Struct Multidisc Optim (2015) 52:613–631
DOI 10.1007/s00158-015-1261-9

REVIEW ARTICLE

Topology and shape optimization methods using evolutionary algorithms: a review

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Received: 16 December 2014 / Revised: 2 April 2015 / Accepted: 23 April 2015 / Published online: 20 May 2015

Abstract Topology optimization has evolved rapidly since the late 1980s. The optimization of the geometry and topology of structures has a great impact on its performance, and the last two decades have seen an exponential increase in publications on structural optimization. This has mainly been due to the success of material distribution methods, originating in 1988, for generating optimal topologies of structural elements. Previous methods suffered from mathematical complexity and a limited scope for applicability, however with the advent of increased computational power and new techniques topology optimization has grown into a design tool used by industry. There are two main fields in structural topology optimization, gradient based, where mathematical models are derived to calculate the sensitivities of the design variables, and non gradient based, where material is removed or included using a sensitivity function. Both fields have been researched in great detail over the last two decades, to the point where structural topology optimization has been applied to real world structures. It is the objective of this review paper to present an overview of the developments in non gradient based structural topology and shape optimization, with a focus on evolutionary algorithms, which began as a non gradient method, but have developed to incorporate gradient based techniques. Starting with the early work and development of the popular algorithms and focusing on the various applications. The sensitivity functions for various optimization tasks are presented and real world applications are analyzed. The article concludes with new applications of topology optimization and applications in various engineering fields.

Keywords Structural optimisation · Non gradient · Evolutionary algorithms · Topology optoimisation · Shape optimisation · Applications

1 Introduction

1.1 Structural optimization

Structural optimization, in particular the layout optimization, has been identified as the most challenging and economically the most rewarding task in structural design (Rozvany 1988). The pioneer of the theory of layout optimization was the Australian inventor, engineer and mathematician A. G. M. Michell in 1904 (Michell 1904). Michell used the work of Maxwell (1872) to determine the first truss solutions of least weight and developed a general theory for deriving them. Michell structures are designed for only one load case and depend on appropriate specification of strain fields (Topping 1983). These structures are also statically determinate and impractical, consisting of an infinite number of bars. Therefore this work was purely academic and did not have any application.

Michell's work was left untouched until the 1950s where it was picked up by Hill (1950), Heyman (1951), Heyman and Prager (1958), Foulkes (1954), Drucker and Shield (1956), Livesley (1956), Prager (1956, 1958), Prager and Shield (1959), Onat et al. (1957), and Hemp (1958). Hill (1950) developed techniques for calculating the form of slip lines. Slip lines are used to construct Hencky nets, which are used to determine the strain field of a problem, a

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method employed by both Hemp (1958) and Prager (1958). Drucker and Shield (1956) developed a condition that states that the dissipation per unit volume must be constant. This theory was applied to the design of circular sandwich plates by Onat et al. (1957). Prager and Shield (1959) apply the theory of constant volume dissipation to more general loading cases. Prager (1956) used a minimum cost function on the limiting plastic moments of beams to determine optimum structures built from beams. Heyman (1951) and Livesley (1956) and Heyman and Prager (1958) used a linear programming technique to determine the least limiting plastic moment at a node, by applying hinges to each nodal member. Foulkes (1954) was the first to give the conditions for an optimal beam structure, expressed in terms of its collapse mechanism.

The 1960s saw Chan (1960a, b), Dorn (1964), Hemp (1964), Cox (1965), Richards and Chan (1966), and Kienzi et al. (1968) add to the field of optimization. Chan (1960a and b) developed techniques to determine suitable strain fields via graphical construction, while Cox (1965) showed several interesting examples of single-stress structures and applications of Maxwell's theorem. Richards and Chan (1966) applied the method of Michell continua to plates, noting the advantage of plates for like signed principal stresses. Marcal and Prager (1964) give the optimization of a beam using a volume cost function. This was generalized by Prager and Shield (1967) to arbitrary one and two-dimensional plastic structures. Kienzi et al. (1968) developed a program that uses the 'simplex method', a linear programming technique developed by Dantzig (1963), to solve framework optimization problems. The simplex method leads to a definite minimum value for volume; this defines an optimum framework of least volume of material designed according to the methods of plastic design. This optimum design is statically determinate, i.e. there are no more supporting joints than necessary to maintain equilibrium, and so it is an elastic design as well as a plastic one. Hemp and Chan (1966) developed a programme that uses dual formulation to solve framework optimization problems. The method is to cover the region of space in which the required structure is allowed to lie by a closely spaced rectangular grid of node points and allowing members to lie along all segments joining node points. Cox (1965) was first to show that the optimum structure of maximum stiffness for frameworks is identical with the structure of least volume designed to carry the same load. However, these works still suffered from the same impracticalities and remained academic.

The field of structural optimization remained strong in the 1970s, Hemp (1973) published a book in which he gave Michell's theory for frameworks, beams, circular sandwich plates and plates with inplane loading. Pope and Schmit (1971) put limitations on both stress and displacement for optimizing elastic designs. Charrett and Rozvany (1972) extended the early work of Foulkes (1954) to multi-component specific cost functions and multiple load conditions. Michell's early optimization theory was extended to grillages, systems of beams, by Rozvany and Adidam (1972). Prager also developed an interest in the topic and teamed up with Rozvany to continue extending optimal layout theory (Prager and Rozvany 1976). An early version of perimeter control was first seen in grillage layout optimization (Rozvany and Prager 1976), by enforcing a finite number of beams. It was noticed that the exact solutions of layout theory usually contain an infinite number of intersecting members with infinitesimal spacing, termed 'grillage-like continua' (Prager 1974). Rozvany and Prager (1979) developed optimal conditions for curved surface structures, consisting of a system of intersecting arches, with given vertical loads. The biggest advancement in the 1970s for the field of structural optimization was the beginning of generalized or variable topology shape optimization by Rossow and Taylor (1973). Although a penalization factor of 1 was used, therefore the optimum results did not result in a black-white topology, Taylor's intention was to obtain cavities in the plate in some areas. Therefore this paper is considered the first conceptualization of generalized shape optimization (Rozvany 2001b).

The 1980s saw a drive for progress in generalized shape optimization. Two key developments emerged from the eighties. Firstly, Cheng and Olhoff (1981) found in an FE-based optimization of solid plates that the optimal plate design develops a system of ribs. Similarly, Kohn and Strang (1983) found solid, porous and empty regions in plastically designed cross-sections for torsion. Rozvany et al. (1982) investigated in great detail the densely ribbed nature of plastically designed optimal solid plates. Secondly, optimal microstructures for perforated plates were established through various mathematical studies, the most popular being rank-2 laminates (Lurie et al. 1982; Kohn and Strang 1986a; Avellaneda 1987). From these developments (Rozvany et al. 1987) derived the homogenized rigidity tensor of rank-2 layered laminates for zero Poisson's ratio and determined optimal topologies for axisymmetric plates in flexure (Rozvany et al. 1987).

