

An extended probabilistic method for reliability analysis under mixed aleatory and epistemic uncertainties with flexible intervals

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Abstract The reliability analysis approach based on combined probability and evidence theory is studied in this paper to address the reliability analysis problem involving both aleatory uncertainties and epistemic uncertainties with flexible intervals (the interval bounds are either fixed or variable as functions of other independent variables). In the standard mathematical formulation of reliability analysis under mixed uncertainties with combined probability and evidence theory, the key is to calculate the failure probability of the upper and lower limits of the system response function as the epistemic uncertainties vary in each focal element. Based on measure theory, in this paper it is proved that the aforementioned upper and lower limits of the system response function are measurable under certain circumstances (the system response function is continuous and the flexible interval bounds satisfy certain conditions), which accordingly can be treated as random variables. Thus the reliability analysis of the system response under mixed uncertainties can be directly treated as probability calculation problems and solved by existing well-developed and efficient probabilistic methods. In this paper the popular probabilistic reliability analysis method FORM (First Order Reliability Method) is taken as an example to illustrate how to extend it to solve the reliability analysis problem in the mixed uncertainty situation. The efficacy of the proposed method is demonstrated with two numerical examples and one practical satellite conceptual design problem.

Keywords Aleatory uncertainty · Epistemic uncertainty · Flexible interval · Measure theory · Reliability analysis

Nomenclature

\emptyset	Empty set
N	Set of natural numbers
Z	Set of integers
\mathbb{R}	Set of real numbers
$\overline{\mathbb{R}}$	Set $\mathbb{R} \cup \{-\infty\} \cup \{\infty\}$
$\overline{\mathbb{R}}^n$	Set $\{\mathbf{x} = (x_1, \dots, x_n) x_i \in \overline{\mathbb{R}}, 1 \leq i \leq n\}$

1 Introduction

In engineering, it is important to analyze the reliability of the product system performance under the effects of uncertainties throughout its life cycle. The uncertainties generally include both the aleatory type (objective uncertainty) arising from an inherent randomness and the epistemic type (subjective uncertainty) resulting from the lack of knowledge (Helton and Johnson 2011; Helton and Pilch 2011). Due to the necessity and importance of treating the aleatory and epistemic uncertainties properly with corresponding mathematical methods rather than simply using the traditional probabilistic methods to treat all the uncertainties as random ones under strong assumptions (Der Kiureghian and Ditlevsen 2009), there emerges increasing literature in recent years to address the reliability analysis problems under both aleatory and epistemic uncertainties, e.g. Fuzzy set theory (Zhang and Huang 2010; Li et al. 2014; He et al. 2015), random set theory (Oberuggenberger 2015) and probabilistic bounding analysis (Sentz and Ferson 2011), combined probabilistic and interval analysis method (Jiang et al. 2013), combined probabilistic

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and evidence theory method (Du 2008; Eldred et al. 2011; Yao et al. 2013b), and other numerical approaches such as double-loop Monte-Carlo Simulation (MCS) (Du et al. 2009), perturbation based method (Gao et al. 2010, 2011), encapsulation based method (Jakeman et al. 2010; Chen et al. 2013), families of Johnson distributions based probabilistic method (Urbina et al. 2011; Zaman et al. 2011), etc. Among these researches, one of the widely used methods is to model the epistemic uncertainties with intervals and generally the interval bounds are fixed. However in reality, the interval bounds may be flexible and varies with different conditions or other independent variables. For example, in municipal solid waste management, the interval bounds of the unit transportation cost are functions of the energy prices (He et al. 2009). In satellite system design, the interval bounds of some subsystem mass estimation coefficients are also variable as functions of payload performance (as shown in section 4.3). Thus it is motivated to study the reliability analysis approach under mixed aleatory and epistemic uncertainties with flexible intervals in this paper.

In this research, the aleatory uncertainties (random variables) are handled with probability theory. The epistemic uncertainties are handled with evidence theory, as it is a more general method dealing with interval information when very limited knowledge is available and only several possible intervals can be given to roughly describe the distribution of the epistemic uncertainty (Helton and Johnson 2011). With combined probability and evidence theory, the calculation of belief and plausibility measures of failure (the lower and upper bounds of the precise probability of failure) involves the probability calculation of the lower and upper limits of the system response function as the epistemic uncertainties vary in each focal element (Du 2008; Yao et al. 2013a, b). These calculations can be solved by nested MCS, nested optimization, or other nested probabilistic and interval method, which are generally computationally expensive and currently only applicable for fixed interval bounds. To address this problem, an Extended Probabilistic method for Reliability Analysis under mixed aleatory and epistemic uncertainties with Flexible Intervals (EPRAFI) is proposed in this paper. Based on the measure theory, it is proved that the upper and the lower limit functions in the preceding mixed uncertainty analysis algorithm are measurable under certain circumstances, e.g. the system response function is continuous and the flexible interval bounds satisfy certain conditions (the bound functions are continuous or have countable values, either finite or infinite, as the independent variables vary in Borel sets). Then the upper and lower limits of the system response function can be treated as random variables, and its probability calculation can be directly solved by existing probabilistic methods.

2 Preliminaries

2.1 The Fundamentals of measure theory for aleatory uncertainty

Let Ω be a nonempty set, and A a σ -algebra over Ω . (Ω, A) is called a measurable space. $\mu : A \rightarrow \overline{\mathbb{R}}$ is called a **measure** over the space A if it satisfies: 1) $\mu(\emptyset) = 0$; 2) $\mu(a) \geq 0$ for $\forall a \in A$; 3) for every countable sequence of mutually disjoint events $\{a_i\}_{i=1}^{\infty} \subset A$, $\mu(\cup_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i)$. If $\mu(\Omega) = 1$, μ is called a **probability measure** and denoted as Pr . The triple (Ω, A, μ) is called a measure space and (Ω, A, Pr) is called a **probability space**. For two measurable spaces (Ω_1, A_1) and (Ω_2, A_2) , the mapping $f : \Omega_1 \rightarrow \Omega_2$ is called a measurable mapping from (Ω_1, A_1) to (Ω_2, A_2) if for $\forall a \in A_2$, $f^{-1}(a) \in A_1$, where $f^{-1}(a) = \{\omega_1 : f(\omega_1) \in a, \omega_1 \in \Omega_1\}$. Let $B(\overline{\mathbb{R}})$ be the Borel algebra of $\overline{\mathbb{R}}$ which is the σ -algebra over all the open (or closed, or half closed half open, etc.) sets in $\overline{\mathbb{R}}$. The measurable mapping from (Ω, A) to $(\overline{\mathbb{R}}, B(\overline{\mathbb{R}}))$ is called a measurable function. Specifically the measurable function $X = f(\cdot)$ from the probability space (Ω, A, Pr) to $(\mathbb{R}, B(\mathbb{R}))$ is called a **random variable**. The following important theorems which are relevant to the research work in this paper are given and the corresponding proofs are referred to (Halmos 1974; Athreya and Lahiri 2006; Liu 2007).

Theorem 1: Any continuous function $f : \overline{\mathbb{R}}^m \rightarrow \overline{\mathbb{R}}^n$ is a measurable function.

Theorem 2: If the function $f : \overline{\mathbb{R}}^m \rightarrow \overline{\mathbb{R}}$ has countable (either finite or infinite) values and can be formulated as $f(\omega) = a_k$ for $\omega \in A_k (k = 1, 2, \dots)$, where A_k is a Borel set, then f is a measurable function.

Theorem 3: The fundamental arithmetic operation (if it is meaningful) of two measurable functions is also measurable.

Theorem 4: Let g be a measurable mapping from the measurable space (Ω_1, A_1) to (Ω_2, A_2) , f a measurable mapping from the measurable space (Ω_2, A_2) to (Ω_3, A_3) . Define $(f \circ g)(\cdot) = f(g(\cdot))$. Then $f \circ g$ is a measurable mapping from (Ω_1, A_1) to (Ω_3, A_3) .

