

Optimal design of fibre reinforced membrane structures

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Abstract A design problem of finding an optimally stiff membrane structure by selecting one-dimensional fiber reinforcements is formulated and solved. The membrane model is derived in a novel manner from a particular three-dimensional linear elastic orthotropic model by appropriate assumptions. The design problem is given in the form of two minimization statements. After finite element discretization, the separate treatment of each of the two statements follows from classical results and methods of structural optimization: the stiffest orientation of reinforcing fibers coincides with principal stresses and the separate selection of density of fibers is a convex problem that can be solved by optimality criteria iterations. Numerical solutions are shown for two particular configurations. The first for a statically determined structure and the second for a statically undetermined one. The latter shows related but non-unique solutions.

Keywords Membrane · Fiber reinforcement · Design optimization

1 Introduction

A finite element membrane shell model was recently derived by Hansbo and Larson (2014) using tangential differential calculus, meaning that the problem is set in a Cartesian three dimensional space as opposed to a parametric plane, thereby generalizing the classical flat facet element shell model to higher order elements. The present study further extends this membrane model by allowing for non-isotropic materials. In particular, one-dimensional fibers are added to a base material, modeling, e.g., the reinforcements seen in modern racing boat sails. The plane stress property, as well as the membrane property of complete out-of-plane shear flexibility, is shown to be exact consequences of certain material parameter selections for a three-dimensional transversely isotropic base material. This together with a displacement assumption reduces the three dimensional model to the surface model. Based on this finite element model we formulate a design problem where we seek to find the best fiber reinforcements of the membrane, meaning that we find the stiffest structure by both rotation and sizing of the fibers. The formulation consists of two minimization statements. Since these two statements relate to rotation and sizing of the fibers, respectively, such a formulation ties directly to the sequential iterative treatment suggested for similar problems previously (Bendsøe and Sigmund 2002). The optimal rotation is found by identifying the material as a so-called low shear material, implying that the optimal orthotropic principal directions coincides with the principal stress directions (Pedersen 1989, 1991), while the optimal thickness distribution is found by a classical optimality criteria iteration formula. On a more general level the approach

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is reminiscent of so called block coordinate descent methods (Bertsekas 1999; Beck and Tretuashvili 2013).

2 The model

We consider a material that is a mixture of a transversely isotropic linear elastic base material and n reinforcing fibre materials. The transversely isotropic material has material constants that satisfy the plane stress assumption as well as the membrane behaviour of having complete flexibility when sheared perpendicularly to the membrane surface.

2.1 Geometry

The geometry of the membrane is defined by an orientable smooth surface Σ with normal vector field \mathbf{n} . For any point $\mathbf{x} \in \mathbb{R}^3$ we denote the signed distance function relative to Σ by $\zeta(\mathbf{x})$. The membrane with thickness t then occupies

$$\Omega_t = \{\mathbf{x} \in \mathbb{R}^3 : |\zeta(\mathbf{x})| < t/2\}.$$

Note that $\nabla\zeta(\mathbf{x}) = \mathbf{n}$ for $\mathbf{x} \in \Sigma$. For a sufficiently small t , the orthogonal projection point $\mathbf{p}(\mathbf{x}) \in \Sigma$ of $\mathbf{x} \in \Omega_t$ is unique and given by

$$\mathbf{p}(\mathbf{x}) = \mathbf{x} - \zeta(\mathbf{x})\nabla\zeta(\mathbf{x}).$$

Moreover, for $\mathbf{x} \in \Omega_t$, the linear projection operator of vectors onto the tangent plane of Σ at $\mathbf{p}(\mathbf{x})$ is

$$\mathbf{P}_\Sigma = \mathbf{I} - \mathbf{n} \otimes \mathbf{n},$$

where \mathbf{I} is the identity tensor and \otimes denotes exterior product. In the sequel we will also need the projection operator onto the one-dimensional subspace spanned by \mathbf{n} , i.e.,

$$\mathbf{N}_\Sigma = \mathbf{n} \otimes \mathbf{n}.$$

Note that $\mathbf{P}_\Sigma\mathbf{N}_\Sigma = \mathbf{N}_\Sigma\mathbf{P}_\Sigma = \mathbf{0}$. The directions of the reinforcing fibers are given by vector fields \mathbf{s}_i , $i = 1, \dots, n$, such that $\mathbf{s}_i \cdot \mathbf{n} = 0$. Projections onto these directions are then defined by

$$\mathbf{S}_i = \mathbf{s}_i \otimes \mathbf{s}_i.$$

Clearly $\mathbf{P}_\Sigma\mathbf{S}_i = \mathbf{S}_i\mathbf{P}_\Sigma = \mathbf{S}_i$.

