RESEARCH PAPER

An important boundary sampling method for reliability-based design optimization using kriging model

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Abstract Reliability-based design optimization (RBDO) combined with metamodel is a powerful tool to deal with variation of system output induced by uncertainties during practical engineering design. In this paper, the importance boundary sampling (IBS) method is proposed to enhance the efficiency of Kriging-model-based RBDO. Rather than fitting all the parts of the limit state constraints precisely within the design region, the proposed IBS mainly selects sample points on the critical parts of the limit state constraints. Two importance coefficients are proposed to identify these critical boundary parts: the first importance coefficient is determined by the objective function value; and the second one is calculated using the joint probability density value of the design variables. The sampling and optimization processes are conducted alternately to select the sample points more rationally. The computation capability of the proposed method is demonstrated using several mathematical RBDO problems and a box girder design application. The comparison results show that the proposed IBS method is very efficient.

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1 Introduction

Uncertainties in practical engineering design are usually inevitable. RBDO considers uncertainties stemming from various sources (e.g., geometrical sizes (Du and Chen 2004), material properties (Chen et al. 2014), operational uncertainties (Sues et al. 2001), environmental uncertainties (Chan et al. 2010), numerical uncertainties (Du et al. 2005), etc.). And it can yield a reliable design (Aoues and Chateauneuf 2008; Hack et al. 2009; Kang et al. 2012); therefore it has received extensive attention in the past several decades.

The most challenging issue in RBDO is the unbearable computational cost while evaluating structure reliabilities. Probabilistic constraints are usually implicit and need to be evaluated through reliability analysis, in which simulation methods and analytical methods can be used. Simulation methods are accurate, such as Monte Carlo simulation (MCS) (Kuczera et al. 2010; Lee et al. 2008a; Lee et al. 2011; Papadrakakis and Lagaros 2002; Valdebenito and Schuëller 2011). However, this kind of method is computationally unaffordable, especially when the probability of failure is low. Analytical methods are commonly used due to their efficiency compared to simulation methods; They are generally gradient-based, such as the worst case analysis, the moment matching method (Du and Chen 2001) and the most probable point (MPP) based method (Hasofer and Lind 1974; Lee et al. 2008b). The worst case analysis method and the moment matching method are earlier reliability analysis methods which suffer from accuracy issue when random variables have large variations.

The strategy of searching the MPP is more efficient. Reliability index approach (RIA) (Enevoldsen and Sørensen 1994;

Gasser and Schuëller 1997; Grandhi and Wang 1998; Jiang et al. 2011; Nikolaidis and Burdisso 1988; Reddy et al. 1994; Tu et al. 1999; Weiji and Li 1994) and performance measure approach (PMA) (Youn et al. 2005a) are the most commonly used MPP-based analytical methods. Many other efficient reliability analysis methods have also been developed, including the advanced mean value method (AMV) (Youn et al. 2005b), the hybrid mean value method (HMV) (Youn et al. 2005b), the arc search method (Du et al. 2004) and the dimension reduction method (DRM) (Lee et al. 2012; Lee et al. 2008a).

The integration strategies of reliability analysis and optimization also have great influence on the accuracy and efficiency of RBDO (Valdebenito and Schuëller 2010). Double-loop method has a nested structure, which is not efficient; Single-loop method (Agarwal et al. 2007; Chen et al. 1997; Kharmanda et al. 2002; Kirjner-Neto et al. 1998; Kogiso et al. 2012; Li et al. 2013; Liang et al. 2008; Shan and Wang 2008) only has one loop through removing the reliability analysis by its Karush–Kuhn–Tucker (KKT) optimality conditions, it is very efficient for the linear and mild nonlinear problems, but is inaccurate for highly nonlinear problems; Decoupled-loop method (Chen et al. 2013a; Li et al. 2013, ; Cheng et al. 2006; Ching and Hsu 2008; Cho and Lee 2011; Du and Chen 2004; Royset et al. 2001; Yi et al. 2008; Chen et al. 2013b; Zou and Mahadevan 2006) performs optimization and reliability analysis sequentially, it has a good balance between the accuracy and efficiency in solving RBDO problems.

Performance functions in practical engineering design are usually implicit and very expensive to evaluate. Therefore, metamodel-based RBDO methods have been developed to reduce the computational cost. Youn and Choi (2004b) used the response surface method (RSM) for RBDO; (Kim et al. 2008) proposed an RSM method with prediction interval estimation; Zhao et al. 2009) (Zhao et al. 2009) used the RSM and sequential sampling for probabilistic design; Papadrakakis and Lagaros proposed RBDO method using neural network and MCS (Papadopoulos et al. 2012; Papadrakakis and Lagaros 2002); Choi, Youn et al. (2001) used the moving least square (MLS) method based on selective interaction sampling for RBDO; (Ching and Li 2008) proposed the reliability analysis method using artificial neural network (ANN) based genetic algorithms; Mourelatos (2005) adopted symmetric optimal Latin hypercube sampling and Kriging model for RBDO; Pretorius, Craig et al. (2004) used Kriging model as both local and global approximation for continuous casting design optimization; Ju and Lee (2008) proposed a RBDO method using the moment method and Kriging model; Basudhar and Missoum (2009) proposed an adaptive sampling technique for RBDO based on support vector machine (SVM);

Metamodels are fitted from space-filling sampling, whose efficiency and accuracy directly depend on how to select the sample points. Several advanced sampling strategies have been developed: Lee and Jung (2008) suggested a constraint boundary sampling (CBS) method and Kriging model for RBDO; (Bect et al. 2012) proposed the sequential design of computer experiments for the estimation of a probability of failure (Bect et al. 2012); (Lee et al. 2011) introduced a sampling-based RBDO method using the stochastic sensitivity analysis and dynamic Kriging method; (Bichon et al. 2008) proposed the efficient global reliability analysis for nonlinear implicit performance functions; Kim and Lee (2010) suggested an improvement of Kriging-based sequential approximate optimization method via extended use of design of experiments; (Echard et al. 2011) proposed an active learning reliability method combining Kriging and Monte Carlo simulation; (Picheny et al. 2010) presented the adaptive designs of experiments for accurate approximation of a target region; Huang and Chan (2010) developed a modified efficient global optimization algorithm for maximal reliability in a probabilistic constrained space; Zhuang and Pan (2012) introduced a new sequential sampling method for design under uncertainty.

