Struct Multidisc Optim (2016) 54:1423–1437
DOI 10.1007/s00158-016-1556-5

EDUCATIONAL PAPER

On the consideration of uncertainty in design: optimization - reliability - robustness

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Received: 18 January 2016 / Revised: 28 July 2016 / Accepted: 2 August 2016 / Published online: 5 September 2016

Abstract Structural designs proposed by engineers aim to ensure that specific defined requirements are satisfied. Structural optimization is also widely used to identify an admissible design with optimal performance. However, it is important to remember that real mechanical problems exhibit uncertainties in practice that might entail challenges when searching for admissible and/or optimal design solutions. One objective of this paper is to discuss the possible formulations of design problems in this type of uncertain context. By considering uncertainties in the constraint and/or objective functions underlying design problems, several design strategies might in fact be implemented. Especially, this paper should contribute to clarifying the use of the concepts of robustness and reliability in design. After an introduction, the featured design strategies are illustrated and discussed via three academic examples which involve multiple random and design variables. Results show that the structural solutions obtained can be quite sensitive to the formulation of the problem. Thus, the latter should be considered as an essential step during the design procedure.

Keywords Optimization · Design · Robustness · Reliability

Electronic supplementary material The online version of this article (doi:10.1007/s00158-016-1556-5) contains supplementary material, which is available to authorized users.

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1 Introduction

During the design of a structure, engineers need to guarantee that it is able to sustain the loads exerted by its environment. Analytical formulas or numerical models may be used to assess the behavior of a structure and determine whether a particular design is admissible (see e.g. Roy and Chakrabarty 2009; Al Nageim et al. 2010). Parameters such as the dimensions of the structure may be adapted by the designer to prevent failure, and the identification of a set of suitable design variables is also referred to as sizing in this contribution.

Structural optimization is widely used by engineers (Gallagher 1973; Haftka and Gürdal 1992). A performance (resp. cost) metric is defined and subsequently maximized (resp. minimized) using appropriate algorithms. Constrained optimization provides an appropriate framework to include engineering requirements, and the procedure identifies the optimum from among the admissible solutions. Structural optimization has been applied with success to solve academic as well as industrial problems (Sobieszczanski-Sobieski and Haftka 1997; Shin et al. 2002; Ermolaeva et al. 2004; Mrzyglod and Zielinski 2006).

Several parameters can be affected by uncertainties which may be caused by a lack of knowledge about the operating conditions, since little information is available in the early design stage. Other parameters may show inherent scatter, and cannot be reduced to a deterministic value (such as, for example, seismic loading, lack of repeatability in manufacturing processes, etc.). Such uncertainties are often dealt with by introducing simplifying hypotheses, for instance using safety factors, considering only average or extreme values, etc. However, such approaches do not enable us to fully capture the variability of the random parameters, and the outcome of a deterministic optimization may be highly sensitive to minor changes in the parameter values. For instance, Guest and Igusa (2008) showed that the consideration of uncertain nodal location and loads may considerably modify the outcome of a topology optimization. Thus, uncertainties should be explicitly accounted for in order to guarantee that optimal performance is achieved.

In many papers on optimization under uncertainty available in the literature, the emphasis is laid on the implementation of efficient numerical procedures (see for instance Valdebenito and Schuëller 2010a), whereas the focus of the present contribution is on the formulation of the problem, i.e. the strategy used to translate engineering requirements into a formal mathematical problem. This paper actually aims to provide a clear and precise synthesis of design formulations commonly discussed in the literature. Implied notions such as robustness and reliability are thus defined and clarified. The problem formulation is effectively a crucial step of design procedure since results depend significantly on these formulations. This paper also introduces methodologies to adapt a deterministic design problem to a design in uncertain conditions. Stochastic formulations of the objective and constraint functions are described. Robustness is habitually associated with the consideration of uncertainties in the objective function, and reliability-based optimization with the introduction of uncertainties in the constraint functions. This classification presented in Table 1 is based on an overview of the literature available on the topic (see e.g. Doltsinis and Kan 2004; Jensen et al. 2009; Shahraki and Noorossama 2014).

The paper is structured as follows: Section 2 discusses the general concepts of uncertainties, detailing their causes and origins and the way they can be taken into account. It also proposes our classification of design formulations. Based on this classification, Section 3 presents three deterministic design formulations. Section 4 introduces five formulations for design under uncertainty. It also discusses the concept of robustness and an approach to take it into account. Three academic examples then compare the results obtained from each formulation on problems involving different numbers of random and design variables in Section 5. Finally, some conclusions are drawn in Section 6.

2 Design under uncertainty

2.1 Sources of uncertainty and their classification

During the design process engineers are likely to face several uncertainties, since the latter might in theory affect any

Table 1 Different design approaches

increase of numerical efforts →

Robustness →

increase of numerical efforts ↓ / Reliability ↓	No objective function	Objective function X, P deterministic	Objective function X, P uncertain
No constraint function		Optimal design (optimization without constraint) (Section 3.2)	Robust design (Section 4.3)
Constraint function X, P deterministic	Admissible design (sizing) (Section 3.1)	Optimal and admissible design (optimization under constraint) (Section 3.3)	Robust and admissible design (Section 4.4)
Constraint function X, P uncertain	Reliable design (reliability) (Section 4.1)	Optimal and reliable design (RBDO) (Section 4.2)	Robust and reliable design (RBRDO) (Section 4.5)

Grey boxes take into account some uncertainties given quantity. In the mechanical engineering community, it is common practice to separate them into two categories, namely *aleatory* or *random* uncertainties and *epistemic* uncertainties, see for example Beyer and Sendhoff (2007) and Der Kiureghian and Ditlevsen (2009). On one hand, aleatory or random uncertainties refer to those that cannot be reduced by introducing additional data or improving the modeling process. They therefore should be viewed as inherent or intrinsic to the considered phenomenon and, as Beyer and Sendhoff (2007) pointed out, "the designer has to "live with them" and optimize his design according to this reality". Fluctuating loads due to the environment of operation, wind velocity, humidity and to some extent material properties are often defined as aleatory uncertainties, i.e. there is no means to control them. On the other hand, some uncertainties are only due to a lack of knowledge of the studied phenomenon or system behavior, and could therefore be reduced if some conceivable efforts are undertaken. These are called epistemic uncertainties. In this category we can include uncertainties associated with modeling errors or imprecision (coming from the choice of particular mathematical models to describe the real behavior of physical phenomena, knowledge regarding the boundary and operational conditions of such models, the adopted scheme for resolution and numerical approximations), those involved when measuring observations and those due to the statistical modeling of parameters (choice of probabilistic models and evaluation of their parameters). From a philosophical standpoint, most aleatory uncertainties could be converted into epistemic ones if we could improve the knowledge related to them and propose appropriate models. This means that any given uncertainty could be considered either as aleatory or epistemic depending on the "model universe" selected by the engineer, as pointed out by Der Kiureghian and Ditlevsen (2009). For example, uncertainties regarding a material property could in fact be classified as aleatory, due to their physical nature. However, they might potentially be reduced with additional knowledge about the production process (or if a better control is possible) and therefore also be viewed as epistemic. Der Kiureghian and Ditlevsen (2009) emphasize that the choice of such a classification may have an impact in design, decision-making and reliability related problems.