2.2 The material

The base material is transversely isotropic with respect to an axis defined by \mathbf{n} . Such a material can be described by an elasticity tensor expressed in terms of five material

constants $\delta_1, \delta_2, \delta_3, \gamma$ and μ according to Lubarda and Chen (2008) and Nardinocchi and Podio-Guidugli (1994), so that the fourth order tensor of elastic moduli of the base material \mathbb{E}^{base} can be written

$$\begin{aligned} \mathbb{E}^{\text{base}} = & \delta_1 \mathbf{N}_\Sigma \otimes \mathbf{N}_\Sigma + \delta_2 (\mathbf{N}_\Sigma \otimes \mathbf{P}_\Sigma + \mathbf{P}_\Sigma \otimes \mathbf{N}_\Sigma) \\ & + \mu (\mathbf{P}_\Sigma \otimes \mathbf{P}_\Sigma + \mathbf{P}_\Sigma \overline{\otimes} \mathbf{P}_\Sigma) + \delta_3 \mathbf{P}_\Sigma \otimes \mathbf{P}_\Sigma + \\ & + \frac{\gamma}{2} (\mathbf{N}_\Sigma \otimes \mathbf{P}_\Sigma + \mathbf{N}_\Sigma \overline{\otimes} \mathbf{P}_\Sigma \\ & + \mathbf{P}_\Sigma \otimes \mathbf{N}_\Sigma + \mathbf{P}_\Sigma \overline{\otimes} \mathbf{N}_\Sigma). \end{aligned} \quad (1)$$

Here dyadic products of second order tensors are defined by their action on a third second order tensor, i.e.,

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})\mathbf{C} &= (\mathbf{B} : \mathbf{C})\mathbf{A}, & (\mathbf{A} \overline{\otimes} \mathbf{B})\mathbf{C} &= \mathbf{A}\mathbf{C}\mathbf{B}^T, \\ (\mathbf{A} \overline{\otimes} \mathbf{B})\mathbf{C} &= \mathbf{A}\mathbf{C}^T\mathbf{B}^T, \end{aligned}$$

where a double dot indicates inner product of second order tensors.

The reinforcing fibers have elasticity tensors of the form

$$\mathbb{E}_i^{\text{fiber}} = \alpha_i \mathbf{S}_i \otimes \mathbf{S}_i, \quad i = 1, \dots, n, \quad (2)$$

where α_i are Young type elasticity coefficients.

The constitutive law of the membrane material is now taken as being composed of a constrained mixture of base material and reinforcing material. The amount of each material is defined by fractions t_b and t_i , $i = 1, \dots, n$, of the membrane thickness t . The total constitutive tensor is assumed as

$$\mathbb{E} = \frac{t_b}{t} \mathbb{E}^{\text{base}} + \sum_{i=1}^n \frac{t_i}{t} \mathbb{E}_i^{\text{fiber}}, \quad t = t_b + \sum_{i=1}^n t_i, \quad (3)$$

and the linear constitutive law is then

$$\boldsymbol{\sigma} = \mathbb{E}[\boldsymbol{\varepsilon}], \quad (4)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, respectively. The linear mixing rule (3), widely used in composite mechanics (Jones 1975; Thomsen 1991), corresponds to the so-called Voigt upper bound on the effective elastic modulus tensor of two-phase materials. More precise bounds of the Hashin-Shtrikman type are given in Parnell and Calvo-Jurado (2015).

2.3 Membrane stress assumptions

We define a membrane material by the requirements that it is always in a state of plane stress and no shear stress perpendicular to the membrane surface exists, i.e.,

$$\mathbf{N}_\Sigma \boldsymbol{\sigma} \mathbf{N}_\Sigma = \mathbf{0}, \quad \mathbf{P}_\Sigma \boldsymbol{\sigma} \mathbf{N}_\Sigma = \mathbf{N}_\Sigma \boldsymbol{\sigma} \mathbf{P}_\Sigma = \mathbf{0}. \quad (5)$$

The zero bending stiffness behaviour of membranes will be a result of a kinematic assumption introduced subsequently. Inserting (4) into (5) gives

$$\begin{aligned} N_\Sigma \sigma N_\Sigma &= \frac{t_b}{t} [\delta_1 N_\Sigma (N_\Sigma : \boldsymbol{\varepsilon}) + \delta_2 N_\Sigma (\mathbf{P}_\Sigma : \boldsymbol{\varepsilon})] \\ &= \mathbf{0}, \end{aligned} \tag{6}$$

$$\mathbf{P}_\Sigma \sigma N_\Sigma = \frac{t_b}{t} [\gamma \mathbf{P}_\Sigma \boldsymbol{\varepsilon} N_\Sigma] = \mathbf{0}. \tag{7}$$

Thus, we conclude that the constitutive constant γ needs to be zero and that the strain perpendicular to the membrane is controlled by the in-plane strain as

$$N_\Sigma : \boldsymbol{\varepsilon} = -\frac{\delta_2}{\delta_1} \mathbf{P}_\Sigma : \boldsymbol{\varepsilon}. \tag{8}$$

Moreover, the in-plane stress can be calculated from (4) as follows:

$$\begin{aligned} \mathbf{P}_\Sigma \sigma \mathbf{P}_\Sigma &= \frac{t_b}{t} [\delta_2 \mathbf{P}_\Sigma (N_\Sigma : \boldsymbol{\varepsilon}) + \delta_3 \mathbf{P}_\Sigma (\mathbf{P}_\Sigma : \boldsymbol{\varepsilon}) \\ &\quad + 2\mu \mathbf{P}_\Sigma \boldsymbol{\varepsilon} \mathbf{P}_\Sigma] + \sum_{i=1}^n \frac{t_i}{t} \alpha_i \mathbf{S}_i (\mathbf{S}_i : \boldsymbol{\varepsilon}), \end{aligned}$$