Sampling strategies that locate sample points evenly within the whole design domain are not efficient, such as Latin hypercube sampling (LHS) and full factorial design which are usually applied in the initialization stage. Sequential sampling strategies are more efficient, such as the constraint boundary sampling (CBS) (Lee and Jung 2008) which selects sample points sequentially on the limit state constraint boundaries within the design domain. However, the target of an optimization problem is to find the minimum/maximum objective function value, while the optimal design satisfies the constraints. Therefore, there is no need to keep all the parts of the limit state constraints within the design region being accurate. (Chen et al. 2014) proposed the local adaptive sampling (LAS) method for RBDO using Kriging model. The LAS method uses the local sampling window strategy, which can improve the accuracy of the local design region. However, the LAS method may fall into the local optimum.

In this paper, the importance boundary sampling (IBS) approach will be proposed to enhance the efficiency of Kriging-model-based RBDO. However, the IBS method in this paper is different from classical importance sampling (IS) method, which is used to calculate the probability of failure P_f in the reliability analysis process. The classical IS method is widely applied to theoretical research and engineering applications to overcome the difficulties in the crude MC method. The basic idea is to carry out MC simulation with those sample points having a higher rate of falling in the failure region, because only these sample points contribute to the

evaluation of P_f (Glynn and Iglehart 1989; Siegmund 1976; Tang et al. 2012). The proposed IBS, by contrast, is used to improve the efficiency of the sampling process in the Kriging-model-based RBDO.

In the proposed IBS, the sampling and optimization processes will be conducted alternatively; and rather than fitting all the parts of the limit state constraints precisely within the design domain, IBS will firstly ensure the accuracy of the critical limit state constraint parts. Two importance coefficients are proposed to select sample points. The first one is calculated using the objective function value, and the second one is determined by the joint probability density value of the design variables. The optimization results from previous iteration are used in the subsequent sampling process to locate the sample points more rationally. The sequential optimization and reliability assessment (SORA) method (Du and Chen 2004) is applied in this paper to solve RBDO problems.

This paper assumes that the original performance functions are implicit, such as expensive physical test and complex finite element simulation. The searching of the MPP is based on the cheap Kriging models. When compared with the implicit performance function calls, the computational effort to find the MPP is very cheap.

The rest of the paper is organized as follows: the commonly used methods in RBDO and Kriging model will be reviewed in Section 2; the proposed IBS method will be explained in Section 3; Section 4 then uses illustrative examples to demonstrate the application of IBS with four comparison experiments; in Section 5, the conclusion will be drawn.

2 Commonly used methods in RBDO and kriging model

2.1 RBDO model

RBDO problem is typically formulated as follows:

$$find: \boldsymbol{d}, \boldsymbol{\mu_X}$$
$$\begin{aligned} min : \quad & f\left(\boldsymbol{d}, \boldsymbol{\mu_X}, \boldsymbol{\mu_P}\right) \\ s.t. : \quad & Prob\left(g_j(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \geq 0\right) \geq R_j, \ j = 1, \cdots, N \\ & \boldsymbol{d}^{Lower} \leq \boldsymbol{d} \leq \boldsymbol{d}^{Upper}, \ \boldsymbol{\mu_X}^{Lower} \leq \boldsymbol{\mu_X} \leq \boldsymbol{\mu_X}^{Upper} \end{aligned} \quad (1)$$

where $f(\boldsymbol{d}, \boldsymbol{\mu_X}, \boldsymbol{\mu_P})$ is the objective function, $Prob(g_j(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P}) \geq 0)$ is the probability function which denotes the probability of satisfying the jth performance function $g_j(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P})$; N is the number of probabilistic constraints. d is the vector of the deterministic design variables; X and P are the vectors of random design variables and random parameters; $\boldsymbol{\mu_X}$ and $\boldsymbol{\mu_P}$ denote the mean vectors of X and P; R_j denotes the desired design probability of satisfying the jth performance function.

2.2 Reliability analysis methods RIA and PMA

Reliability analysis is usually performed in the standard normal space. When the random variables X and P are not statistically independent or do not have normal distributions, transformation: $\boldsymbol{u} = T(\boldsymbol{X}, \boldsymbol{P})$, and $(\boldsymbol{X}, \boldsymbol{P}) = T^{-1}(\boldsymbol{u})$ will be needed, such as the Rosenblatt transformation (Rosenblatt 1952) or Nataf transformation (Liu and Der Kiureghian 1986), u is the vector of random variables in the standard normal space. Then the probabilistic constraint $g(\boldsymbol{d}, \boldsymbol{X}, \boldsymbol{P})$ becomes $G(\boldsymbol{u})$. The vector d of deterministic design variables will be ignored since it is regarded as constant during the reliability analysis process.

