_{ext}}(x) = \exp\left[-\lambda_0 - \sum_{k=1}^{n} \lambda_k x^k\right] \tag{55}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_N$ are the Lagrange multipliers by solving the following set equations

$$\lambda_0 = \log\left\{\int \exp\left[-\lambda_0 - \sum_{k=1}^{n} \lambda_k x^k\right] dx\right\} \tag{56}$$

$$u_k = \int x^k \exp\left[-\lambda_0 - \sum_{k=1}^{n} \lambda_k x^k\right] dx; k = 1, 2, \ldots, n \tag{57}$$

However, a relatively large number of integer moments are required to achieve a reasonable accuracy in the modeling of the distribution tail, which is of great importance to reliability assessment (Zhang and Pandey 2013). However, the numerical algorithm may become unstable as the number of integer moments constraints increases.

4.3 Fractional moments evaluation based on the proposed methodology

The MEM with fractional moments as constraints has drawn increasing attention for reliability analysis from the research community. In such an approach, a finite number of fractional moments, instead of integer moments, are applied such that

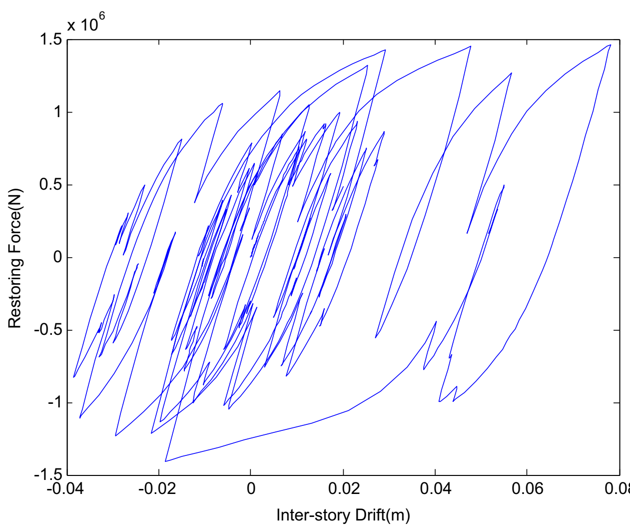


Fig. 13 Restoring force v.s. inter-storey drift

$$u_{\alpha_k} = \int x^{\alpha_k} \exp\left[-\lambda_0 - \sum_{k=1}^{n} \lambda_k x^{\alpha_k}\right] dx; k = 1, 2, \ldots, n \tag{58}$$

where α_k is the k th fractional order and u_{α_k} is the corresponding fractional moment.

The EVD can be rewritten as

$$p_{X_{ext}}(x) = \exp\left[-\lambda_0 - \sum_{k=1}^{n} \lambda_k x^{\alpha_k}\right] \tag{59}$$

The reason for using fractional moments is because a single fractional moment embodies a large number of central moments, i.e. (Zhang et al. 2014; Xu et al. 2017; Xu 2016).

$$u_{\alpha_k} = \sum_{i=0}^{\infty} \binom{\alpha_k}{i} {X_0}^{\alpha_k - i} E\left[(X_{ext} - X_0)^i\right] \tag{60}$$

where X_0 denotes the mean of X_{ext}. Thus, a few of fractional moments can be used to recover the EVD more clearly than a finite number of integer moments do.

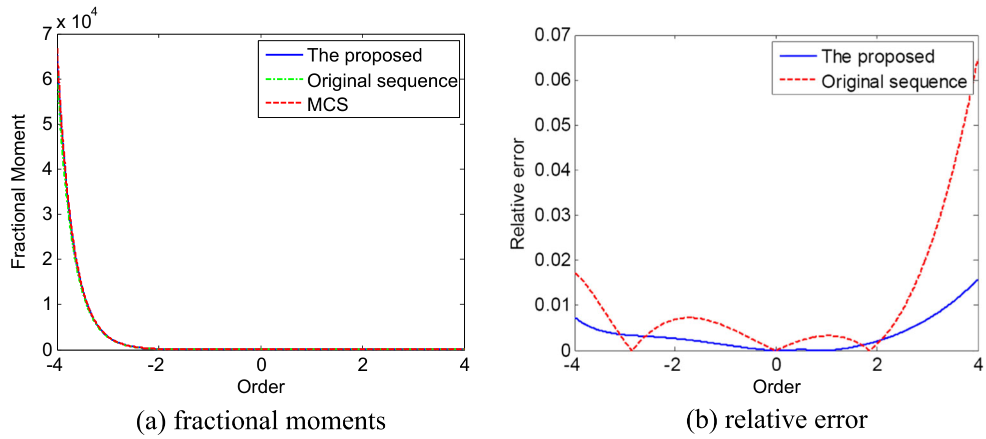
After a series mathematical manipulations, the parameters $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]^T$ and $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_n]^T$ can be determined by (Zhang and Pandey 2013)

