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Sensitivity reanalysis of vibration problem using combined approximations method

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Abstract Sensitivity is indispensable to structural modification and optimization. This paper focuses on the analytical sensitivity reanalysis for vibration problem in the framework of combined approximations (CA) method. The sensitivity reanalysis formulations of eigenvalues and eigenvectors are derived from the vibration equation reduced by CA method, where the eigenvector sensitivity is solved by Nelson's method. Numerical examples demonstrate the accuracy and efficiency of the proposed reanalysis method. Especially, this method can greatly improve the efficiency of sensitivity analysis and can accelerate the gradient-based structural optimization constrained with frequencies and modal shapes.

Keywords Sensitivity reanalysis · Vibration reanalysis · Structural optimization · Combined approximations method

1 Introduction

Repetitive analysis is usually needed when the structure is modified and optimized. One of the main obstacle of repetitive analysis is the high computational cost in the large scale problems solution. Reanalysis technology, which aims to evaluate the structural response for changed structures, such as displacements, stresses, frequencies and so on, without solving the complete set of modified equations, came into being from

the 1970s (Arora 1976; Phansalkar 1974). Many approximate (Kirsch and Rubinstein 1972) and exact (Level et al. 1996) reanalysis methods were proposed to analyze structures which are modified due to changes in design and optimization. Approximate methods are more efficient and popular. Combined approximations (CA) method proposed by Kirsch is one of the effective approximate methods for solving static displacement equations (Kirsch 2000, 2003b). Responses of the original structures are used in the solution procedures, and then the computational efficiency is improved compared to the repetitive analysis.

Based on CA method, extended CA methods were proposed to solve the structural static reanalysis. A hybrid Fox and CA method combined the merits of Fox's polynomial fitting reanalysis and CA reanalysis was suggested, which had the advantage of globe-local approximation (Zuo et al. 2012). To determine the number of basis vectors of CA method, an adaptive technique is used (Sun et al. 2014). Based on the symmetric successive over-relaxation and corresponding sparse raw, Wang et al. improved the efficiency of CA method on GPU platform (Huang et al. 2014; Wang et al. 2013).

Research of vibration reanalysis methods have been discussed since the early 2000s (Chen et al. 2000; Kirsch and Bogomolni 2007). Kirsch firstly transplanted the CA approach to solve eigenproblems (Kirsch 2003a), where approximation results can be obtained by solving smaller eigenproblems in a reduced Krylov subspace composed of several approximation vectors. CA method could not give enough accuracy in many large modification cases of structural design. To improve the accuracy of the eigenvalues reanalysis of large modifications of structural parameters, combining CA and Rayleigh quotient, Chen developed an extended CA method (Chen and Yang 2000). Then Epsilon algorithm was then applied in

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the eigenproblem reanalysis associated with the Neumann series expansion (Chen et al. 2006). A modified combined approximations method for reanalysis of dynamic problems with many dominant mode shapes was discussed (Zhang et al. 2009). Using a suitable frequency-shift method, the frequency-shift combined approximations approach allowed to calculate higher modes accurately (Xu et al. 2011). Combined with CA method and Newmark-β method, an adaptive reanalysis algorithm was suggested for dynamic analysis (Gao et al. 2013). Then, with the multi-GPU acceleration (He et al. 2015), real large-scale structural design problem can be reanalyzed and the vehicle design cycle was reduced (Wang et al. 2016).

In structural design process, the sensitivity, which can provide search direction, is indispensable to the engineer. Based on reanalysis of displacements, efficient sensitivity of displacement was first presented by Kirsch (Kirsch 1994). A unified approach for accurate approximations of displacement derivatives with respect to design variables was then presented (Kirsch and Papalambros 2001). Currently, analytical sensitivity reanalysis of static displacement was investigated by using Taylor series expansion and CA method to improve the efficiency (Zuo et al. 2016). Finite-difference derivatives of static, vibration and dynamic response were given by Kirsch (Kirsch and Bogomolni 2005; Kirsch et al. 2007). Topology optimization for specified eigenfrequency was accelerated by using CA method and corresponding frequency sensitivity analysis (Bogomolni 2010). Reanalysis and sensitivity reanalysis by CA method was summarized by Kirsch in his article (Kirsch 2010). However, rather less attention has been paid to the analytical sensitivity reanalysis of vibration problem, which is also essential information in many flexible structural optimizations.