In a design framework, another classification for uncertainties might be proposed, as suggested by several authors (Chen et al. 1996; Allen et al. 2006; Rolander et al. 2006; Beyer and Sendhoff 2007; Baudoui et al. 2012). From their standpoint, uncertainties can be distinguished depending on whether they can be acted upon or not in the design phase:

- *Type I uncertainties* are primitively linked to the environment and conditions of use. The variables that show this type of uncertainty are hereafter noted in the series $P_j(\omega)$, $j = 1, ..., m$ and stored in vector $\mathrm{P}(\omega)$. They do not play a role in the design procedure, i.e. they are independent from it, and as such they are not design variables. These variables can be known either by characteristic values, noted $P_i^{(k)}$, or ranges of values described for example by probability distributions obtained through measurements.
- *System function uncertainties* are those linked to the evaluation of the performance (or output) of the system, i.e. modeling, measurement and/or statistical uncertainties. They were previously referred to as epistemic. In this paper, these uncertainties are also gathered in vector $\mathrm{P}(\omega)$, i.e. with the type I variables.
- *Feasibility uncertainties* are associated with uncertainties on the constraint functions, that is to say with the design space. These uncertainties are also considered as model uncertainties and grouped in vector $\mathrm{P}(\omega)$.
- *Type II uncertainties* are those connected with the production/manufacturing process. Geometrical variables noted $X_i(\omega)$, $i = 1, ..., n$ (and stored in vector $\mathrm{X}(\omega)$) are usually linked to this type of uncertainty. They are part of the design variables which might affect the performance of the system. In practice, the designer knows their nominal values, noted $\bar{X}_i$, which he can modify to some extent in order to obtain a reliable and robust design.

2.2 Consideration of uncertainties in engineering

In the design process, two approaches can be used to take uncertainties into account. The first approach consists of converting each uncertain parameter into a reference value. In this way the stochastic problem is transformed into a deterministic problem. The so-called worst-case method may be used to identify this reference value. Uncertain parameters are assumed to be bounded, and for each variable the bound associated with the worst scenario is identified. This method is commonly employed in tolerance analysis (see e.g. Greenwood and Chase 1987; Scholtz 1995). For example, the quantity of material is minimized (resp. maximized), and for each variable the bound associated with this configuration is identified. These two extreme cases are often referred to as the maximum material condition and the least material condition (see e.g. Simmons et al. 2012). In tolerance analysis, the dimensions are uncertain and the tolerance intervals are predefined. They provide suitable bounds for each parameter. In general, the identification of appropriate bounds is a challenging issue. The second method applicable to convert a random variable into a deterministic parameter consists of multiplying or dividing each uncertain parameter by safety factors. This method is appropriate for unbounded variables as well. Indeed, in design, resistance variables have to be reduced; these variables must hence be divided by safety factors, and load variables have to be amplified and must be multiplied by safety factors (see e.g. Faber and Sørensen 2002). However, this method has limitations discussed in many references, such as by Lemaire (2009). Alternative strategies are available in the literature to transform a stochastic problem into a deterministic problem such as the Corner space evaluation or the variation patterns formulation presented by Yao et al. (2011).

The second approach consists of explicitly taking into account uncertainties which are characterized by a mathematical representation. It allows us to consider entirely the variation in uncertain parameters and to propagate uncertainties to the response of the structure. This approach is more challenging than using a unique reference value for each parameter, since it requires that uncertainties be entirely modeled, which may necessitate much preliminary work to characterize each variable. Several approaches may be employed to perform this characterization. The probabilistic approach defines each uncertain variable by a distribution function (such as normal distribution or Gumbel distribution) and its theory is detailed e.g. by Nowak and Collins (2000). It is commonly used in mechanical design (see e.g. Meng Li et al. 2015a). An alternative approach is imprecise probability. The possibilistic approach defines uncertain parameters by their possibility distributions, as detailed e.g. by Zadeh (1978). This method has been benchmarked by e.g. Inuiguchi and Ramík (2000) or Mousazadeh et al. (2015). Another method is using fuzzy sets to characterize uncertainties (see e.g. Dubois and Prade 1980). The probabilistic approach may be used for aleatory uncertainties and the imprecise probability approach for epistemic uncertainties (see Pedroni et al. 2013). In the rest of this paper, the probabilistic approach is used to consider explicitly all uncertainties.

2.3 Design requirements

In design processes, a mechanical system can be characterized by two different output functions, characterizing the state of the system's performance. First, the *objective-type functions* $f(\mathrm{X}(\omega), \mathrm{P}(\omega))$ aim to quantify the performances of the mechanical system. The goal of all optimizations is to maximize the performances of the system (e.g. quality level) or to minimize the cost. A target value can also be assigned to the objective functions. Other design-related functions are the *constraint-type functions* $g(\mathrm{X}(\omega), \mathrm{P}(\omega))$ which must be satisfied in all operating conditions. In fact, if these functions are not respected, the system is not functional. it is frequent that these constraint functions must be achieved as closely as possible in order not to degrade the objective functions. The admissible space is written $\{\mathrm{X} \in \mathbb{R}^n \mid g(\mathrm{X}(\omega), \mathrm{P}(\omega)) \geq 0\}$.

2.4 Proposed classification of design formulations

When designing, several strategies can be considered to account for the uncertainties previously defined, which leads to multiple design problem formulations and introduces specific notions such as reliability and robustness. Gang et al. (2015) proposed a classification of five design problem formulations. Our classification is introduced in Table 1, where the distinction is made by taking into account uncertainties in the objective and/or the constraint functions. Robustness is defined by the faculty of a system's response to be insensitive to small variations in system parameters, and is thus associated with the objective function. A deterministic optimization also needs to be admissible with respect to the design requirements. When uncertainties are taken into consideration, non-admissible solutions are tolerated as long as they remain rare. Reliability is therefore associated with the constraint function; it characterizes the ability of a system to ensure its functions in a given context.

Three possible states can be defined for each output function: no function is used, a deterministic function is employed or a function subject to uncertainties is considered. Nine combinations are thus introduced according to the state of objective and constraint functions in Table 1. Hence, reliability-based robust optimization takes into consideration uncertainties in both output functions.