Theorem 5: Let $\{f_i, i = 1, 2, \dots\}$ be a measurable function sequence from (Ω, A) to $(\mathbb{R}, B(\mathbb{R}))$. Then $\inf_{1 \leq i < \infty} f_i$, $\sup_{1 \leq i < \infty} f_i$, $\liminf_{i \rightarrow \infty} f_i$, and $\limsup_{i \rightarrow \infty} f_i$ are also measurable functions. If $\lim_{i \rightarrow \infty} f_i$ exists, $\lim_{i \rightarrow \infty} f_i$ is also a measurable function.

Theorem 6: Let the vector $\mathbf{X} = (X_1 \ X_2 \ \dots \ X_n)$ be an n -dimensional random vector, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a measurable function. Then $f(\mathbf{X})$ is a random variable.

For a random variable X , its cumulative distribution function (CDF), probability density function (PDF), and other

important definitions and theorems are referred to (Halmos 1974; Athreya and Lahiri 2006; Liu 2007). Given a measurable system response function $g(\mathbf{X})$ with random uncertain input vector \mathbf{X} , $g(\mathbf{X})$ is also a random variable (Theorem 6). With the failure region $F = \{\mathbf{x} | g(\mathbf{x}) < a\}$, the probability of failure is $p_f = \Pr\{\mathbf{X} \in F\} = \int_F p(\xi) d\xi$, where $p(\cdot)$ is the joint probability density function of \mathbf{X} (Melchers 1999). This integral is generally difficult to calculate analytically and various approximate calculation methods have been developed, among which FORM (First Order Reliability Method) is very popular for its simplicity and efficiency (Hohenbichler et al. 1987; Zhao and Ono 1999; Rackwitz 2001). In FORM, the random vector \mathbf{X} is first transformed into an uncorrelated Gaussian random vector \mathbf{U} in the standard normal space \mathbb{U} by the transformation $\mathbf{U} = T(\mathbf{X})$. The failure domain in the \mathbb{U} space is defined by $g(\mathbf{X}) = g(T^{-1}(\mathbf{U})) = G(\mathbf{U}) < a$. The Most Probable Point (MPP) is searched through the following optimization:

$$\min_{\mathbf{U}} \|\mathbf{U}\| \quad \text{s.t.} \quad G(\mathbf{U}) = a \tag{1}$$

Denote the optimum as \mathbf{u}^* and define the reliability index as $\beta = \|\mathbf{u}^*\|$. Then p_f can be estimated by $P_f \approx \Phi(-\beta)$ if $P_f \leq 0.5$ or $P_f \approx \Phi(\beta)$ if $P_f > 0.5$.

2.2 The fundamentals of evidence theory for epistemic uncertainty

Denote the epistemic uncertain variable as Y and describe its distribution with a triple $(2^\Psi, \mathcal{Y}, m)$, which is called the **evidence space**. Ψ is the universal set containing all the finite elementary propositions for the possible values of Y . The propositions can be intervals that are consonant or non-consonant and continuous or discrete. 2^Ψ is the power set of Ψ . m is the basic probability assignment (BPA) function which maps 2^Ψ to $[0, 1]$. It satisfies the following axioms: 1) $\forall A \in 2^\Psi, m(A) \geq 0$; 2) for the empty set $\emptyset, m(\emptyset) = 0$; 3) for all the $A \in 2^\Psi, \sum m(A) = 1$. The set A which satisfies $m(A) > 0$ is called a **focal element**. \mathcal{Y} is the set of all the focal elements. For an epistemic uncertain vector $\mathbf{Y} = (Y_1 \ Y_2 \ \dots \ Y_{N_y})$, its evidence space (C, \mathcal{Y}, m) is defined by the evidence space $(C_i, \mathcal{Y}_i, m_i)$ of each element Y_i as

$$C = C_1 \times C_2 \times \dots \times C_{N_y} \\ = \{c_k = (b_1, b_2, \dots, b_{N_y}) | b_i \in C_i, 1 \leq i \leq N_y\} \tag{2}$$

$$m(c_k) = \prod_{i=1}^{N_y} m_i(b_i), \quad c_k = (b_1, b_2, \dots, b_{N_y}) \tag{3}$$

For a system response function $g(\mathbf{Y})$ with the epistemic uncertain input \mathbf{Y} defined by (C, \mathcal{Y}, m) and the failure region defined by $F = \{\mathbf{y} | g(\mathbf{y}) < a\}$, the precise probability of failure $p_f = \Pr\{\mathbf{Y} \in F\}$ cannot be obtained due to the lack of knowledge about the precise probability distribution of \mathbf{Y} . In

evidence theory, the belief measure (Bel) and plausibility measure (Pl) are defined to bracket the precise p_f as $\text{Bel}\{\mathbf{Y} \in F\} \leq p_f \leq \text{Pl}\{\mathbf{Y} \in F\}$, which are defined by

$$\text{Bel}\{\mathbf{Y} \in F\} = \sum_{A|A \subseteq F} m(A), \quad \text{Pl}\{\mathbf{Y} \in F\} = \sum_{A|A \cap F \neq \emptyset} m(A) \tag{4}$$

To calculate Bel and Pl, the extreme values $[g_{\min}(\mathbf{Y}), g_{\max}(\mathbf{Y})]$ in each focal element should be calculated and compared with the limit value a . To identify the response extremes, vertex method, sampling method, and optimization based method can be used (Bae et al. 2004a, b). The relation between the belief and plausibility measures is $\text{Pl}\{F\} = 1 - \text{Bel}\{\bar{F}\}$, where \bar{F} is the complement set of F . As the information about \mathbf{Y} increases, $\text{Bel}\{\mathbf{Y} \in F\}$ and $\text{Pl}\{\mathbf{Y} \in F\}$ will gradually get close and finally converge to $\Pr\{\mathbf{Y} \in F\}$. By varying the value of a , the **cumulative belief function (CBF)** and **cumulative plausibility function (CPF)** of $g(\mathbf{Y})$ can be obtained as

$$\text{CBF}(a) = \text{Bel}\{g < a\} \quad \text{CPF}(a) = \text{Pl}\{g < a\} \tag{5}$$

The fundamental knowledge of evidence theory which is the basic to understand this paper is briefly introduced above. For more detailed tutorials, readers are referred to (Shafer 1976; Oberkampf and Helton 2002).

2.3 Reliability analysis under mixed aleatory and epistemic uncertainties

When the system is affected by both aleatory and epistemic uncertainties, the system response function can be denoted as $g(\mathbf{X}, \mathbf{Y})$ with the aleatory uncertain input \mathbf{X} defined by (Ω, A, Pr) and the epistemic uncertain input vector \mathbf{Y} described by (C, \mathcal{Y}, m) with N_C focal elements. Denote the failure region as $F = \{(\mathbf{x}, \mathbf{y}) | g(\mathbf{x}, \mathbf{y}) < a\}$. The probability of failure $p_f = \Pr\{(\mathbf{X}, \mathbf{Y}) \in F\}$ cannot be precisely calculated due to the existence of epistemic uncertainties \mathbf{Y} , and its possible value range is an interval instead. With combined probability and evidence theory, p_f are bounded by Bel and Pl which are derived as (Du 2006; Yao et al. 2013a, b)

$$\text{Bel}\{(\mathbf{X}, \mathbf{Y}) \in F\} = \sum_{k=1}^{N_C} (m(c_k) \cdot \text{Bel}_k\{(\mathbf{X}, \mathbf{Y}) \in F\}) \\ \text{Bel}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = \Pr\{\mathbf{X} | \forall \mathbf{Y} \in c_k, g(\mathbf{X}, \mathbf{Y}) < a\} \\ = \Pr\left\{\mathbf{X} \mid \sup_{\mathbf{Y} \in c_k} g(\mathbf{X}, \mathbf{Y}) < a\right\} \tag{6}$$

$$\text{Pl}\{(\mathbf{X}, \mathbf{Y}) \in F\} = \sum_{k=1}^{N_C} (m(c_k) \cdot \text{Pl}_k\{(\mathbf{X}, \mathbf{Y}) \in F\}) \\ \text{Pl}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = \Pr\{\mathbf{X} | \exists \mathbf{Y} \in c_k, g(\mathbf{X}, \mathbf{Y}) < a\} \\ = \Pr\left\{\mathbf{X} \mid \inf_{\mathbf{Y} \in c_k} g(\mathbf{X}, \mathbf{Y}) < a\right\} \tag{7}$$