In this study, a novel sensitivity reanalysis method of vibration problem is proposed to acquire the derivatives of the eigenvalues and eigenvectors. In this method, only a much smaller eigenproblem and its derivative problem are needed to be calculated. The formulas of sensitivity reanalysis of vibration problem are expressed in Section 2, and then three numerical examples of 8-bar truss structure, truss grid and a monocoque bus frame are demonstrated for the accuracy in Section 3. Finally, efficiency consideration and conclusion are summarized in Section 4.

2 Sensitivity reanalysis formulations

In this paper, sensitivity reanalysis of vibration problem is proposed in the framework of CA method. Therefore, the reanalysis of eigenvalue and eigenvector is firstly introduced here from CA method.

2.1 Kirsch's reanalysis of vibration problem

Given an initial design with stiffness matrix $\mathbf{K}_0$ and mass matrix $\mathbf{M}_0$. The corresponding eigenvector $\mathbf{r}_0$ and eigenvalue λ_0 are calculated by solving the set of initial equations

$$\mathbf{K}_0\mathbf{r}_0 = \lambda_0\mathbf{M}_0\mathbf{r}_0 \tag{1}$$

where $\mathbf{K}_0$ is given from the initial analysis in the decomposed form

$$\mathbf{K}_0 = \mathbf{U}_0^\mathbf{T}\mathbf{U}_0 \tag{2}$$

and $\mathbf{U}_0$ is an upper triangular matrix. $\lambda_0 = \omega_0^2$, ω_0 is the circle free-vibration frequency.

Assuming there is a change in the design, then corresponding changes in the stiffness matrix and mass matrix can be expressed respectively

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K}, \quad \mathbf{M} = \mathbf{M}_0 + \Delta\mathbf{M} \tag{3}$$

Then the modified equations are given by

$$\mathbf{K}\mathbf{r} = \lambda\mathbf{M}\mathbf{r} \tag{4}$$

For new eigenpairs λ and $\mathbf{r}$, the CA method can obtain them efficiently and accurately. Calculation of $\mathbf{r}$ in (4) by the CA method involves the following steps (Kirsch 2003a):

(a) Calculate the basis vectors by steps of iteration. The detailed explanation has been demonstrated in CA method. Here we directly give the form of the basis vectors

$$\mathbf{r}_1 = \lambda_0\mathbf{K}_0^{-1}\mathbf{M}\mathbf{r}_0 \tag{5}$$

For $k = 2$ to s

$$\mathbf{r}_k = -\mathbf{B}\mathbf{r}_{k-1} \tag{6}$$

where s is the number of basis vector which is much smaller than the structural degrees of freedom (DoFs), and $\mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K}$. Notice that we have acquired the decomposed form of $\mathbf{K}_0$ in process of solving the original equations, so the calculations of basis vectors involve only forward and back substitutions of (6).

(b) Calculate the new eigenvector $\mathbf{r}$. Assume $\mathbf{r}$ can be approximated by the linear combination of s preselected linearly independent basis vectors $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_s$, i.e.