However, alternative definitions of the optimization problem under uncertainty exist. For instance, the optimization of maintenance schedules often associates reliability with the objective function (see e.g. Kharmanda et al. 2002; Valdebenito and Schuëller 2010b; Beaurepaire et al. 2012); A. Beck and his co-workers introduced risk-based optimization (Beck and de Santana Gomes 2012; Beck et al. 2015); performance-based optimization provides an alternative framework for the consideration of uncertainties in structural design (see e.g. Möller et al. 2009; Beck et al. 2014).

The next sections detail each design problem formulation presented in Table 1. Indexes of vectors $\bar{\mathrm{X}}$ are used to define the design problem formulation to which they refer.

3 Deterministic design formulation

3.1 Admissible design - sizing

An admissible design space corresponds to a set of solutions (e.g. nominal values) that respect constraints. The underlying design formulation is completely deterministic (1).

$$\text{Find } \bar{\mathrm{X}}_{Adm} \text{ such that: } g(\bar{\mathrm{X}}_{Adm}, \mathrm{P}^{(k)}) \geq 0 \tag{1}$$

This scheme is very often used (e.g. in Strength of Materials) because of its ease of implementation and because it gives the admissible solution space. If we want to deal with uncertainties in an implicit manner, the worst-case approach or the safety factors presented in Section 2.2 can be used beforehand to transform an initial stochastic problem into such a deterministic problem.

3.2 Optimal design - optimization without constraint

Optimal design consists of optimizing an objective function which is not subject to constraints. This problem, which does not take into account uncertainties, is formulated in (2).

$$\text{Find } \bar{\mathrm{X}}_{Opt} \text{ such that: } \bar{\mathrm{X}}_{Opt} = \arg\max_{\bar{\mathrm{X}}} f(\bar{\mathrm{X}}, \mathrm{P}^{(k)}) \tag{2}$$

This design formulation, being an unconstrained optimization, is not used for industrial problems since it can lead to non-admissible solutions (see Section 3.1). Thus it can be applied only for mathematical applications such as linear and non-linear regressions solved by the least-square method. Many methods have been devoted to solving such optimization problems and are detailed by e.g. Huang and Lin (2014), Zhang and Ni (2015) or Andrei (2016).

3.3 Optimal and admissible design - optimization under constraint

The formulations presented in Sections 3.1 and 3.2 are generally insufficient for designers. The combination of these two formulations is more relevant and leads to the definition of a design which is both optimal with respect to the objective function and admissible with respect to the constraint functions (3).

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{Opt\,Adm} \text{ such that:} \\ &\bar{\mathrm{X}}_{Opt\,Adm} = \arg\max_{\bar{\mathrm{X}}} f(\bar{\mathrm{X}}, \mathrm{P}^{(k)}) \\ &\text{Subject to (s.t.): } g(\bar{\mathrm{X}}, \mathrm{P}^{(k)}) \geq 0 \end{aligned} \tag{3}$$

This is a classic problem of optimization under constraint without considering uncertainties, which is also called deterministic constrained optimization. It can be solved by a wide range of standard methods, detailed e.g. by Rao (2009). Deterministic constrained optimization has been widely used by engineers and may feature several strategies such as topology optimization (Mei et al. 2008), shape optimization (Ghabraie et al. 2010) and multidisciplinary and structural optimization (see e.g. Lee et al. 2014).

4 Formulation of design in an uncertain context

4.1 Reliable design

Reliable design formulation (4) can be viewed as an extension of the admissible design problem introduced in Section 3.1. In this approach, uncertainties are explicitly taken into account. They may be modeled by the probabilistic approach briefly recalled in Section 2.2, thereby leading to an assessment of the failure probability of a system or structure and guaranteeing that it is below a given threshold value P_{target}. $\bar{\mathrm{X}}$ denotes therefore the mean of the probability distribution of X.

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{Rel} \text{ such that:} \\ &\text{Prob}\left(g(\mathrm{X}(\bar{\mathrm{X}}_{Rel}, \omega), \mathrm{P}(\omega)) \leq 0\right) \leq P_{target} \end{aligned} \tag{4}$$

The failure probability threshold value may be fixed using a risk-based approach (e.g. Rackwitz 2002) for which consequences of failure are evaluated. In the case of major impact failure events P_{target} needs to be very low, whereas its value can be higher if failure events have minor consequences.

4.2 Optimal and reliable design - reliability-based design optimization

Optimal and reliable design formulation is the search for a deterministic optimum, i.e. no uncertainties are involved in the objective function but uncertainties in the constraint functions are explicitly considered, as described in Section 4.1. This design, commonly named reliability-based design optimization (RBDO), is formulated in (5) as suggested by Vanmarcke (1973), Aoues and Chateauneuf (2010), Shahraki and Noorossama (2014), Meng et al. (2015b), and Meng et al. (2015a). In this paper, RBDO is always associated with deterministic objective and probabilistic constraints.

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{Opt\,Rel} \text{ such that:} \\ &\bar{\mathrm{X}}_{Opt\,Rel} = \arg\max_{\bar{\mathrm{X}}} f(\bar{\mathrm{X}}, \mathrm{P}^{(k)}) \\ &\text{s.t.: Prob}\left(g(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \leq 0\right) \leq P_{target} \end{aligned} \tag{5}$$

RBDO has been applied with success to solve industrial problems. For instance Acar and Solanki (2009) optimized the design of a car subjected to crashes. Obtained RBDO solutions may be significantly different from those resulting from optimization and admissible design problems. This has been highlighted as well by Dubourg et al. (2011) and Beck and de Santana Gomes (2012).

Reliability-based design optimization has been widely discussed in the literature and alternative formulations are proposed, such as those which consider uncertainties in both objective and constraint functions (Nikolaidis and Burdisso 1988; Enevoldsen and Sørensen 1994; Gasser and Schuëller 1997; Schuëller and Jensen 2008; Xia et al. 2015).

4.3 Robust design

Robust design formulation consists of maximizing the objective function, which is affected by uncertainties, without taken any constraints into account (6).

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{Rob} \text{ such that:} \\ &\bar{\mathrm{X}}_{Rob} = \arg\max_{\bar{\mathrm{X}}} \Psi(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \end{aligned} \tag{6}$$

Industrial problems often include constraints which cannot be considered in this formulation. Hence, (6) cannot be used int this context. The choice of formulation for the robust function is essential, since it can significantly modify the result of the optimization. Many formulations can be used for the robust function, from basic ones to more elaborate formulas. Papadrakakis et al. (2005) suggested to consider "an additional objective function which is related to the influence of the random nature of some structural parameters on the performance of the structure". To perform this, a multi-objective formulation (7) can be used (Mourelatos and Liang 2006; Rathod et al. 2013).

$$\begin{cases} \text{Minimize } E\left[f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))\right] \\ \text{Minimize } Var\left[f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))\right] \end{cases} \tag{7}$$

where $E[\circ]$ is the expectation and $Var[\circ]$ is the variance.