The boundary variation method for shape optimization dominated the literature in the 1980s. Boundary variation can be implemented in a number of ways, Pironneau (1984) employed mesh moving schemes where the design variables are the co-ordinates of the nodal points of a finite element model (Pironneau 1984). Kikuchi et al. (1986) showed the importance of 'regularity' of the finite element model near the design boundary so as to yield a physically sound optimal shape (Kikuchi et al. 1986). Another popular way to implement the boundary variation method is to employ a boundary segment idea; this describes the boundary by a set of simple segments such as straight lines, circular arcs, elliptic arcs and splines. The optimum is then determined within this restricted definition of the boundary. This method has been employed by Botkin and Bennett (1985) and Braibant and Fleury (1984). The mathematical foundation of boundary variation methods for optimal shape design and design sensitivity analysis was thoroughly investigated in the 1980s, for example, Simon (1980), Zolesio (1981), Choi and Haug (1983), Rousselet and Haug (1983), Haug et al. (1986), and Choi and Seong (1986) and Haber (1987). These methods could be used for shape optimization under the assumption that the initial topology is fixed during the iterative design optimization. Thus, it is not possible to find the optimal topology and optimal shape of the structure using these boundary variation techniques. To do this shape optimization problems must be transformed to material distribution problems. One method of doing this is the microstructure approach, this was first demonstrated by Cheng and Olhoff (1981, 1982) on optimal thickness distribution for elastic plates. This work led to a series of works on optimal design problems with microstructures in the formulation of the problem. For applications to plate problems see Bendsoe (1986) and Gibiansky and Cherkaev (1984). For the optimum design of torsion bars constructed from two dissimilar materials see Goodman et al. (1986) and Kohn and Strang (1986b). These studies concluded that laminated structures give more efficient designs and therefore microstrucutres have to be built in order to determine the strongest structure. Mathematically the introduction of microstructures to the formulation of structural design problems is a relaxation of the variational problem that is formulated for the design optimization, for examples of this see Kohn and Strang (1986a) and Bonnetier and Vogelius (1986). For more information on boundary variation methods the reader is advised to seek out the surveys by Ding (1986) and Haftka and Gandhi (1986).

The boundary variation techniques are not straightforward to implement and usually require some method for FEM-remeshing which is normally done several times during an iterative optimization scheme. It is clear that these early techniques suffered from complicated mathematical formulations restricting their implementation to real structures and applications. Therefore leaving them as an academic exercise only.

In the late 1980s two methods for structural optimization were introduced, the homogenization method by Bendsoe and Kikuchi (1988) and the Solid Isotropic Microstructure with Penalization Method (SIMP) by Bendsoe (1989). The full SIMP algorithms was developed by Zhou and Rozvany (1991) and Rozvany and Zhou (1991); however, the term SIMP was not coined until 1992 by Rozvany et al. (1992). These two methods revolutionized the field of structural optimization, making it applicable to real world problems, instead of just being an academic exercise. This is where this review paper begins. Its purpose is to analyze the developments made and the applications of structural optimization, namely evolutionary algorithms. The following section outlines the structure of the paper.

1.2 Layout of review paper

The present review is dedicated to the evolutionary topology and shape optimization methods and there applications. This area of research has been extremely active since the late 1980s due in-part to the publication of two papers (Bendsoe and Kikuchi 1988; Bendsoe 1989). In recent years hundreds of publications have emerged including a number of books, for example, Bendsoe (1995), Xie and Steven (1997), Hassani and Hinton (1999), Bendsoe and Sigmund (2003), Huang and Xie (2010a), and Rozvany and Lewinski (2013). The current comprehensive review articles on the topic are Rozvany, Bendsoe and Kirsch review paper of 1995 (Rozvany et al. 1995). This review covers the early work in the field. Eschenauer and Olhoff's review paper in 2001 (Eschenauer and Olhoff 2001). This review captures the activity of the 1990s, however does not look into the non gradient based methods with detail. It concentrates on topology optimization, not shape or topography optimization. A lot of the European contribution has been discussed, however not the full international contribution. Rozvany's review paper in 2001 (Rozvany 2001b), focuses on computer aided topology optimization, such as the homogenization and SIMP methods. Rozvany's latest review paper (Rozvany 2009), compares the popular established methods used in topology optimization. Sigmund and Maute's comparative review paper in 2013 (Sigmund and Maute 2013), aimed not to perform a comprehensive review but to observe the trends and directions and to tie these developments together (Sigmund and Maute 2013). The authors categorize the ESO method as a 'discrete SIMP approach' arguing that little difference exists between the ESO and SIMP methods. The authors of this review believe this view oversimplifies the theoretical formulation of the ESO method, although the procedure may only differ marginally the implementation allows for extra complexity and computational efficiency, with possible reduction in optimality of less than 1 % (Huang and Xie 2010a). Furthermore, SIMP is purely a gradient based method taking the derivative of the sensitivity numbers, whereas evolutionary algorithms rank the sensitivity numbers based on their magnitudes. This allows for a more flexible objective function and ease of implementation, which have greatly benefited the application of topology optimization in fields such as architecture, Felicetti (2009), and the manufacturing, Kim et al. (2000b), industries. The most recent is the review paper by Deaton and Grandhi (2014), this review is an update of Eschenaur and Olhoff's review, highlighting the advancements in topology optimization in continuum structures from 2000 to 2012.

Section 2 of this review paper deals with the variety of non gradient based algorithms, these are algorithms that do not calculate the Jacobean or Hessian matrix for the design variables. A brief discussion of the popular algorithms is given, with references to other papers for more detail.

Section 3 is devoted to the previous methods and developments of the evolutionary structural optimization algorithm. A comprehensive review of the algorithm is given, with the variety of applications and advances made. The suggested shortcomings of the algorithm is also analyzed followed by the recent developments to alleviate these pitfalls.

Section 4 focuses on the various sensitivity functions that have been developed for a variety of optimization problems. The earlier sensitivity functions and more recent are given to further highlight the advancements in evolutionary topological optimization.

Section 5 looks at the use of evolutionary algorithms in shape optimization. This section is similar to Section 3, however looking at shape optimization instead of structural optimization. The developments and applications of shape optimization are discussed.

Section 6 discusses the applications of evolutionary algorithms to real world engineering problems. The development of structural topology optimization has allowed such tools to be used by architects and engineers in industry. This section discusses the use of such tools in industry applications.

Section 7 presents the possible future developments of evolutionary topology optimization, looking at the most recent algorithms and applications.

Section 8, finally, presents the conclusions of this article.

2 Non gradient based algorithms

In the literature there are several non gradient based algorithms that have been applied to structural optimization. Such methods include Genetic Algorithms (GAs) Balamurugan et al. (2008, 2011), Jain and Saxena (2010), Madeira et al. (2010), Wang and Tai (2005), Zhou (2010), Wang et al. (2006), Guest and Genut (2010), Bureerat and Limtragool (2006), Manan et al. (2010), Artificial Immune Algorithms (Luh and Chueh 2004), Ant Colonies (Kaveh et al. 2008; Luh and Lin 2009), Particle Swarms (Luh et al. 2011), Simulated Annealing (Shim and Manoochehri 1997), Harmony Search (Lee and Geem 2004), Differential Evolution Schemes (Wu and Tseng 2010), Bacterial Foraging (Georgiou et al. 2014) and many others. For further non gradient based algorithms the reader is referred to the book written by Yang (2010). The aforementioned methods have been directly applied to topology optimization, their binary nature intuitively lends itself to the determination of solid/void material, however they have not experienced significant acceptance. This can be partly attributed to the fact that it is difficult to ensure structural connectivity because of the stochastic search procedure and partly due to the excessive computation power required. Despite these shortcomings one technique that began as a non gradient based algorithm, the Evolutionary Structural Optimization (ESO) method, has gained widespread popularity among researches in structural optimization and practitioners in engineering and architecture (Huang and Xie 2010c). ESO has been used for a wide range of applications and well over one hundred publications and thousands of citations have been produced by researchers from around the world. For these reasons the authors have decided to focus on the ESO technique in this review paper.