Bel_k and Pl_k are called the sub-belief and sub-plausibility of the focal element $c_k(1 \leq k \leq N_C)$. Traditionally the bounds of c_k are fixed, and the corresponding methods to calculate Bel_k and Pl_k are studied by Du (Du 2008) and Yao (Yao et al. 2013b). However in practical engineering, the bounds of c_k may be variable and the corresponding reliability analysis method has not been studied yet. In this paper, it will be thoroughly studied in the following sections, where an Extended Probabilistic method for Reliability Analysis under mixed aleatory and epistemic uncertainties with Flexible Intervals (EPRAFI) is proposed.

3 Extended probabilistic method for reliability analysis under mixed uncertainties

The approaches to calculate Bel_k and Pl_k for each focal element $c_k(1 \leq k \leq N_C)$ are the same regardless of the focal element index k . For simplicity, in the following discussion, we directly eliminate the subscript k . The interval bounds of the focal element are considered to be variable and assumed to be the functions of the random variables \mathbf{X} in this paper. Then the focal element is denoted as $C(\mathbf{X})$. Thus in the following discussion, the fixed interval c_k in (6) and (7) are replaced by the variable interval $C(\mathbf{X})$.

To calculate $\Pr\left\{\mathbf{X} \mid \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\}$ in (6) and $\Pr\left\{\mathbf{X} \mid \inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\}$ in (7), the method EPRAFI proposed in this paper will be developed in two steps. First, it will be proved that the upper and lower limit functions $\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and $\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ are measurable functions from the probability space (Ω, A, \Pr) to $(\mathbb{R}, B(\mathbb{R}))$ under the following conditions: 1) the system response function g is continuous; and 2) the flexible interval bounds of \mathbf{Y} are continuous or can be formulated as equation (9). Then the limit function responses can be treated as random variables. Second, since $\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and $\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ are random variables, then (6) and (7) can be directly calculated with existing probabilistic methods. In this paper, the extension of the popular reliability analysis method FORM is explained for exemplification. The preceding two steps will be elaborated in Section 3.1 and 3.2 respectively.

3.1 Measurability of the upper and lower limit functions

Let the system response function $g(\mathbf{X}, \mathbf{Y})$ be a continuous function $g: \mathbb{R}^{N_x+N_y} \rightarrow \mathbb{R}$ with the random uncertain input vector $\mathbf{X}=(X_i, 1 \leq i \leq N_x)$ and the epistemic uncertain input vector $\mathbf{Y}=(Y_i, 1 \leq i \leq N_y)$. The random vector \mathbf{X} is defined on the

probability space (Ω, A, \Pr) . The value y_i of each element Y_i is located in the interval $C_i(\mathbf{X})=[c_{i_min}(\mathbf{X}), c_{i_max}(\mathbf{X})]$, the bounds of which are variable and functions with respect to \mathbf{X} . The interval can also be open or half closed half open. Then the possible value set of \mathbf{Y} can be defined by the Cartesian product of $C_i(\mathbf{X})(1 \leq i \leq N_y)$ as

$$C(\mathbf{X}) = C_1 \times C_2 \times \dots \times C_{N_y}$$

$$= \{ \mathbf{c}_k = (b_1, b_2, \dots, b_{N_y}) \mid b_i \in C_i(\mathbf{X}), 1 \leq i \leq N_y \} \quad (8)$$

Let c_{i_*} represent c_{i_min} or c_{i_max} . Two situations of the variable bounds are considered in this paper. First, c_{i_*} is continuous. Second, c_{i_*} has countable (either finite or infinite) values and can be formulated as

$$c_{i_*}(\mathbf{X}) = a_{ik_*} \quad \forall \mathbf{X} \in A_{ik_*} (k = 1, 2, \dots) \quad (9)$$

where A_{ik_*} is a Borel set. In the aforementioned two variable situations, from Section 2.1 we know that both c_{i_min} and c_{i_max} are measurable functions from $(\mathbb{R}^{N_x}, B(\mathbb{R}^{N_x}))$ to $(\mathbb{R}, B(\mathbb{R}))$. Obviously the fixed interval case can be regarded as a special case of the variable interval wherein $c_{i_*}(\mathbf{X}) \equiv a_{i_*}$. Although only the preceding two situations are considered, they can already cover a large application field. Other forms of variable intervals will be studied in the future according to the application needs.

For any integer $n \in \mathbb{N}$, define the variable

$$\eta_{i_ki}(\mathbf{X}) = c_{i_min}(\mathbf{X}) + \frac{k_i}{n}(c_{i_max}(\mathbf{X}) - c_{i_min}(\mathbf{X})), \quad 0 \leq k_i \leq n \quad (10)$$

Denote $\mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}) = (\eta_{1_k_1}(\mathbf{X}), \eta_{2_k_2}(\mathbf{X}), \dots, \eta_{n_k_{N_y}}(\mathbf{X}))$ where $0 \leq k_i \leq n$ and $1 \leq i \leq N_y$. Denote the point set $P = \{ \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}), 0 \leq k_i \leq n, 1 \leq i \leq N_y \}$. It can be regarded that the space $C(\mathbf{X})$ is uniformly divided into n^{N_y} cubes and all the corner points compose the set P . The subscript $k_i(1 \leq i \leq N_y)$ represents the position index in each dimension. At each point of P , the value of $Y_i(1 \leq i \leq N_y)$ is set as $y_i = \eta_{i_k_i}(\mathbf{X})(0 \leq k_i \leq n)$, and the function response is $g(\mathbf{X}, \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}))$. Since c_{i_min} and c_{i_max} are measurable functions (in both situations) of \mathbf{X} , each element of $\mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X})$ is a random variable as it is defined by the linear arithmetic calculation of c_{i_min} and c_{i_max} , as shown in (10). Thus both \mathbf{X} and $\mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X})$ are random vectors defined on the probability space (Ω, A, \Pr) and the output of the continuous function $g(\mathbf{X}, \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}))$ is also a random variable (according to Theorem 6).