One of the major disadvantages is that no compromise is suggested. Indeed, the minimum of the first objective function is not necessary on the minimum of the second objective function. A Pareto front draws the whole possible compromises between the two objectives. These compromises are the points where if one of the two objectives is improved the other will be worsened. Additional criteria need to be introduced to select the optimum on the Pareto front.

Another technique to formulate a robust objective function is the linear combination between the quality and the variation of the performance. This is a method to transform the multi-objective problem into a problem with only one objective function, and can also be a technique to choose the desired compromise on the Pareto front. Papadrakakis et al. (2005) introduce a linear combination between the expectation ($E[\circ]$) and the standard deviation ($\sigma[\circ]$) of the initial objective function (8).

$$\begin{aligned} \phi(\bar{\mathrm{X}}) &= \lambda \times E[f(\mathrm{X}(\bar{\mathrm{X}}, \omega)), \mathrm{P}(\omega)] \\ &\quad + (1-\lambda) \times \sigma[f(\mathrm{X}(\bar{\mathrm{X}}, \omega)), \mathrm{P}(\omega)] \\ \lambda &\in [0, 1] \end{aligned} \tag{8}$$

This formulation is only used when the function f must be minimized; the robust function ϕ must therefore be minimized. Other references (Du et al. 2004; Youn et al. 2005; Lee et al. 2008; Yadav et al. 2010; Zaman et al. 2011) suggest a few alternative formulations. The linear combination is an easy formula to model the robust objective function. Moreover, it enables designers to choose which parameter (quality or variation) they want to highlight, since the parameter λ is arbitrarily selected.

Trosset (1997) and Beyer and Sendhoff (2007) present the Taguchi approach. The robustness of a system can be measured by the loss of quality, quantified by Taguchi's Mean Square Deviation (MSD) function, defined by (9).

$$\begin{aligned} MSD(\bar{\mathrm{X}}) &= E\left[(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) - \bar{f})^2\right] \\ &\simeq \frac{1}{p}\sum_{j=1}^{p}(f(\mathrm{X}(\bar{\mathrm{X}}, \omega_j), \mathrm{P}(\omega_j)) - \bar{f})^2 \end{aligned} \tag{9}$$

where $\bar{f}$ is the system's performance objective (defined by the specifications) and $f(\mathrm{X}(\bar{\mathrm{X}}, \omega_j), \mathrm{P}(\omega_j))$ are the product performance measurements affected by uncertainties. The MSD formula is different if the designer wants to minimize, to maximize or to be as close to a target as possible. All of these cases are introduced by Zang et al. (2005). Taguchi's approach is closely linked to that of experimental design. Equation (9) contains a discrete sum of the results of the available experiments. Equation (10) is an alternative continuous expression for Taguchi's Mean Square Deviation.

$$\begin{aligned} MSD(\bar{\mathrm{X}}) &= \sigma^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \\ &\quad + \delta^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \end{aligned} \tag{10}$$

where $\sigma^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))$ is the performance variance and $\delta^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) = \left(E(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) - \bar{f}\right)^2$ is called the average shift, $E(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))$ being the average performance of the system. To reduce the noise of the response, the Signal-to-Noise-Ratio (SNR) defined in (11) is used.

$$SNR(\bar{\mathrm{X}}) = -10 log_{10}(MSD(\bar{\mathrm{X}})) \tag{11}$$

One criticism concerns the fact that the same weight is assigned to $\sigma^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))$ and $\delta^2(f(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))$ in the definition of MSD. Indeed, contrary to a linear combination approach, it is not possible to modify the weight of the quality parameter with respect to the variation parameter. This approach still enables us to have an indicator of robustness in percentage terms. In fact, if the MSD is divided by the target performance $\bar{f}$, the indicator will be dimensionless.

In the next section, the robust objective function is noted $\Psi(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega))$ and could be associated with any of the functions defined in (8–11).

4.4 Robust and admissible design

As discussed by Otto and Antonsson (1991), the previous formulation is generally insufficient for designers, the constraint functions being discarded. Robust and admissible design consists of searching for the solution that optimizes the robust objective function subject to deterministic constraint functions (12). This formulation is also known as feasibility robustness (Chen et al. 1996; Allen et al. 2006).

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{RobAdm} \text{ such that:} \\ &\bar{\mathrm{X}}_{RobAdm} = \arg\max_{\bar{\mathrm{X}}} \Psi(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \\ &\text{s.t.: } g(\bar{\mathrm{X}}, \mathrm{P}^{(k)}) \geq 0 \end{aligned} \tag{12}$$

This formulation involves uncertainties in the objective functions but the constraint functions remain deterministic. This allows us to obtain an optimum performance which is rather insensitive with respect to moderate variations of the variables. However, as the uncertainties in the constraint functions are not accounted for, the failure probability of the structure cannot be controlled.

4.5 Robust and reliable design

In this last formulation, uncertainties are taken explicitly into account in both objective and constraint functions (13). Solutions are both robust and reliable. This formulation is sometimes called reliability-based robust design optimization (RBRDO) (Lee et al. 2008; Yadav et al. 2010; Rathod et al. 2013; Shahraki and Noorossama 2014).

$$\begin{aligned} &\text{Find } \bar{\mathrm{X}}_{RobRel} \text{ such that:} \\ &\bar{\mathrm{X}}_{RobRel} = \arg\max_{\bar{\mathrm{X}}} \Psi(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \\ &\text{s.t.: } \mathrm{Prob}\left(g(\mathrm{X}(\bar{\mathrm{X}}, \omega), \mathrm{P}(\omega)) \leq 0\right) \leq P_{target} \end{aligned} \tag{13}$$

This formulation might be viewed as the most complete formulation since it allows to identify the design solution which is insensitive to moderate fluctuations of variables for objective and constraint functions (Du and Chen 2000). Lee et al. (2008) apply the RBRDO formulation to solve a problem involving a side impact on a car and compare three methods which are the Dimension Reduction Method, Performance Moment Integration and the Percentile Difference Method. Youn and Xi (2009) propose the eigenvector dimension reduction method to solve the RBDO problem.