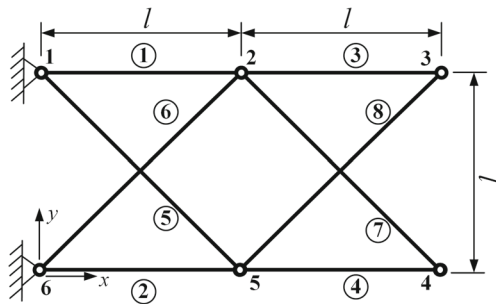$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \cdots + y_s\mathbf{r}_s = \mathbf{r}_B\mathbf{y} \tag{7}$$

Fig. 1 8-bar truss structure

where $\mathbf{r}_B = [\mathbf{r}_1, \quad \mathbf{r}_2, \quad \cdots, \quad \mathbf{r}_s]$ and $\mathbf{y} = [y_1, y_2, \cdots, y_s]^T$.
Now substitute (7) into (4) and pre-multiply $\mathbf{r}_B^T$

$$\mathbf{r}_B^T \mathbf{K} \mathbf{r}_B \mathbf{y} = \mathbf{r}_B^T \lambda \mathbf{M} \mathbf{r}_B \mathbf{y} \tag{8}$$

With the notation

$$\mathbf{K}_R = \mathbf{r}_B^T \mathbf{K} \mathbf{r}_B, \mathbf{M}_R = \mathbf{r}_B^T \mathbf{M} \mathbf{r}_B \tag{9}$$

(8) is reduced as a $s \times s$ eigenproblem

$$\mathbf{K}_R \mathbf{y} = \lambda \mathbf{M}_R \mathbf{y} \tag{10}$$

This small-scale equation can be efficiently solved by Jacobi method. And then substituting the obtained λ and $\mathbf{y}$ into (7), $\mathbf{r}$ is yielded.

2.2 The proposed sensitivity reanalysis of vibration problem

In the following, we derive the sensitivity of eigenvector $\mathbf{r}$ with respect to the design variable x_i. Differentiating (7) directly with respect to x_i gives the first-order derivative equations at the modified design

$$\frac{\partial \mathbf{r}}{\partial x_i} = \frac{\partial \mathbf{r}_B}{\partial x_i} \mathbf{y} + \mathbf{r}_B \frac{\partial \mathbf{y}}{\partial x_i} \tag{11}$$

In order to acquire $\partial \mathbf{r} / \partial x_i$, we should firstly calculate $\partial \mathbf{r}_B / \partial x_i$ and $\partial \mathbf{y} / \partial x_i$. The stiffness matrix $\mathbf{K}$ is usually explicit functions of the design variables. $\mathbf{K}_0$, $\mathbf{r}_0$ and λ_0 are constant for arbitrary change of design variable x_i, so $\partial \mathbf{r}_1 / \partial x_i$ can be expressed as

$$\frac{\partial \mathbf{r}_1}{\partial x_i} = \mathbf{K}_0^{-1} \lambda_0 \frac{\partial \mathbf{M}}{\partial x_i} \mathbf{r}_0 \tag{12}$$

For $k = 2$ to s

$$\begin{aligned}\frac{\partial \mathbf{r}_k}{\partial x_i} &= -\mathbf{K}_0^{-1} \frac{\partial \Delta \mathbf{K}}{\partial x_i} \mathbf{r}_{k-1} - \mathbf{K}_0^{-1} \Delta \mathbf{K} \frac{\partial \mathbf{r}_{k-1}}{\partial x_i} \\ &= -\mathbf{K}_0^{-1} \left(\frac{\partial \Delta \mathbf{K}}{\partial x_i} \mathbf{r}_{k-1} + \Delta \mathbf{K} \frac{\partial \mathbf{r}_{k-1}}{\partial x_i} \right)\end{aligned} \tag{13}$$

Accordingly, we obtain

$$\frac{\partial \mathbf{r}_B}{\partial x_i} = \left[\frac{\partial \mathbf{r}_1}{\partial x_i}, \quad \frac{\partial \mathbf{r}_2}{\partial x_i}, \quad \cdots, \quad \frac{\partial \mathbf{r}_s}{\partial x_i} \right] \tag{14}$$