Given $n \in \mathbb{N}$, denote $g_{n_sup} = \sup_{0 \leq k_i \leq n, 1 \leq i \leq N_y} g(\mathbf{X}, \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}))$ and $g_{n_inf} = \inf_{0 \leq k_i \leq n, 1 \leq i \leq N_y} g(\mathbf{X}, \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}))$. According to

Theorem 5, g_{n_sup} and g_{n_inf} are also measurable functions of \mathbf{X} . For the measurable function sequences $\{g_{n_sup}\}_{n=1}^\infty$ and $\{g_{n_inf}\}_{n=1}^\infty$, it is proved that $\lim_{n \rightarrow \infty} g_{n_sup} = \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and

$$\lim_{n \rightarrow \infty} g_{n_inf} = \inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) \text{ as follows.}$$

Proof:

Given n and \mathbf{X} , denote the point \mathbf{y}^* which satisfies $g(\mathbf{X}, \mathbf{y}^*) = \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$. In the point set $P =$

$\{\mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X}), 0 \leq k_i \leq n, 1 \leq i \leq N_y\}$, denote the point nearest to \mathbf{y}^* as \mathbf{y}_n^* . It is obvious that \mathbf{y}^* is located either at the point \mathbf{y}_n^* or in the cube surrounded by \mathbf{y}_n^* and some other corner points of this cube. Thus the distance between \mathbf{y}_n^* and \mathbf{y}^* , denoted as $\|\mathbf{y}_n^* - \mathbf{y}^*\|$, must be no larger than the largest distance between the two corner points of the cube, i.e.

$$\|\mathbf{y}_n^* - \mathbf{y}^*\| \leq \sqrt{\sum_{i=1}^{N_y} \frac{1}{n^2} (c_{i_max}(\mathbf{X}) - c_{i_min}(\mathbf{X}))^2} \tag{11}$$

Since $g(\mathbf{X}, \mathbf{Y})$ is a continuous function, given $\forall \varepsilon > 0, \exists \delta > 0$ satisfies

$$\|\mathbf{Y}_1 - \mathbf{Y}_2\| < \delta \Rightarrow \|g(\mathbf{X}, \mathbf{Y}_1) - g(\mathbf{X}, \mathbf{Y}_2)\| < \varepsilon \tag{12}$$

$$\text{Thus if } n > \sqrt{\sum_{i=1}^{N_x} \frac{1}{\delta^2} (c_{i_max}(\mathbf{X}) - c_{i_min}(\mathbf{X}))^2},$$

$$\|\mathbf{y}_n^* - \mathbf{y}^*\| < \sqrt{\sum_{i=1}^{N_x} \frac{1}{n^2} (c_{i_max}(\mathbf{X}) - c_{i_min}(\mathbf{X}))^2} < \delta \tag{13}$$

From (12) we have $\|g(\mathbf{X}, \mathbf{y}_n^*) - g(\mathbf{X}, \mathbf{y}^*)\| < \varepsilon$, thus $g(\mathbf{X}, \mathbf{y}^*) - \varepsilon < g(\mathbf{X}, \mathbf{y}_n^*)$. Because for any n in the process $n \rightarrow \infty$, we always have

$$g(\mathbf{X}, \mathbf{y}_n^*) \leq \sup_{0 \leq k_i \leq n, 1 \leq i \leq N_y} g(\mathbf{X}, \mathbf{y}_{k_1 k_2 \dots k_{N_y}}(\mathbf{X})) = g_{n_sup} \leq g(\mathbf{X}, \mathbf{y}^*) \tag{14}$$

Denote $N_\delta = \sqrt{\sum_{i=1}^{N_x} \frac{1}{\delta^2} (c_{i_max}(\mathbf{X}) - c_{i_min}(\mathbf{X}))^2}$. Therefore given $\forall \varepsilon > 0, \exists n > N_\delta$ satisfies

$$g(\mathbf{X}, \mathbf{y}^*) - \varepsilon < g(\mathbf{X}, \mathbf{y}_n^*) \leq g_{n_sup} \leq g(\mathbf{X}, \mathbf{y}^*) < g(\mathbf{X}, \mathbf{y}^*) + \varepsilon \Rightarrow \|g_{n_sup} - g(\mathbf{X}, \mathbf{y}^*)\| < \varepsilon \tag{15}$$

Thus the limit of the sequence $\{g_{n_sup}\}_{n=1}^\infty$ exists and $\lim_{n \rightarrow \infty} g_{n_sup} = g(\mathbf{X}, \mathbf{y}^*) = \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$. With the similar process

it can be proved that $\lim_{n \rightarrow \infty} g_{n_inf} = \inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$. \square

According to Theorem 5, the limit of the measurable function sequence (if it exists) is also a measurable function. Thus $\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and $\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ are measurable functions

of \mathbf{X} , i.e. the functions response limits $\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and

$\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ are random variables. Thus the belief and

plausibility measures in (6) and (7) can be obtained directly by the probability calculation of these random variables, which can be formulated as follows:

$$\begin{aligned} \text{Bel} &= \Pr\left\{\mathbf{X} \mid \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\} = \Pr\left\{\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\} \\ \text{Pl} &= \Pr\left\{\mathbf{X} \mid \inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\} = \Pr\left\{\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\} \end{aligned} \tag{16}$$

3.2 The extension of the probabilistic method to mixed uncertainty problem

The function $\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ can be reformulated as

$$\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \mathbf{Y}^*), \quad \mathbf{Y}^* = \arg \max_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) \tag{17}$$

It is obvious that only the vector \mathbf{X} is independent variables. Thus we can denote $g_{sup}(\mathbf{X}) = \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$. For

the random variable $g_{sup}(\mathbf{X})$, the traditional probabilistic reliability analysis method can be directly applied to calculate $\Pr(g_{sup}(\mathbf{X}) < a)$. In the rest of this section, the efficient probabilistic reliability analysis method FORM will be taken as an example to illustrate how to extend it to solve the reliability analysis problem under mixed uncertainty situation.

Replace the function $g(\mathbf{X})$ in the FORM method (Section 2.1) as $g_{sup}(\mathbf{X}) = \sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$, then in the \mathbf{U}

space the failure domain is defined by

$$g_{sup}(T^{-1}(\mathbf{U})) = \sup_{\mathbf{Y} \in C(T^{-1}(\mathbf{U}))} g(T^{-1}(\mathbf{U}), \mathbf{Y}) = \sup_{\mathbf{Y} \in C(T^{-1}(\mathbf{U}))} G(\mathbf{U}, \mathbf{Y}) < a \tag{18}$$

Then (1) can be reformulated as

$$\begin{aligned} \min_{\mathbf{U}} \quad & \|\mathbf{U}\| \\ \text{s.t.} \quad & \sup_{\mathbf{Y} \in C(T^{-1}(\mathbf{U}))} G(\mathbf{U}, \mathbf{Y}) = a \end{aligned} \tag{19}$$

According to (17), (19) can be reformulated as

$$\begin{aligned} \min_{\mathbf{U}} \quad & \|\mathbf{U}\| \\ \text{s.t.} \quad & G(\mathbf{U}, \mathbf{Y}^*) = a, \quad \mathbf{Y}^* = \arg \max_{\mathbf{Y} \in C(T^{-1}(\mathbf{U}))} G(\mathbf{U}, \mathbf{Y}) \end{aligned} \tag{20}$$

Denote the optimum of (20) as \mathbf{u}_{\max}^* and define $\beta = \|\mathbf{u}_{\max}^*\|$. Then $\text{Bel}\{(\mathbf{X}, \mathbf{Y}) \in F\} = p_{f_min} = \Pr\left\{\sup_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\}$ can be estimated by $p_{f_min} \approx \Phi(-\mathbf{u}_{\max}^*)$ if $p_{f_min} \leq 0.5$ or $p_{f_min} \approx \Phi(\mathbf{u}_{\max}^*)$ if $p_{f_min} > 0.5$.