and when using (8) we get

$$\begin{aligned} \mathbf{P}_\Sigma \sigma \mathbf{P}_\Sigma &= \frac{t_b}{t} [\delta \mathbf{P}_\Sigma (\mathbf{P}_\Sigma : \boldsymbol{\varepsilon}) + 2\mu \mathbf{P}_\Sigma \boldsymbol{\varepsilon} \mathbf{P}_\Sigma] \\ &\quad + \sum_{i=1}^n \frac{t_i}{t} \alpha_i \mathbf{S}_i (\mathbf{S}_i : \boldsymbol{\varepsilon}), \end{aligned} \tag{9}$$

where

$$\delta = \delta_3 - \frac{\delta_2^2}{\delta_1}.$$

The elasticity coefficient μ equals the in-plane shear modulus, while δ is a plane stress Lamé coefficient. The two elasticity moduli δ and μ can be expressed in terms of in-plane Young and Poisson moduli E and ν as

$$\delta = \frac{\nu E}{1 - \nu^2}, \quad \mu = \frac{E}{2(1 + \nu)}.$$

The volumetric specific strain energy can, due to (5) be written as

$$W_s = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2} (\mathbf{P}_\Sigma \sigma \mathbf{P}_\Sigma) : \boldsymbol{\varepsilon}.$$

Inserting (9) we get

$$W_s = \frac{1}{2} (\mathbb{E}^{\text{memb}}[\boldsymbol{\varepsilon}]) : \boldsymbol{\varepsilon},$$

where the membrane elasticity tensor is defined by

$$\begin{aligned} \mathbb{E}^{\text{memb}} &= \frac{t_b}{t} [\delta \mathbf{P}_\Sigma \otimes \mathbf{P}_\Sigma + \mu (\mathbf{P}_\Sigma \underline{\otimes} \mathbf{P}_\Sigma + \mathbf{P}_\Sigma \overline{\otimes} \mathbf{P}_\Sigma)] \\ &\quad + \sum_{i=1}^n \frac{t_i}{t} \alpha_i \mathbf{S}_i \otimes \mathbf{S}_i. \end{aligned}$$

2.4 Potential energy

The strain is derived as usual as the symmetrized gradient of the displacement vector \mathbf{u} :

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T).$$

Therefore, we can regard the volume specific strain energy as a function of the displacement field, i.e., $W_s = W_s(\mathbf{u})$.

We now introduce the basic kinematic assumption that all material points in Ω_t that lie along a normal to the surface Σ have the same displacement vector, i.e.,

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{p}(\mathbf{x})), \quad \mathbf{x} \in \Omega_t.$$

This kinematics imply that bending of the membrane is essentially eliminated and no bending stiffness, despite the finite thickness, is present.

The total strain energy, which is the volume integral of W_s can then be written:

$$W = \int_{-t/2}^{t/2} \int_\Sigma W_s(\mathbf{u}(\mathbf{p}(\mathbf{x}))) d\Sigma_\zeta d\zeta,$$

where $d\Sigma_\zeta$ is an area element for a surface parallel to Σ at the distance ζ , which reads

$$d\Sigma_\zeta = d\Sigma(1 + \zeta H + \zeta^2 K),$$

where $d\Sigma$ is the area element of Σ , and H and K are the mean curvature and Gaussian curvature, respectively. For a membrane that is thin compared to its curvature we can use the approximation

$$d\Sigma_\zeta \approx d\Sigma.$$

The total potential energy is now taken as

$$\Pi = \frac{t}{2} \int_\Sigma W_s(\mathbf{u}(\mathbf{x})) d\Sigma - \langle \mathbf{F}, \mathbf{u} \rangle_\Sigma,$$

where the force \mathbf{F} is a member of the dual space of displacement fields on Σ and $\langle \cdot, \cdot \rangle_\Sigma$ is a duality pairing.

3 Equilibrium

We define the membrane forces (per unit length) as

$$\mathbf{M} = t \mathbf{P}_\Sigma \sigma \mathbf{P}_\Sigma.$$

Stationarity of the potential energy gives the following principle of virtual work:

$$\int_\Sigma \mathbf{M} : \boldsymbol{\varepsilon}(\mathbf{v}) d\Sigma = \langle \mathbf{F}, \mathbf{v} \rangle_\Sigma, \tag{10}$$

for all kinematically admissible fields \mathbf{v} . Such fields will generally be restricted in the tangential direction on a subset

of $\partial\Sigma$. We will assume that loading on the membrane can be written as

$$\langle \mathbf{F}, \mathbf{v} \rangle_\Sigma = \int_\Sigma \mathbf{f} \cdot \mathbf{v} \, d\Sigma + \int_S \mathbf{p} \cdot \mathbf{v} \, dS,$$

where \mathbf{f} is a force per area over Σ , and \mathbf{p} is a force per unit length over the part S of $\partial\Sigma$ where the displacement is not prescribed. Using now Lemma 2.1 of Gurtin and Murdoch (1975), i.e., an integral theorem for surfaces, we obtain the equilibrium equations

$$-\operatorname{div}_\Sigma \mathbf{M} = \mathbf{f}, \tag{11}$$

$$\mathbf{M}\mathbf{v} = \mathbf{p}, \tag{12}$$

where $\operatorname{div}_\Sigma$ is the surface divergence, and \mathbf{v} is a unit vector of $\partial\Sigma$, tangential to Σ . Since $\mathbf{M}\mathbf{v}$ will also be a vector tangent to Σ we conclude that \mathbf{p} can have no component perpendicular to the surface.

