

The ESO method revisited

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Abstract This paper examines the evolutionary structural optimisation (ESO) method and its shortcomings. By proposing a problem statement for ESO followed by an accurate sensitivity analysis a framework is presented in which ESO is mathematically justifiable. It is shown that when using a sufficiently accurate sensitivity analysis, ESO method is not prone to the problem proposed by Zhou and Rozvany (Struct Multidiscip Optim 21(1):80–83, 2001). A complementary discussion on previous communications about ESO and strategies to overcome the Zhou-Rozvany problem is also presented. Finally it is shown that even the proposed rigorous ESO approach can result in highly inefficient local optima. It is discussed that the reason behind this shortcoming is ESO's inherent unidirectional approach. It is thus concluded that the ESO method should only be used on a very limited class of optimisation problems where the problem's constraints demand a unidirectional approach to final solutions. It is also discussed that the Bidirectional ESO (BESO) method is not prone to this shortcoming and it is suggested that the two methods be considered as completely separate optimisation techniques.

Keywords Evolutionary structural optimization (ESO) · Sensitivity analysis · Compliance design · Bidirectional evolutionary structural optimization (BESO) · Sequential element rejection and admission (SERA) · Hard-kill methods

1 Introduction

The evolutionary structural optimization (ESO) method was initially introduced by Xie and Steven (1993) based on the simple idea of gradually removing inefficient parts of structures to reach optimised designs. A bidirectional version of ESO (BESO) was proposed later by Querin et al. (1998) and Yang et al. (1999) which also permits adding new elements to the efficient parts of structures. The term SERA (sequential element rejection and admission) was later proposed by Rozvany and Querin (2002) for this method to distinguish it from Darwinian evolutionary-based methods.

In ESO/BESO terminology, the measure of an element's efficiency is its *sensitivity number*. In earlier works on ESO/BESO, not enough attention was paid to mathematically stating the optimisation problem and the sensitivity numbers used to be defined rather intuitively. For example, to obtain a fully stressed structure, a stress invariant of elements such as von Mises stress was considered as sensitivity number (Xie and Steven 1993).

In more recent works on BESO, some researchers have adopted a rigorous approach of clearly stating the optimisation problem and defining sensitivity numbers based on sensitivity analysis of the objective function. For example, one can refer to Huang and Xie (2007, 2009, 2010a, b, Ghabraie (2009), Ghabraie et al. (2010), and Nguyen et al. (2014). Despite the apparent similarity between ESO and BESO, however, to the best knowledge of the author, no satisfying mathematical problem statement has been proposed for ESO. As explained by Rozvany (2009), the problem statements proposed earlier (e.g. by Tanskanen 2002) are not justifiable.

Another serious criticism on ESO/BESO methods was proposed by Zhou and Rozvany (2001). Through a simple example, Zhou and Rozvany showed that ESO method can

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lead to non-optimal solutions. Suggestions for changes in ESO algorithm to answer this criticism have been proposed by a number of authors. Rozvany (2009) reviewed some of the most important proposed discussions and solutions, including Tanskanen (2002), Rozvany and Querin (2002), Rozvany et al. (2004), Edwards et al. (2007), and Huang and Xie (2008). Rozvany discusses that some of these proposed “solutions” (e.g. Edwards et al. 2007) are unjustifiable. For other suggestions, such as mesh refinement and “freezing” critical boundary elements (proposed by Huang and Xie 2008), he discusses that the solution is not always reliable.

The problem can be fixed if a *soft-kill BESO* approach is employed. In soft-kill BESO, also known as the “virtual material” approach (e.g. in Rozvany and Querin 2002), the inefficient elements are weakened by using a very soft material rather than being completely removed from the design domain (hard-kill). However, as Rozvany correctly summarises, none of the proposed solutions completely rectifies this problem for the ESO method. He thus concludes that “ESO is presently fully heuristic, computationally rather inefficient, methodologically lacking rationality, occasionally unreliable, with highly chaotic convergence curves.”

This paper presents a framework in which the ESO approach can be mathematically justifiable. We start by proposing a problem statement for ESO (Section 2) followed by an accurate sensitivity analysis which overcomes the Zhou-Rozvany problem (Section 2.2). A complementary discussion on some other proposed strategies to overcome the Zhou-Rozvany problem and relevant communications is also presented (Section 4). It is discussed that the proposed accurate sensitivity analysis can ensure that ESO always reaches a local optimum. Finally we will show that even this rigorous ESO approach can result in highly inefficient local optima (Section 5). The reasons behind this behaviour are discussed. It is concluded that the ESO method should only be used on a very limited class of optimisation problems. It is also discussed that the BESO method is not prone to this problem.

2 A problem statement for ESO

In order to investigate the theoretical background of the ESO method, it is necessary to know what sort of optimisation problems are actually solved when implementing this method.

Consider a design domain which is discretised using finite element method into N elements. To be able to modify the topology of the design domain, consider the following linear material interpolation scheme,

$$\mathbf{K}_e(x_e) = \mathbf{K}_e^{(0)} + x_e \left(\mathbf{K}_e^{(1)} - \mathbf{K}_e^{(0)} \right), \quad e = 1, \dots, N \quad (1)$$

where \mathbf{K}_e is the stiffness matrix of element e and x_e is the design variable of this element which can vary continuously between 0 and 1. $\mathbf{K}_e^{(0)}$ and $\mathbf{K}_e^{(1)}$ are the stiffness matrices of the same element if it was made of two different materials, namely, material 0 and material 1 respectively. For single material-void designs, $\mathbf{K}_e^{(0)}$ is a zero matrix of the same size as $\mathbf{K}_e^{(1)}$. In this case, (1) simplifies to $\mathbf{K}_e = x_e \mathbf{K}_e^{(1)}$.

To simplify the matters hereafter we limit our study to compliance-based ESO. Starting from the full design domain, in each step ESO removes one or more elements which are considered to be inefficient. Throughout this procedure the function which is constantly reduced is the volume of the structure. It is thus reasonable to think of volume as the objective function for ESO. Based on this fact, we propose the following problem statement for the compliance-based ESO method:

$$\begin{aligned} \min_{x_1, \dots, x_N} \quad & V = \sum_{e=1}^N v_e x_e \\ \text{such that} \quad & c \leq \bar{c} \\ \text{and} \quad & x_e \in \{0, 1\}, \quad e = 1, \dots, N \end{aligned} \quad (2)$$

where V is the total volume of the design domain, v_e is the volume of element e , and c is the mean compliance of the structure defined as

$$c = \mathbf{f}^T \mathbf{u} \quad (3)$$

with \mathbf{f} and \mathbf{u} representing the nodal force and displacement vectors respectively. \bar{c} is a predefined upper limit for the mean compliance. It should be noted that the only allowable values of design variables in problem (2) are the binary values of 0 and 1.