The idea of having a continuously evolving design, that slowly improves until it reaches the optimum, is best seen in nature. Over time all living things have evolved to become better suited for survival. This idea was first used by Xie and Steven (1993) for structural optimization in 1993 and is known as Evolutionary Structural Optimization (ESO). Thus moving away from the traditional mathematical programming techniques originating from Rozvany in the early 1970s (Rozvany 1972a, b) and then Prager in the late 1970s and early 1980s (Prager and Rozvany 1977a, b).

The concept behind ESO is that by slowly removing inefficient material from a structure, the shape of the structure evolves towards an optimum (Xie and Steven 1997). The idea of element removal has been tried by other researchers, prior to 1993, including (Maier 1973; Rodriguez-Velazquez and Seireg 1985), Schnack (1978, 1988), Schnack et al. (1988), Schnack and Iancu (1989, 1991, 1993) and Atrek (1989), these studies however did not result in a generalized method. The design domain is constructed by the finite element method. The criteria for determining the inefficient material in a structure is varied and depends on the types of design constraints, this will be discussed in detail in Section 4. An initial rejection rate (RR) is set at the beginning of the optimization, material that fall below this rejection rate is periodically removed, until a steady-state is reached. At this point the rejection rate is updated, and the process is repeated. The optimization is completed once a certain criteria, i.e. volume fraction, minimum stress level, etc., has been met. Evolutionary algorithms are known to be robust, versatile and capable of solving global optimization problems (Tanskanen 2002).

3 Previous methods and developments

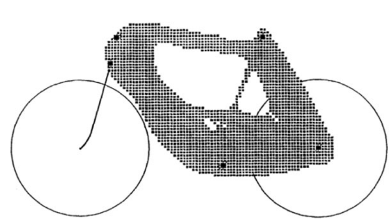
The early methods of the ESO technique were initially applied to natural structures, such as bones, to show that the optimum topology and shape of such structures are achieved over time, by following an evolutionary path (Xie and Steven 1993). Xie and Steven give the example of a metal implant for a broken bone (Xie and Steven 1993). After the repair the bone diminishes as calcium is absorbed in the bloodstream. It is shown that if any local bone does not have sufficient stress applied that part diminishes, these results were mirrored in the optimization. These early evolutionary techniques only removed inefficient material, therefore the initial model used an over sized structure, that slowly was reduced. The early methods also used what is called a fully stressed structure (Gallagher 1977), this method attempts to only use structural components that are subjected to their maximum allowable stress. Therefore the design criteria is the von Mises stress of each component. The ESO technique has been integrated into interactive computer programs, some to note are ESFD developed by Hinton and Sienz (1995) and Evolve97 by Xie and Steven (1997).

Following the early development of the ESO method on single loaded structures, the method was further applied to multiple load case structures by Xie and Steven (1994a) and Steven et al. (1995). This development allows the ESO method to be used on more realistic structures, since most structures in a real environment are subjected to multiple load cases. One of the most significant early examples was the application of the evolutionary procedure to the bicycle frame, see Fig. 1a, the final model currently hangs outside the Lawrence Hargrave Professor's office at the University of Sydney, see Fig. 1b. This method was further applied to frame structures by Manickarajah et al. (2000).

The next development of the ESO method was to optimize a structure for resonance (Xie and Steven 1994b), instead of looking for a fully stressed design. This is an extension from the previous ESO methods to eigenvalue problems. The evolutionary algorithms began to adopt gradient based techniques, such as the sensitivity number of the eigenvalue problem, to determine which elements to remove; however, elements were still explicitly defined as existent or absent, resulting in structures free of intermediate densities. Xie and Steven developed an ESO technique to optimize the shape and topology of a general structure that maximized or minimized the natural frequency. This allowed excessive vibrations due to resonance to be avoided. This method was further developed by Xie and Steven to include keeping a chosen natural frequency constant, maximizing the gap between two natural frequencies and multiple frequency constraints (Xie and Steven 1996). This allowed the ESO method to be applied to dynamic problems. Some examples can be found in Zhao et al. (1995, 1996). More recently the 'soft-kill' penalty based BESO method was applied to frequency optimization by (Huang et al. 2010).

The element strain energy based criterion was first utilized for optimization using the ESO method by Chu et al. (1996, 1997, 1999). They showed a method using the

Fig. 1 ESO for Multiple Load Case



(a) Optimal Design of a Bicycle Frame Xie and Steven (1994a)

(b) Manufactured Bicycle Frame Outside Lawrence Hargrave Professor's Office

ESO technique that would minimize the weight of the structure while satisfying stiffness requirements. The process is to determine the sensitivity of each element on the stiffness of the overall structure and then remove the elements which make the least change in the stiffness of the structure. Querin applied this approach to nonlinear problems where material and geometric nonlinearities are considered (Querin et al. 1996). Liang moved away from a heuristic approach, creating an element removal criterion based on a sensitivity number (Liang et al. 2000a).

For a slender member the load carrying capacity is often determined by its buckling load. Optimum structural design for buckling can be achieved by moving material from the strongest locations to the weakest, such that the final design will have a higher buckling load than that of the initial design of the same weight. Optimization against buckling for columns, frames, and plates was first implemented using the ESO technique by Manickarajah et al. (1995) and Manickarajah and Xie (1998).

Other load conditions such as thermal stresses were incorporated into the ESO technique by Li et al. (1997, 2000). A fully stressed design under thermal loading conditions was developed. This work was further developed to include optimization for a structure with as even a temperature or flux distribution as possible (Li et al. 1999b). Research involving optimization under heat loads using the ESO technique was applied to conducting fields to reduce the temperature at a specific position, by varying the conducting material distribution in other regions (Li et al. 2004). Thermoelasticity was investigated using the ESO technique by Li et al. (2001c). Patel et al. solved extremal conductivity microstructures by employing the ESO technique (Patil et al. 2008). Heat conducting structures with design dependent heat loads were optimized using the BESO technique by Gao et al. (2008). Ansola et al. developed a BESO technique for the design of thermal compliant actuators under a uniform temperature distribution and non-uniform heating including conduction and convection (Ansola et al. 2010, 2012). For a good paper on the application of the ESO technique on different physical field problems the reader should see Steven et al. (2000).