4.6 Numerical efforts of the consideration of uncertainties

The consideration of uncertainties in design procedure involves an increase of the computational efforts, as stochastic simulations require numerous evaluations of a numerical model. Optimization under uncertainty demands thus more efforts than deterministic optimization. Trosset and Torczon (1997) and Jin et al. (2003) have proposed efficient numerical strategies to perform these optimizations. Generally in the literature, research efforts about robust optimization are geared towards the identification of appropriate robustness criteria. In Reliability-based Design Optimization, the numerical efforts are critical issues and multiple efficient strategies are proposed in the literature (see for instance Valdebenito and Schuëller 2010a, for an in-depth review). The reference method to estimate probabilities is the Monte Carlo method (see e.g. Caflisch 1998) as it converges towards the exact solution. However, a large number of evaluations of the numerical model are required, and the numerical efforts become prohibitive. Some methods can be employed to approximate the limit state such as the First and Second-Order Reliability Method (see e.g. Du 2005) with reduced numerical efforts. However, these methods are based on significant assumptions and limitations have been identified (see e.g. Rackwitz 2001). Two other schemes allow the designer to reduce considerably design numerical efforts associated with the RBDO. First, the calculation of the failure probability gradient can be coupled with a gradient optimization algorithm (Royset and Polak 2004; Valdebenito and Schuëller 2011) at this goal. Jensen et al. (2009) show that this scheme provides a significant numerical effort reduction. The second approach is the calibration of a metamodel of the time-consuming model (involving for instance a finite element analysis) to be able to evaluate the performance function with reduced efforts (see e.g. Papadrakakis and Lagaros 2002; Agarwal and Renaud 2004; Jensen 2005; Echard et al. 2011; Bourinet 2016; Díaz et al. 2016). Dubourg and Sudret (2014) propose a coupling of these two approaches in their Meta-IS method.

Even though efficient advanced methods have been proposed, stochastic design methods are much more time-consuming than their deterministic counterparts.

5 Application examples

This section aims to implement and illustrate each design formulation of the classification (Table 1) on three academic applications, with different numbers of random variables and design variables. The solutions obtained are then compared and discussed, showing the crucial influence of the formulation on the results.

5.1 Mathematical example

The first example studied is a mathematical application based only on one parameter. The performance is here defined by the following objective function, which is plotted in Fig. 1:

$$f(X) = 9 + 6\exp(-14 + X)\sin(15 - X) - \frac{1}{8}(8 - X)^2 \tag{14}$$

This function is supposed to be maximized to increase the performance of the system. The Nelder-Mead algorithm (Nelder and Mead 1965) is used for the optimization. The

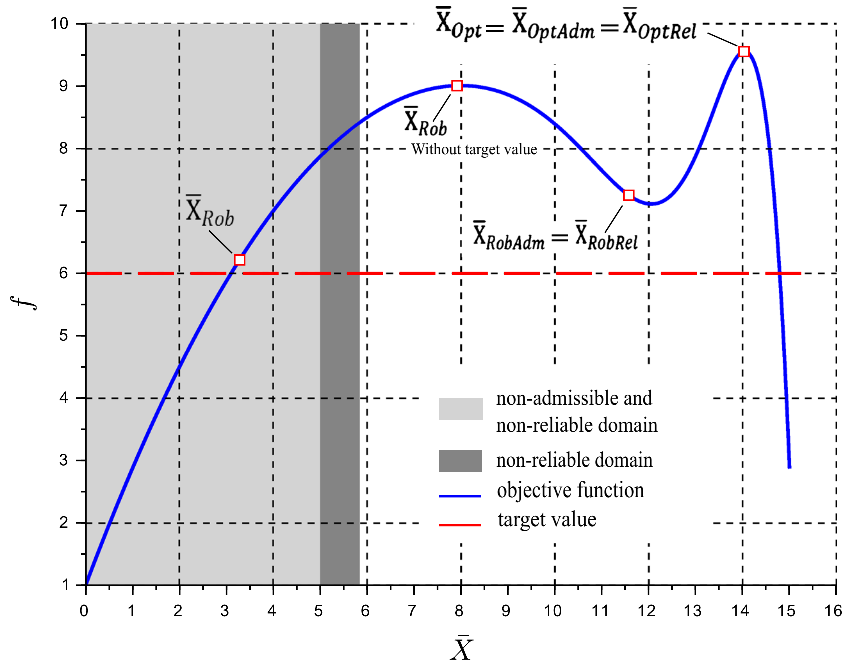

Fig. 1 Results of the mathematical application for different design formulations

potential solutions are subjected to a constraint function defined by (see Fig. 1):

$$g(X) = X - 5 \tag{15}$$

This function must be positive to be admissible. Solutions resulting from featured design formulations in Table 1 are summarized in Table 2.

Deterministic design The admissible domain (Fig. 1) is defined by the constraint function. All the nominal values $\bar{X}$ verifying $\bar{X} \geq 5$ are admissible. The deterministic optimum of $f(X)$ is obtained for $\bar{X}_{Opt} = 14.04$. This corresponds to the global maximum presented in Fig. 1. This solution is admissible regarding the constraint function; it is consequently the solution for optimization under constraint which corresponds to the optimal and admissible design.

Design in an uncertain context The variable X is now considered to be uncertain and modeled by a Gaussian distribution centered on its nominal value $\bar{X}$ and with a standard deviation $\sigma_X = 1$. A maximum target failure probability $P_{target} = 0.1$ is assumed. The reliable domain is

Table 2 Summary of results for the mathematical application

		Robustness ——>		
Reliability ↓		No objective function	Objective function X, P deterministic	Objective function X, P uncertain
	No constraint function		Optimal design $\bar{X}_{Opt} = 14.04$ $f = 9.56$	Robust design $\bar{X}_{Rob} = 3.42$ $f = 6.38$
	Constraint function X, P deterministic	Admissible design $\bar{X}_{Adm} \geq 5$	Optimal and admissible design $\bar{X}_{OptAdm} = 14.04$ $f = 9.56$	Robust and admissible design $\bar{X}_{RobAdm} = 11.63$ $f = 7.23$
	Constraint function X, P uncertain	Reliable design $\bar{X}_{Rel} \geq 5.7$	Optimal and reliable design (RBDO) $\bar{X}_{OptRel} = 14.04$ $f = 9.56$	Robust and reliable design (RBRDO) $\bar{X}_{RobRel} = 11.63$ $f = 7.23$

therefore defined by the set of nominal values $\bar{X}$ verifying $P_f(\bar{X}) \leq P_{target}$, i.e. $\bar{X} \geq 5.7$. The failure probability is determined using Monte Carlo simulation with 10^6 samples.

The Taguchi's Signal to Noise Ratio (11) is selected to be the robustness metrics when uncertainties are considered in the objective function. Two cases are studied. First, no target value for the objective function is considered. In this case the solution is $\bar{X}_{Rob} = 7.93$, corresponding to the local maximum of the objective function. The difference with the deterministic optimization is explained by the fact that the deterministic optimum is on the highest peak. A little variation in the variable X here implies a large variation in the system response f. The curve on the robust optimum is smoother; the variation is thus weaker. In the second case, a target value $\bar{f} = 6$ is assigned to the objective function; the global optimum of the SNR function is now the solution, i.e. $\bar{X}_{Rob} = 3.42$.