4 Design problem

From now on we will consider the special case of an orthotropic material consisting of two orthogonal families of fibers, consisting of the same material, i.e., $\alpha_1 = \alpha_2 = \alpha$. We use the notation $\mathbf{s} = \mathbf{s}_1$ and $\mathbf{s}^\perp = \mathbf{s}_2$.

The orientation of the fibers in the tangent plane of the membrane, i.e., \mathbf{s} and \mathbf{s}^\perp , can be defined by an angle θ belonging to

$$\Theta = \{\theta \mid 0 \leq \theta \leq 2\pi\}.$$

This angle will be a design variable in the optimal design problem. Other such design variables are t_1 and t_2 , i.e., the fiber contents in the two orthogonal directions. The field $\mathbf{t} = (t_1, t_2)$ belongs to the set

$$T = \left\{ \mathbf{t} = (t_1, t_2) \mid \underline{t}_\alpha \leq t_\alpha \leq \bar{t}_\alpha, \alpha = 1, 2, \int_\Sigma (t_1 + t_2) d\Sigma \leq V \right\},$$

where \underline{t}_α and \bar{t}_α are non-negative upper and lower bounds and V is a limit for the total amount of material that can be used for the fibers.

The potential energy is seen as a function

$$\Pi : V \times T \times \Theta \rightarrow \mathbb{R},$$

where V is the set of kinematically admissible displacements. Minimizing Π with respect to the first argument gives the equilibrium displacement as a function of the design variables, i.e., $\mathbf{u} = \mathbf{u}(\mathbf{t}, \theta)$. As a measure of stiffness we use the so called compliance

$$\begin{aligned} C(\mathbf{t}, \theta) &:= \frac{1}{2} \langle \mathbf{F}, \mathbf{u}(\mathbf{t}, \theta) \rangle_\Sigma = -\Pi(\mathbf{u}(\mathbf{t}, \theta), \mathbf{t}, \theta) \\ &= -\min_{\mathbf{v} \in V} \Pi(\mathbf{v}, \mathbf{t}, \theta). \end{aligned}$$

Our design goal is to find a design that minimizes the compliance. We choose to split this into two parts as follows: find $\mathbf{t}^* \in T$ and $\theta^* \in \Theta$ such that

$$(\mathbb{P}) \quad \begin{cases} C(\mathbf{t}^*, \theta^*) = \min_{\mathbf{t} \in T} C(\mathbf{t}, \theta^*) \\ C(\mathbf{t}^*, \theta^*) = \min_{\theta \in \Theta} C(\mathbf{t}^*, \theta). \end{cases}$$

The splitting into two minimization statements is partly motivated by the numerical treatment, where the two sub-problems are solved in sequence, reminiscent of a block coordinate descent method, Bertsekas (1999) and Beck and Tetruashvili (2013). However, it is also motivated by the fact that each of the two problems are well-posed in the sense that existence of solutions can be proved, as discussed below.

The second sub-problem of (\mathbb{P}) , i.e., finding an optimal orientation for an orthotropic material, has been extensively discussed by Pedersen (1989, 1991) and Hammer (1999), where it is treated in its present form, using θ as variable. However, the problem can also be rewritten in terms of so-called lamination parameters, see Hammer et al. (1997) and Bendsøe and Sigmund (2002). Such a rewriting gives an objective that is linear, making a proof of existence of solutions possible and thereby indicating that regularization, e.g., by filters, is not necessary. However, checkerboard-like patterns were found in the finite element scheme used in Thomsen and Olhoff (1990) and Thomsen (1991) for a particular choice of material, interpreted as having one fiber direction only. Such anomalies have not been seen in our calculations and following Pedersen (1989, 1991) the second sub-problem of (\mathbb{P}) is solved locally, i.e., the orientation of the material is determined by the local stress state only, and in particular the orientation of principal stresses and strains. Due to the plane stress assumption there are only two possibly non-zero principal components of the stress tensor $\boldsymbol{\sigma}$, denoted σ_I and σ_{II} , such that $|\sigma_I| \geq |\sigma_{II}|$. The corresponding principal directions (eigenvectors) are tangent to the membrane plane. Obviously, these facts also holds for the principal components of \mathbf{M} , i.e., M_I and M_{II} , such that $|M_I| \geq |M_{II}|$. For a so-called low shear orthotropic material, the solution θ^* of the second sub-problem of (\mathbb{P}) represents an orientation where the orthotropic principal directions coincide with the principal stress or membrane force directions, which are also the principal strain directions. Moreover, the orthotropic principal direction having the highest stiffness should be in the direction corresponding to σ_I and M_I . In the Appendix we show that the particular orthotropic material defined above, having two families of fibers in orthogonal directions \mathbf{s} and \mathbf{s}^\perp , is indeed a low shear material and, therefore, the optimal directions of \mathbf{s} and \mathbf{s}^\perp are in the directions of principal stress. Moreover, if $t_1 > t_2$ then \mathbf{s} is in the direction of σ_I .