Based on the fact that removal of any element will increase the value of mean compliance ($\frac{\partial c}{\partial x_e} < 0$), it is obvious that the problem stated in (2) does not always have a solution. Solutions can only exist if \bar{c} is not smaller than the compliance of the full structure. Also, it is obvious that the optimal solution requires the compliance to attain its maximum feasible value because otherwise it is possible to obtain a better solution by removing more elements.

2.1 Sensitivity numbers in ESO

The ESO procedure approaches the optimal point from one side. In order to find a solution for problem (2), in each step ESO needs to remove the element (or elements) which results in the smallest increment to c . The procedure should be stopped when the condition becomes active (i.e. $c = \bar{c}$) or when even the smallest increment to c caused by removal of the least efficient element violates the condition.

The sensitivity number defined for each element in this problem should thus reflect the effect of removing that element on the mean compliance of the system. Considering a

single material-void design, the sensitivity number used by ESO for this problem is defined as

$$\alpha_e = \mathbf{u} \mathbf{K}_e^{(1)} \mathbf{u} \tag{4}$$

This sensitivity number is consistent with the results of first order sensitivity analysis. Using Taylor’s series, the change in c due to a change in the value of x_e can be expressed as

$$\begin{aligned} \Delta c = & \frac{\partial c}{\partial x_e} \Delta x_e + \frac{1}{2!} \frac{\partial^2 c}{\partial x_e^2} \Delta x_e^2 \\ & + \dots + \frac{1}{m!} \frac{\partial^m c}{\partial x_e^m} \Delta x_e^m + \dots \end{aligned} \tag{5}$$

In ESO, like nearly all other gradient-based topology optimisation methods, only the first term in the above series is considered and it is assumed that

$$\Delta c = \frac{\partial c}{\partial x_e} \Delta x_e \tag{6}$$

Because the elements can only be totally removed in ESO, we always have $\Delta x_e = 0 - 1 = -1$. Hence, we have

$$\Delta c = -\frac{\partial c}{\partial x_e} = \mathbf{u} \mathbf{K}_e^{(1)} \mathbf{u} = \alpha_e$$

The problem in the above sensitivity analysis is the assumption made in (6). Looking at (5), for $|\Delta x_e| \ll 1$, the higher powers of Δx_e can be neglected and the change in c is approximately equivalent to the first term. In this case, assumption (6) is valid. In the ESO method, however, $\Delta x_e = -1$ and the higher powers of Δx_e can not be neglected. In fact, in some cases, when $\Delta x_e \rightarrow \pm 1$ the contribution of higher order terms becomes quite considerable. This effect is demonstrated in Section 3.1.

2.2 Accurate sensitivity numbers

As discussed above, from (5), it is clear that when $\Delta x_e \rightarrow \pm 1$, for obtaining an accurate estimate of Δc , one has to consider the higher order terms.

In order to calculate $\frac{\partial^m c}{\partial x_e^m}$, we differentiate (3) several times. Noting that the force vector does not depend on design variables, we have

$$\frac{\partial^m c}{\partial x_e^m} = \mathbf{f}^T \frac{\partial^m \mathbf{u}}{\partial x_e^m} \tag{7}$$

To calculate $\frac{\partial^m \mathbf{u}}{\partial x_e^m}$, we need to use the equilibrium equation. In linear systems, this equation can be expressed as

$$\mathbf{K} \mathbf{u} = \mathbf{f} \tag{8}$$

where \mathbf{K} is the global stiffness matrix of the system which can be obtained by assembling the elements’ stiffness matrices in the following form:

$$\mathbf{K} = \sum_{e=1}^N \mathbf{K}_e(x_e) \tag{9}$$

Rearranging the equilibrium equation and differentiating it several times, noting that the force vector is not dependent on design variables, we obtain¹

$$\frac{\partial^m \mathbf{u}}{\partial x_e^m} = \frac{\partial^m \mathbf{K}^{-1}}{\partial x_e^m} \mathbf{f} \tag{10}$$

On the other hand, using the definition $\mathbf{K} \mathbf{K}^{-1} = \mathbf{I}$ and differentiating, we have

$$\frac{\partial \mathbf{K}^{-1}}{\partial x_e} = -\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial x_e} \mathbf{K}^{-1} \tag{11}$$

The term $\frac{\partial \mathbf{K}}{\partial x_e}$ is calculable from (1). Noting that a linear relationship is used in (1), $\frac{\partial^m \mathbf{K}}{\partial x_e^m} = 0$ for any $m > 1$. Now by further differentiation of (11) we can obtain the following general expression

$$\frac{\partial^m \mathbf{K}^{-1}}{\partial x_e^m} = m! \left(-\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial x_e} \right)^m \mathbf{K}^{-1} \tag{12}$$

Substituting from (12) into (10) and then into (7), we can write

$$\frac{\partial^m c}{\partial x_e^m} = m! \mathbf{f}^T \left(-\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial x_e} \right)^m \mathbf{K}^{-1} \mathbf{f} \tag{13}$$

which can be written in terms of displacement as

$$\frac{\partial^m c}{\partial x_e^m} = (-1)^m m! \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial x_e} \left(\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial x_e} \right)^{m-1} \mathbf{u} \tag{14}$$

Noting that $\Delta x_e = -1$, the m -th term in the Taylor series (5), is

$$\delta_e^m c = \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial x_e} \left(\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial x_e} \right)^{m-1} \mathbf{u}, \quad m \in \mathbb{N} \tag{15}$$

Noting the positive-definiteness of $\frac{\partial \mathbf{K}}{\partial x_e}$ and \mathbf{K}^{-1} , it is clear that for any m the above term is positive. This means that in ESO, using the first order sensitivity number always results in an underestimation of the actual change in c . Moreover, it is also clear that the variation of c with respect to any single design variable is completely monotonic and thus the Taylor’s series is converging. In mathematical terms we have²

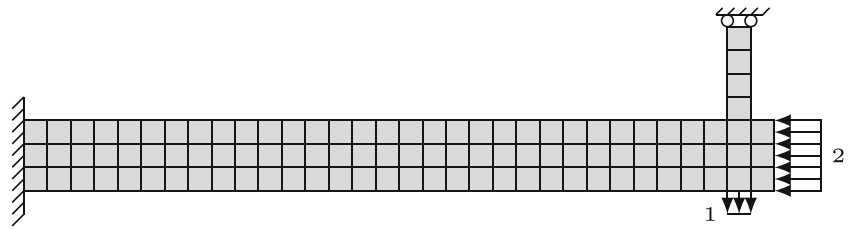
$$\delta_e^m c > \delta_e^{m+n} c > 0, \quad \forall e \in \{1, \dots, N\}, m, n \in \mathbb{N} \tag{16}$$

The convergence of the series in (5) will be also demonstrated numerically in Section 3.1. Due to convergence of the Taylor’s series, it is clear that by using enough terms in the Taylor’s series, one can approach the most accurate sensitivity numbers with any arbitrary tolerance.