The three non-robust maximizations (deterministic objective function) give the same result, i.e. $\bar{X} = 14.04$. This maximum is in both the feasible and reliable domains, and thus is not affected by the strategy used to consider uncertainty in the constraint functions. The robust design with the target value is not admissible with respect to the deterministic and reliable domains. The admissible and robust optimum corresponds to a second peak of the SNR function which is a local optimum, i.e. $\bar{X}_{RobAdm} = \bar{X}_{RobRel} = 11.63$. The robust and admissible solution falls in the reliable domain, and is thus equal to the robust and reliable solution. Explicitly modeling the uncertainties in the constraint function here does not affect the result.

Even if this example is a mathematical application with only one design variable, it shows the main differences between the formulations and the influence of the choice of formulation on the optimization results.

5.2 Application to a container

The second application used to highlight the effect of design formulation on the solution is the study of a cylindrical container (like a soft drink can). This container is defined by a radius R and a height h and must be able to contain a minimum of 33 cm^3 whilst using a minimum of material. The latter is metal sheet with the same thickness for the top, bottom and sides. The constraint function is thus defined by (16).

$$g(R, h) = V(R, h) - 33 = \pi R^2 h - 33 \tag{16}$$

This function must be positive for the container to have a sufficient volume. The two design variables (R, h) are sought to minimize the quantity of container material. Thus, the objective function (in grayscale in Fig. 2) is defined in (17) by the area of sheet metal used.

$$f(R, h) = 2\pi Rh + 2\pi R^2 \tag{17}$$

This function has to be minimized. The Nelder-Mead algorithm is used for the optimization. As shown in Figs. 2 and 3, the optimization problem is convex and the only local optimum is the global optimum. Hence, the selection of the optimization algorithm is not a critical issue in this particular example. However, for problems of greater complexity, the optimization algorithm needs to be selected carefully, as it may have an influence on the outcome of the optimization procedure. The search of each design is bounded to $1 \leq R \leq 4$ and $1 \leq h \leq 10$ (cm) and results are given in Table 3.

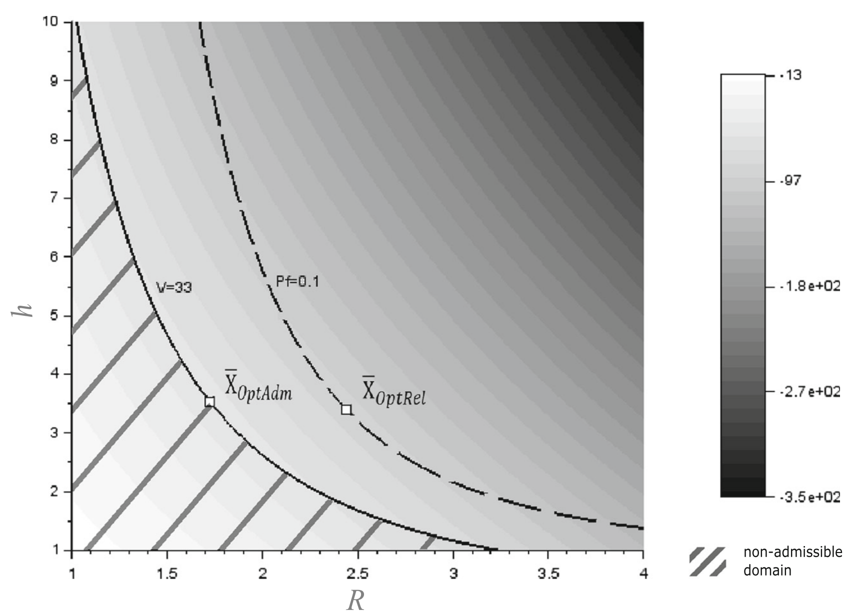

Fig. 2 Results for Container test case objective function ($-f(R, h)$ to be maximized) in grayscale

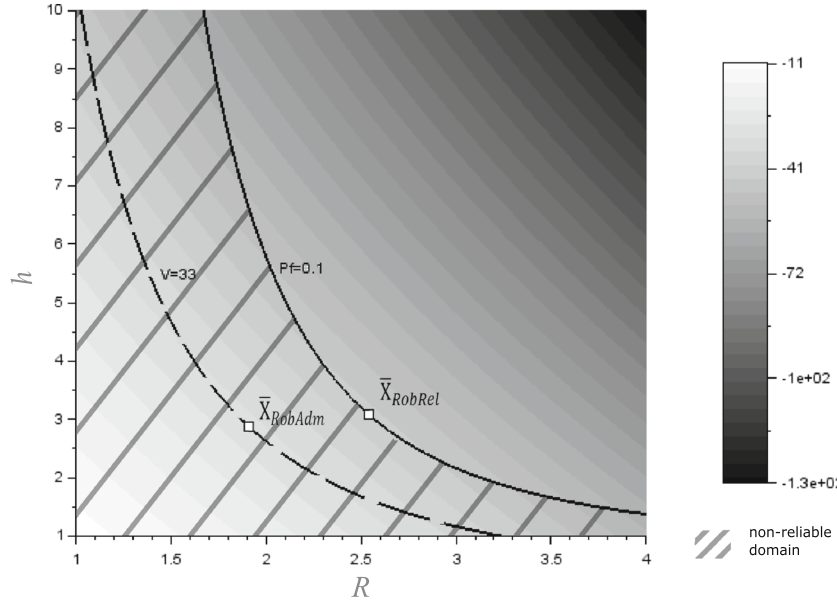

Fig. 3 Results for Container test case objective function ($-\Psi(R, h)$ to be maximized) in grayscale

Deterministic design The non-admissible domain is defined by the constraint function (16) and corresponds to designs which cannot contain 33 cm^3. It is represented by a hatched area in Fig. 2. The optimal design is the zero solution ($R = 0$ and $h = 0$) since no constraint is taken into account; the quantity of material is obviously minimized when no material is used. To best respect the constraint, the optimal and admissible solution ($R = 1.73$ and $h = 3.53$) is at the limit between the admissible and non-admissible domains (see continuous curve Fig. 2).