$$\begin{cases} find & \boldsymbol{\alpha}, \boldsymbol{\lambda} \\ Min & W(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \ln\left[\int \exp\left[-\sum_{k=1}^{n} \lambda_k x^{\alpha_k}\right] dx\right] + \sum_{k=1}^{n} \lambda_k u_{\alpha_k} \end{cases} \tag{61}$$

Therefore, the EVD $p_{X_{ext}}(x)$ can be reconstructed to investigate the reliability by substituting $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ into (59).

It is noted that the fractional moments evaluation is of paramount importance to derive the EVD $p_{X_{ext}}(x)$. This approach has been studied for static reliability analysis of structures in Ref. (Zhang and Pandey 2013). In that study, the response function is decomposed by the dimension reduction method (Zhang and Pandey 2013). Then, the fractional moments can be evaluated in a multiplicative form very efficiently. In this regard, the PDF of the response function is captured by the MEM and therefore the reliability is obtained accordingly. Nevertheless, direct employment of this technique to structural dynamic systems may yield spurious results for reliability assessment. The response function (extreme value) of a complex structural dynamic system is often very complicated, implicit and highly nonlinear with respect to the basic random variables. In this regard, the first-order dimension reduction modeling of the response function may be not

Fig. 14 Fractional moments of extreme value (MDOF nonlinear structure)



adequate, which could induce large error in the moments evaluation.

Instead, the fractional moment of EVD involves a multidimensional integral such that

$$u_{\alpha_k} = \int_{\Omega_{\tilde{\Theta}}} x\left(\tilde{\theta}\right)^{\alpha_k} p_{\tilde{\Theta}}(\theta) d\tilde{\theta}; k = 1, 2, \ldots, n \tag{62}$$

where $x\left(\tilde{\theta}\right)$ actually refers to the implicit response function (extreme value) with respect to the randomness.

This integral can be simply evaluated based on the rearranged low-discrepancy points and their unequal weights proposed in Section 3 such that

$$\begin{aligned} u_{\alpha_k} &= \int_{\Omega_{\tilde{\Theta}}} x\left(\tilde{\theta}\right)^{\alpha_k} p_{\tilde{\Theta}}(\theta) d\tilde{\theta} \\ &= \sum_{q=1}^{N} \omega_q x\left(\tilde{\theta}_q\right)^{\alpha_k}; \begin{array}{l} k = 1, 2, \ldots, n, \\ q = 1, 2, \ldots, N \end{array} \end{aligned} \tag{63}$$

Thus, the EVD $p_{X_{ed}}(x)$ can be evaluated and therefore the reliability of structural dynamic systems is assessed. It is noted that

Fig. 15 Reliability assessment (MDOF nonlinear structure)