The first sub-problem of (\mathbb{P}) is a classical stiffness optimization problem, albeit having two design fields, one for

each fiber orientation. This is a convex problem that can be proved well-posed (see, e.g., Petersson 1996) and regularization by filters is not needed. It can be solved by satisfying the optimality conditions, leading to a fixed point iteration formula. The surface elasticity tensor $\mathbb{S}^{\text{memb}} = t \mathbb{E}^{\text{memb}}$ is regarded as a function of the design, i.e., $\mathbb{S}^{\text{memb}} = \mathbb{S}^{\text{memb}}(\mathbf{t}, \theta)$. The optimality conditions of the first sub-problem of (\mathbb{P}) become (Bendsøe and Sigmund 2002; Christensen and Klarbring 2009):

$$\frac{\partial \mathbb{S}^{\text{memb}}}{\partial t_\alpha} [\boldsymbol{\varepsilon}(\mathbf{u})] : \boldsymbol{\varepsilon}(\mathbf{u}) = \Lambda + \lambda_\alpha^+ - \lambda_\alpha^-, \quad \alpha = 1, 2, \quad (13)$$

$$\Lambda \geq 0, \quad \Lambda \left(\int_\Sigma (t_1 + t_2) d\Sigma - V \right) = 0, \quad (14)$$

$$\begin{aligned} \lambda_\alpha^+ &\geq 0, \quad \lambda_\alpha^- \geq 0, \quad \lambda_\alpha^- (t_\alpha - \bar{t}_\alpha) = 0, \\ \lambda_\alpha^+ (t_\alpha - \bar{t}_\alpha) &= 0, \quad \alpha = 1, 2, \end{aligned} \quad (15)$$

where $\Lambda, \lambda_\alpha^+$ and λ_α^- are Lagrangian multipliers, $\mathbf{t} \in T$ and $\mathbf{u} = \mathbf{u}(\mathbf{t}, \theta)$ is the displacement solution, i.e., the minimum field with respect to \mathbf{v} of $\Pi(\mathbf{v}, \mathbf{t}, \theta)$.

Note that

$$\frac{\partial \mathbb{S}^{\text{memb}}}{\partial t_\alpha} = \alpha \mathbf{S}_\alpha \otimes \mathbf{S}_\alpha.$$

Concerning existence of solutions to the full problem (\mathbb{P}) , this, as well as proof of convergence of the iterative treatment, seems to be open questions. However, experience in similar problems, e.g., in Bendsøe and Sigmund (2002) and Thomsen and Olhoff (1990), indicates that the approach is viable.

5 Discretization and algorithm

For the numerical treatment of (\mathbb{P}) we need to introduce a discrete approximation. The discretization of the state problem, i.e., the problem of finding the minimum displacement $\mathbf{u} \in V$ of the potential energy Π for a given design $\theta \in \Theta$ and $\mathbf{t} \in T$, follows Hansbo and Larson (2014). This implies introducing a triangulation of Σ resulting in a discrete surface, with corresponding discrete normal vector field and projections. The displacement field is approximated using the same triangulation but is possibly of different order.

In addition to the approximation of the state problem we also need to approximate the design fields $\mathbf{t} \in T$ and $\theta \in \Theta$. This is achieved by using point values: these are denoted $\mathbf{t}_i = (t_{1i}, t_{2i})$ and θ_i for point i . In particular, we use superconvergence points of the finite elements (Barlow 1976). Such a discretization means that (13) and (15) are imposed at these evaluation points and the integral in (14) is replaced by a sum.

Let

$$A_{\alpha i}^k = \left(\frac{\partial \mathbb{S}^{\text{memb}}}{\partial t_\alpha} [\boldsymbol{\varepsilon}(\mathbf{u}^k)] : \boldsymbol{\varepsilon}(\mathbf{u}^k) \right)_i,$$

be the left hand side of (13) evaluated at point i and for a displacement field \mathbf{u}^k . Also, let $B_{\alpha i}^k = (\Lambda^k)^{-1} A_{\alpha i}^k$ where Λ^k is a current iterate of the Lagrangian multiplier Λ . For a given displacement iterate \mathbf{u}^k and rotation θ^k the following fixed point iteration formula is suggested by the optimality conditions (13) through (15):

$$t_{\alpha i}^{k+1} = \begin{cases} \underline{t}_{\alpha i} & \text{if } t_{\alpha i}^k (B_{\alpha i}^k)^\eta \leq \underline{t}_{\alpha i} \\ \bar{t}_{\alpha i} & \text{if } t_{\alpha i}^k (B_{\alpha i}^k)^\eta \geq \bar{t}_{\alpha i} \\ t_{\alpha i}^k (B_{\alpha i}^k)^\eta & \text{otherwise,} \end{cases} \quad (16)$$

where $\underline{t}_{\alpha i}$ and $\bar{t}_{\alpha i}$ are point values of the upper and lower bounds and $0 < \eta \leq 1$ is a damping coefficient.

The following algorithmic steps, the convergence of which gives satisfaction of a discrete version of the optimality conditions of (\mathbb{P}) , are now suggested:

1. For a given design θ^k and \mathbf{t}^k , solve the state problem, i.e., find the minimum displacement field of $\Pi(\mathbf{v}, \mathbf{t}^k, \theta^k)$ so as to obtain the current displacement iterate \mathbf{u}^k .
2. Obtain new fiber thickness distributions by the optimality criteria formula (16) where
 - Λ_k is determined such that

$$\sum_i (t_{1i}^{k+1} + t_{2i}^{k+1}) d\Sigma \leq V.$$

A local iteration is needed for this.