Additionally, $\partial \mathbf{y} / \partial x_i$ is the derivative of the eigenvector from the reduced eigenproblem (10). Since we have calculated λ and $\mathbf{y}$, $\partial \mathbf{y} / \partial x_i$ can be solved by using Nelson method (Nelson 1976). Rewrite (10) as

$$(\mathbf{K}_R - \lambda \mathbf{M}_R) \mathbf{y} = 0 \tag{15}$$

Besides, $\mathbf{y}$ and $\mathbf{M}_R$ satisfy the following normal equation

$$\mathbf{y}^T \mathbf{M}_R \mathbf{y} = 1 \tag{16}$$

Differentiate (15) and (16) directly with respect to x_i and rearrange them as

$$(\mathbf{K}_R - \lambda \mathbf{M}_R) \frac{\partial \mathbf{y}}{\partial x_i} = -\left(\frac{\partial \mathbf{K}_R}{\partial x_i} - \frac{\partial \lambda}{\partial x_i} \mathbf{M}_R - \lambda \frac{\partial \mathbf{M}_R}{\partial x_i} \right) \mathbf{y} \tag{17}$$

$$\mathbf{y}^T \mathbf{M}_R \frac{\partial \mathbf{y}}{\partial x_i} = -\frac{1}{2} \mathbf{y}^T \frac{\partial \mathbf{M}_R}{\partial x_i} \mathbf{y} \tag{18}$$

Table 1 Eigenvalue and eigenvector sensitivities of 8-bar truss structure

Variables	$\partial \lambda_1 / \partial x_i$			$\partial u_1^{4y} / \partial x_i$			$\partial \lambda_2 / \partial x_i$			$\partial u_2^{4y} / \partial x_i$		
	Exact ($\times 10^3$)	Appr. ($\times 10^3$)	Error (%)	Exact ($\times 10^{-3}$)	Appr. ($\times 10^{-3}$)	Error (%)	Exact ($\times 10^3$)	Appr. ($\times 10^3$)	Error (%)	Exact ($\times 10^{-4}$)	Appr. ($\times 10^{-4}$)	Error (%)
x_1	−15.48	−15.48	0.0	7.940	7.935	0.1	−28.82	−28.82	0.0	358.9	359.0	0.0
x_2	−20.79	−20.79	0.0	−19.84	−19.73	0.6	−23.49	−23.51	0.1	−199.7	−198.1	0.8
x_3	−2.365	−2.359	0.3	−3.576	−3.536	1.1	−2.745	−2.748	0.1	−44.30	−43.85	1.0
x_4	−1.851	−1.850	0.1	2.537	2.545	0.3	−3.224	−3.223	0.0	8.705	8.445	3.0
x_5	−10.42	−10.43	0.1	−19.86	−19.93	0.4	−48.18	−48.15	0.1	−227.0	−231.6	2.0
x_6	−7.289	−7.295	0.1	8.939	8.890	0.5	−48.12	−48.12	0.0	390.8	391.8	0.3
x_7	−2.583	−2.586	0.1	5.312	5.250	1.2	−9.630	−9.634	0.0	8.984	9.637	7.3
x_8	−2.786	−2.783	0.1	−4.751	−4.721	0.6	−7.171	−7.174	0.0	−61.76	−61.08	1.1

u_1^{4y} and u_2^{4y} denote modal displacements of the 1st and 2nd order at the 4th node in the y direction, respectively
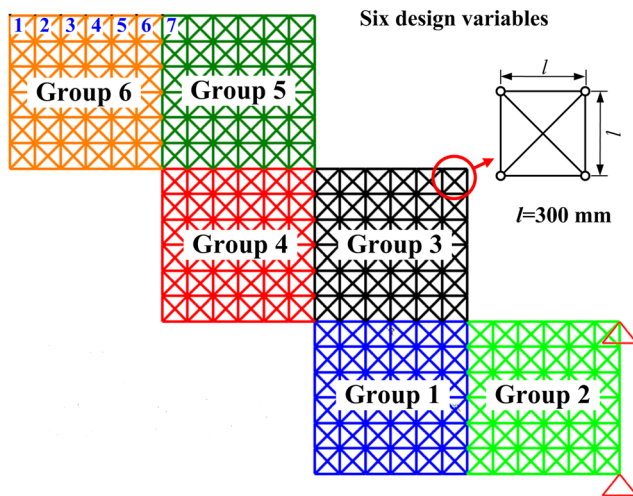
Fig. 2 6-groups truss grid structure