The elastic contact problem was solved by Li et al. (1998) where the contact profile of several separate bodies is optimized to reduce the maximum contact stress. This method was extended to three-dimensional contact surfaces, where contact profiles are iteratively and slowly modified based on the relative levels of contact stress (Li et al. 2001d). Contact constraints in beam/truss structures were developed using the ESO technique (Li et al. 2003a). Work on multiple contact constraints was also considered by Li et al. (2003b, 2005), where the example of a washer shows that this method is effective for design problems consisting of single- or multiple-contact regions in mechanical systems.

The early ESO methods are limited by only removing material from the structure, the consequence of this is that the initial model must be significantly over-designed and that if structure is prematurely removed it cannot be recovered. To overcome this an improved model of the early ESO techniques was developed by Querin et al. (1998) known as bi-directional ESO (BESO). This technique allows elements to be readmitted to the structure. BESO was further developed to include stiffness optimization (Yang et al. 1999). A further extension of this method to three-dimensional structures can be found in Young et al. (1999). A reverse method to the original ESO algorithm was proposed, whereby the structure evolves from the base which is the minimum structural form required to carry the load regardless of the magnitude of the stress levels (Querin et al. 2000a). The addition of a perimeter constraint to the BESO technique was developed by Yang et al. (2003). Kim et al. introduced cavity control techniques into BESO (Kim et al. 2000a, 2002a). Along with these developments, fixed grids were introduced, to reduce the computational time (Kim et al. 2000b, 2002b). The optimality of the bi-directional ESO method was validated by Querin et al. (2000b).

Rozvany and Zhou examined the ESO/BESO technique in 2001, Zhou and Rozvany (2001), and concluded that both methods are not able to always guarantee an optimal design. As a result, a large number of solutions generated during the optimization process have to be compared to generate the optimal solution (Rozvany 2009). Secondly, Sigmund and Petersson pointed out that the ESO/BESO procedure cannot easily be extended to other constraints such as displacement (Sigmund and Petersson 1998). Following the early criticism from Rozvany and Zhou; Zhu, Zhang and Qiu developed a BESO method where an element replaceable method is proposed for better representing the element status (Zhu et al. 2007). Huang and Xie proposed a new version of the BESO technique that is proven to produce convergent solutions (Huang and Xie 2007b; 2008a). The improvement involved increasing the accuracy of the elemental sensitivity numbers by using its historical information. In addition Edwards et al. compared the ESO and SIMP methods in support of the ESO technique (Edwards et al. 2007).

The extension of the ESO method to include multicriteria design problems was first achieved by Proos et al. (2001a, b). They used a weighting method to optimize simultaneously, the maximization of the first mode of natural frequency and the minimization of the mean compliance of the structure. Maximum stiffness with minimum stress was first developed by Steven et al. (2002). Kim et al. developed a combined static/dynamic objective parameter for use in the ESO technique for thermal stress and frequency criteria (Kim et al. 2006).

Checkerboarding, the repetition of material and void elements in a structure, is a common issue in optimization algorithms. Li et al. (2001b) developed a simple algorithm to alleviate these numerical difficulties in the ESO method. Another issue associated with optimization is whether the solution is a ‘global’ or a ‘local’ optimum. To solve this problem Liang et al. introduced performance based methods where an ESO type element removal technique is performed and the optimal solution is determined from iteration histories (Liang et al. 2001; Liang and Steven 2002).

The literature has shown significant developments in the ESO procedure since the idea was brought to fruition in 1992 (Xie and Steven 1992). However a lot of the recent developments have come out of the criticism of the algorithms ability to find the optimum solution efficiently and effectively (Rozvany and Querin 2002; Zhou and Rozvany 2001; Sigmund and Petersson 1998; Rozvany 2009, 2001a). This lead to the development of a ‘soft-kill’ ESO/BESO technique (Deaton and Grandhi 2014). The idea, first introduced by Rozvany and Querin (2002), does not completely remove inefficient elements. Doing this allows for the computations to re-admit material to be based directly on the void elements, as opposed to the surrounding elements. Rozvany labeled this technique the Sequential Element Rejection and Admission (SERA) method (Rozvany and Querin 2002).

More recent developments to stabilize the convergence of the ESO/BESO method have been made by Huang and Xie (2007b) and Huang and Burry (2006). Here a modified BESO is proposed where the nodal sensitivity numbers, instead of the element sensitivity, are used for compliance minimization. A mesh dependency filter is used, similar to the filter schemes that were introduced by Sigmund (1997), Sigmund and Petersson (1998), and Sigmund (2001) for the SIMP method, to produce stable convergence to mesh independent and checkerboard free solutions. A similar penalized density measure to the SIMP model, developed by Bendsoe (1989), Bendsoe and Sigmund (2003), Rietz (2001), Zhou and Rozvany (1991), and Bendsoe and Sigmund (1999), was employed in the ESO technique independently by both Zhu et al. (2007) and Huang and Xie (2009). Huang and Xie demonstrated that the previous ’hard-kill’ BESO methods are a special case of the ’soft-kill’ penalty based BESO methods with a infinite penalty (Huang and Xie 2009). The ’soft-kill’ BESO method was extended by Huang and Xie to include local displacement constraints (Huang and Xie 2010b, c). Abolbashari and Keshavarzmanesh investigated the sensitivity of the initial and evolutionary rejection ratio on the optimum structure of the ESO technique on 2D and 3D structures (Abolbashari and Keshavarzmanesh 2006). They found that the evolutionary rejection ratio had a bigger impact on the final design, this emphasizes the slow nature of the evolution process.

Structures with a non-linearity are not easily optimized by ’soft-kill’ methods due to low density elements creating numerical difficulty. Huang and Xie first applied BESO to structures with both material and geometric non-linearities, initially using the traditional BESO technique (Huang and Xie 2007a) followed by the improved ’hard-kill’ method with filtering (Huang and Xie 2008b).

The ESO/BESO technique was first modified for design dependent loads by Yang et al. (2005). The sensitivity number was modified to show that the BESO procedure can be used for transmissible loads, surface loading and self weight body loads. Another modified sensitivity number for self weight body loads was proposed by Ansola et al. (2006). More recently Huang and Xie proposed a formulation for self weight loading using the soft-kill penalty-based BESO method (Huang and Xie 2011).

Finally, there has been some work on reliability-based topology optimization (RBTO) utilizing ESO/BESO methods. Kim et al. was first to use ESO for use in first-order reliability methods (Kim et al. 2007). More recently the improved ’hard-kill’ BESO along with a response surface method was used for RBTO by Eom et al. (2011). Cho et al. extended these methods to include stiffness, applied load and dimensions for multi-objective problems, using BESO and the performance measure approach (Cho et al. 2011).

4 Sensitivity functions and optimization criteria

Over the past three decades structural topology optimization has matured to a level where multiple objectives, such as frequency and buckling, can be considered in the optimization algorithm. For example vibration response is a crucial design consideration for structures subjected to dynamic loads. It is advantageous to keep the natural frequency of the structure away from any driving frequencies that may be applied to the structure. Also, structures with a high fundamental frequency result in a stiff design, which is good for static loads (Krog and Olhoff 1999). Designers have got it wrong in the past, the most famous example being the Tacoma Narrows bridge in 1940, which collapsed due to resonance (von Karman 2005). The problem being the frequency of the wind gust differing little from the natural modes and frequencies of the bridge deck (Blevins 2001). This section outlines the sensitivity functions and optimization criteria used for structural topology optimization of evolutionary algorithms.