Similarly, replace the function $g(\mathbf{X})$ in the FORM method as $g_{\inf}(\mathbf{X}) = \inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$, it can be derived that the plausibility measure $\text{Pl}\{(\mathbf{X}, \mathbf{Y}) \in F\} = p_{f_max} = \Pr\left\{\inf_{\mathbf{Y} \in C(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\}$ can be estimated by $p_{f_max} \approx \Phi(-\mathbf{u}_{\min}^*)$ if $p_{f_max} \leq 0.5$ or $p_{f_max} \approx \Phi(\mathbf{u}_{\min}^*)$ if $p_{f_max} > 0.5$, where \mathbf{u}_{\min}^* is the optimum of the following optimization problem.

$$\begin{aligned} & \min_{\mathbf{U}} \|\mathbf{U}\| \\ \text{s.t.} \quad & G(\mathbf{U}, \mathbf{Y}^*) = a, \quad \mathbf{Y}^* = \arg \min_{\mathbf{Y} \in C(T^{-1}(\mathbf{U}))} G(\mathbf{U}, \mathbf{Y}) \end{aligned} \tag{21}$$

When the interval is fixed as $C(T^{-1}(\mathbf{U})) = C(\mathbf{X}) = C_{fix}$, then (20) is

$$\begin{aligned} & \min_{\mathbf{U}} \|\mathbf{U}\| \\ \text{s.t.} \quad & G(\mathbf{U}, \mathbf{Y}^*) = a, \quad \mathbf{Y}^* = \arg \max_{\mathbf{Y} \in C_{fix}} G(\mathbf{U}, \mathbf{Y}) \end{aligned} \tag{22}$$

And (21) can be stated as

$$\begin{aligned} & \min_{\mathbf{U}} \|\mathbf{U}\| \\ \text{s.t.} \quad & G(\mathbf{U}, \mathbf{Y}^*) = a, \quad \mathbf{Y}^* = \arg \min_{\mathbf{Y} \in C_{fix}} G(\mathbf{U}, \mathbf{Y}) \end{aligned} \tag{23}$$

The formulations of (22) and (23) are totally the same with the FORM based unified uncertainty analysis method (FORM-UUA) proposed by Du under mixed random and epistemic uncertainties with fixed focal elements (Du 2008) and the probability-interval (with fixed bounds) hybrid reliability analysis method proposed by Jiang (Jiang et al. 2013). Thus the research work in this paper also provides a strict mathematical proof for the correctness of the proposed methods in (Jiang et al. 2013) and (Du 2008).

3.3 The EPRAFI algorithm

To sum up, the EPRAFI method for mixed aleatory and epistemic uncertainties with flexible intervals is as follows.

Step 0: Initialization. Denote the failure region as $F = \{(\mathbf{X}, \mathbf{Y}) | g(\mathbf{X}, \mathbf{Y}) < a\}$ and denote the focal element index $k = 1$.

Step 1: Conduct a global optimization to find the minimum function response $g_{\min}(\mathbf{X}, \mathbf{Y})$ for $\mathbf{X} \in \Omega$ and $\mathbf{Y} \in C_k(\mathbf{X})$. If $g_{\min} \geq a$, C_k is fully contained in the safe

region and $\text{Bel}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = \text{Pl}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = 0$; go to Step 5. Otherwise go to Step 2.

Step 2: Conduct a global optimization to find the maximum function response $g_{\max}(\mathbf{X}, \mathbf{Y})$ for $\mathbf{X} \in \Omega$ and $\mathbf{Y} \in C_k(\mathbf{X})$. If $g_{\max} < a$, C_k is fully contained in the failure region and $\text{Bel}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = \text{Pl}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = 1$; go to Step 5. Otherwise go to Step 3.

Step 3: If $g_{\min} \leq a \leq g_{\max}$, check whether the two conditions to apply EPRAFI are satisfied. If $g(\cdot)$ is continuous, and the flexible interval bounds of C_k are continuous functions of \mathbf{X} or can be formulated as (9), then the EPRAFI method can be applied. Go to Step 4.

Step 4: Select the probabilistic analysis method to be extended to solve the reliability analysis problem under mixed uncertainties. Replace the original response function $g(\mathbf{X})$ in that probabilistic method with the upper and lower limit functions $\sup_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and $\inf_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ respectively. Directly run the probabilistic analysis process by treating $\sup_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ and $\inf_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y})$ as random variables, and calculate Bel_k

$$\{(\mathbf{X}, \mathbf{Y}) \in F\} = \Pr\left\{\sup_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\} \text{ and } \text{Pl}_k\{(\mathbf{X}, \mathbf{Y}) \in F\} = \Pr\left\{\inf_{\mathbf{Y} \in C_k(\mathbf{X})} g(\mathbf{X}, \mathbf{Y}) < a\right\}.$$

For example, if FORM is selected, then Bel_k and Pl_k can be obtained by solving (20) and (21). Go to Step 5.

Step 5: If $k < N_C$, $k = k + 1$ and go to Step 1. If $k = N_C$, calculate $\text{Bel}\{g < a\}$ and $\text{Pl}\{g < a\}$ with (6) and (7) respectively.

4 Numerical examples

4.1 Example 1: a simple numerical example

First a simple numerical example is used to exemplify the proposed method. Define a continuous function as

$$g(X, Y) = (Y - 0.5)^2 - X + 0.5 \tag{24}$$

where the single random variable X is subject to the uniform distribution within the range $[-1, 1]$, denoted as $U(-1, 1)$. The BPA of the single epistemic variable Y is defined as

$$C = \begin{cases} [-1, 0], X < 0 \\ [0, 1], X \geq 0 \end{cases}, \quad m(C) = 1 \tag{25}$$

It is obvious that Y has only one focal element, and its lower and upper bounds vary with the value of X . Define the failure region as $F = \{(X, Y) | g(X, Y) \leq 0\}$. The function response

distribution of $g(X, Y)$ with respect to X and Y is presented in Fig. 1. The failure region boundary is described with a solid line.

According to (6), the belief of failure $\text{Bel}\{(X, Y) \in F\}$ can be calculated by integrating the probability density function of X over the regions where the response $g(X, Y)$ is fully contained in the failure region for any value of Y in the focal element C . From Fig. 1 it is obvious that when $X \geq 0.75$, $g(X, Y) \leq 0$ for any $Y \in C$. Since the random variable X is subject to $U(-1, 1)$, thus

$$\text{Bel}\{(X, Y) \in F\} = \frac{1-0.75}{1-(-1)} = 0.125 \tag{26}$$

According to (7), the plausibility of failure $\text{Pl}\{(X, Y) \in F\}$ can be calculated by integrating the probability density function of X over the regions where the responses of $g(X, Y)$ are fully or partially contained in the failure region when Y varies in the focal element C . In Fig. 1 it is shown that when $X \geq 0.5$, there exists $Y \in C$ such that $g(X, Y) \leq 0$. Thus

$$\text{Pl}\{(X, Y) \in F\} = \frac{1-0.5}{1-(-1)} = 0.25 \tag{27}$$

Next, a more complicated situation with a refined BPA description of Y is considered, which is defined as follows:

$$\begin{aligned} C_1 &= \begin{cases} [-1, -0.7], X < 0 \\ [0, 0.3], X \geq 0 \end{cases}, & C_2 &= \begin{cases} [-0.7, -0.3], X < 0 \\ [0.3, 0.7], X \geq 0 \end{cases} \\ C_3 &= \begin{cases} [-0.3, 0], X < 0 \\ [0.7, 1], X \geq 0 \end{cases} \\ m(C_1) &= 0.3, \quad m(C_2) = 0.3, \quad m(C_3) = 0.4 \end{aligned} \tag{28}$$

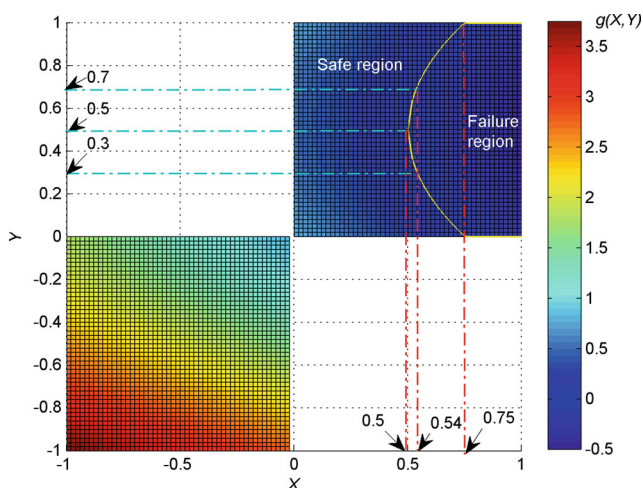


Fig. 1 Function response distribution of $g(X, Y)$ in example 1

Table 1 Belief and plausibility of $F = \{g < 0\}$ in example 1

	BPA Setting 1		BPA Setting 2	
	Belief	Plausibility	Belief	Plausibility
EPRAFI	0.125	0.25	0.1565	0.236
Theoretical value	0.125	0.25	0.1565	0.236