Premultiply (17) by $\mathbf{y}^T$, and noting (15) and (16), we obtains the derivative of eigenvalue

$$\frac{\partial \lambda}{\partial x_i} = \mathbf{y}^T \left(\frac{\partial \mathbf{K}_R}{\partial x_i} - \lambda \frac{\partial \mathbf{M}_R}{\partial x_i} \right) \mathbf{y} \qquad (19)$$

Noting (9), one can obtain the derivative of reduced matrix $\mathbf{K}_R$ and $\mathbf{M}_R$, respectively

$$\frac{\partial \mathbf{K}_R}{\partial x_i} = \frac{\partial \mathbf{r}_B^T}{\partial x_i} \mathbf{K} \mathbf{r}_B + \mathbf{r}_B^T \frac{\partial \mathbf{K}}{\partial x_i} \mathbf{r}_B + \mathbf{r}_B^T \mathbf{K} \frac{\partial \mathbf{r}_B}{\partial x_i} \qquad (20)$$

$$\frac{\partial \mathbf{M}_R}{\partial x_i} = \frac{\partial \mathbf{r}_B^T}{\partial x_i} \mathbf{M} \mathbf{r}_B + \mathbf{r}_B^T \frac{\partial \mathbf{M}}{\partial x_i} \mathbf{r}_B + \mathbf{r}_B^T \mathbf{M} \frac{\partial \mathbf{r}_B}{\partial x_i} \qquad (21)$$

Because $\mathbf{y} = [y_1, y_2, \cdots, y_s]^T$ is obtained in (10), we select the maximum element of vector $\mathbf{y}$, for example y_k. Then, introducing the notation

$$\mathbf{R} = -\left(\frac{\partial \mathbf{K}_R}{\partial x_i} - \frac{\partial \lambda}{\partial x_i} \mathbf{M}_R - \lambda \frac{\partial \mathbf{M}_R}{\partial x_i} \right) \mathbf{y} \qquad (22)$$

we let the elements of matrix $\mathbf{K}_R - \lambda \mathbf{M}_R$ and $\mathbf{R}$ are, respectively

$$(\mathbf{K}_R - \lambda \mathbf{M}_R)_{ki} = 0, \ (\mathbf{K}_R - \lambda \mathbf{M}_R)_{ik} = 0, \ i = 1, 2, \cdots, s, \ i \neq k$$
$$(\mathbf{K}_R - \lambda \mathbf{M}_R)_{kk} = 1, \ \mathbf{R}_k = 0 \qquad (23)$$

Solve (17) to obtain vector $\mathbf{v}$, and

$$\frac{\partial \mathbf{y}}{\partial x_i} = \mathbf{v} + c\mathbf{y} \qquad (24)$$

where c is an unknown parameter which can be determine by substituting (24) into (18)

$$c = -\mathbf{y}^T \left(\frac{1}{2} \frac{\partial \mathbf{M}_R}{\partial x_i} \mathbf{y} + \mathbf{M}_R \mathbf{v} \right) \qquad (25)$$

In summary, the DoFs of the reduced (17) is much smaller than that of the corresponding equation of complete structure, so it is efficient to calculate $\partial \mathbf{y} / \partial x_i$ by the Nelson method.

3 Numerical examples

Three examples are used to demonstrate the efficiency and accuracy of the proposed sensitivity reanalysis method in application of structural modifications.