RIA searches the most probable point (MPP) $\boldsymbol{u}_{RIA}^*$ in the standard normal space that has the minimum distance from the origin to the limit state constraint. The RIA model can be expressed as follows:

$$\begin{aligned} find : \quad & \boldsymbol{u} \\ min : \quad & \|\boldsymbol{u}\| \\ s.t. \quad & G(\boldsymbol{u}) = 0 \end{aligned} \quad (2)$$

Here, u is vector of the random variables in the standard normal space.

PMA searches the minimum value of the constraint function, while the design satisfies the target reliability index (e.g. $\|\boldsymbol{u}\| = \beta^t$).

$$\begin{aligned} find : \quad & \boldsymbol{u} \\ min : \quad & G(\boldsymbol{u}) \\ s.t. \quad & \|\boldsymbol{u}\| = \beta^t \end{aligned} \quad (3)$$

PMA is also called the inverse reliability analysis method. The optimal design $\boldsymbol{u}_{PMA}^*$ obtained from (3) is called the inverse most probable point (IMPP).

2.3 Sequential optimization and reliability assessment (SORA)

For simplicity, the vector of random parameters P in (1) is ignored, that is because the vector P is more easily to be evaluated than these random variables. In this paper, the Kriging models are used to substitute the constraint functions, and the IBS method can be easily extended to account for random parameters P.

SORA is one of the most promising decoupled approaches. It is widely used in practical applications. In SORA, deterministic optimization and reliability analysis are performed sequentially. The probabilistic optimization is converted into a deterministic optimization by

shifting the limit state constraints to the feasible region as seen in (4). The shifting vector s^{k+1} is obtained from reliability analysis.

$$
\begin{aligned}
&find : d, \mu_X \\
&\min \quad f(d, \mu_X) \\
&s.t. \quad g_j(d, \mu_X - s_X^{k+1}) \geq 0, \ j = 1, 2, \cdots, N \\
&\quad\quad d^{Lower} \leq d \leq d^{Upper}, \mu_X^{Lower} \leq \mu_X \leq \mu_X^{Upper}
\end{aligned}
\tag{4}
$$

where

$$
s_x^{k+1} = \mu_X^k - X_{IMPP}^k
$$
$$
X_{IMPP}^k = T^{-1}(u_{IMPP}^k)
$$

where N is the number of probabilistic constraints, μ_X^k denotes the optimal design in the kth iteration, X_{IMPP}^k is the IMPP in the original design space corresponding to the IMPP u_{IMPP}^k obtained from (3).

2.4 Kriging model

Kriging model (also called Gaussian process model) is proposed by a South African geostatistician (Matheron 1963; Sacks et al. 1989). In Kriging model, the response at a certain sample point not only depends on the design parameters but is also affected by the points in its neighborhood. The spatial correlation between design points is considered (Zhuang and Pan 2012).

The notations for constructing Kriging models of the constraint function $g(x)$ are used in the description. The corresponding Kriging approximations are denoted as $\widehat{g}(x)$. Kriging is based on the assumption that the response function $g(x)$ is composed of a regression model $F(x)^T\beta$ and stochastic process $Z(x)$ as follows (Huang and Chan 2010; Kim et al. 2009a; Picheny et al. 2010):

$$
g(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})^T \boldsymbol{\beta} + Z(\boldsymbol{x})
\tag{5}
$$

where $F(x)$ is the trend function which consist of a vector of regression functions; β is the trend coefficient vector; $Z(x)$ is assumed to have a zero mean and a spatial covariance function between $Z(x)$ and $Z(w)$ as follows:

$$
Cov[Z(\boldsymbol{x}), Z(\boldsymbol{w})] = \boldsymbol{\sigma}_Z^2 \boldsymbol{R}(\boldsymbol{\theta}, \ \boldsymbol{x}, \ \boldsymbol{w})
\tag{6}
$$

where w is a point different from the point x; σ_Z^2 is the process variance and R is the correlation function defined by its set of parameters θ (Echard et al. 2011; Kim et al. 2009b).

Several models exist to define the correlation function, but the squared-exponential function (also called anisotropic Gaussian model) is commonly used (Bichon et al. 2008;

Rasmussen and Carl Edward 2004; Sacks et al. 1989), and is selected here for R:

$$
\boldsymbol{R}(\boldsymbol{\theta}, \boldsymbol{x}, \boldsymbol{w}) = \prod_{i=1}^{n} \exp\left[-\theta_i(x_i - w_i)^2\right]
\tag{7}
$$

where x_i and w_i are the ith coordinates of the points x and w, n is the number of coordinates in the points x and w, and θ_i is a scalar which gives the multiplicative inverse of the correlation length in the ith direction. An anisotropic correlation function is preferred here, as in reliability studies the random variables are often of different natures (Echard et al. 2011).

Given a set of sample points $[\boldsymbol{x}^1, \cdots \boldsymbol{x}^m]$, $\boldsymbol{x}^i \in \mathfrak{R}^n$, and the responses $\boldsymbol{g}$, $g^i = g(\boldsymbol{x}^i) \in \mathfrak{R}$, m is the number of samples, then the expected value $\widehat{g}(\boldsymbol{x})$ and variance $\sigma_g^2(\boldsymbol{x})$ of the Kriging model prediction at point x are (Bichon et al. 2008): σ_Z^2

$$
\widehat{g}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x})^T \boldsymbol{\beta} + \boldsymbol{r}(\boldsymbol{x}) \boldsymbol{R}^{-1}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{\beta})
\tag{8}
$$

$$
\boldsymbol{\sigma}_g^2(\boldsymbol{x}) = \boldsymbol{\sigma}_Z^2 - [\boldsymbol{F}(\boldsymbol{x})\boldsymbol{r}(\boldsymbol{x})] \begin{bmatrix} \boldsymbol{0} & \boldsymbol{H}^T \\ \boldsymbol{H} & \boldsymbol{R} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{F}(\boldsymbol{x}) \\ \boldsymbol{r}(\boldsymbol{x}) \end{bmatrix}
\tag{9}
$$

where $r(x)$ is a vector containing the covariance between x and each of the m training points (defined by σ_Z^2 (6)), R is a $m \times m$ matrix containing the correlations between each pair of training points, g is the vector of response outputs at each of the training points, and H is a $m \times q$ matrix with rows $F(x^i)^T$ (the trend function for the ith training point containing q terms; for a constant trend, $q=1$). This form of the variance accounts for the uncertainty in the trend coefficients β, but assumes that the parameters governing the covariance function (and θ) have known values (Bichon et al. 2008).