4.1 Stress optimization

The first objectives of structural topology optimization methods was the minimization of stress, where for the evolutionary methods the sensitivity function for stress minimization is the von Mises stress. For the original evolutionary structural optimization technique the sensitivity function for stress minimization is defined by:

$$\sigma_{remove}^{vm} = RR\sigma_{max}^{vm} \tag{1}$$

where RR is the rejection ratio of the iteration (Section 2). Elements with a von Mises stress that fall below σ_{remove}^{vm} are removed. Li et al. (1999c) and McKeown (1997) showed that such a fully stressed design is equivalent to that of the stiffness criterion, i.e. it cannot always minimize the highest stress in the structure.

4.2 Stiffness optimization

The mean compliance, the inverse measure of the overall stiffness of a structure, is used for optimizing structures with stiffness constraints. The mean compliance is defined as the total strain energy of the structure or the external work done by the applied loads. The sensitivity function for the mean compliance is defined as Huang and Xie (2010a):

$$\alpha_i^e = \frac{1}{2}\mathbf{u_i^T K_i u_i} \tag{2}$$

where $\mathbf{u_i}$ is the displacement vector of the i^{th} element and $\mathbf{K_i}$ is the stiffness matrix of the i^{th} element. The sensitivity function indicates the increase in mean compliance as a result of the removal of the i^{th} element. The increase in mean compliance due to the removal of the i^{th} element is equal to the elemental strain energy of the i^{th} element. Therefore, the elements with the lowest strain energy are removed. Li et al. (1999c) showed that structures that are designed for maximum stiffness result in similar topologies to structures that are designed for minimum stress. Papadrakakis et al. (1996) compare resulting topologies using different objective functions for stress and stiffness optimization.

4.3 Stiffness optimization with multiple materials

An extension of the previous section is to optimize structures composed of multiple materials with a stiffness constraint. To interpolate the material properties between two neighboring phases, E_j and E_{j+1}, the interpolation method devised by Bendsoe and Sigmund is used (Bendsoe and Sigmund 1999).

$$E(x_{ij}) = x_{ij}^p E_j + (1 - x_{ij}^p)E_{j+1} \tag{3}$$

where p is the penalty exponent. Therefore the sensitivity number is defined by:

$$\alpha_{ij} = \frac{1}{2}x_{ij}^{p-1}\left(u_i^T K_i^j u_i - u_i^T K_i^{j+1} u_i\right) \tag{4}$$

where K_i^j and K_i^{j+1} are the element stiffness matrices calculated using E_j and E_{j+1} respectively. Note that the sensitivity number α_{ij} is defined in the entire design domain, however it is only used for making adjustments between materials j and $j+1$. The sensitivity number for the 'soft-kill' BESO method can be written as:

$$\alpha_{ij} = \begin{cases} \frac{1}{2}\left[1 - \frac{E_{j+1}}{E_j}\right]u_i^T K_i^j u_i & \text{for materials} = 1, \dots, j \\ \frac{1}{2}\frac{x_{min}^{p-1}(E_j - E_{j+1})}{2x_{min}^p E_j + (1 - x_{min}^p)E_{j+1}}u_i^T K_i^{j+1} u_i & \text{for materials} = j+1, \dots, n \end{cases} \tag{5}$$

where K_i^{j+1} is found using the material interpolation method (3) to calculate the elastic modulus. For the 'hard-kill' BESO method the penalty number is set to infinity giving:

$$\alpha_{ij} = \begin{cases} \frac{1}{2}\left[1 - \frac{E_{j+1}}{E_j}\right]u_i^T K_i^j u_i & \text{for materials} = 1, \dots, j \\ 0 & \text{for materials} = j+1, \dots, n \end{cases} \tag{6}$$

The optimization problem is converged when the volume fractions for all the materials are met.

4.4 Displacement constraints

It is often required that the maximum displacement of a structure or the displacement at a specific location be within a prescribed limit. To find the change in displacement at a specific point, j, due to the removal of an element, i, we introduce a unit load vector, $\mathbf{F^j}$, in which only the j^{th} component is equal to unity and all the others are equal to zero. The sensitivity function for the displacement constraint can then by calculated by:

$$\alpha_{ij} = \mathbf{u^{ij} K^i u^i} \tag{7}$$

where $\mathbf{u^{ij}}$ is the i^{th} element displacement vector due to the unit load, j, and $\mathbf{u^i}$ is the i^{th} displacement vector. This sensitivity function gives the change in the specified displacement component u_j due to the removal of the i^{th} element. To minimize the change in displacement due to element removal the elements with α_{ij} closest to zero are removed. Kocvara (1997) develop a bi-level programming method for the topology optimization of a truss structure with a linear displacement constraint.

4.5 Frequency optimization

A structure's response to a dynamic loading is heavily dependent on the first few natural frequencies of the structure. Frequency optimization of structures is the second most common optimization criteria, after compliance minimization, due to dynamic stability. A good survey paper by Grandhi (1993) on frequency optimization, before the ESO method, covers most of the early developments in this area. The early ESO methods, where inefficient material was simply removed such that no intermediate material was present (i.e. 0-1 solution), solved the following eigenvalue problem for its sensitivity function (Xie and Steven 1997):

$$(\mathbf{K} - \omega_{\mathbf{n}}^{\mathbf{2}}\mathbf{M})\mathbf{u_n} = 0 \tag{8}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{M}$ is the global mass matrix, ω_n is the n^{th} natural frequency and $\mathbf{u_n}$ is the eigenvector related to the n^{th} natural frequency. The sensitivity function is then defined by:

$$\alpha_n^i = \frac{1}{m_n}\mathbf{u_n^i}^{\mathbf{T}}(\omega_{\mathbf{n}}^{\mathbf{2}}\mathbf{M^i} - \mathbf{K^i})\mathbf{u_n^i} \tag{9}$$

where m_n is the modal mass corresponding to the n^{th} natural frequency. This sensitivity function gives the change in ω_n^2 as a result of the removal of the i^{th} element.

More recent ESO methods use a 'soft-kill' algorithm where the material is not completely removed, but applies a material interpolation scheme similar to the SIMP method. It has been demonstrated that these 'soft-kill' methods are unsuitable for frequency optimization due to artificial localized modes in low density regions (Pederesen 2000). However, Huang et al. (2009) developed a new 'soft-kill' BESO method for frequency optimization. The sensitivity function was determined by:

$$\alpha_i = \frac{1}{p}\frac{d\omega_j}{dx_i}\begin{cases} \frac{1}{2\omega_j}u_j^T\left(\frac{1-x_{min}}{1-x_{min}^p}\mathbf{K_i^0} - \frac{\omega_{\mathbf{j}}^{\mathbf{2}}}{\mathbf{p}}\mathbf{M_i^0}\right)u_j & x_i = 1 \\ \frac{1}{2\omega_j}u_j^T\left(\frac{x_{min}^{p-1}-x_{min}^p}{1-x_{min}^p}\mathbf{K_i^0} - \frac{\omega_{\mathbf{j}}^{\mathbf{2}}}{\mathbf{p}}\mathbf{M_i^0}\right)u_j & x_i = x_{min} \end{cases} \tag{10}$$

The main problem is that the extremely high ratio between mass and stiffness for small values of x_i with penalty exponents, p, greater than 1 causes artificial localized vibration modes in the low density regions (Du and Olhoff 2007). A material interpolation scheme that keeps the ratio between mass and stiffness constant when $x_i = x_{min}$ is given by:

$$\rho(x_i) = x_i\rho^0 \tag{11}$$

$$E(x_i) = \left[\frac{x_{min} - x_{min}^p}{1 - x_{min}^p}\left(1 - x_i^p\right) + x_i^p\right]E^0 \qquad (0 < x_{min} \leq x_i \leq 1) \tag{12}$$

where ρ^0 and E^0 are the density and stiffness for solid material.