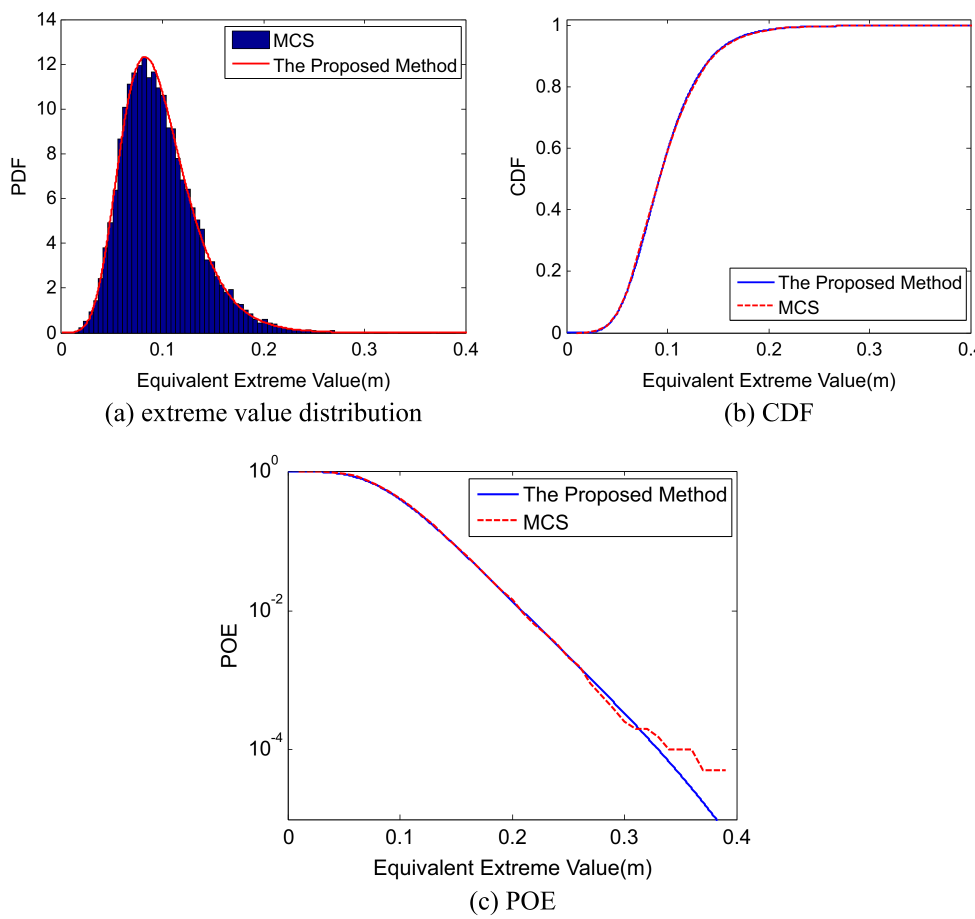
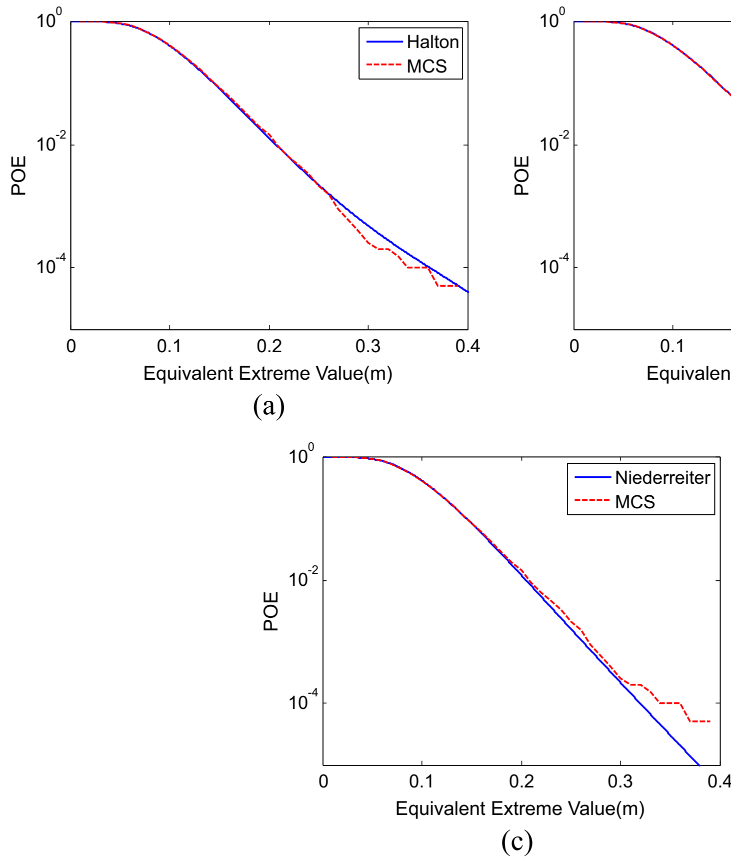


Fig. 16 POE curves by other low-discrepancy sequences (MDOF nonlinear structure)



the cost of deriving the EVD $p_{X_{ext}}(x)$ is negligible compared to the cost of performing the series of deterministic dynamic analyses of structures to obtain $x(\tilde{\theta}_q)$, $q = 1, 2, \ldots, N$,which is reduced using the rearranged low-discrepancy points.