In Fig. 1 it is shown that when $0.5 \leq X \leq 0.54$, C_2 is partially in the failure region. When $0.54 \leq X \leq 0.75$, C_2 is fully in the failure region, and C_1 and C_3 are partially in the failure region. When $X \geq 0.75$, all of C_1 , C_2 , and C_3 are fully in the failure region. Thus

$$\begin{aligned} \text{Bel}\{(X, Y) \in F\} &= \frac{1-0.54}{1-(-1)}m(C_2) + \frac{1-0.75}{1-(-1)}(m(C_1) + m(C_3)) = 0.1565 \\ \text{Pl}\{(X, Y) \in F\} &= \frac{1-0.5}{1-(-1)}m(C_2) + \frac{1-0.54}{1-(-1)}(m(C_1) + m(C_3)) = 0.236 \end{aligned} \tag{29}$$

The Bel and Pl calculations above are directly based on the standard reliability analysis formulation under mixed uncertainties. The obtained results are accurate theoretical analysis values and can be used as the benchmark to verify the accuracy of the proposed EPRAFI method. In this test, the response function $g(X, Y)$ is continuous, and the variable bounds of the focal elements of Y can be formulated as (9). Thus the two conditions to apply EPRAFI are satisfied. In EPRAFI, the probabilistic method FORM is used. $\text{Bel}_k\{(X, Y) \in F\}$ and $\text{Pl}_k\{(X, Y) \in F\}$ are obtained by solving (20) and (21). The analysis results are presented and compared with the theoretical values in Table 1 for both BPA setting 1 defined by (25) and BPA setting 2 defined by (28). It is shown that the EPRAFI analysis results are exactly the same with the theoretical values,

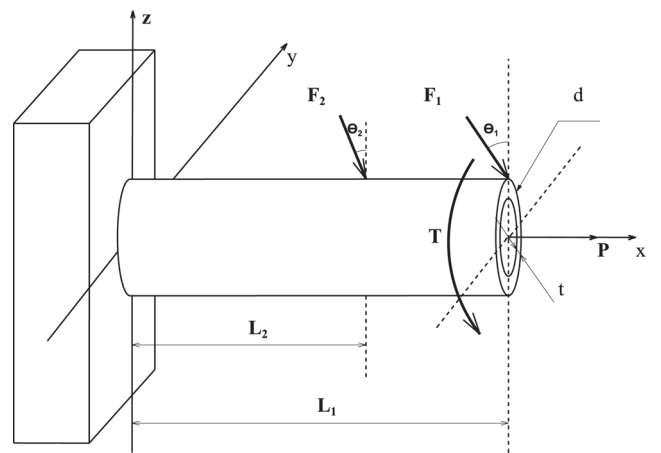


Fig. 2 The cantilever tube problem in example 2

Table 2 Random variables in example 2

Variable	Distribution	Parameters	
t	Normal	Mean 5 mm	Std 0.1 mm
d	Normal	Mean 42 mm	Std 0.5 mm
L_1	Uniform	Lower bound 119.75 mm	Upper bound 120.25 mm
L_2	Uniform	Lower bound 59.75 mm	Upper bound 60.25 mm
F_1	Normal	Mean 3.0 kN	Std 0.3 kN
F_2	Normal	Mean 3.0 kN	Std 0.3 kN
P	Normal	Mean 12.0 kN	Std 1.2 kN
T	Normal	Mean 90.0 Nm	Std 9.0 Nm
S_y	Normal	Mean 220 MPa	Std 22.0 MPa

which validates the accuracy of EPRAFI. Besides, it can be observed that the gap between the belief and plausibility measures in BPA setting 2 is smaller than that in BPA setting 1. This demonstrates that the refinement of the distribution information of the epistemic uncertainty Y can enhance the description accuracy of the uncertain distribution of $g(X, Y)$ with narrower range bounding the precise probability.

4.2 Example 2: the cantilever tube example

The cantilever tube example (Du 2008) is used in (Yao et al. 2013b) to testify the reliability analysis method under mixed uncertainties with fixed intervals. Herein it

is taken to verify the EPRAFI method, and to compare the reliability analysis results between the situations with fixed and flexible intervals. The system response function is defined as

$$g(\mathbf{X}, \mathbf{Y}) = S_y - \sqrt{\sigma_x^2 + 3\tau_{zx}^2}$$

$$\mathbf{X} = [t \ d \ L_1 \ L_2 \ F_1 \ F_2 \ P \ T \ S_y], \quad \mathbf{Y} = [\theta_1 \ \theta_2]$$

$$\sigma_x = \frac{P + F_1 \sin(\theta_1) + F_2 \sin(\theta_2)}{A} + \frac{Mh}{I}, \quad \tau_{zx} = \frac{Td}{2J}$$

$$M = F_1 L_1 \cos(\theta_1) + F_2 L_2 \cos(\theta_2), \quad A = \frac{\pi}{4} [d^2 - (d-2t)^2]$$

$$h = d / 2, \quad I = \frac{\pi}{64} [d^4 - (d-2t)^4], \quad J = 2I \tag{30}$$

Table 3 Epistemic uncertain variables in example 2

	Fixed interval	Variable interval	BPA
Setting 1	θ_1 $C_{\theta_1} = [C_{\theta_1_fixlow}, C_{\theta_1_fixup}] = [0, 10] \text{deg}$	$C_{\theta_1_varlow} = C_{\theta_1_fixlow} + 10 \times \frac{L_1-120}{0.25}$ $C_{\theta_1_varup} = C_{\theta_1_fixup} + 10 \times \frac{L_1-120}{0.25}$	$m_1(C_{\theta_1}) = 1$
	θ_2 $C_{\theta_2} = [C_{\theta_2_fixlow}, C_{\theta_2_fixup}] = [5, 15] \text{deg}$	$C_{\theta_2_varlow} = C_{\theta_2_fixlow} + 10 \times \frac{L_2-60}{0.25}$ $C_{\theta_2_varup} = C_{\theta_2_fixup} + 10 \times \frac{L_2-60}{0.25}$	$m_2(C_{\theta_2}) = 1$
Setting 2	θ_1 $C_{\theta_{1_1}} = [C_{\theta_{1_fixlow1}}, C_{\theta_{1_fixup1}}] = [0, 3] \text{deg}$	$C_{\theta_{1_varlowi}} = C_{\theta_{1_fixlowi}} + 10 \times \frac{L_1-120}{0.25}$ $C_{\theta_{1_varupi}} = C_{\theta_{1_fixupi}} + 10 \times \frac{L_1-120}{0.25}$ $i = 1, 2, 3$	$m_1(C_{\theta_{1_1}}) = 0.3$
	$C_{\theta_{1_2}} = [C_{\theta_{1_fixlow2}}, C_{\theta_{1_fixup2}}] = [3, 6] \text{deg}$		$m_1(C_{\theta_{1_2}}) = 0.3$
	$C_{\theta_{1_3}} = [C_{\theta_{1_fixlow3}}, C_{\theta_{1_fixup3}}] = [6, 10] \text{deg}$		$m_1(C_{\theta_{1_3}}) = 0.4$
	θ_2 $C_{\theta_{2_1}} = [C_{\theta_{2_fixlow1}}, C_{\theta_{2_fixup1}}] = [5, 8] \text{deg}$	$C_{\theta_{2_varlowi}} = C_{\theta_{2_fixlowi}} + 10 \times \frac{L_2-60}{0.25}$ $C_{\theta_{2_varupi}} = C_{\theta_{2_fixupi}} + 10 \times \frac{L_2-60}{0.25}$ $i = 1, 2, 3$	$m_2(C_{\theta_{2_1}}) = 0.3$
	$C_{\theta_{2_2}} = [C_{\theta_{2_fixlow2}}, C_{\theta_{2_fixup2}}] = [8, 11] \text{deg}$		$m_2(C_{\theta_{2_2}}) = 0.3$
	$C_{\theta_{2_3}} = [C_{\theta_{2_fixlow3}}, C_{\theta_{2_fixup3}}] = [11, 15] \text{deg}$		$m_2(C_{\theta_{2_3}}) = 0.4$