4.6 Optimization for buckling

With the development of high strength materials, many structural elements are becoming thinner making them more susceptible to buckling. Optimum structural design against buckling is achieved by shifting material from the strongest locations to the weakest. This allows the final design to have a much higher buckling load, but with the same weight as the initial design. In spite of the extensive work performed in structural optimization, only a small amount of this work has been directed towards buckling optimization. The first being (Keller 1960) presenting the shape of the strongest column with simply supported ends. Tadjbakhsh and Keller (1962) extended this to several other boundary conditions. Simitses et al. (1973) use the finite element procedure to optimize the shape of columns. This work is extended by Szyszkowski and Watson (1988) to include the optimization of frames (Szyszkowski et al. 1989). Pandey and Sherbourne (1992) extend the classic solution by Keller (1960) to a simply supported plate. Manickarajah et al. (1995) developed the ESO technique for buckling optimization.

The linear buckling behavior of a structure is determined by the following eigenvalue problem:

$$\left([K] + \lambda_n\left[K_g\right]\right)\{u_n\} = 0 \tag{13}$$

where $[K]$ is the global stiffness matrix, $\left[K_g\right]$ is the global stress/geometric stiffness matrix, λ_n is the n^{th} eigenvalue and $\{u_n\}$ is the corresponding eigenvector. The most critical buckling load is the lowest one, which is equal to the first eigenvalue, λ_1, multiplied by the applied load. λ_1 is referred to as the buckling load factor. The objective of buckling optimization is to raise the buckling load factor, λ_1, so that the buckling load may be maximized. The sensitivity number for the buckling load is defined as:

$$\alpha_i = -\left\{u_1^i\right\}^T\left[\Delta K^i\right]\left\{u_1^i\right\} \tag{14}$$

where $\left\{u_1^i\right\}$ is the eigenvector of the i^{th} element and $\left[\Delta K^i\right]$ is the change in the i^{th} element stiffness matrix. This sensitivity number is a measure of the effect of changing the cross-sectional area of the i^{th} element on the buckling load factor. For a plate structure this sensitivity number, α_i, is only valid for the case of gradually varying thicknesses of plate elements. Therefore an element cannot be removed from the structure, as is the convention in the classical ESO methods, because of the significant changes in the membrane or axial stress resultants in its surrounding elements. Buckling optimization is not performed by element removal, but by changing the cross-sectional areas. For the case of an increase in cross-sectional area the change in the element stiffness matrix is given by:

$$\left[\Delta K^i\right] = \left[\Delta K^i\right]^+ = \left[K^i(A + \Delta A)\right] - \left[K^i(A)\right] \tag{15}$$

and for reduction in cross-sectional area:

$$\left[\Delta K^i\right] = \left[\Delta K^i\right]^- = \left[K^i(A - \Delta A)\right] - \left[K^i(A)\right] \tag{16}$$

Hence for the effect of cross-sectional area changes on the buckling load factor two sensitivity numbers are calculated, one for area increase:

$$\alpha_i^+ = -\left\{u_1^i\right\}^T \left[\Delta K^i\right]^+ \left\{u_1^i\right\} \tag{17}$$

and another for area reduction:

$$\alpha_i^- = -\left\{u_1^i\right\}^T \left[\Delta K^i\right]^- \left\{u_1^i\right\} \tag{18}$$

To raise the buckling load factor it is most effective to increase the cross-sectional areas of elements with highest α_i^+ values and reduce those with highest α_i^- values. This technique is the original ESO buckling optimization method. Rong et al. (2001) extended the ESO method to include the sensitivity of closely-spaced eigenvalues and repeated eigenvalues. This type of problem is known as bi-modal or multimodal. Extensive research has been carried out to solve bi-modal and multimodal problems (Olhoff and Rasmussen 1977; Haug 1982; Szyszkowski et al. 1989; Rodrigues et al. 1995). When the first eigenvalue becomes close to the subsequent eigenvalues, there will be interference between the first and the subsequent buckling modes. When λ_1 and λ_2 become close, the first two buckling modes may swap as a result of the structural modifications during the optimization iterations. To effectively increase the buckling load factor in these circumstances, both λ_1 and λ_2 have to be increased. To achieve this the average of λ_1 and λ_2 is increased. Therefore the sensitivity number becomes:

$$\alpha_i = \Delta\left(\frac{\lambda_1 + \lambda_2}{2}\right) = -\frac{1}{2}\left(\left\{u_1^i\right\}^T \left[\Delta K^i\right] \left\{u_1^i\right\} + \left\{u_2^i\right\}^T \left[\Delta K^i\right] \left\{u_2^i\right\}\right) \tag{19}$$

Similarly for the multimodal case, when the distance of the m^{th} eigenvalue, λ_m, and the first eigenvalue, λ_1, is within a certain limit and the distance between λ_{m+1} and λ_1 is greater than that limit, the sensitivity number becomes:

$$\alpha_i = \Delta\left(\frac{1}{m}\sum_{j=1}^{m}\lambda_j\right) = -\frac{1}{m}\sum_{j=1}^{m}\left(\left\{u_j^i\right\}^T \left[\Delta K^i\right] \left\{u_j^i\right\}\right) \tag{20}$$

These sensitivity numbers are the average values of the individual sensitivity numbers of all participating modes.

As mentioned in Section 4.5 numerical instabilities occur when solving eigenvalue problems with low density elements. Huang et al. (2010) developed an alternative material model to avoid numerical instabilities (see Section 4.5), which can also be applied to the buckling optimization problem.

4.7 Optimization for design dependent loading

Yang et al. (2005) showed that for compliance optimization of structures with design dependent loads, such as self-weight and surface loads, the conventional BESO procedure can be applied with a modified sensitivity number. When the self weight load is included in the finite element analysis, modifying an element leads to changes in the load vector, which needs to be considered in the sensitivity analysis. The sensitivity number becomes:

$$\alpha_i = \frac{u_i^T K_i u_i + 2\Delta P_i^T u_i}{W_i} \tag{21}$$

where ΔP_i is the change in the load vector due to the removal of the i^{th} element. Ansola et al. (2006) also developed a modified sensitivity number for self weight loads in ESO. More recently (Huang and Xie 2011) applied the 'soft-kill' penalty based BESO method to self-weight loading compliance optimization.