¹It should be noted that \mathbf{K} is only invertible when the components corresponding to the restrained degrees of freedom are eliminated from it. Hereafter, by \mathbf{K} we are actually referring to this reduced matrix.

²In general we have $\delta^m c \geq \delta^{m+n} c$, but the equal sign is only applicable when the whole structure is removed resulting in $\Delta c \rightarrow \infty$.

Fig. 1 The Zhou-Rozvany problem



3 The Zhou-Rozvany problem

The Zhou-Rozvany problem is shown in Fig. 1. As noted by Zhou and Rozvany (2001), when applied on this problem, both stress-based and compliance-based ESO approaches will eliminate the top-most element resulting in a non-optimal solution (Fig. 2). Because ESO cannot reintroduce the removed elements, obviously the problem cannot be fixed by further iterations.

Zhou and Rozvany (2001) correctly mentioned that in ESO, such a failure can occur “if the sensitivity for the rejected element increases significantly as its normalized density (t_i) [(here x_i)] varies from 1 to zero”. It is also shown in the previous section that in ESO the first order sensitivity analysis may not be sufficiently accurate. In the next section, we study the effects of considering higher order terms in sensitivity analysis of some of the elements in the Zhou-Rozvany problem.

3.1 Sensitivity numbers of some of the elements in the Zhou-Rozvany problem

To observe the effects of higher order terms in (5), in this section we study the variation of the mean compliance of the Zhou-Rozvany system due to gradual changes in design variable of some elements. Three elements are considered individually and their design variable is gradually reduced from 1.

For each value of the design variable the actual value of the mean compliance is calculated using finite element analysis. For each value, we also estimated the value of c using the Taylor’s series (5) with different number of terms. Using M terms, for example, we have the following approximation

$$c = c_0 + \sum_{m=1}^M \delta_e^m c$$

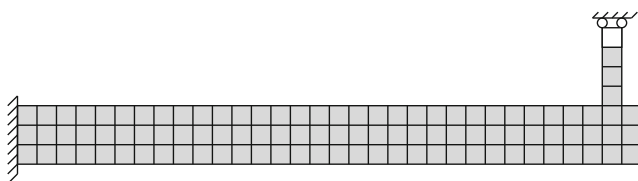


Fig. 2 The solution obtained by ESO for the Zhou-Rozvany problem after removing 1 element

where c_0 is the mean compliance of the full design where $x_e = 1, \forall e \in \{1, \dots, N\}$.

The results are shown in Fig. 3. The 100th element is the top most element which is removed by ESO in its first iteration. It can be seen in Fig. 3 that for this element the value of c shows a significantly steep increase when the design variable approaches zero. Only after considering 3000 terms in the Taylor’s series, the relative error between the predicted and actual values of $c|_{x_{100}=0.001}$ becomes negligible. For $x_{100} < 0.001$, even more terms need to be considered.

The values of the first 10 terms of the Taylor’s series (5) for the three considered elements are reported in Table 1. Considering the first term only, the 100th element shows the lowest value (and hence is considered the least efficient by ESO if only first order sensitivity numbers are considered). From this table, and also from Fig. 3, it is obvious that even second-order sensitivities (as suggested by Rozvany 2009) would not rectify ESO’s problem. It is only after considering 7 terms, that it is revealed that the 95th element is in fact the correct choice for removal.

Despite the fact that several thousand terms are needed to correctly calculate $\frac{\Delta c}{100}$ from (5), it should be noted that no further calculations is required once the Taylor’s series for at least one of the elements is converged. For example, considering a relative error tolerance of 1% for Taylor’s series convergence, by calculating 10 terms of the series one already ensures that element 95 is the suitable candidate for removal.

Solutions obtained by using ESO with an accurate sensitivity numbers on the Zhou-Rozvany problem are reported in Section 5.1. Before presenting those results, however, in the next section some comments are made on some of the other treatments suggested for this problem.

4 Complementary comments on treatments suggested for ESO failure in the Zhou-Rozvany problem

Rozvany (2009) reviewed some of the most important suggested treatments for ESO failure in the Zhou-Rozvany problem. Apart from using soft-kill BESO, which is actually solving the problem using a different optimisation method, the other key suggested solutions involve using accurate sensitivity numbers (as was followed herein) and mesh

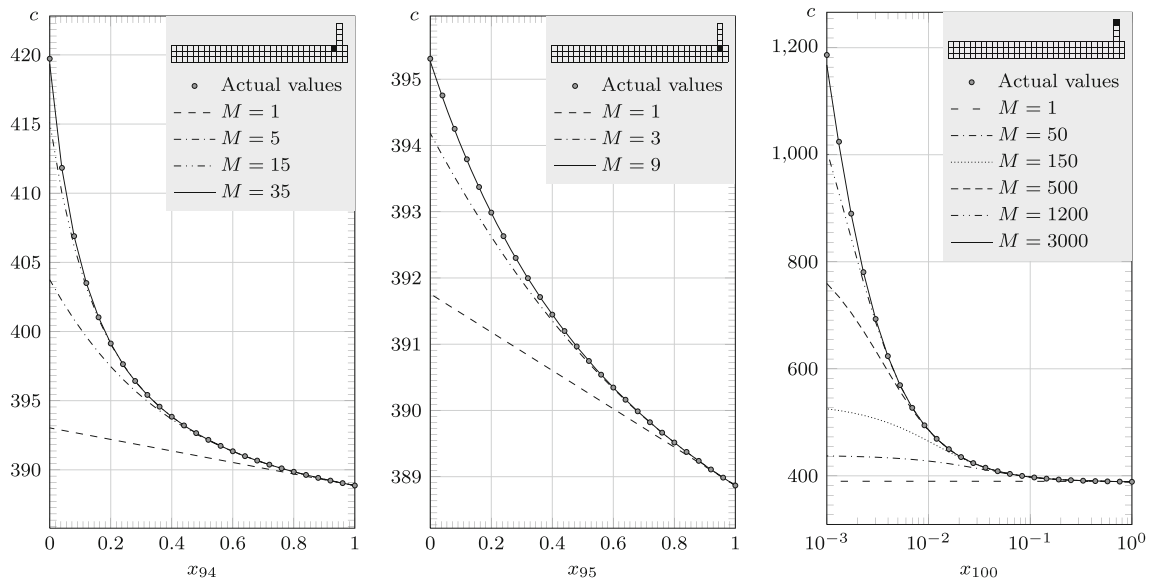


Fig. 3 Variation of the actual and predicted values of c using different number of terms (M) in (5) with respect to changes in x_e for $e \in \{94, 95, 100\}$ in the Zhou-Rozvany problem. The considered elements