3.1 8-bar truss structure

A 8-bar truss structure is shown in Fig. 1, where $l = 36$cm, elastic modulus is 72,000 MPa, and density is 2700 kg/m³. Design variables are the cross-sectional area of bar members, i.e. $\mathbf{x} = [x_1, \ x_2, \ \cdots, x_8]^T$. The initial cross-sectional areas are $\mathbf{x}_0 = [10, 10, 10, \ 10, 10, \ 10, 10, 10]^T$ whereas the arbitrarily modified cross-sectional areas are listed in vector $\mathbf{x} = [11, \ 10, \ 8, \ 9, \ 10, \ 11, \ 10, \ 10]^T$. Three basis vectors are used to approximately solve the modified structure. The

Table 2 First order eigenvalue and eigenvector sensitivities of truss grid structure

Variables	$\partial \lambda_1 / \partial x_i$					$\partial u_1^{1y} / \partial x_i$				
	Exact ($\times 10^4$)	Approximate method ($\times 10^4$)				Exact ($\times 10^1$)	Approximate method ($\times 10^1$)			
		2 basic vector	Error (%)	3 basic vector	Error (%)		2 basic vector	Error (%)	3 basic vector	Error (%)
x_1	−2.565	−2.565	0.0	−2.565	0.0	−48.02	−47.94	0	−48.01	0.0
x_2	−55.89	−55.89	0.0	−55.89	0.0	−131.7	−132.0	0.2	−131.8	0.1
x_3	−16.28	−16.28	0.0	−16.28	0.0	−23.57	−23.56	0	−23.51	0.0
x_4	−10.54	−10.54	0.0	−10.54	0.0	4.741	5.108	7.7	4.812	1.5
x_5	−3.975	−3.975	0.0	−3.975	0.0	10.38	10.33	0.5	10.38	0
x_6	−1.167	−1.167	0.0	−1.167	0.0	14.13	14.18	0.4	14.03	0.7

u_1^{1y} denotes the modal displacement of the 1st order at the 1st node in the y direction

Fig. 3 Monocoque bus frame
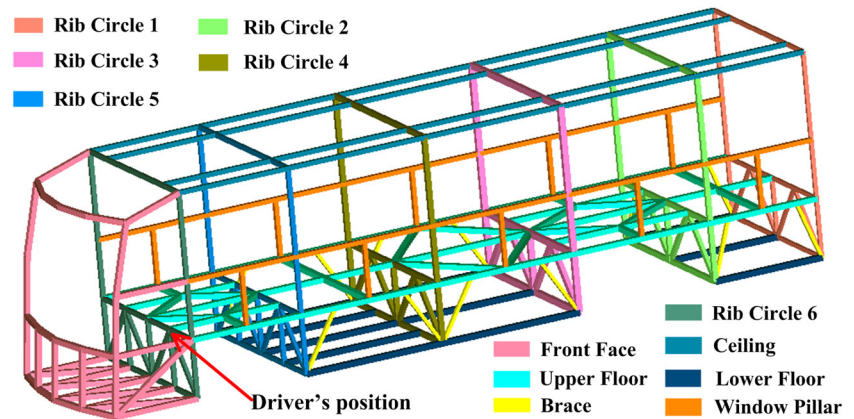
approximate and accurate eigenvalue and eigenvector sensitivities are compared in Table 1. To concisely present the results, u_1^{4y} and u_2^{4y} are selected to list in Table 1, which denote modal displacements of the 1st and 2nd orders at the 4th node in the y direction, respectively. It can be seen that the reanalysis error of eigenvalue sensitivity is less than 1%. For the eigenvector sensitivity, the accuracy of the 1st order is higher than that of the 2nd order. However, the accuracy of eigenvector sensitivity is lower than that of eigenvalue sensitivity. Therefore, the proposed method can provide highly accurate frequency sensitivity for structural optimization with the constraint of frequencies. Next, the efficiency of this method is further investigated by large-scale structures.