Table 4 Belief and plausibility of $F = \{g < 0\}$ in example 2

	BPA Setting 1		BPA Setting 2	
	Belief	Plausibility	Belief	Plausibility
Variable interval	1.55×10^{-4}	1.73×10^{-4}	1.63×10^{-4}	1.72×10^{-4}
MCS (Variable interval)	1.57×10^{-4}	1.72×10^{-4}	1.63×10^{-4}	1.71×10^{-4}
Fixed interval	1.53×10^{-4}	1.73×10^{-4}	1.62×10^{-4}	1.72×10^{-4}
MCS (Fixed interval)	1.56×10^{-4}	1.71×10^{-4}	1.63×10^{-4}	1.70×10^{-4}

The graph of this cantilever tube problem is shown in Fig. 2. The distributions of aleatory uncertainties \mathbf{X} and epistemic uncertainties \mathbf{Y} are described in Tables 2 and 3 respectively. Two different BPA settings of \mathbf{Y} are considered, and in each setting both the fixed and variable bounds of focal elements are studied for comparison. In the variable case, the focal element bounds of θ_2 (the input angle of force F_2) varies with L_2 (the distance between the force F_2 input point and the cantilever fixed point). The focal element bounds of θ_1 (the input angle of force F_1) varies with L_1 (the distance between the force F_1 input point and the cantilever fixed point). It is obvious that $g(\mathbf{X}, \mathbf{Y})$ is continuous and the variable bounds of the focal elements of \mathbf{Y} are also continuous with respect to \mathbf{X} . Thus the two conditions to apply EPRAFI are satisfied. In EPRAFI, the probabilistic method FORM is used, and $\text{Bel}_k\{(X, Y) \in F\}$ and $\text{Pl}_k\{(X, Y) \in F\}$ are obtained by solving (20) and (21), where the failure region is defined as $F = \{(\mathbf{X}, \mathbf{Y}) | g(\mathbf{X}, \mathbf{Y}) \leq 0\}$.

The analysis results are presented in Table 4. It is shown that in both BPA settings, the belief and plausibility measures of the variable case are slightly larger than the fixed case. It is because in the variable case, the lower and upper bounds of θ_1 and θ_2 decrease with the decrease of L_1 and L_2 when they vary in the probability space, as shown in (30), which can provide more chances to maintain the values of M and correspondingly the values of σ_x in a high level so as to make $g < 0$ when L_1 and L_2 decrease. But in the fixed case, the chance of the σ_x value to make $g < 0$ will decrease when L_1 and L_2 decrease with the fixed bounds of θ_1 and θ_2 . To further demonstrate the accuracy of EPRAFI, it is compared with the benchmark MCS method. In implementing MCS, 10^6 random variable samples are generated and for each random sample point 2000 epistemic uncertain variable samples are further generated in each focal element. The results are shown in Table 4. It can be observed that the analysis results of the proposed method are very close to the MCS results and the

relative difference is less than 1.3 % in this case, which clearly verifies the accuracy of the proposed method.

4.3 Example 3: a satellite conceptual design example

The conceptual design problem of a hypothetical earth-observation small satellite is chosen as a practical application test for the proposed EPRAFI method. This example is previously used in (Yao et al. 2013b) to testify the reliability analysis method for the mixed uncertainty problem with fixed intervals. In this conceptual design problem, the system response under study is the satellite mass $M_{sat}(\mathbf{X}, \mathbf{Y})$ which is estimated by empirical equations with five input variables $\mathbf{X} = [h, f_c, b, l, t]$, including the orbit altitude h , the CCD (Charge Coupled Device) camera focal length f_c , the body width b , the body height l , and the side wall thickness t (Wertz and Larson 1999). These five variables are subject to aleatory uncertainties and the distributions are illustrated in Table 5. Besides, in the mass estimation model, the scaling coefficients s_{dh_m} and s_{ttc_m} for the mass estimation of the subsystems OBDH (onboard data handling) and TTC (telemetry, tracking, and command) can be hardly defined in the conceptual design phase. These parameters are treated as epistemic uncertainties and denoted as $\mathbf{Y} = [s_{dh_m}, s_{ttc_m}]$. Two different BPA settings of \mathbf{Y} are considered. In each BPA setting, both the fixed and variable interval bounds are studied for comparison. In the variable case, as shown in Table 6, the interval

Table 5 Random variables in example 3

Variable	Description	Distribution	Parameters	
b	Body width (mm)	Normal	Mean 800	Std 10
l	Body height (mm)	Normal	Mean 800	Std 10
t	Body thickness (mm)	Normal	Mean 5	Std 0.1
h	Orbit altitude (km)	Normal	Mean 650	Std 6.5
f_c	Focal length (mm)	Normal	Mean 300	Std 1.0

Table 6 Epistemic uncertain variables in example 3

	Fixed interval	Variable interval ^a	BPA
Setting 1	s_{dh_m} $C_{dh} = [C_{dh_fixlow}, C_{dh_fixup}] = [0.04, 0.06]$	$C^*_{_varlow} = C^*_{_fixlow} + \lambda \times (f_c - 300)$	$m_1(C_{dh}) = 1$
	s_{ttc_m} $C_{ttc} = [C_{ttc_fixlow}, C_{ttc_fixup}] = [0.04, 0.06]$	$C^*_{_varup} = C^*_{_fixup} + \lambda \times (f_c - 300)$	$m_2(C_{ttc}) = 1$
Setting 2	s_{dh_m} $C_{dh_1} = [C_{dh_fixlow1}, C_{dh_fixup1}] = [0.04, 0.05]$	$C^*_{_varlowi} = C^*_{_fixlowi} + \lambda \times (f_c - 300)$ $C^*_{_varupi} = C^*_{_fixupi} + \lambda \times (f_c - 300)$ $i = 1, 2$	$m_1(C_{dh_1}) = 0.5$
	$C_{dh_2} = [C_{dh_fixlow2}, C_{dh_fixup2}] = [0.05, 0.06]$		$m_1(C_{dh_2}) = 0.5$
	s_{ttc_m} $C_{ttc_1} = [C_{ttc_fixlow1}, C_{ttc_fixup1}] = [0.04, 0.05]$		$m_2(C_{ttc_1}) = 0.5$
	$C_{ttc_2} = [C_{ttc_fixlow2}, C_{ttc_fixup2}] = [0.05, 0.06]$		$m_2(C_{ttc_2}) = 0.5$

^aThe asterisk symbol in the subscript represents *dh* or *ttc*

bounds of the mass scaling coefficients s_{dh_m} and s_{ttc_m} increase as f_c increases because more on-board data handling capacity and data transmission capacity from the satellite to the ground are required if the camera performance is enhanced, which in turn leads to larger scaling coefficients to estimate the OBDH and TTC subsystem mass. The interval bounds are assumed to be linear functions of f_c and the linear coefficient is denoted as λ , which is a positive real number. Two different λ values representing different variable degrees are studied for comparison.