5 Numerical examples

5.1 A SDOF nonlinear system

Consider a SDOF structure with random parameters subjected to seismic excitations with random amplitudes. The mass,

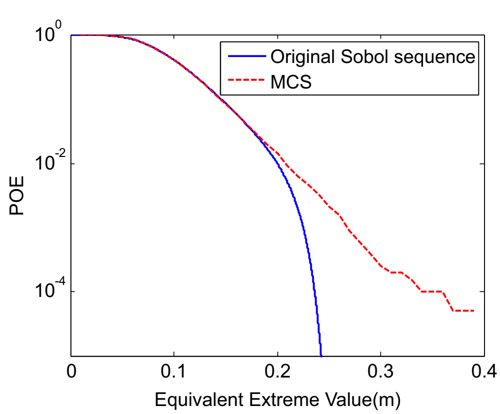


Fig. 17 POE curve without Step 2 (MDOF nonlinear structure)

stiffness and damping are all taken as dimensionless quantities. The equation of motion can be expressed as

$$m\ddot{x} + c\dot{x} + G(x) = F(t) \tag{64}$$

where $G(x) = k(x - x^3)$; m, c, k are assumed to be normally distributed random variables with the mean $m_0 = 1$, $c_0 = 0.63$, $k_0 = 39.48$ and coefficients of variation $\delta_m = \delta_c = \delta_k = 0.15$; and the external excitation $F(t)$ takes the form such that

$$F(t) = -m\left[\xi_1 \ddot{x}_g^{N-S}(t) + \xi_2 \ddot{x}_g^{E-W}(t)\right] \tag{65}$$

where $\ddot{x}_g^{N-S}(t)$ and $\ddot{x}_g^{E-W}(t)$ denote El Centro accelerogram in the N-S and W-E direction, respectively, ξ_1 and ξ_2 are the random amplitudes, which are assumed to be normal random variables with the means $2m/s^2$, $1.5m/s^2$ and the standard deviations of $0.6m/s^2$, $0.3m/s^2$. Thus, five random variables are involved in predicting the structural response $x(t)$.

The extreme value X_{ext} in (51) could be denoted as $X_{ext} = \max_{t \in [0,T]} (|x(t)|)$. To derive the probability density distribution $p_{X_{ext}}(x)$ for reliability analysis, the fractional moments evaluation in (63) is implemented. For this purpose, the proposed two-step algorithm is employed to generate the point set. According to Table 1, Hammersley sequence produces the lowest CL2 discrepancy among these sequences with the same number of points in this example. This feature reveals that Hammersley sequence is the most uniformly scattered sequence in this case, which is preferred in the following rearrangement of points and weights calculation.

Then, the Hammersley sequence $\wp_2$ is rearranged by the GF discrepancy based optimization process, where the rearranged point set is denoted as $\tilde{\wp}_2$. The fractional moments of the EVD $p_{X_{ext}}(x)$ is assessed by employing the rearranged point set $\tilde{\wp}_2$ when the number of samples is specifically adopted as $N = 400$. The relative error is defined as

$$e = \frac{\left| u_{\alpha_k}\left(\tilde{\wp}_2\right) - u_{\alpha_k}(MCS) \right|}{\left| u_{\alpha_k}(MCS) \right|} \tag{66}$$

where $u_{\alpha_k}(\tilde{\wp}_2)$ and $u_{\alpha_k}(MCS)$ are the fractional moments obtained by the rearranged point set $\tilde{\wp}_2$ and MCS (10^5 runs), respectively.

Figure 4a shows the comparison of the fractional moments of EVD, where the fractional order varies between [−4,4]. It is seen the results evaluated by using the rearranged sequence $\tilde{\wp}_2$ (proposed methodology) and the original sequence $\wp_2$ (without Step 2) accord quite well with that by MCS. As is known, for reliability problems of structural dynamic systems, the total computational effort is always governed by the necessary number of deterministic dynamic response analyses of structures. The computation time of the proposed methodology is much less than that of the repeated time-consuming deterministic dynamic response analyses of structures. In this regard, the computational time of using the rearranged sequence $\tilde{\wp}_2$ for fractional moments evaluation is almost the same with that of using the original sequence $\wp_2$, which is around 4/1000 of that by MCS. However, when the relative errors are concerned, it is seen in Fig. 4b that the more accurate fractional moments could be obtained by using the rearranged sequence $\tilde{\wp}_2$, where the relative error of using $\tilde{\wp}_2$ is obviously smaller than that of using $\wp_2$. This manifests that the proposed methodology can be applied to efficiently and accurately obtain the fractional moments of EVD which are required in the reliability assessment procedure.