3. For each integration point, calculate principal stresses (and/or principal membrane forces). Take s to correspond to the main material direction, i.e., to t_{1i} , such that $t_{1i} \geq t_{2i}$, and chose θ^{k+1} so that this s aligns with the main principal stress direction.
4. Let $k = k + 1$ and return to the first step.

Steps 1 and 2 can be iterated several times before continuing with calculation of fiber directions in Step 3. In fact, in the examples the fixed point iteration (16), for newly calculated displacement \mathbf{u}^k , is repeated until convergence before continuing with the fiber directions in Step 3.

Note that step 3 assumes distinct principal stresses. Numerically coalescence of such stresses occur with close to zero probability but may show up as non-convergence issues. For statically determined structures, i.e., when \mathbf{M} is uniquely determined by (11) and (12), this may be of particular concern. For such cases that have distinct principal stresses, step 3 above needs to be performed only once since

these principal stresses are independent of t . Such problems essentially become convex since the first part of (P) is a convex problem. The first problem of Section 6 is statically determinate but has not everywhere distinct principal stresses.

Clearly other algorithms than the optimality criteria algorithm are available for solving the first part of problem (P). General purpose sequential convex approximation methods (Christensen and Klarbring 2009) like the Method of Moving Asymptotes (MMA) (Svanberg 1987) could be directly applied. A variable thickness sheet problem, that has the same structure and convexity property as the first part of (P) was solved by MMA in Christensen and Klarbring (2009), generally needing more iterations than the optimality criteria algorithm. However, since convergence properties could be sensitive to parameter values it is hard to draw general conclusions based on this experience. Moreover, the optimality criteria algorithm can in fact itself be seen as a particular first order sequential convex approximation method (Christensen and Klarbring 2009) and, thereby, shows similar properties as MMA. In Thomsen and Olhoff (1990) and Thomsen (1991) the first part of (P) was solved by the CONLIN method, see, e.g., Christensen and Klarbring (2009), which is yet another first order method. The same range of number of iterations as in this paper were needed for convergence. Again indicating that essentially any first order method shows similar behaviour.

A distinctively different algorithmic treatment of (P) is to solve both types of variables - rotation and sizing - simultaneously by a general purpose method. However, such an approach would not utilize that the second part of (P) has a known solution in terms of given stresses, and, in particular, would not use the special property of statically determinate problems. Moreover, an explicit parametrization of the rotation is needed for calculation of sensitivities. Such a parametrization would involve a cyclic variable, which together with non-convexity may make a simultaneous approach more likely to end up in local minima.

6 Examples

6.1 Oblate spheroid

An oblate spheroid, where Σ is defined by

$$x^2 + y^2 + (2z)^2 = 1,$$

was solved by different finite elements and triangulations in Hansbo and Larson (2014). Here we treat the same geometry but use an internal pressure p as loading. We seek for optimal fiber distribution as described in previous sections. The data are $E = 1$, $\nu = 0.3$, $t_b = 0.005$, $p = 10$, $V = 0.01$, $\bar{t}_1 = \bar{t}_2 = 0.004$, $\underline{t}_1 = \underline{t}_2 = 0$ and $\alpha = 1$. The

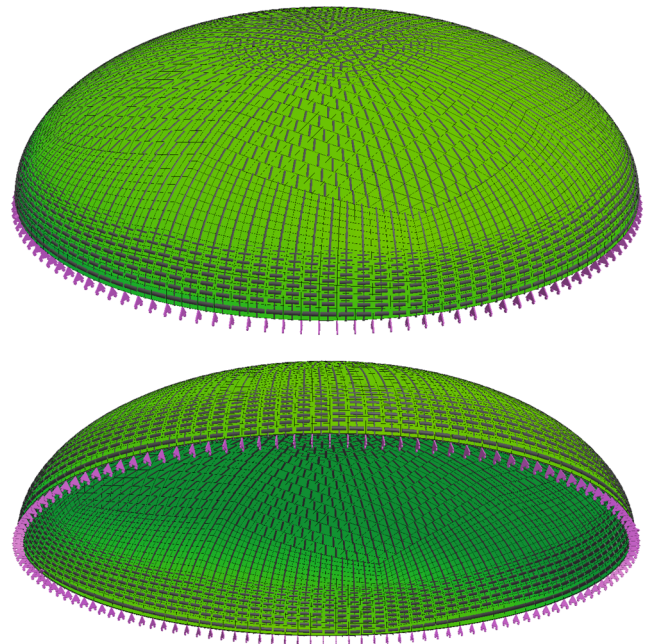


Fig. 1 Optimal fiber distribution of an oblate spheroid, loaded by internal pressure

initial fiber thickness is uniform and chosen so as to satisfy the volume constraint as an equality. We use 3072 bilinear 4-node fully integrated isoparametric elements, implying one superconvergent point per element and, thus, three design variables per element. Symmetry is utilized and only half of the spheroid is modeled. The problem converged in 36 optimality criteria updates and 7 updates of the fiber orientations. As convergence criteria an objective value change below 0.001% and a change $\Delta\theta$ of θ such that $\cos \Delta\theta > 0.999$ are used. Note that the problem is statically determinate, but at the poles of the spheroid symmetry implies that the principal stresses coincide for an exact solution. This is the reason for the need of several updates of fiber