5 Shape optimization

The maximum stress present in a structure provides the basis of the design limit, even though the bulk of the surface of the structure often has a much lower stress. The ESO technique has been used to provide structural shapes where the boundary is evolved to make the surface form as evenly stressed as possible. Shape optimization has been an important topic well before the advent of ESO, the articles by Haftka and Grandhi (1985) and Ding (1986) give excellent surveys on the early shape optimization techniques.

For shape optimization only a small region of a structure is of interest, thus only this region is available for the ESO process and the rest of the structure is designated as non-design. Material is only removed form the structural boundaries in shape optimization, i.e. cut-outs and holes are not created. Early shape optimization using non gradient based algorithms are seen in Mattheck (1989), and Baumgartner et al. (1992), Mattheck and Burkhardt (1990, 1992), Mattheck and Erb (1991), Huber-Betzer and Mattheck (1991), and Chen and Tsai (1993).

An early use of ESO for shape optimization was in the design of adhesive fillets (Rispler et al. 2000). The ESO method was employed to optimize the shape of adhesive fillets found in tabs of tensile test specimens. Shape optimization of cutouts in laminated carbon-fiber composite panels using ESO was demonstrated by Falzon et al. (1996). An initial small cutout was introduced into the finite element model and elements were removed from around the cutout based on a rejection criteria. An interesting example of shape optimization using the ESO technique is given by Querin (1997), who seeks to find the optimal shape for an object hanging in the air under its own weight. By removing elements from the boundaries based on stress level, a shape with uniform stress is obtained. The results of the shape optimization are indicative of shapes in nature such as, plums and cherries, giving confidence to the algorithm as it is known that nature evolves shapes that lead to uniform surface stress (Mattheck 1990a, b; Mattheck et al. 1992).

Li et al. (1999a) applied the ESO method for shape optimization with stress minimization to a fillet in plane stress under tension. Tekkaya and Guneri (1996) perform a parametric study on the shape optimization of a plate with a hole using a biological growth approach. Woon and Querin performed shape optimization using genetic algorithms (Woon and Querin 1999). Woon et al. (2000) extended this analysis to incorporate fixed-grid methods. Garcia and Gonzalez (2004) adopt evolutionary strategies to perform shape optimization on continuum structures with a fixed grid finite element method. They give the example of a spanner, to show that it is the optimum shape for its loading conditions.

Shape optimization with other constraints, such as frequency constraints, has been considered. Wang et al. (2004) analyze shape and sizing optimization for truss structures with frequency constraints. Lingyun et al. (2005) perform the shape and sizing optimization of truss structures with frequency constraints. Gomes (2011) use the particle swarm algorithm to optimize the shape of truss structures with dynamic constraints. Miguel (2012) compares the harmony search and firefly algorithms for the shape and size optimization of truss structures considering dynamic constraints.

More recent efforts in shape optimization have focused on aerodynamic shape optimization. Arias-Montano et al. (2011) give an overview of Evolutionary algorithms applied to multi-objective aerodynamic shape optimization. Carrigan et al. (2012) developed a fully automated process for optimizing the aerofoil cross-section of a wind turbine.

6 Applications

ESO/BESO have found a wide range of applications in industrial topology optimization. The Akutagwa river side project office building in Japan was designed using a BESO method (Ohmori et al. 2005). The sensitivity function used for the office building design is a stress level. The land size of the building is approximately $10m \times 6m$. The BESO procedure was applied to the south, west and north side walls simultaneously, while the east side wall and floor slabs were kept unchanged. The optimization was run for both dead weight in the vertical direction and earthquake loading in the horizontal direction. The topology of the three walls evolved as material was removed from areas of low stress and added to areas of high stress.

Cui et al. (2003) in collaboration with the well renowned architect Arata Isozaki, created a tree-like structure using a BESO method for the Florence New Station project in Italy. The structure had a length of 400 m a width of 40 m and height of 20 m. During the optimization process the structure transformed from an initial design of a deck with legs simply supported at the bottom to the final organic form. Sasaki (2005) commented on the structure saying 'The structural elements were optimally formed within a three-dimensional space while satisfying the given design conditions, and the structural shape thus obtained manifests maximum mechanical efficacy with a minimum use of materials.' Although this proposal was not chosen as the design for the Florence New Station, the same design was chosen for the 250 m long entrance to the Qatar National Convention Center.

Burry et al. (2005) carried out a series of studies which revealed remarkable similarities between Gaudi's designs and ESO solutions. Antoni Gaudi's Sagrada Familia Church in Spain show parallels to natural growth and morphogenesis which has much in common with ESO/BESO techniques. Gaudi had an unusual insight into optimal structural forms, creating many structures that proved to be highly efficient. Burry developed several models based off Gaudi's original drawings. After these prototype models were created, structural optimization of columns on a sloping surface was performed. There was a great resemblance between the ESO results and the actual columns.

Black Kosloff Knott (BKK) architects, working in collaboration with the Innovative Structures Group, are designing pedestrian bridges for use in major metropolitan freeways in Australia. The design team employed the BESO method to create structurally efficient and elegant forms. The structure must have a width of 65 *m* and height of 5.7 *m*, with a maximum ramp slope of 1 : 20. Currently a number of optimized solutions, calculated with different constraints and element types, are being investigated (Zuo 2009).

Other interesting applications of ESO/BESO found in civil engineering include reinforced concrete strut-and-tie models (Liang et al. 2000b), connection patterns for joints (Li et al. 2001a), bridge design (Guan et al. 2003), underground mining cavity design (Ren et al. 2005) and tunnel engineering (Liu et al. 2008).

Applications in other fields of engineering include the use of a conventional ESO method to design aircraft bulkheads for strength and low weight (Das and Jones 2011). Biomedical design applications such as tissue scaffolds using a BESO algorithm and a wall shear stress criterion (Chen et al. 2011). Naceur et al. (2004) demonstrated the use of ESO for sheet metal optimization. Ansola et al. (2007) used ESO for compliant mechanism design, developing similar topologies to those obtained using SIMP. More recently, Huang et al. (2012) used BESO to design multifunctional periodic composites having both extremal magnetic permeability and electrical permittivity and Yang et al. (2011) did the same, however for stiffness and thermal conductivity. There have been a number of recent publications in applying BESO methods to topology optimization of material microstructures (Yang et al. 2013; Huang et al. 2013a; Zuo et al. 2013).

7 Future developments

This section looks at the direction that the research of evolutionary algorithms is heading. A recent method that has been developed in the field of topology optimization is boundary variation techniques. Boundary variation methods originated in shape optimization, where an explicit function, such as the strain energy, defines a structural boundary as shown in Fig. 2.

Figure 2 shows a UAV spar where the domain exists as an explicit parameterization of variables between 0 and 1. The structural boundary exists at the interface of regions between 0 and 1. There are two main boundary variation methods in literature, level set (Sethian and Weigmann 2000; van Dijk et al. 2013) and phase-field method (Bourdin and Chambolle 2003, 2006). The recent publications in evolutionary based optimization has involved coupling algorithms such as ESO/BESO with level set methods (Jia et al. 2011; Zhu et al. 2014). The idea is to take the merits of both the evolutionary algorithm and the level set algorithms. Since traditional level-set methods are largely dependent on the initial guess topology, ESO/BESO has been added to avoid this Jia et al. (2011) and Zhu et al. (2014).