Figure 5a, b depicts the EVD $p_{X_{ext}}(x)$ and its cumulative density function (CDF) by using the MEM, where the initial fractional orders are adopted as 2.223,-3.127,0.23 and 0.78. The proposed methodology is adopted to evaluate the fractional moments. Likewise, all the results are compared with those given by MCS. Figure 5c pictures the probability of exceedance (POE) curves in logarithmic coordinate by the proposed methodology and MCS. It should be noted that the value of the CDF at a specified abscissa actually gives the reliability in terms of the prescribed threshold (Chen and Li 2007). Similarly, the value of the POE at the abscissa provides the corresponding failure probability. Very good accordance between the results by the proposed methodology and those of MCS can be observed, indicating the accuracy of the proposed methodology for reliability assessment of structural dynamic systems based on the selected low-discrepancy sequence. If Step 2 is not considered, the comparison of the POE curves in logarithmic scale obtained from the original sequence $\wp_2$ and MCS is pictured in Fig. 6. Obvious deviation could be noticed between these curves, which indicates the great necessity of Step 2 in the proposed methodology. Further, Fig. 7 shows the POE curves when other low-discrepancy sequences are applied without the first step in the proposed methodology. It is seen that these POE curves may deviate from the result by MCS at the region of small failure probabilities, i.e. $p_f < 10^{-3}$. However, the POE curve obtained by the proposed methodology always accords well with that of MCS (Fig. 5c). This also reveals the great necessity of the first step in the proposed methodology for reliability analysis of structural dynamic systems.

5.2 MDOF systems

To further validate the proposed methodology, the reliability of a 13-storey shear structure subjected to ground motions is studied (Fig. 8). The mass and the stiffness of the structure are listed in Table 3. Herein, the stiffness of each storey is regarded as independent normal random variable with coefficient of variation 0.15. The Rayleigh damping is adopted such that $\mathbf{C} = a\mathbf{M} + b\mathbf{K}$, where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the initial mass, stiffness and damping matrices, respectively, and $a = 0.2512s^{-1}$, $b = 0.0082s$. The ground motion $\ddot{x}_g(t)$ still reads

$$\ddot{x}_g(t) = \xi_1 \ddot{x}_g^{N-S}(t) + \xi_2 \ddot{x}_g^{E-W}(t) \tag{67}$$

In this case, ξ_1 and ξ_2 are assumed to be normal random variables with the means $2.5m/s^2$, $2m/s^2$ and the standard deviations of $0.75m/s^2$,$0.6m/s^2$.

In this case, a total of 15 independent random variables are considered. Listed in Table 4 are the CL2 discrepancies with different numbers of sample size in low-discrepancy sequences. From the table, it is seen that Sobol sequence results in the lowest CL2 discrepancy with the same number of samples in these sequences. Thus, the rearranged point set $\tilde{\wp}_3$ based on Sobol sequence is preferred in this case to evaluate the fractional moments of EVD. In addition, it is noted that Faure sequence gives the worst CL2 discrepancy, which may produce some points with infinite values in $\tilde{\wp}_4$. Thus, the rearranged point set $\tilde{\wp}_4$ is not applicable in this case. In this case, the first inter-storey drift, denoted as $X_{0-1}(t)$ is of great concern for reliability assessment.

5.2.1 Linear structure

First, a linear structure is considered. Then, the rearranged Sobol sequence with the number of samples $N = 500$, denoted as $\tilde{\wp}_3$, is adopted for fractional moments assessment of EVD of $X_{0-1}(t)$. The comparisons of the fractional moments of EVD and the relative errors, where the fractional order varies between [−4,4] are shown in Fig. 9, where 10^5 times MCS are also carried out for comparisons. Similar conclusions in terms of accuracy of using the proposed methodology for fractional moments evaluation. Remarkably, it is seen that the proposed methodology possesses the property that the computational effort is almost irrelevant with dimension. This feature is quite attractive in engineering practice, which means even for high-dimensional cases, say $d > 10$, only hundreds of deterministic dynamic response analyses are adequate to accurately obtain the fractional moments and therefore to assess the reliability. Besides, the accuracy has been significantly improved by using the proposed methodology compared with that by the original Sobol sequence (Fig. 9b) with the same number of points (without Step 2). It is seen that the relative error of using the proposed methodology is less than 2% whereas the relative error of only using Step 1 is as large as 6%.