are highlighted at the top of the legends. For $e = 100$, due to extremely steep changes, the variations are only plotted for $x_{100} \in [0.001, 1]$ and with logarithmic scale

refinement. Although Rozvany covered many aspects of the suggested treatments and discussions on ESO, this author identifies some important points which need to be addressed about these communications.

4.1 ESO breakdown in statically determinate problems

The treatments suggested by Huang and Xie (2008) are based on monitoring the boundary conditions to detect breaking of supports. In their paper, Huang and Xie (2008) stated that “failure of ESO may occur when a prescribed boundary support is broken for a statically indeterminate structure. When a boundary support is broken, the structural

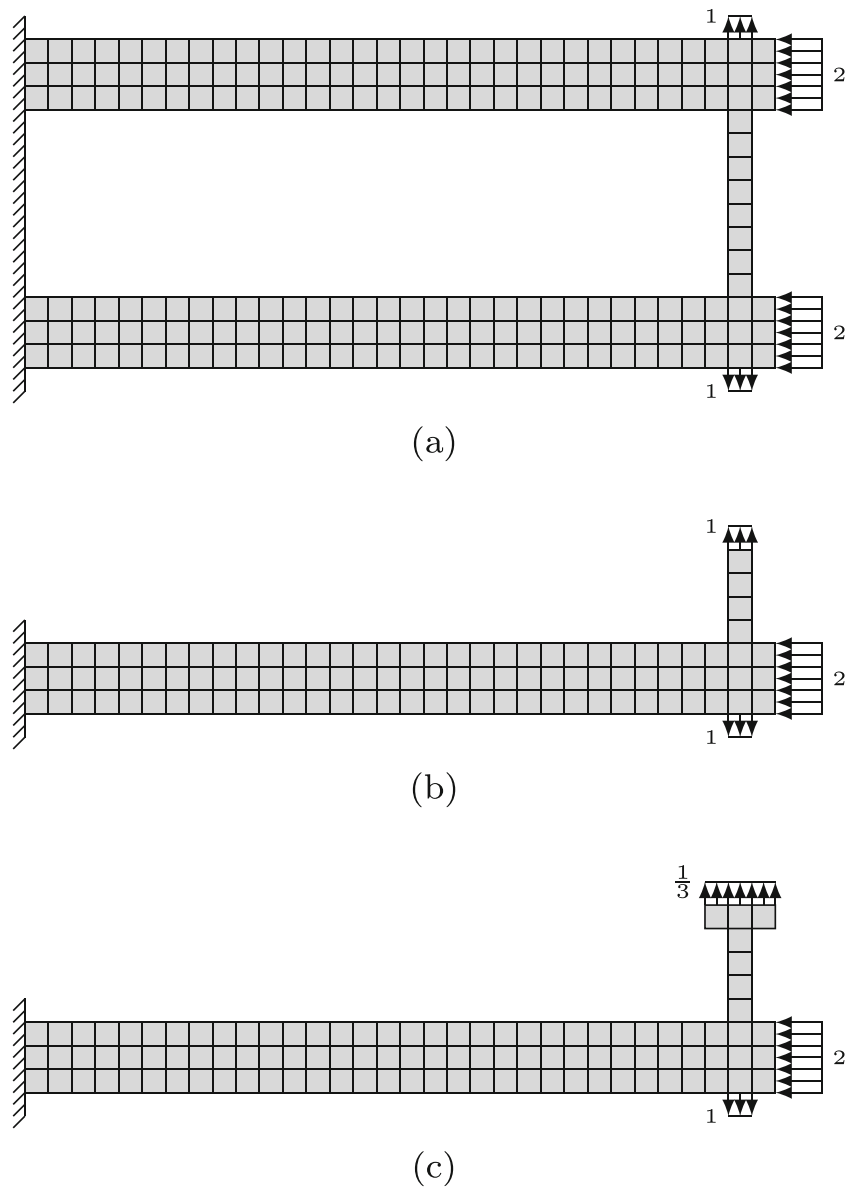
system could be completely changed from the one originally defined in the initial design and even BESO would not be able to rectify the nonoptimal design. To avoid this problem, it is imperative that the prescribed boundary conditions for the structure be checked and maintained at each iteration during the optimization process.” Presumably based on the nature of the Zhou-Rozvany problem, it seems to be believed that the ESO breakdown can only happen in similar statically indeterminate problems. Here we show that ESO failure can also happen in statically determinate problems.

The finite element model of the Zhou-Rozvany problem can be considered to be a half domain of a symmetric statically determinate problem shown in Fig. 4a. Obviously the

Table 1 The values of first 10 terms of the series (5) for elements 94, 95, and 100

M	$e = 94$		$e = 95$		$e = 100$	
	$\delta_{94}^M c$	$\sum_{m=1}^M \delta_{94}^m c$	$\delta_{95}^M c$	$\sum_{m=1}^M \delta_{95}^m c$	$\delta_{100}^M c$	$\sum_{m=1}^M \delta_{100}^m c$
1	4.165	4.165	2.894	2.894	0.996	0.996
2	3.329	7.494	1.560	4.453	0.995	1.992
3	2.815	10.309	0.864	5.317	0.994	2.985
4	2.437	12.746	0.485	5.803	0.993	3.978
5	2.131	14.877	0.275	6.078	0.991	4.969
6	1.873	16.750	0.156	6.234	0.990	5.960
7	1.649	18.399	0.089	6.323	0.989	6.948
8	1.454	19.853	0.050	6.374	0.988	7.936
9	1.283	21.137	0.029	6.403	0.986	8.923
10	1.133	22.270	0.017	6.420	0.985	9.908

Fig. 4 Some statically determinate problems in which ESO fails



same problem will occur if ESO is tried on this problem. Figure 4b, c illustrate other statically determinate problems in which ESO will fail.

Clearly the technique suggested by Huang and Xie (2008) cannot detect the failure of ESO in these statically determinate problems.