3.2 Truss grid

A truss grid with six groups is shown in Fig. 2, whose elastic modulus is $E = 2.1 \times 10^5$ MPa and density is $7.85 \times 10^{-9} t/mm^3$. It consists of 906 bars, 259 nodes and 518 DOFs. All these 906 bar members are classified into six groups and each bar which belongs to the same group has the identical cross-sectional area, so there are six design variables $\mathbf{x} = [x_1, \quad x_2, \quad \cdots, x_6]^T$ for this structure. The initial and modified design variables are $\mathbf{x}_0 = [120, \quad 160, \quad 100, \quad 80, 60, 40]^T$ and

$\mathbf{x} = [123, \quad 158.5, \quad 103.5, \quad 77.5, 59, 38.5]^T$, respectively. We choose the first order eigenvalue λ_1 and the u_1^{1y} component of the corresponding eigenvector $\mathbf{u}_1$ with respect to six design variables to concisely demonstrate the results, as listed in Table 2. For eigenvalue sensitivity, all the approximate solution is equal to the exact solution. The accuracy of eigenvector sensitivity increases with the number of basis vector. In the case of 3 basis vector, the error of eigenvector sensitivity is reduced to 1.5%, which can be accepted in engineering design. Computational cost spent on the sensitivity reanalysis of λ_1 and $\mathbf{u}_1$ with respect to six design variables is 3.2 s, while that of exact analysis is 15.6 s. The proposed approximate method has improved the efficiency by almost 5 times, compared with the exact method.

3.3 Bus frame

A monocoque bus frame, as shown in Fig. 3, is used as an example to demonstrate the effectiveness of the proposed method for real large-scale structure. This bus frame has 172 nodes, 1032 DoFs and 333 thin-walled box beam elements which are grouped into 12 components. The thin-walled box cross section of each component has three design variables height h, breadth b and thickness t, as marked in Fig. 4. The material is Q235 steel with elastic modulus 2.05×10^5 MPa, poisson ratio 0.27 and density 7.85×10^{-9} t/mm³. The initial

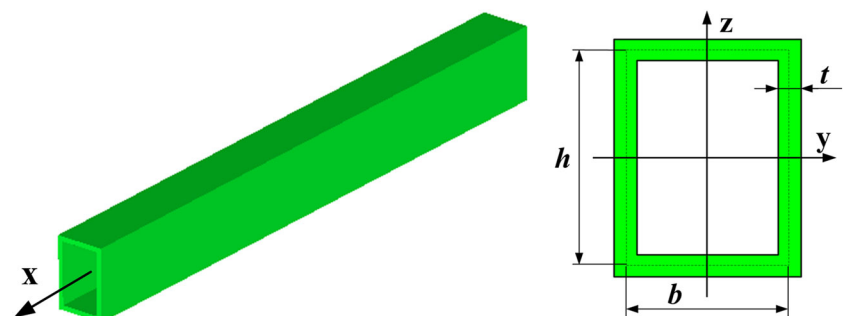**Fig. 4** Thin-walled box section and its design variables

Table 3 Comparisons between the initial and modified cross-sectional sizes

Component name	Cross-sectional sizes (mm)					
	Initial			Modified		
	Breadth	Height	Thickness	Breadth	Height	Thickness
Ceiling	70	80	2	69	79	1.95
Upper floor	100	90	4	101	91	4.05
Lower floor	85	100	4.5	86	101	4.55

and modified cross-sectional sizes of three components are listed in Table 3. The first 20 orders eigenvectors of the initial bus frame is selected as the basic vectors. In this numerical example, we calculate the sensitivities of eigenvalues and eigenvectors for the first two orders with respect to the cross-sectional variables of the ceiling component (denoted as b_1, h_1, t_1) and the lower floor component (denoted as b_2, h_2, t_2). In the design of bus body structure, the vibration of driver's position should be controlled to satisfy the demand of comfort. Accordingly, the first two order eigenvector sensitivities $\partial u_1^z/\partial x_i$ and $\partial u_2^z/\partial x_i$ in the z direction at the driver's position are calculated and listed in Table 4. The maximum error is 13.0%, which may be completely accepted at the conceptual design stage of bus frame. During the structural optimization, move limit technology is extensively used to calculate the step size of design variables, so as to control the magnitude of design modification. The design variables approach to the optimal solution step by step, and each step can belong to small modification (Thomas et al. 2002).