The trend coefficients β is estimated according to the references (Lophaven et al. 2002) by:

$$
\boldsymbol{\beta} = \left(\boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H}\right)^{-1} \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{g}
\tag{10}
$$

The parameters σ_Z^2 and θ are determined through maximum likelihood estimation. This involves taking the log of the probability of observing the response values g given the covariance matrix R, which can be written as (Bichon et al. 2008; Jones et al. 1998; Lophaven et al. 2002; Sacks et al. 1989):

$$
\log[Prob(\boldsymbol{g}|\boldsymbol{R})] = -\frac{1}{m} \log|\boldsymbol{R}| - \log\left(\widehat{\boldsymbol{\sigma}}_Z^2\right)
\tag{11}
$$

where $|\boldsymbol{R}|$ indicates the determinant of R, and $\widehat{\sigma}_Z^2$ is the optimal value of the variance given $\widehat{g}(x^i)$ an estimate of θ and is defined by

$$
\widehat{\boldsymbol{\sigma}}_Z^2 = \frac{1}{m}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{\beta})^T \boldsymbol{R}^{-1}(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{\beta})
\tag{12}
$$

maximizing (11) gives the maximum likelihood estimate of θ, which in turn defines $\widehat{\sigma}_Z^2$ (Bichon et al. 2008).

Kriging model is an exact interpolation method. The prediction in a point x^i of the design of experiments is exact, i.e. $\widehat{g}(x^i) = g(x^i)$. Therefore, the Kriging variance is null in these points and it becomes important in unexplored areas. This enables to quantify the uncertainty of local predictions with an easily computable analytical function. These properties are interesting in reliability studies and metamodels as the Kriging variance represents a good index to improve a design of experiments (Echard et al. 2011).

3 Importance boundary sampling method

In this paper, the proposed IBS uses the important coefficients to identify the critical parts of limit state constraint boundaries. The objective function values and the joint probability density values of the design variables in the standard normal space are also applied to calculate the important coefficients. In addition, the sampling and optimization processes are conducted alternately in the proposed IBS method.

The proposed sampling criteria in IBS will be discussed in Section 3.1; then the procedures and flowcharts of IBS will be introduced in Section 3.2.

3.1 Importance boundary sampling (IBS) criteria

The proposed IBS method mainly selects sample points on the important parts of the limit state constraints which have relatively smaller objective function values or shorter distances to the current design point in RBDO. So in IBS, two sampling criteria will be proposed; and they will work together to improve the efficiency of Kriging-model-based RBDO. The constraint boundary sampling criterion will be first discussed, and then IBS criteria for RBDO will be introduced.

3.2 Constraint boundary sampling (CBS) criterion

In the RBDO processes, the design points and MPPs are usually located on the limit state constraints. Therefore, the accuracy of the limit state constraint boundaries should first be ensured.

The CBS criterion was proposed by Lee and Jung (Lee and Jung 2008). When there is sufficient sample data to construct the Kriging model, the prediction of approaches the normal distribution with mean $\widehat{g}(x)$ and standard deviation $\sigma_g(x)$.

If the limit state constraint is defined as $g(x)=0$, then the probability of the Kriging prediction satisfying the constraint $g(x) \geq 0$ is as follows:

$$Prob(x) = 1 - \Phi\left(\frac{0 - \widehat{g}(x)}{\sigma_g(x)}\right) = \Phi\left(\frac{\widehat{g}(x)}{\sigma_g(x)}\right) \quad (13)$$

where, Φ is the cumulative density function of the standard normal distribution.

Then the standard normal probability density function φ $\left(\widehat{g}(x)/\sigma_g(x)\right)$ can be used to measure the closeness of the Kriging prediction $\widehat{g}(x)$ to the limit state constraint $g(x)=0$, as seen in Fig. 1. The CBS criterion is defined as follows:

$$CBS = \begin{cases} \sum_{j=1}^{N} \varphi\left(\frac{\widehat{g}_j(x)}{\sigma_{g_j}(x)}\right).D(x) & if \ \widehat{g}_j(x) \geq 0, \ \forall j \\ 0 & otherwise \end{cases} \quad (14)$$

where, N is the number of constraints, $D(x)$ is the minimal distance from the current sample point x to the existing sample points as defined in (15), m is the number of existing sample points.

$$D(x) = \frac{\min\|x - x^i\|}{\max\|x^j - x^i\|}, \quad i, j = 1, \cdots, m \quad (15)$$

When all the constraints are satisfied, the CBS criterion will have a large value at $\widehat{g}_j(x)/\sigma_{g_j}(x) \approx 0$, where the sample point x is close to the limit state constraints $\left(\widehat{g}_j(x) \approx 0\right)$ or the Kriging prediction has large variance $\sigma_{g_j}(x)$. Therefore, CBS can locate sample points efficiently along the limit state constraint boundaries.