4.2 Detecting ESO failure by comparing predicted change to actual change

Restating the conclusions from Rozvany and Querin (2002), Rozvany (2009) argues that “ESO would give a correct ‘iteration-wise optimal element change’, if for all rejected elements of that iteration the relevant sensitivities did not change significantly as their thickness varies from unity to zero.” Similar point was also mentioned by Zhou and

Rozvany (2001). At least for compliance-based ESO, the correctness of this statement is apparent from (5) and (16). If, after removing an element, the actual change in the mean compliance is not significantly different from the first order sensitivity number used by ESO, it means that the higher order terms in (5) were in fact negligible.

Based on this, Rozvany suggests that this difference is “checked in each iteration by comparing the sensitivity value with the actual change caused by a unit change in the density of the rejected elements. If the difference is large, the corresponding elements could be stopped from being eliminated.” Two points need to be mentioned here about this proposal:

1. Although this approach is overall reasonable, it is arguable that how can one ensure that the difference

between the sensitivities and actual change in the objective function is *significant enough* to consider the elimination as erroneous? Is there any guarantee that a certain threshold value which works for one specific problem can work well on other problems as well?

- Moreover one can think of some problems in which this approach fails to work (or prevents ESO to proceed). Consider, for example, the problem depicted in Fig. 5b. In this problem elimination of any element will result in a significant variation between the predicted and actual values of the objective function. Nevertheless, one can obtain a design with smaller volume if a sufficiently large upper limit for compliance is adopted.

We will come across this class of problems again in Section 5.1. In these problems, removing any element will change the connectivity of the system. We denote this class of problems by \mathcal{S} . The \mathcal{S} problems can be divided further into two subclasses. In some \mathcal{S} problems, removal of any element results in instability of the system. In other words the mean compliance of the system approaches infinity by removing any element from it. We denote this subclass by \mathcal{S}_0 . The other subclass, denoted hereafter by \mathcal{S}_1 , includes all \mathcal{S} designs which do not belong to \mathcal{S}_0 . Figure 5 illustrates examples of both subclasses.

It is clear that once ESO reaches a \mathcal{S}_0 , no further solutions can be obtained. However, if a \mathcal{S}_1 design is reached, depending on the condition of the problem, it is possible to obtain further solutions by removing elements until reaching a \mathcal{S}_0 . The problem with the approach suggested by Rozvany and Querin (2002) and Rozvany (2009) is that if the initial problem is a \mathcal{S}_1 design, no solution can be found using this approach.

4.3 Non-optimal or local optimal solution?

In an interesting observation, Huang and Xie (2010b) noted that the solution obtained by ESO for the Zhou-Rozvany problem after eliminating four elements (Fig. 6) is a “highly inefficient local optimum” rather than a non-optimal solution.

To prove their statement, Huang and Xie (2010b) used the Solid Isotropic Microstructure with Penalisation (SIMP)

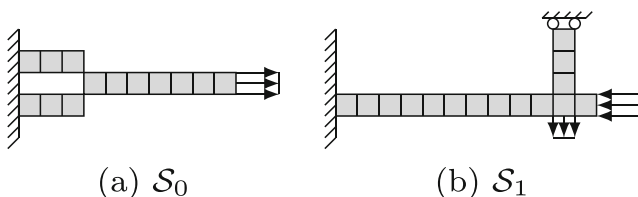


Fig. 5 Examples of problems in which the first order sensitivity of compliance in all the elements are significantly smaller than the actual change caused by their removal (\mathcal{S} problems)

method (Bendsøe 1989; Rozvany and Zhou 1991; Rozvany et al. 1992) to solve the Zhou-Rozvany problem starting from the initial design shown in Fig. 6 “with $x_i = 1$ for all elements in the horizontal beam and $x_i = x_{min} = 0.001$ for the four elements in the vertical tie”. They report that with a penalty factor of $p \geq 3.1$ the SIMP method cannot improve this initial design any further and then concluded that because “the SIMP method with continuous design variables guarantees that its solution should be at least a local optimum” this design is a local minimum. There are a number of points which need to be mentioned about this approach and conclusion:

- Although the design shown in Fig. 6 is in fact a local minimum (as will be demonstrated soon), the approach used by the authors to prove this is arguable. The results obtained by the SIMP method depend on its algorithmic parameters; most importantly the penalty factor (p), the minimum allowable value of design variables (x_{min}), the move limit (m), and the regulating power (η) (here we adopted the notation used by Sigmund 2001). By playing with these parameters, one can force the SIMP method to lock on many clearly non-optimal initial designs.
- More importantly, the problems solved by SIMP and ESO are different from each other. Even if a particular design is a local minimum in SIMP, one cannot readily conclude that it is a local minimum for ESO.
- As will be soon demonstrated, some of the solutions obtained by the ESO method cannot be considered as local minimum. Thus the conclusion made by Huang and Xie (2010b) cannot be generalised.

In the following we elaborate more on points 2 and 3 above.

4.3.1 Local minima for problems with continuous variables

A compliance minimisation problem with continuous variables can be expressed as follows.

$$\begin{aligned}
 & \min_{x_1, \dots, x_N} && c \\
 & \text{such that} && V \leq \bar{V} \\
 & \text{and} && x_e \in [0, 1], \quad e = 1, \dots, N
 \end{aligned} \tag{17}$$

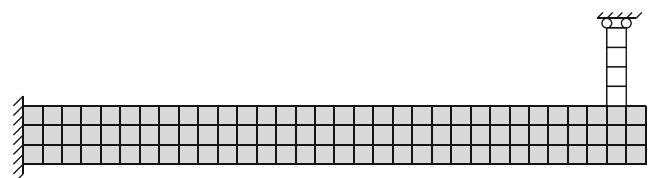


Fig. 6 The solution obtained by ESO for the Zhou-Rozvany problem after removing 4 elements

where \bar{V} is a predefined upper limit on the volume of the structure. The typical problem solved by the SIMP method is a penalised version of the above form with a typical power-law interpolation scheme in the form

$$E_e(x_e) = x_e^p E, \quad p > 1$$

where E_e is the Young's modulus of element e , and E is the Young's modulus of a base material.