Denote the limit state value as a and calculate the belief and plausibility measures of the failure $F = \{(\mathbf{X}, \mathbf{Y}) | M_{sat}(\mathbf{X}, \mathbf{Y}) \leq a\}$. As the mass estimation function $M_{sat}(\mathbf{X}, \mathbf{Y})$ is continuous, and the variable bounds of the focal elements of \mathbf{Y} are also continuous with respect to \mathbf{X} , thus the two conditions to apply EPRAFI are satisfied. In EPRAFI, the probabilistic method FORM is used. The reliability analysis results are presented in Table 7. It can be noticed that if $\lambda = 8 \times 10^{-5}$, which represents relatively small variable degree, the reliability analysis results between the variable and fixed bounds are very small. But

if λ is increased to 8×10^{-4} , the analysis difference becomes much larger. For example, with BPA setting 2, the plausibility of the satellite mass $M_{sat} < 188\text{kg}$ is 0.8309 with fixed intervals. But in the variable interval case it is 0.8308 with $\lambda = 8 \times 10^{-5}$ and 0.8865 with $\lambda = 8 \times 10^{-4}$ respectively. With larger value of λ , the analysis difference between fixed and variable intervals can reach 6.7 % in this example, which is not negligible especially in the reliability analysis. By varying the value of a , the CPF and CBF of the satellite mass with fixed and variable interval bounds ($\lambda = 8 \times 10^{-4}$) can be obtained and are compared in Fig. 3. It is obvious that in the range of satellite mass from 182 to 190 kg, the plausibility difference between the fixed and variable conditions is large. Thus in this case if we use the traditional evidence theory and simply treat the bounds as fixed ones, it may lead to large error in reliability estimation and consequently affect the reliability-based design optimization.

To illustrate the effect of different refinement degree of BPA settings on the reliability analysis results, the graphs of CPF and CBF of the satellite mass in the variable interval case ($\lambda = 8 \times 10^{-4}$) with BPA setting 1 and 2 are

Table 7 Belief and plausibility of $F = \{M_{sat} < 188\}$ and $F = \{M_{sat} < 196\}$ in example 3

BPA settings	Interval bound conditions	$F = \{M_{sat} < 188\}$		$F = \{M_{sat} < 196\}$	
		Belief	Plausibility	Belief	Plausibility
BPA Setting 1	Fixed interval	0.1674	0.9849	0.9789	1.0000
	Variable interval ($\lambda = 8 \times 10^{-5}$)	0.1675	0.9849	0.9788	1.0000
	Variable interval ($\lambda = 8 \times 10^{-4}$)	0.1712	0.9837	0.9772	1.0000
BPA Setting 2	Fixed interval	0.4751	0.8309	0.9935	1.0000
	Variable interval ($\lambda = 8 \times 10^{-5}$)	0.4751	0.8308	0.9935	1.0000
	Variable interval ($\lambda = 8 \times 10^{-4}$)	0.4753	0.8865	0.9929	0.9999

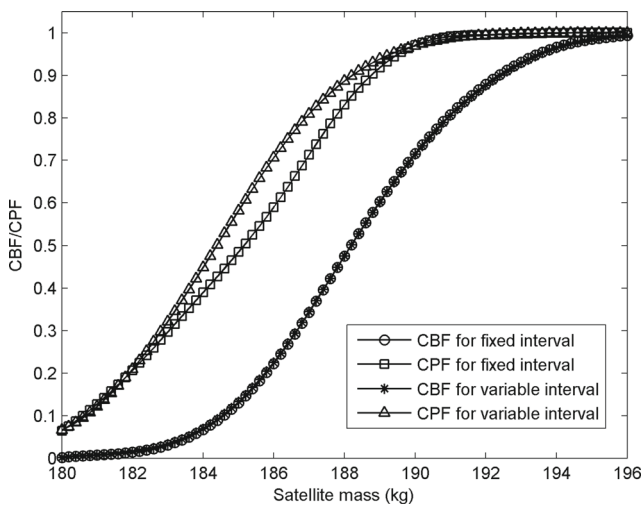


Fig. 3 Comparison of CBF and CPF of satellite mass between fixed and variable interval bounds ($\lambda = 8 \times 10^{-4}$) with BPA setting 2 in example 3

drawn and compared in Fig. 4. It is obvious that the gap between CPF and CBF can be reduced with the refinement of BPA of Y , which accordingly can describe the uncertain distribution of the satellite mass more precisely. Thus in practical engineering, as the design phase goes forward, more knowledge about the design object can be learned and its performance can be described more precisely.

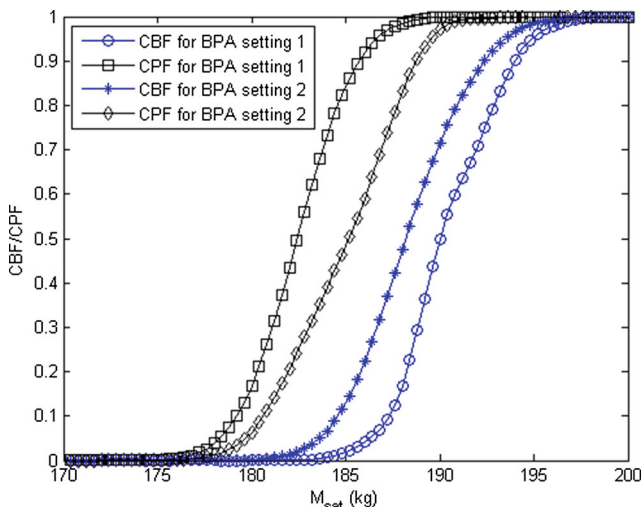


Fig. 4 Comparison of CBF and CPF of satellite mass between BPA setting 1 and 2 (both with variable interval bounds, $\lambda = 8 \times 10^{-5}$) in example 3

5 Conclusions

In this paper, to address the reliability analysis problem under both aleatory uncertainties and epistemic uncertainties with flexible intervals (the interval bounds are either fixed or variable as functions of other independent variables), the reliability analysis approach EPRAFI is proposed based on combined probability and evidence theory. Based on the measure theory, it is proved that the upper and lower limits of the system response function are measurable upon the following conditions: 1) the system response function is continuous; and 2) the functions of the flexible interval bounds of the epistemic uncertainties are continuous or have countable values as the independent variables vary in Borel sets. Then the upper and lower limits of the system response function can be treated as random variables, and their reliability analysis can be directly solved by existing probabilistic methods. The significant advantage of this method is that the well-developed and efficient probabilistic methods can be directly applied to handle the mixed uncertainty situation upon the specified conditions. As the fixed interval can be regarded as a special flexible interval, the EPRAFI method can also be directly applied to the fixed interval situation. FORM is taken as an example to illustrate the extension of the probabilistic method to solve reliability analysis problem in the mixed uncertainty situation. The derived mathematical formulations of EPRAFI based on FORM for the fixed interval situation are totally the same with those reported in the literature. Thus the research work in this paper also provides another strict mathematical proof for the correctness of those early developed methods with fixed intervals.

The efficacy of the proposed method is demonstrated with two numerical examples and one practical satellite conceptual design example. In Example 3, the results showed that if the variable degree of the interval bounds is relatively small, there will not be much difference in the reliability analysis results whether the bounds are treated as fixed ones with traditional evidence theory or as variable ones. But if the variable degree is relatively large, the difference will not be negligible. Thus the interval bounds should be properly treated according to specific conditions in practical application, especially when the variable degree is large. However in this research, EPRAFI is developed with the prerequisite that the system response function is continuous and only two flexible interval bound situations are considered. The applicability of EPRAFI for other forms of flexible intervals should be studied in the future according to application needs.

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