This paper constrains samples to lie in the predicted feasible region in CBS and IBS, as seen in (14), that is because when the design space is very large and constraint boundries are very long, but the feasible region is relatively small; then the constraint boundaries lacated outside the feasible region are not $IBS_1(I_1)$ necessaryly to be refined accurately.

3.3 IBS criterion 1 for RBDO

Assume that in RBDO, the target is to find the minimum objective function value. Then the design regions that have relatively smaller objective function values will be more important than other regions, and the limit state constraint boundaries within these design regions will be more critical. Therefore, these critical limit state constraint boundaries should first be fitted accurately.
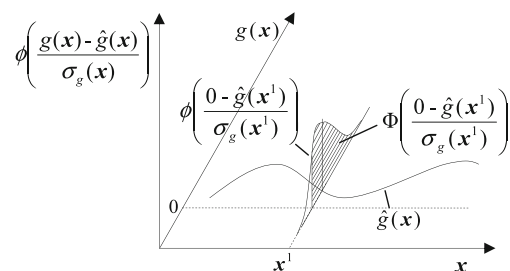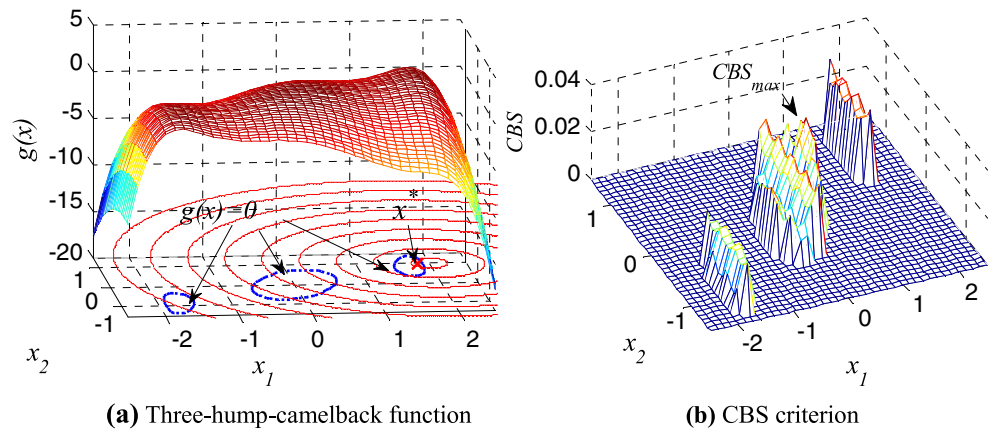

Fig. 1 Feasible probability

(a) Three-hump-camelback function   **(b)** CBS criterion

The proposed IBS uses the objective function value to evaluate the importance coefficient for the limit state constraint boundaries as follows:

$$I_1(\boldsymbol{x}) = \frac{\max\{f(\boldsymbol{x}^i)\}_{i=1}^m - f(\boldsymbol{x})}{\max\{f(\boldsymbol{x}^i)\}_{i=1}^m - \min\{f(\boldsymbol{x}^i)\}_{i=1}^m} \qquad (16)$$

where $f(\boldsymbol{x})$ is the objective function, $\boldsymbol{x}^i$ denotes the ith sample point, m is the number of existing sample points.

From (16), it can be seen that the smaller the objective function value $f(\boldsymbol{x})$ at the sample point x is, the higher the importance coefficient $I_1(\boldsymbol{x})$ will be; And if the value $f(\boldsymbol{x})$ is in the range of $[\min(f(\boldsymbol{x}^i)), \max(f(\boldsymbol{x}^j))]$, $i,j=1,\cdots,m$, then the importance coefficient $I_1(\boldsymbol{x})$ will be in the range of $[0, 1]$.

The IBS criterion 1 for RBDO $f(\boldsymbol{x})$ can be expressed as follows:

$$IBS_1(I_1) = \begin{cases} \sum_{j=1}^N \varphi\left(\dfrac{\widehat{g}_j(\boldsymbol{x})}{\sigma_{g_j}(\boldsymbol{x})}\right) \cdot D(\boldsymbol{x}) \cdot I_1(\boldsymbol{x}) & if \ \widehat{g}_j(\boldsymbol{x}) \geq 0, \quad \forall j \\ 0 & otherwise \end{cases}$$

$$(17)$$

where $I_1(\boldsymbol{x})$ is the importance coefficient $I_1(\boldsymbol{x})$ obtained from (16).

The $IBS_1(I_1)$ in (17) combines the probability density value of the Kriging predictor $\widehat{g}_j(\boldsymbol{x})$, the minimum distance from the current design point to the existing sample points and the importance coefficient $I_1(\boldsymbol{x})$. If the objective function $f(\boldsymbol{x})$ has a small value at the sample point x, then the value of the IBS criterion 1 $IBS_1(I_1)$ will be enlarged, and this point x will be more likely to be chosen as the next new sample point.

However, because the importance coefficient $I_1(\boldsymbol{x})$ in (16) has a linear relation with the objective function value, its effect on the value $IBS_1(I_1)$ is limited. In other words, the importance coefficient $I_1(\boldsymbol{x})$ may not clearly identify the critical parts of the limit state constraint boundaries. In this paper, this problem will be overcome through using the importance coefficient $I_2(\boldsymbol{x})$ as follows:

$$I_2(\boldsymbol{x}) = e^{\tau \cdot I_1(\boldsymbol{x})} \qquad (18)$$

where τ is a constant, "e" is the "na $f(\boldsymbol{x})$ tural" exponential.