A feasible neighbourhood with radius $\epsilon > 0$ about a feasible point like \mathbf{x} for problem (17) takes the following simple form.

$$\mathcal{N}_\epsilon(\mathbf{x}) = \left\{ \mathbf{x} + \Delta\mathbf{x} \in [0, 1]^N \mid \|\Delta\mathbf{x}\| < \epsilon, \sum_{e=1}^N \Delta x_e = 0 \right\} \quad (18)$$

This set defines an open hyper-disk³ formed by intersection of a hyper-ball with radius ϵ and a hyper-plane in an N -dimensional space. A point $\bar{\mathbf{x}}$ (with $c(\bar{\mathbf{x}}) = \bar{c}$ and $V(\bar{\mathbf{x}}) = \bar{V}$) is a local minimum of problem (17) if

$$\exists \epsilon > 0 : (\forall \mathbf{x} \in \mathcal{N}_\epsilon(\bar{\mathbf{x}}) : c(\mathbf{x}) \geq \bar{c}) \quad (19)$$

4.3.2 Local minima for ESO

The big difference between problems (17) and (2) is in the last condition which changes from a continuous boxing condition in SIMP to a binary condition in ESO. The feasible domain of ESO problem is not continuous. Thus the concept of neighbourhood needs to be considered carefully.

Consider the following problem which is the binary form of problem (17).

$$\begin{aligned} \min_{x_1, \dots, x_N} \quad & c \\ \text{such that} \quad & V \leq \bar{V} \\ \text{and} \quad & x_e \in \{0, 1\}, \quad e = 1, \dots, N \end{aligned} \quad (20)$$

A feasible neighbourhood about a feasible point \mathbf{x} for this problem can be defined based on (18) as

$$\mathcal{M}_\epsilon(\mathbf{x}) = \mathcal{N}_\epsilon(\mathbf{x}) \cap \{0, 1\}^N \quad (21)$$

which only contains a finite number of points of the aforementioned hyper-disk.

Because of its discrete nature, trivially the smallest neighbourhood around each point contains only that point ($0 < \epsilon < \sqrt{2}$). Obviously we cannot accept this neighbourhood when assessing whether a point is a local minimum or not. Neglecting this trivial case, the next smallest neighbourhood around each point is obtained when $\sqrt{2} < \epsilon < 2$, i.e.

when only two components of $\Delta\mathbf{x}$ are non-zero⁴. For simplicity we show this neighbourhood by $\mathcal{M}(\bar{\mathbf{x}})$. Noting that this is the smallest non-trivial feasible neighbourhood, we can call a point $\bar{\mathbf{x}}$ a local minimum for problem 20 if

$$\forall \mathbf{x} \in \mathcal{M}(\bar{\mathbf{x}}) : c(\mathbf{x}) \geq \bar{c} \quad (22)$$

It can be easily shown that any local minimum point $\bar{\mathbf{x}}$ of problem 20 is also a local minimum point of problem 2. Otherwise, problem 2 has a different local minimum \mathbf{x}^* in the smallest non-trivial feasible neighbourhood with $V^* < \bar{V}$ and $c^* \leq \bar{c}$. Now if we add an element to \mathbf{x}^* to increase its volume (up to \bar{V}), its compliance will decrease and we obtain a solution for problem 20 with $V^* < V \leq \bar{V}$ and $c < c^* \leq \bar{c}$ in $\mathcal{M}(\bar{\mathbf{x}})$ which contradicts our assumption that $\bar{\mathbf{x}}$ is a local minimum of problem 20.

Now we can argue that a point $\bar{\mathbf{x}}$ is a local minimum for ESO if (22) holds. In simple words, an ESO solution is a local minimum if it yields the minimum value of c among all designs obtained by switching one solid element to void and one void element to solid.

It is now clear that the solution depicted in Fig. 6 is in fact a local minimum for ESO. Because removing any of the beam elements, increases the mean compliance to the extent that turning none of the tie elements into solid can sufficiently decrease it down to its initial value.

It should be noted however, that not all the solutions obtained by ESO are local minimum. For example, it is clear that the solution obtained after removal of the first element from the Zhou-Rozvany problem (Fig. 2) is not a local minimum and is in fact a non-optimal solution.

Based on this discussion, we can also readily conclude that the solutions obtained by ESO through using accurate sensitivity analysis are always locally minimum. Although, as we will demonstrate in the next section, they may be highly inefficient.

5 Another shortcoming of ESO

5.1 Solution to the Zhou-Rozvany problem using accurate sensitivity numbers

The results obtained for the Zhou-Rozvany problem after applying the ESO method with high-order sensitivity analysis are illustrated in Fig. 7.

The result obtained after removing 1 element expectedly matches the global optimum for $V = 99$ (or for $c = 395.31$) as reported by Stolpe and Bendsoe (2011). After that ESO results slightly deviate from the global optima. Again, after

³A hyper-ball in $(N - 1)$ -dimensional space

⁴Due to the binary nature of design variables, the only feasible non-zero values in $\Delta\mathbf{x}$ are ± 1 . Keeping the volume constant requires the sum of the components in $\Delta\mathbf{x}$ to vanish. Thus the smallest feasible positive value of $\|\Delta\mathbf{x}\|$ is $\sqrt{2}$.

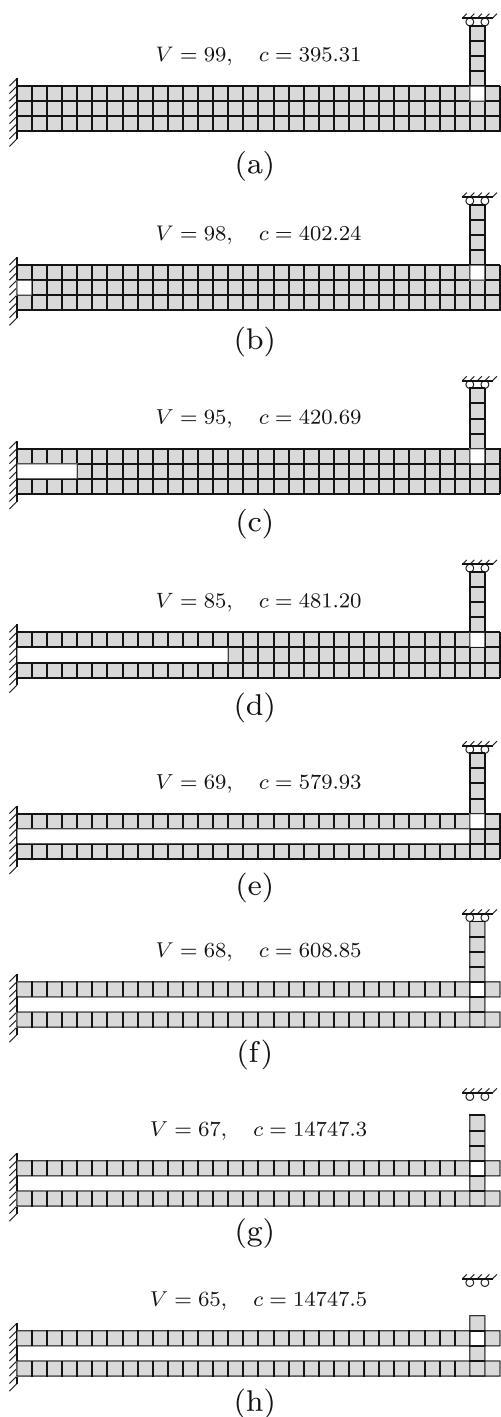


Fig. 7 Solutions to the Zhou-Rozvany problem using accurate sensitivity numbers with different compliance limits \bar{c}