A recent structural topology optimization algorithm, developed by Tong and Lin (2011), called the Moving Iso-Surface Threshold (MIST) technique is a hybrid of the ESO method, as it uses a physics based function, such as von Mises stress, to drive the optimization. Vasista and Tong (2014) recently applied the MIST topology optimization method to aircraft structural design and extended the method to three-dimensional 'block' design. Munk et al. (2014) extended the method to complex three-dimensional geometries, such as the internal configuration of aircraft wings, with structural cross-coupling.

Victoria et al. (2009) developed a novel optimization algorithm, isolines topology design (ITD), which uses an iterative algorithm similar to the ESO/BESO. The shape and design depend on material which is removed and added according to the shape and distribution of the isolines. Victoria et al. (2014) extended the ITD algorithm to include multi-material designs.

Section 3 showed the vast developments made in the ESO/BESO optimization algorithm, however isotropic materials were only considered during optimization. Sun et al. (2011) extended the BESO algorithm to include anisotropic material in the design process. Huang et al. (2013b) applied a BESO algorithm to the microstructure topology optimization of cellular materials and composites with periodic microstructures. Their objective was to maximize the stiffness of the macrostructure's design. Huang et al. (2011) performed a similar analysis, however with the objective of maximizing the bulk or shear modulus of the macrostructure with a volume constraint. Yang et al. (2013) applied the BESO method to the design of three-dimensional orthotropic materials with predefined ratios for effective Young's moduli. The recent developments of the ESO/BESO algorithm are to apply such methods to microstructural optimization, which have typically been the focus of gradient based methods such as: homogenization and SIMP. Alonso et al. (2014) recently developed a Sequential Element Rejection and Admission (SERA) method, another term for ESO (see Section 3), that allows material to flow between different material models. Separate criteria for the rejection and admission of elements allows the material to change between the different predefined material models and efficiently achieve the optimum design.

The two streams of topology optimization, macroscopic (ESO/BESO) and microscopic (Homogenization), have

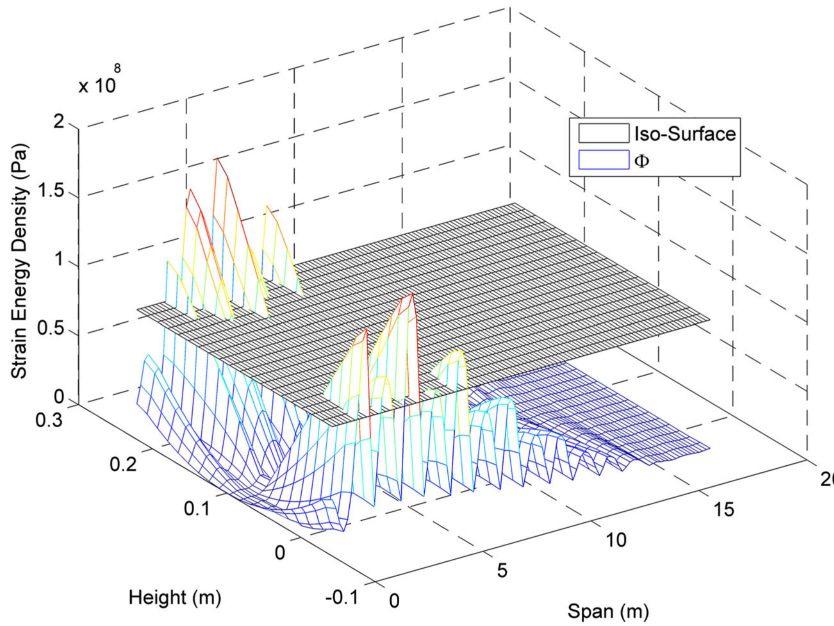


Fig. 2 Physical Response Function taken from Munk et al. (2014)

developed individually. However, recently (Yan et al. 2014) have coupled the ESO/BESO algorithm for the macroscopic optimization and the homogenization algorithm for the microscopic optimization of structures. A similar approach was performed by Coelho et al. (2008), they used a hierarchical model for topology optimization of the macroscopic structure and a SIMP based method for concurrent optimization of the microscopic structure.

Coupling between evolutionary and purely non gradient based techniques has also been implemented to improve the convergence of the ESO methods. A popular recent development termed GESO Genetic Evolutionary Structural Optimization use the ESO sensitivity number as the fitness function and less fit elements die off throughout the evolutionary procedure. GESO has advantages over the conventional ESO technique as the probabilistic criteria involved in the genetic algorithm helps to avoid convergence to local optima. Examples of applications of GESO are found in Liu and Yi (2010). Zuo et al. (2009) developed a genetic BESO technique that is similar to GESO, however utilizing a BESO formulation similar to the hard-kill BESO method.

A new approach that has been developed from optimization in CFD design problems is called Nash-evolutionary algorithms. The algorithm was first introduced in Sefrioui and Periaux (2000) for solving computational fluid dynamics problems. They include the mathematical concepts of Nash equilibrium (Nash 1950; 1951), competitive game theory where players maximize their payoffs while considering the strategies of their competitors, in the evolutionary search. The idea is that the subpopulations interact to evolve towards the equilibrium (Greiner et al. 2015).

8 Conclusion

This review article presented a comprehensive overview of the developments made in evolutionary algorithms for structural topology and shape optimization. An introduction on the early 'foundations' of structural topology and shape optimization is given, followed by some prominent references for non gradient based techniques. Due to the initial resistance for the non gradient based methods, the article focuses on the evolutionary algorithms, which have matured to the point of industry application. The article attempts to emphasis the broad range of topics and developments made in the evolutionary algorithms, similar to the review articles for gradient based methods (Eschenauer and Olhoff 2001; Deaton and Grandhi 2014). The review article looks at some of the different mathematical formulations for the variety of optimization problems that the evolutionary algorithms have been applied to. Application of evolutionary algorithms to real world problems have been investigated in this review article, showing the maturity of the current state of the art. Finally, the most recent developments and directions of the research in evolutionary algorithms have been discussed.

The early ESO optimization algorithms were criticized for being slow and computationally inefficient as they were not driven by a mathematical formulation. The early methods also seemed to fail for certain types of problems, where the gradient based methods produced optimal results. However, this criticism lead to significant developments, outlined in this article, which allowed the ESO method to be applied to industrial problems.

Current criticism directed towards evolutionary algorithms include: lack of convergence and no guarantee of finding the global solution. The current evolutionary algorithms include sensitivity functions that are said to be converged when no material can be removed or added without breaking the constraints of the problem. The latest research have coupled the SIMP material model and level set methods to the ESO/BESO algorithm to improve the algorithms convergence and efficiency.

The final section of the review paper analyzed the most recent research undertaken in evolutionary algorithms for structural topology optimization. Furthermore, new avenues in the field have been discussed in combining gradient and non gradient based algorithms to improve efficiency. Finally, approaches in running evolutionary algorithms concurrently with homogenization and SIMP methods for optimization of the macrostructure and microstructure has been examined.

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