In (18), the importance coefficient $I_2(\boldsymbol{x})$ has an exponential relation with the objective function value . When using $I_2(\boldsymbol{x})$, the critical parts of the design region which have relatively small objective function values will be clearly identified, and the other parts will be filtered. As mentioned above, $I_1(\boldsymbol{x})$ is in the range of $[0, 1]$, so the value of $I_2(\boldsymbol{x})$ will be in the interval $[1, e^\tau]$, the constant τ in this paper is chosen as 2~4, which can

Fig. 3 Mesh plot of the IBS
criterion 1 **a** IBS criterion 1 using
b IBS criterion 1 using $I_2(\boldsymbol{x})$

(a) IBS criterion 1 using $I_1(\boldsymbol{x})$ **(b)** IBS criterion 1 using $I_2(\boldsymbol{x})$
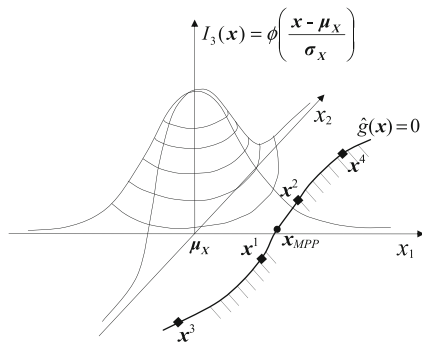
Fig. 4 Importance coefficient $I_3(x)$ for RBDO

make $I_2(x)$ possess enough effect on . This paper has tried to choose different values to study the sensitivity of the results to its variations. It found that the value of the parameter τ had very weak effect to the results of IBS_1. Unless the value of τ was too small as 0.01, or too large as 100, the IBS_1 (I_2) would be able to find the new sample point. This paper uses the nonlinear relationship between the objective function value and the importance coefficient in (18), so the value of the parameter τ is not very critical in the sampling criterion of $IBS_1(I_2)$.

The IBS criterion 1 using $I_2(x)$ can be expressed as follows:

$$IBS_1(I_2) = \begin{cases} \sum_{j=1}^{N} \varphi\left(\frac{\widehat{g}_j(x)}{\sigma_{g_j}(x)}\right) \cdot D(x) \cdot I_2(x) & if \ \widehat{g}_j(x) \geq 0, \quad \forall j \\ 0 & otherwise \end{cases}$$

(19)

To illustrate the behavior of IBS criterion 1, a mathematical problem (Lee and Jung 2008) is tested as follows:

$$
\begin{aligned}
min : \ & f(x) = (x_1\text{-}2)^2\text{-}(x_2\text{-}1)^2 \\
s.t. \ & g(x) \geq 0 \\
& g(x) = -2x_1^2 + 1.05x_1^4 - \frac{1}{6}x_1^6 + x_1x_2\text{-}x_2^2 + 0.5 \\
& \text{-}2.5 \leq x_1 \leq 2.5, \qquad \text{-}1.5 \leq x_2 \leq 1.5
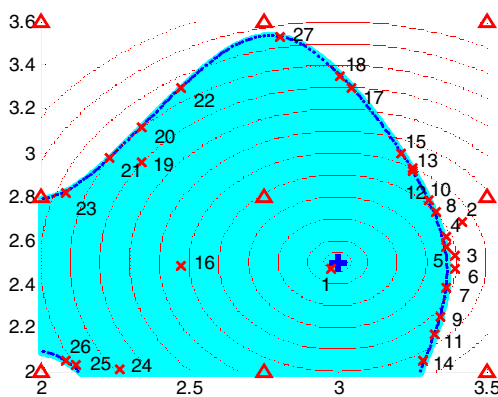\end{aligned}
$$

(20)

Fig. 5 Sampling with the IBS criterion 2

The constraint $g(x)$ is a three-hump-camelback function as seen in Fig. 2 (a). The dotted-lines are the limit state constraint $g(x)=0$, and there are three disconnected feasible regions $g(x)>0$. The objective function value $f(x)$ decreases up and to the right within the design region, x^* is the optimal design point.

Figure 2 (b) shows the values of the *CBS* criterion based on the Kriging model $\widehat{g}(x)$ which is constructed using 50 uniform sample points. The *CBS* values in all the three feasible regions are noticeably large, and the maximum value CBS_{max} is located in the middle feasible region, which means this point w $I_1(x)$ ill be chosen as a new sample in the CBS method. However, the target of this example is to find the minimum objective function value, so the limit state constraint boundary around the upper right feasible region should first be fitted accurately.