removing 31 elements (Fig. 7e) the ESO method reaches a global optimum at $V = 69, c = 579.93$. The next point at $V = 68, c = 608.85$ (Fig. 7f) also matches a global optimum. This solution is clearly a \mathcal{S}_1 design. By further removal of elements, ESO jumps to a \mathcal{S}_0 design at $V = 65, c = 14747.5$. Beyond this point, any further element removal will result in an unstable system.

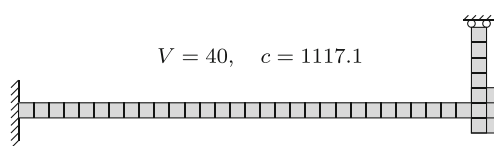


Fig. 8 An intuitively suggested solution by Zhou and Rozvany (2001)

The designs obtained after the \mathcal{S}_1 design (Fig. 7g–h) are highly inefficient. In fact the intuitively suggested design by Zhou and Rozvany (2001) depicted in Fig. 8 provides a much better result of $V = 40$ at $c = 1117^5$.

Noting that accurate sensitivity numbers have been used to obtain these results, it is obvious that this time ESO’s problem is not due to using incorrect or inaccurate sensitivity numbers.

5.2 The reason behind this shortcoming of ESO

As seen in the previous section, ESO method can lead to highly inefficient (locally optimum) solutions even if accurate sensitivity numbers are used. Perhaps this shortcoming of ESO is even more serious than the problem caused by using inaccurate sensitivity numbers.

This shortcoming is due to the fact that ESO is restricted to move in only one direction. In fact, ESO modifies the problem as it proceeds. So, for example, the solution obtained by removing 32 elements (Fig. 7f) is actually a solution to the problem with the initial design depicted in Fig. 7e, and likewise Fig. 7g is a solution to Fig. 7f. Using accurate sensitivity analysis, one can ensure that the solution obtained in iteration $i + 1$ is the optimal solution to the problem with the initial design equivalent to the solution obtained in iteration i . But this is not enough to ensure that the solution obtained is an optimal solution to the initial problem. Although Fig. 7g is the optimal solution to a problem with the design domain depicted in Fig. 7f, it is not an efficient solution to the Zhou-Rozvany problem.

Even for statically determinate problems, ESO’s solution may be far from optimum. Consider for example, the statically determinate problem shown in Fig. 4b. After removing 32 elements, for this problem ESO reaches the design illustrated in Fig. 9. As this is a \mathcal{S}_0 design, no further elements can be removed by ESO. This is a good solution for $\bar{c} = 616$ (yielding the volume of 68 elements) but it is a highly inefficient solution, for example, for $\bar{c} = 2000$. For this value of \bar{c} , an intuitive design similar to the one depicted in Fig. 8 (without the top rollers) would yield a considerably lower volume of $V = 40$.

⁵The difference between this number and the value of $c = 1121$ reported for this design by Zhou and Rozvany (2001) is due to using analytically integrated stiffness matrices for finite elements in this paper. Also see Stolpe and Bendsøe (2011).

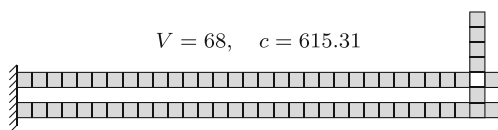


Fig. 9 The solution obtained by ESO for the problem depicted in Fig. 4b for all $\bar{c} \geq 615.31$

5.3 Unreliable behaviours of ESO

Noting that ESO changes the problem as it proceeds, it can be concluded that the solutions obtained by ESO are only (locally) optimal in one branch of possible solutions. It can then be expected that the overall behaviour of ESO is extremely problem-dependent. In the following we mention two interesting observations.

5.3.1 A more restricted version of the problem may lead to better results

It is possible to obtain a better solution for the same problem if we force ESO to stick to another branch of solutions. For example, if we start with the initial design depicted in Fig. 10a (which is a subset of the Zhou-Rozvany problem), we can obtain the result shown in Fig. 10b with $V = 54, c = 931.81$ which is obviously better than Fig. 7g and h. This simple example shows that allowing more elements in the initial design domain does not necessarily mean a better solution can be obtained by ESO.

5.3.2 Using accurate sensitivity analysis might lead to worse results

Due to the above observation, it is expected that at some point, an inaccurate first-order sensitivity analysis leads to a better solution compared to more accurate higher-order sensitivity analyses. In other words there is no guarantee that a more accurate approach yields a better result.

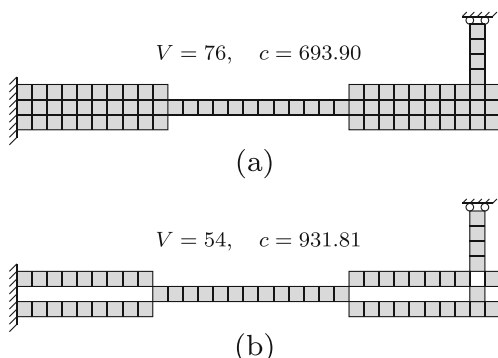


Fig. 10 A modified version of the Zhou-Rozvany problem: **a** initial design, and **b** a solution found by ESO using accurate sensitivity numbers

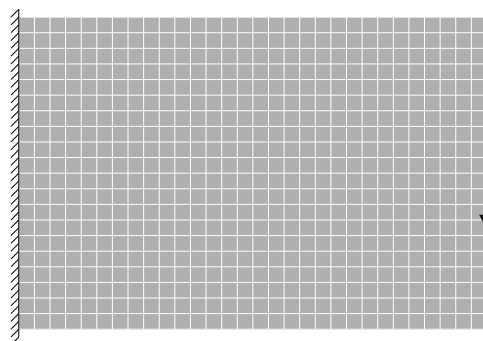


Fig. 11 A short cantilever beam problem

To illustrate this a short cantilever beam is considered which is discretised into a finite element mesh of 30×20 square 9-node elements (Fig. 11). Some of the solutions obtained by ESO using first order sensitivity numbers and accurate sensitivity numbers are shown in Fig. 12.