Design in an uncertain context The variables R and h are affected by type-II uncertainties and modeled by Gaussian distributions with the same standard deviation of 0.5 cm, independently of their nominal value. The probability that the volume is less than 33 cm^3 constitutes the design failure probability, set at 0.1, i.e. $P_{target} = 0.1$. Monte Carlo simulation with 10^6 samples is used to estimate failure probabilities. The Monte Carlo simulations are directly nested within the optimization procedure to perform optimization under uncertainty; this technique is sometimes referred to as the double loop. The method converges towards the optimum, but it requires a large number of evaluations of the performance function. In case large scale numerical models are involved, it is advised to used advanced procedures, as discussed in Section 4.6. The reliable design domain is bounded by this limit and the non-reliable domain is represented by the hatched area in Fig. 3. This reliable domain

Table 3 Summary of results for the container application

	Robustness ——>			
Reliability ↓		No objective function	Objective function X, P deterministic	Objective function X, P uncertain
	No constraint function		Optimal design $\bar{X}_{Opt} = [0; 0]$ $f = 0$	Robust design $\bar{X}_{Rob} = [0; 0]$ $f = 0$
	Constraint function X, P deterministic	Admissible design see Figure 2	Optimal and admissible design $\bar{X}_{OptAdm} = [1.73; 3.53]$ $f = 57.18$	Robust and admissible design $\bar{X}_{RobAdm} = [1.91; 2.89]$ $f = 57.60$
	Constraint function X, P uncertain	Reliable design see Figure 2	Optimal and reliable design (RBDO) $\bar{X}_{OptRel} = [2.44; 3.40]$ $f = 89.53$	Robust and reliable design (RBRDO) $\bar{X}_{RobRel} = [2.54; 3.09]$ $f = 89.85$

is more restrictive than the admissible domain, since neither the worst-case approach nor safety factors are used for deterministic designs.

For robust optimization, uncertainties must be considered in the objective function, therefore requiring us to resort to a new objective function. We here choose to define the robustness criterion (in grayscale in Fig. 3) by a linear combination (18) of the expectation ($E(\circ)$) and the standard deviation ($\sigma(\circ)$) of the initial objective function.

$$\Psi(R,h) = 0.25 \times E(f(R,h)) + 0.75 \times \sigma(f(R,h)) \quad (18)$$

Since no constraints are taken into account, the robust design solution is $R = 0$ and $h = 0$, as in the case of the deterministic solution. It is hence not an admissible design but corresponds to the global optimum of the objective function.

As mentioned previously, the reliable domain is more restrictive than the admissible domain. By optimizing the objective function in such a framework, the obtained RBDO solution ($R = 2.44$ and $h = 3.4$) appears at the limit between the reliable and non-reliable domains (see dotted curve in Fig. 2). As with optimal and admissible design (resp. optimal and reliable design), the robust and admissible design (resp. robust and reliable design) is at the limit between the admissible and non-admissible domains (resp. reliable and non-reliable design). However, their positions (see Fig. 3) have changed to be more insensitive to variations in the problem parameters.

The container application considering two design variables leads to the same conclusions as the first example. Indeed, the results obtained depend on the choice of design formulation, which therefore should be considered as an essential component of the design process in order to obtain the desired design.

5.3 Application to a bracket structure

This application is inspired by Chateauneuf and Aoues (2008) and aims to compare the different design formulations for a structural application with four design variables and five random variables. The bracket structure sketched in Fig. 4 will now be studied.

It is loaded by its own weight and by a vertical load applied at its free tip. Two failure modes are taken into account.

– The maximum bending stress in the CD-beam should not exceed the yield strength of the material

$$g_1(x) = \sigma_y - \sigma_B(x) \geq 0 \quad (19)$$

where

$$\sigma_B = \frac{6M_B}{w_{CD}t^2} \quad \text{with} \quad M_B = \frac{PL}{3} + \frac{\rho g w_{CD} t L^2}{18} \quad (20)$$

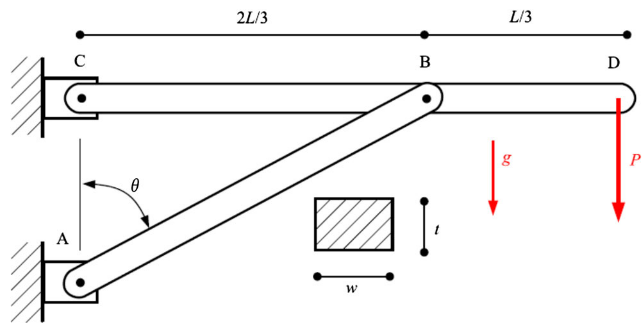

Fig. 4 Bracket structure, reproduced from Dubourg et al. (2011)

– The maximum axial load in the AB-beam should not exceed the Euler critical buckling load

$$g_2 = F_{Buckling}(x) - F_{AB}(x) \geq 0 \quad (21)$$

where

$$F_{Buckling}(x) = \frac{\pi^2 E t w_{AB}^3 9 \sin^2\theta}{48L^2} \quad (22)$$

$$F_{AB} = \frac{1}{\cos\theta}\left(\frac{3P}{2} + \frac{3\rho g w_{CD} t L}{4}\right) \quad (23)$$

The objective function is defined by (24) and corresponds to the weight of the structure to be minimized. As in the previous examples, the Nelder-Mead algorithm is used for the optimization. The design variables are highlighted in grey in Table 5. The domain of these variables is bounded: $50 \leq \mu_{w_{AB}}, \mu_{w_{CD}}, \mu_t \leq 300$ (mm) and $45° \leq \mu_\theta \leq 80°$. Results of each formulation are summarized in Table 4 and illustrated in Fig. 5.

$$c(w_{AB}, w_{CD}, t, \theta) = \rho t L\left(\frac{2w_{AB}}{3\sin\theta} + w_{CD}\right) \quad (24)$$

Deterministic design Deterministic optimization without constraints results in the combination which most minimizes the weight of the structure, such as the minimum for the length and the maximum for the angle. For deterministic optimization under constraints, safety factors are used here. For such purposes, force P is transformed into $\gamma_s P$, the unit mass ρ becomes $\gamma_p \rho$ and the yield stress σ_y changes to σ_y/γ_r with $\gamma_s = \gamma_r = 1.5$ and $\gamma_p = 1.35$. The resulting performance is 1785kg, which is certain to be admissible thanks to the safety factors.