Figure 3 shows the mesh plot of the IBS criterion 1. When using the importance coefficient $I_1(x)$ as seen in Fig. 3 (a), the value of $IBS_1(I_1)$ decreases down and to the left in the design region, and the maximum value IBS_{1max} is located in the upper right feasible region. However, the $IBS_1(I_1)$ value in the middle feasible region is still very large, which shows that the effect of the importance coefficient $I_1(x)$ on the $IBS_1(I_1)$ is limited. When using the importance coefficient $I_2(x)$ as seen in Fig. 3 (b), the IBS values $IBS_1(I_2)$ in the upper right feasible region are extremely high; but in the other two feasible regions, the IBS values are very small. Therefore, these unimportant feasible regions are filtered, and the sample
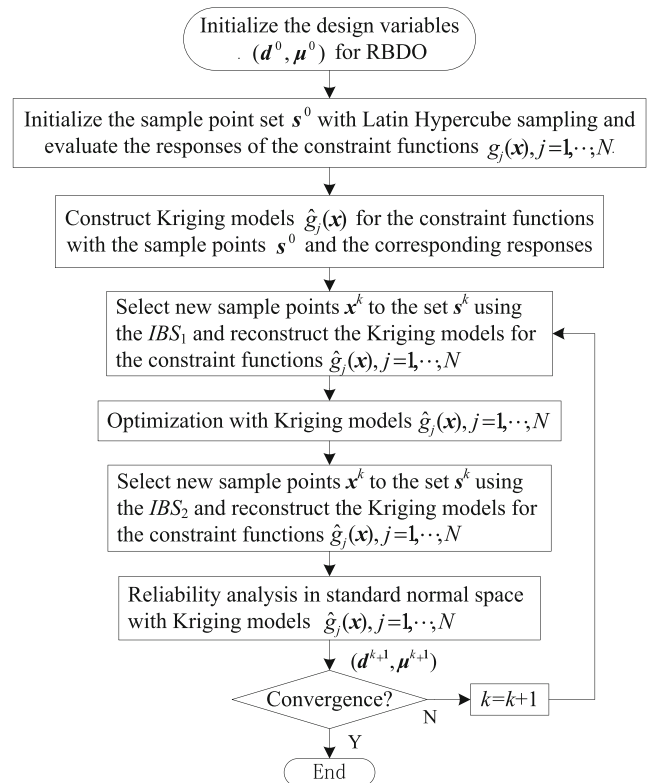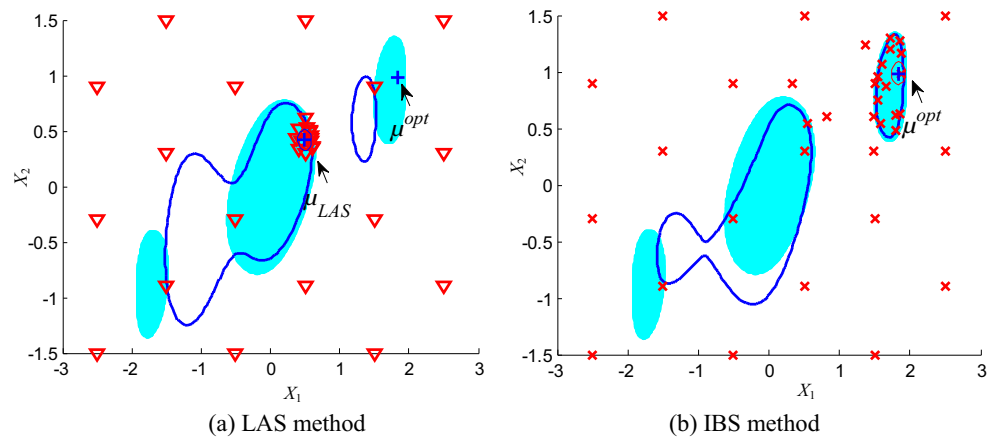

Fig. 6 Flowchart of the IBS method

Fig. 7 Limit state constraints and sample points in LAS and IBS for example 1 **a** LAS method **b** IBS method

(a) LAS method

(b) IBS method

points will be mainly selected on the limit state constraint boundary around the upper right feasible region which is more critical in the optimization process.

3.4 IBS criterion 2 for RBDO

When using analytical method in RBDO, the target of reliability analysis will be to find the minimum distance from the current optimal design to the limit state constraints in the standard normal space, and this point is called MPP as described in (2). Therefore, the region in the vicinity of the current design point is more critical, and the limit state constraint boundaries within this region should first be fitted accurately (Wang et al. 2005). In this paper, the joint probability density value of the random variables is applied to calculate the importance coefficient.

If the random variables have normal distributions: $x \sim N(\mu_X, \sigma_X)$, then the importance coefficient is defined as follows:

$$I_3(x) = \varphi\left(\frac{x - \mu_X}{\sigma_X}\right) \quad (21)$$

where "φ" is the joint probability density function of standard normal distribution. Then the IBS criterion 2 for RBDO can be expressed as follows:

$$IBS_2(x) = \begin{cases} \sum_{j=1}^{N} \varphi\left(\frac{\widehat{g}_j(x)}{\sigma_{g_j}(x)}\right) \cdot D(x) \cdot I_3(x) & if \ \widehat{g}_j(x) \geq 0, \quad \forall j \\ 0 & otherwise \end{cases} \quad (22)$$

If the random variables x do not have normal distributions, then the transformation: $u = T(x)$, and $x = T^{-1}(u)$ will be needed, such as Rosenblatt transformation (Rosenblatt 1952) or Nataf transformation (Liu and Der Kiureghian 1986). The transformed random variables u have normal distributions $u \sim N(\mu_U, \sigma_U)$, then the importance coefficient can be calculated as follows:

$$I_3(x) = I_3\left(T^{-1}(u)\right) = \varphi\left(\frac{u - \mu_U}{\sigma_U}\right) \quad (23)$$

As seen in Fig. 4, μ_X is the current optimal design, x_{MPP} denotes the MPP, x^1-x^4 are four potential sample points on the limit state constraint boundaries. In order to get the MPP x_{MPP}