Besides, the average time spent on the exact analysis is 52s, while the approximate method only needs 14s, which saves 73.1% of computational cost. There are 12 components in the bus frame, meanwhile each component has 3 design variables. During the structural optimization, more than 200 iterations often occur. If that, the sensitivity analysis will cost 200×52 s (about 3 h) for the exact analysis, while it needs only 200×14 s (about 45 min) by using the proposed method. The efficiency of optimization may be obviously improved.

4 Conclusion

This paper proposes a sensitivity reanalysis of vibration problem in the framework of CA method. Three numerical examples verify the accuracy and efficiency of the presented method. For the eigenvector sensitivity, the accuracy of the 1st order is higher than that of the 2nd order. Also, the accuracy of eigenvector sensitivity increases with the number of basis vector. However, the accuracy of eigenvector sensitivity is lower than that of eigenvalue sensitivity. Additionally, the proposed approximate method has greatly improved the efficiency, compared with the exact method. In terms of the accuracy of sensitivity reanalysis, truss structure is superior to frame structure. It is because that each bar element in truss structure has only one design variable, while each box beam element in frame structure has three. Therefore, the accuracy of sensitivity reanalysis is determined by both modification and the number of design variables.

The proposed method can be applied in any structural optimization problem constrained with frequency or modal shape responses. Sensitivity calculation occupies a large part of the optimization process. Using the proposed method, almost three quarters of calculating time can be reduced. During the process of structural optimization, exact analysis and approximate reanalysis should be alternately used to calculate the responses and its derivatives, where exact solution can provide the initial values for approximate reanalysis. In this way, the accumulated reanalysis error can be truncated and independent of the iteration number of structural optimization.

Table 4 Eigenvalue and eigenvector sensitivities of bus frame

Variables	$\partial \lambda_1/\partial x_i$			$\partial u_1^z/\partial x_i$			$\partial \lambda_2/\partial x_i$			$\partial u_2^z/\partial x_i$		
	Exact	Appr.	Error (%)	Exact	Appr.	Error (%)	Exact	Appr.	Error (%)	Exact	Appr.	Error (%)
b_1	11.46	11.44	0.2	1.498×10^{-4}	1.383×10^{-4}	7.6	4.28	4.271	0.2	-2.900×10^{-3}	-2.900×10^{-3}	0.0
h_1	13.58	13.55	0.2	1.339×10^{-4}	1.295×10^{-4}	3.3	9.236	9.212	0.3	-3.200×10^{-3}	-3.200×10^{-3}	0.0
t_1	-91.85	-92.72	0.9	2.300×10^{-3}	2.600×10^{-3}	13.0	-129.7	-128.7	0.8	2.180×10^{-2}	2.390×10^{-2}	9.6
b_2	4.349	4.33	0.4	2.827×10^{-5}	2.643×10^{-5}	6.5	-2.08	-2.101	1.0	-1.634×10^{-7}	-1.700×10^{-7}	3.9
h_2	4.656	4.638	0.4	6.198×10^{-8}	5.747×10^{-8}	7.2	-1.275	-1.294	1.5	5.905×10^{-4}	6.659×10^{-4}	12.7
t_2	-11.32	-12.13	7.2	2.681×10^{-6}	2.370×10^{-6}	11.6	-107.4	-108.5	1.0	-6.278×10^{-6}	-6.407×10^{-6}	2.1

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