Figure 10 shows the EVD $p_{X_{ext}}(x)$, its CDF and POE curves in logarithmic scale, where $X_{ext} = \max\limits_{t\in[0,T]}(|X_{0-1}(t)|)$, and the initial fractional orders in MEM are adopted as 0.214, −1.567, 1.123 and 2.12, respectively. Obviously, the results evaluated by the proposed methodology still has very good agreements with those by MCS, which manifests the efficacy of the proposed methodology for reliability assessment of MDOF linear structural dynamic systems. Likewise, when the Halton, the Hammersley and the Niederreiter sequences are employed to generate the basic point set, the POE curves in logarithmic scale are shown in Fig. 11. Again, it is noted that these POE curves may deviate from the result by MCS at the region of small failure probabilities. In this regard, the great necessity of the first step in the proposed methodology is elucidated again.

Further, when the original Sobol sequence (without Step 2) is applied to evaluate the fractional moments of EVD in MEM, the comparison of the POE curves in logarithmic scale obtained from the original Sobol sequence and MCS is shown in Fig. 12. It is seen that the POE curves do not align, demonstrating the Step 2 in the proposed methodology can significantly improve the accuracy of reliability calculation.

5.2.2 Nonlinear structure

Here, a nonlinear MDOF structure is investigated. The Bouc-Wen model (Wen 1976) is adopted to characterize the nonlinear restoring force, the parameters of which are listed in Table 5. A typical sample of restoring force v.s. inter-storey drift is shown in Fig. 13. It is clearly seen that strong nonlinearity occurs in the mechanical behavior of the structure under seismic ground motions. Similarly, the rearranged point set $\tilde{\wp}_3$ in the linear counterpart is still adopted to evaluate the fractional moments. Figure 14 shows the fractional moments of EVD and the corresponding relative errors by the proposed methodology, where the fractional moments and relative errors by the original sequence without Step 2 in the proposed methodology is also pictured. It is also noted that the proposed methodology obviously gives more accurate results than those of the original sequence. This also validates the efficacy of the proposed two-step methodology to accurately capture the fractional moments for reliability assessment.

Pictured in Fig. 15 are the EVD, the CDF and the POE curves in logarithmic scale by the proposed methodology, where the initial fractional orders are 0.567, −2.123, 1.124 and 2.12, respectively. Again, very good accordance between the results by the proposed methodology and MCS could be observed, which demonstrates the high accuracy and efficiency of the proposed methodology. Figure 16 shows the POE curves by using other low-discrepancy sequences as the basic point set without using Step 1 in the proposed methodology to find the point set with the minimum CL2 discrepancy. Clearly, it is noticeable that the accuracy of these POE curves in logarithmic scale is slightly lower than that of the proposed methodology, especially in the region of small failure probabilities. This again verifies the great necessity of Step 1 in the proposed methodology. Likewise, if the original Sobol sequence without Step 2 in the proposed methodology is employed to obtain the fractional moments of EVD in MEM, the POE curve in logarithmic scale could severely deviate from that of MCS, which is shown in Fig. 17. In this regard, the Step 2 is indeed of paramount importance in the proposed methodology to improve the accuracy for tail distribution estimation. It could be concluded that the proposed two-step methodology can even achieve the tradeoff of accuracy and efficiency for reliability assessment of strongly nonlinear MDOF structural dynamic systems.

6 Concluding remarks

A new methodology, based on the well-developed low-discrepancy sequences, has been put forward for reliability assessment of structural dynamic systems in the present paper. Five low-discrepancy sequences- Halton, Hammersley, Sobol, Faure, and Niederreiter sequences have been revisited. Among these five sequences, the sequence with the lowest CL2 discrepancy, that demonstrates the most uniformity was selected. Then, the selected sequence was rearranged to minimize the GF discrepancy, where the Voronoi cell partition was adopted to calculate the weight for each point. Therefore, a two-step methodology was developed to select and rearrange the point set based on low-discrepancy sequences.

The proposed methodology was demonstrated using the maximum entropy method with the fractional moments as constraints to derive the extreme value distribution for reliability assessment of structural dynamic systems. The conclusions include:

(1) The proposed methodology gives noticeably higher accuracy for fractional moments evaluation than the original low-discrepancy point set with the same number of samples. Besides, the proposed methodology seems to be irrelevant with dimension to some extent. This means the fractional moments needed for reliability evaluation could be efficiently and accurately obtained even for high-dimensional cases.
(2) The numerical results validate the efficacy of the proposed methodology for reliability assessment of structural dynamic systems, even for very complicated case, e.g. MDOF hysteretic system subjected to seismic ground motions. Thus, the proposed methodology is applicable for general dynamic systems. On the other hand, the proposed method can be also applied in static cases. Therefore, the proposed methodology qualifies as a comprehensive tool in the structural reliability community.

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