Design in an uncertain context All the parameters involved in the problem are now assumed to be uncertain except variable g, which remains deterministic (see Table 5). The mean values of the random variables highlighted in grey are assumed to be adjustable and correspond to the considered design variables. They are thus affected by type-II uncertainties, whereas the other random

Table 4 Summary of results for the bracket structure application

		Robustness ——>		
Reliability ↓		No objective function	Objective function X, P deterministic	Objective function X, P uncertain
	No constraint function		Optimal design (a) $\bar{\mathrm{X}}_{Opt} =$ [0.05; 0.05; 0.05; 80] $f = 165$	Robust design (d) $\bar{\mathrm{X}}_{Rob} =$ [0.05; 0.05; 0.05; 65.7] $f = 170$
	Constraint function X, P deterministic	Admissible design *Domain defined by* $g_1(x) \geq 0$ *and* $g_2(x) \geq 0$	Optimal and admissible design (b) $\bar{\mathrm{X}}_{OptAdm} =$ [0.055; 0.112; 0.3; 63.9] $f = 1785$	Robust and admissible design (e) $\bar{\mathrm{X}}_{RobAdm} =$ [0.058; 0.112; 0.3; 62.1] $f = 1825$
	Constraint function X, P uncertain	Reliable design *Domain defined by* $\text{Prob}(g_i(x)) \leq P_{target}$ $i = [1, 2]$	Optimal and reliable design (RBDO) (c) $\bar{\mathrm{X}}_{OptRel} =$ [0.055; 0.072; 0.3; 64.9] $f = 1361$	Robust and reliable design (RBRDO) (f) $\bar{\mathrm{X}}_{RobRel} =$ [0.056; 0.072; 0.3; 62] $f = 1396$

variables are affected by type-I uncertainties. The probabilistic approach enables us to characterize these variables by their distribution, presented in Table 5. The coefficients of variation are maintained constant throughout the optimization process. Thus, increasing the mean value of a variable results in an increase in its standard deviation. The distribution of the other random variables is also defined in Table 5. Their parameters remain constant during the optimization procedure. The target failure probability of each constraint function is taken as being $P_{target} = 0.023$, i.e. $\text{Prob}(\{g_i \leq 0\}) \leq 0.023$ for $i = [1, 2]$. Monte Carlo simulation with 10^6 samples is employed to calculate failure probabilities.

An aggregation approach (25) is used here when uncertainties are taken into account in the objective function.

$$\Psi(c) = 0.25 \times E(c) + 0.75 \times \sigma(c) \tag{25}$$

The robust design (Fig. 5d) is slightly different from the optimal design (Fig. 5a) without constraints, but is still non-admissible since the constraints are not considered.

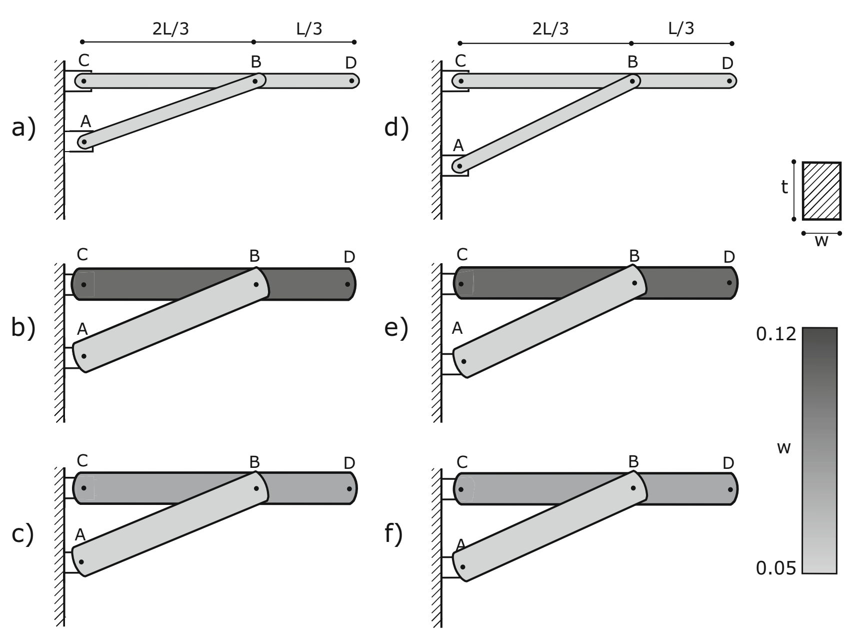

Fig. 5 Illustration of the results of the bracket structure application; (**a**) Optimal design, (**b**) Optimal and admissible design, (**c**) Optimal and reliable design, (**d**) Robust design, (**e**) Robust and admissible design, (**f**) Robust and reliable design

Table 5 Probabilistic model for the bracket structure

Variable		Distribution	Mean	C.o.V.
P	(kN)	Gumbel	100	14%
E	(GPa)	Gumbel	200	8%
σ_y	(MPa)	Lognormal	225	8%
ρ	(kg/m^3)	Weibull	7860	10%
L	(m)	Normal	5	5%
g	(m/s^2)	—	9.81	—
w_{AB}	(mm)	Normal	$\mu_{w_{AB}}$	5%
w_{CD}	(mm)	Normal	$\mu_{w_{CD}}$	5%
w_t	(mm)	Normal	μ_{w_t}	5%
θ	(mm)	Normal	μ_θ	10%

A robust and admissible design (Fig. 5e), taking uncertainties into account in the objective function, results in an increase in the average weight of the structure from 1785kg (for the optimal and admissible design (Fig. 5b)) to 1825kg but it ensures a reduced sensitivity with respect to parameter variations. The solutions obtained by the RBDO (Fig. 5c) and by the robust and reliable design (Fig. 5f) are quite close, even if the average weight is once again increased by the consideration of uncertainties in the objective function: 1361kg for the RBDO (Fig. 5c) compared to 1396kg for the robust and reliable design (Fig. 5f). It is worth noting that there are major differences between the results obtained from deterministic constraint formulations and those produced using uncertain constraint formulations. In fact, the safety factors excessively restrict the admissible domain by increasing the obtained weight by 400kg, which represents 30 % of the result provided by the reliable optimization formulation.

Finally, it appears that the design is strongly influenced by the formulation of the problem. The use of safety factors appears to be too conservative, and increases the weight of the structure. This example also shows that increasing the number of design variables and random variables does not change the conclusions, which is that major differences exist between design formulations.

6 Conclusions

A classification of design problems (Table 1) is proposed in this paper, and different definitions and formulations are clarified. A comprehensive summary of design formulations for designers is discussed. Designing aspires to identify admissible solutions subject to constraints, and optimizing aims to maximize performance metrics. During these procedures, uncertainties can be taken into consideration in two main ways: each parameter can be characterized by a reference value and a deterministic optimization is then solved, or uncertainties can be taken into account explicitly and a stochastic optimization is then solved. Robustness is linked to the objective whereas reliability is associated with the constraints. Thus, a reliable and robust design considers both objective and constraint uncertainties. A consensus has been reached for the formulation of uncertainties in constraint functions: failure probability is used. Numerous strategies exist for robustness and some of them are reviewed in this paper. Applications show that significant differences exist between formulations. The choice of design formulation is hence an essential step in the design procedure and requires careful consideration since it must be a trade-off between the accuracy of results and the numerical effort.

Acknowledgments The first author was funded by a ministerial grant from Blaise Pascal University. The contribution of SNECMA Villaroche is also gratefully acknowledged.

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