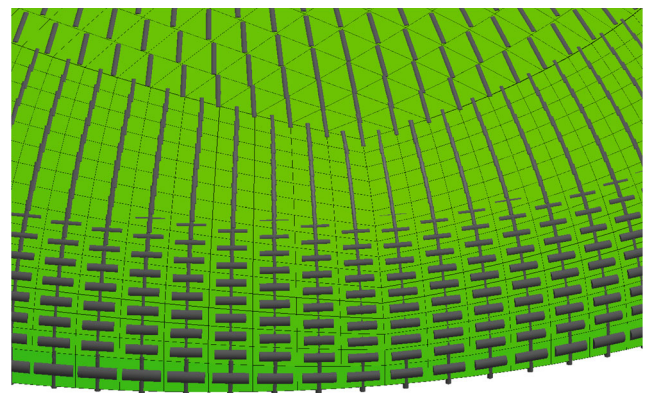


Fig. 2 Zoomed optimal fiber distribution for the oblate spheroid in Fig. 1

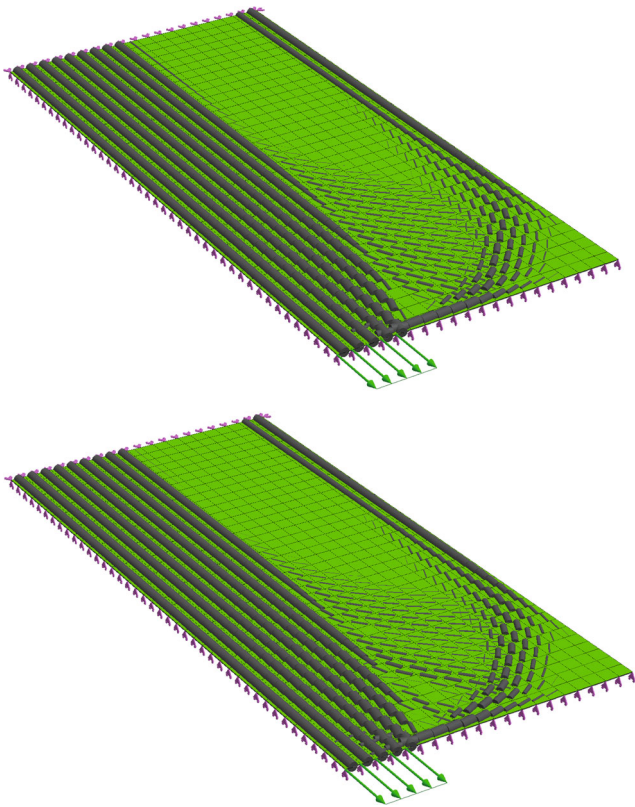


Fig. 3 Optimal fiber distribution for a rectangular membrane using two different initial fiber directions

orientations before convergence, despite the problem being statically determinate.

What concerns the general features of the solution one finds, on examination of Figs. 1 and 2, that close to the equator both fiber families are present, with a compressive stress in the latitudinal direction. As we move towards the poles only the longitudinal fiber family is present, while at the very poles the principal stresses coincide and the direction of fibers becomes indeterminate.

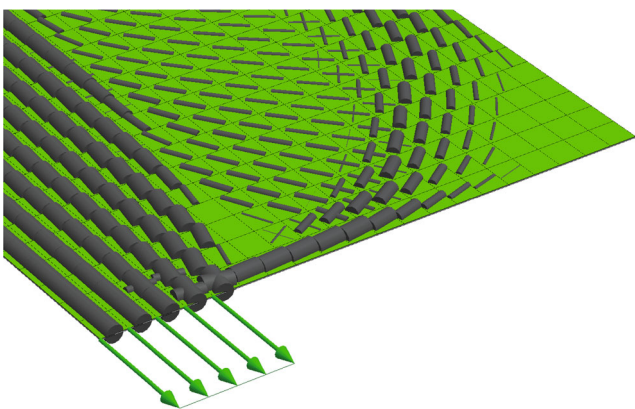


Fig. 4 Zoomed optimal fiber distribution for the upper rectangular membrane in Fig. 3

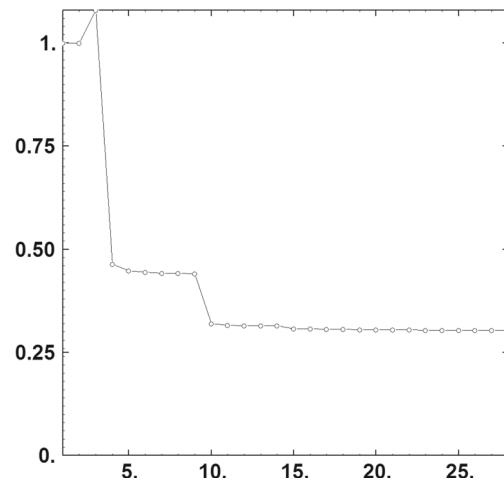


Fig. 5 Iteration history for the upper rectangular membrane in Fig. 3. Relative objective function values are shown on the y-axis and iteration numbers are shown on the x-axis

6.2 Membrane strip

A rectangular membrane of shape 1×0.5 is fixed along one of its short sides and loaded by a force q per unit length on a part of length 0.1 of the other short side, as shown in Fig. 3. The data are $E = 1, \nu = 0, t_b = 0.005, q = 0.001, V = 0.01, \bar{t}_1 = \bar{t}_2 = 0.008, t_1 = t_2 = 0$ and $\alpha = 2$. As in the previous example, the initial fiber thickness is uniform and chosen so as to satisfy the volume constraint as an equality.