When the condition $c < \bar{c} = 40$ is imposed, the results obtained using accurate sensitivity numbers are better than the ones obtained using first order sensitivity numbers. But the first order sensitivity numbers yield better results when the condition is changed to $c < \bar{c} = 140$.

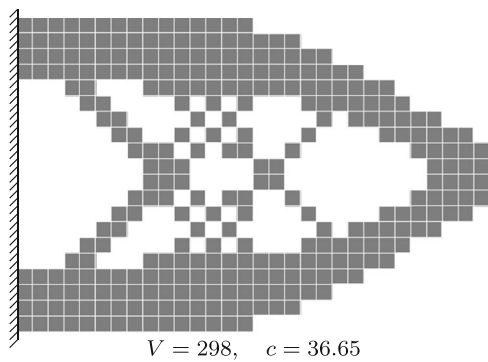
The graph in Fig. 13 shows the relationship between volume (V) and compliance (c) for the solutions obtained using the two sets of sensitivity numbers. It can be seen that for almost all values of $\bar{c} > 43$ the first order sensitivity numbers lead to better results. As explained before, such a behaviour can be expected because ESO changes the problem as it proceeds. In the earlier stages, the problem is not modified very much so the higher order sensitivity numbers work better. But there is no guarantee that staying in the branch followed by the accurate sensitivity numbers always lead to better results.

Based on these unreliable behaviours, even when accurate sensitivity numbers are employed, perhaps one can argue that ESO method should be generally avoided.

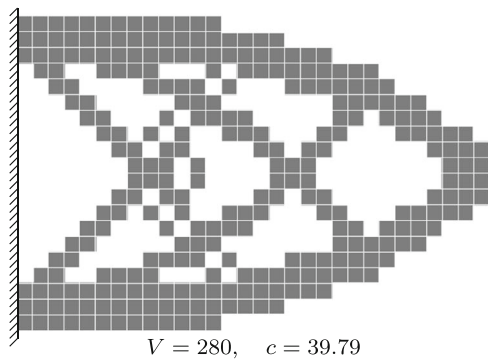
6 What about BESO?

It should be noted here that BESO is essentially different from its predecessor. By allowing elements to be added as well, the initial problem is not modified at least for soft-kill BESO.⁶ It is thus possible for BESO to move across

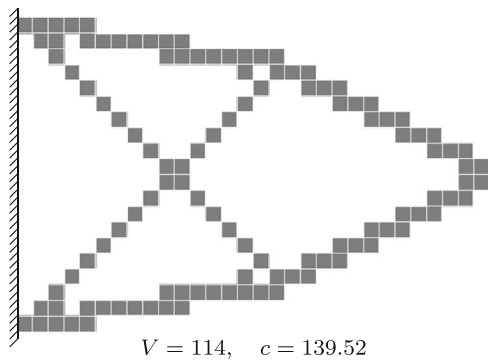
⁶In contrast to what seems to be generally believed, it can be shown that the same can be true even for hard-kill BESO. This is however beyond the scope of this work and the author wishes to address this matter in a separate communication. For the current discussion, it is enough to accept that soft-kill BESO will not suffer from this shortcoming of ESO.



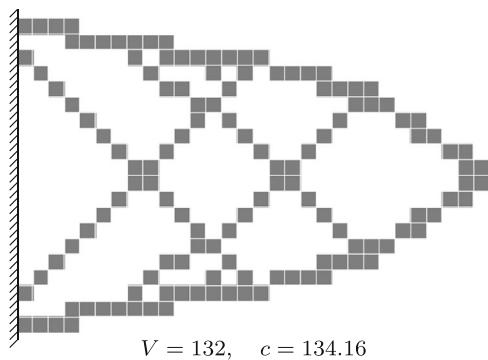
(a) First order sensitivities with $c < 40$.



(b) High order sensitivities with $c < 40$.



(c) First order sensitivities with $c < 140$.



(d) High order sensitivities with $c < 140$.

Fig. 12 ESO solutions obtained for the short cantilever beam problem for different values of \bar{c} with first and higher order sensitivity numbers

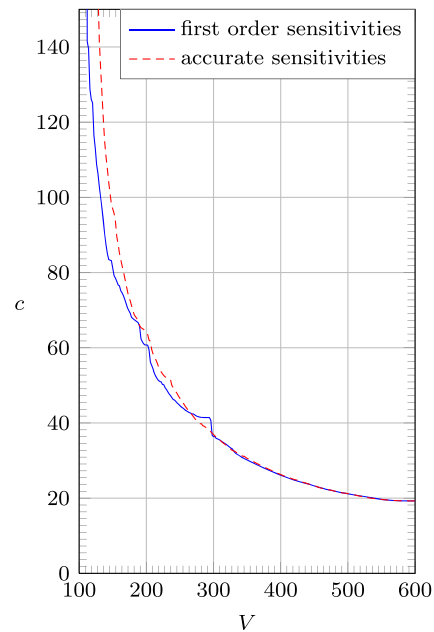


Fig. 13 A comparison between ESO results obtained using first and higher order sensitivity numbers

different branches of solutions and the method is not prone to this shortcoming of ESO.

The other important difference between the two methods is the range of problems that can be solved with them. Due to its nature, ESO can only minimise the volume (weight) of a structure but BESO (like SIMP) can be formulated to minimise a wide range of objective functions.

Based on this discussion, this author suggests that ESO and BESO methods are treated as completely separate and distinct methods despite their historical relationship.

6.1 Is there any problem for which using ESO is preferred?

Generally for all problems using a bidirectional method such as BESO (or SIMP) is preferable. Nevertheless, there are some specific types of problems where the limitations of the problem justify a unidirectional approach. A good example is the problem of finding the next piece of ground to be removed in an excavation project. In this case, once one part of the domain is removed it cannot be physically reintroduced so the unidirectional approach of ESO fits well to this problem (Ghabraie et al. 2008).

7 Conclusion

In this paper the ESO method and its shortcomings are studied. A problem statement is proposed for ESO. An accurate sensitivity analysis is also proposed and accurate sensitivity numbers are calculated for compliance-based ESO. It

is shown that the proposed approach can solve the Zhou-Rozvany problem. It is then demonstrated that due to its unidirectional approach to optimal points, even when using accurate sensitivity numbers, the ESO method can lead to highly inefficient solutions. Based on the observations and discussions, it is concluded that

- ESO should only be used in problems where the limitations justify a unidirectional approach to the solution.
- A distinction should be made between BESO and ESO, and these two methods should be considered separately.

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