The upper solution of Fig. 3 (see also Fig. 4) is found using initial fiber directions defined by the rectangle sides. The lower solution, on the other hand, uses initial directions defined by principal stress directions found in an initial calculation where fibers are excluded. The upper problem converged, using the same tolerances as in the previous problem, in 28 optimality criteria iterations and 12 updates of the fiber directions. The lower problem converged in 15 optimality criteria iterations and 6 updates of the fiber directions. The slight difference between the two solutions is likely the result of a possible non-uniqueness of the solution of problem (P). However, the objective function values for the two cases are essentially the same.

The iteration history for the upper the solution in Fig. 3 is shown in Fig. 5. The marked jumps in the curve correspond to updates of the fiber directions. Note that such updates may result in higher objective function values since fiber fractions are not optimal for these new fiber orientations.

7 Conclusions

The classical facet approach to membrane shells was recently extended to curved elements by Hansbo and Larson (2014). Here we make a further extension by showing how

orthotropic material, of fiber type, can be treated in a similar way, partly inspired by exact plate theory of Nardinocchi and Podio-Guidugli (1994). Based on this orthotropic membrane shell theory we formulate a stiffness design problem, where we seek an optimal structure by both rotation and sizing of reinforcing fibers. The two design variables - representing rotation and sizing - naturally split the formulation into two minimum statements, which suggests a sequential numerical treatment, previously used for similar problems (Bendsøe and Sigmund 2002). This type of formulation also makes clear the distinct character of statically determined problems, which occur for large classes of membrane shells (Ciarlet 2000). For such problems, the material independent stress state implies that the two minimization statements of (\mathbb{P}) become decoupled, and the full problem then means solving a convex sizing problem and separated local orientation problems, each with known analytical global minima.

The approach presented in this paper has several intriguing extensions, that would be important for applications such as the design of racing boat sails. Inclusion of pre-stress and wrinkling states related to negative stresses are examples of this. Extension to large deformations, based on the model of Hansbo et al. (2015), should also be of clear interest.

It may also be noted that, while the large design freedom allowed when optimizing orientation and thickness independently at each point may seem unrealistic from a manufacturability point of view, restrictions involving patches of equal designs can easily be added to the formulation. However, new manufacturing methods, like 3D printing, constantly reduces the need for such constraints.

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Appendix

As a special case of the fiber material defined by \mathbb{E}^{memb} , consider the orthotropic material consisting of two orthogonal families of mechanically equal fibers, i.e., $\alpha_1 = \alpha_2 = \alpha$. We will represent the constitutive law of such a material in the orthogonal base $\{s, s^\perp, n\}$, where $s = s_1$ and $s^\perp = s_2$.

The non-zero part of the stress tensor is $P_\Sigma \sigma P_\Sigma$ and in the indicated base we have:

$$\sigma_{11} := s \cdot (P_\Sigma \sigma P_\Sigma) s = S : (P_\Sigma \sigma P_\Sigma) = A \varepsilon_{11} + B \varepsilon_{22}, \tag{17}$$

$$\sigma_{22} := s^\perp \cdot (P_\Sigma \sigma P_\Sigma) s^\perp = C \varepsilon_{22} + B \varepsilon_{11}, \tag{18}$$

$$\sigma_{12} := s \cdot (P_\Sigma \sigma P_\Sigma) s^\perp = D(\varepsilon_{12} + \varepsilon_{21}) = 2D \varepsilon_{12}, \tag{19}$$

where

$$\begin{aligned} \varepsilon_{11} &= s \cdot (\boldsymbol{\varepsilon} s) = S : \boldsymbol{\varepsilon}, & \varepsilon_{22} &= s^\perp \cdot (\boldsymbol{\varepsilon} s^\perp), \\ \varepsilon_{12} &= s \cdot (\boldsymbol{\varepsilon} s^\perp), & \varepsilon_{21} &= s^\perp \cdot (\boldsymbol{\varepsilon} s) \end{aligned}$$

and

$$\begin{aligned} A &= \frac{t_b}{t}(\delta + 2\mu) + \frac{t_1}{t}\alpha, & B &= \frac{t_b}{t}\delta, \\ C &= \frac{t_b}{t}(\delta + 2\mu) + \frac{t_2}{t}\alpha, & D &= \frac{t_b}{t}\mu. \end{aligned}$$

Since there is no coupling between normal and shear stresses, one concludes that the principal material directions are given by s and s^\perp . Moreover, the condition defining a so-called low shear material is that the constant β below is non-negative, which is indeed the case:

$$\beta = A + C - 2B - 4D = \frac{t_1 + t_2}{t}\alpha \geq 0.$$

Moreover, $A \geq C$ obviously follows from $t_1 \geq t_2$.

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