Fig. 8 Plot of the feasible region and the LHS for example 2 **a** True constraint boundaries and the feasible region **b** Latin hypercube sampling
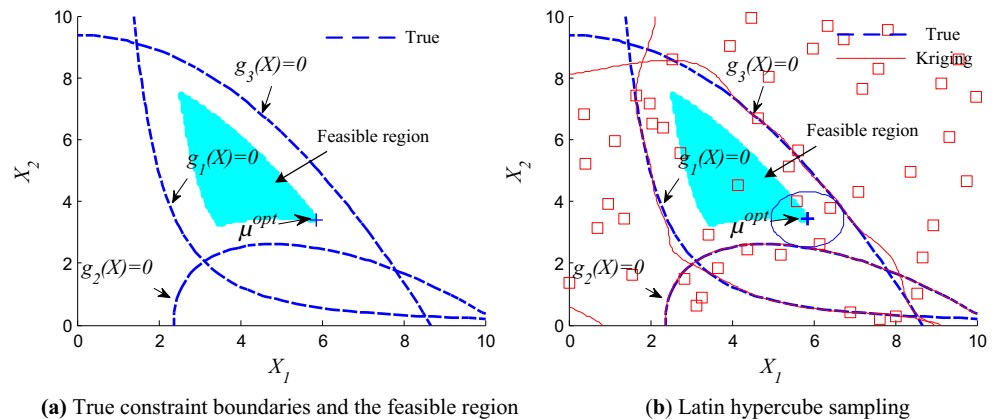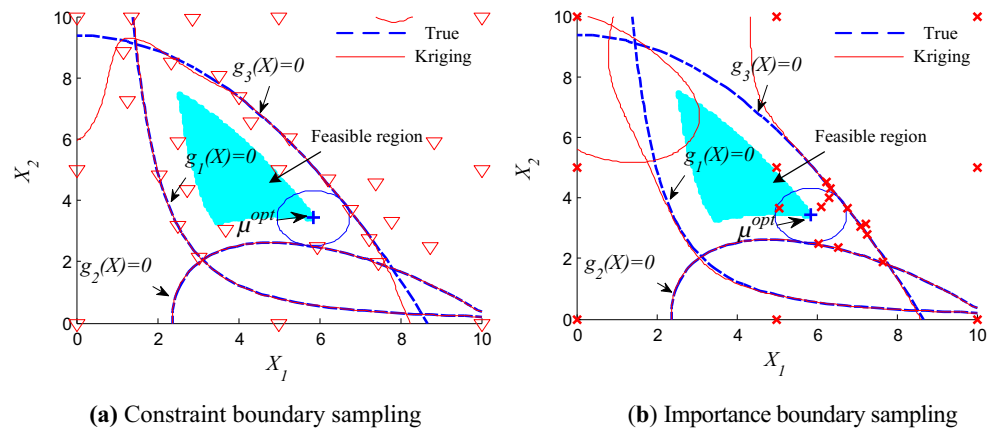

(a) True constraint boundaries and the feasible region

(b) Latin hypercube sampling

Fig. 9 Sample points in CBS and IBS for example 2 **a** Constraint boundary sampling **b** Importance boundary sampling

(a) Constraint boundary sampling

(b) Importance boundary sampling

more accurately, the limit state constraint boundary $\widehat{g}(x) = 0$ in the vicinity of the MPP should first be fitted precisely. However, the MPP x_{MPP} is unknown until the reliability analysis process is finished. This issue can be overcome through using the importance coefficient $I_3(x)$, which will have a large value when x is close to the current optimal design μ_X. In Fig. 4, the potential sample points x^1 and x^2 will first be selected, but the sample points x^3 and x^4 which are far away from the current optimal design μ_X will not be chosen. Also by using the minimum distance $D(x)$ in (15), the sample points will not be too close to each other. So the proposed importance coefficient $I_3(x)$ in (23) can make the selection of the sample points more rational for RBDO.

To illustrate the behavior of IBS criterion 2, a mathematical problem is tested as follows.

$$
\begin{aligned}
&g(X) = -X_1\sin(4X_1) - 1.1X_2\sin(2X_2) \\
&X_i \sim N\left(\mu_i, 0.1^2\right), \quad i = 1, 2 \\
&\mu = [3.00, 2.50]^T
\end{aligned}
\tag{24}
$$

here, the solid line circles are the contour lines for the IBS criterion 2; triangle and fork points denote the initial sample
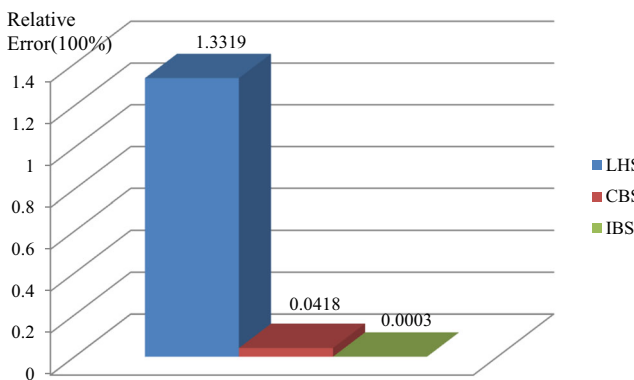

Fig. 10 Relative errors of the LHS、CBS、IBS methods for example 2

points and new sample points using the IBS criterion 2 respectively.

In Fig. 5, nine samples are selected as initial sampling, denoted by triangles; 27 sequential samples are selected by using the IBS criterion 2. It can be seen that the samples that are close to the design variables are firstly selected. Although due to the lack of accuracy of the Kriging models in the early stage of the sequential sampling process, the first four samples 1–4 are not selected strictly according to the distances to the design variables; the rest 23 samples are chosen according to their contributions to the refining the critical MPP searching region. So, the IBS criterion 2 is very effective in the reliability analysis process.

3.5 Procedures and flowchart o (d^0, μ^0) f the IBS method

The flowchart of the IBS method is shown in Fig. 6, and the procedures are described as follows:

1) Initialize design variables for RBDO.
2) Initialize the sample point set s^0 using Latin Hypercube sampling method. The number of samples required to define a quadratic polynomial $(n+1)(n+2)/2$ is a convenient rule of thumb (Bichon et al. 2008). So in this paper, this initial selection is a little larger than the required value $(n+1)(n+2)/2$. Then evaluate the responses of the constraint functions $g_j(x)$, $j=1,\cdots,N$. For simplicity, the vector of design variables (d, μ) is replaced by x.
3) Construct the Kriging models $\widehat{g}_j(x)$, $j = 1,\cdots,N$ for the constraint functions using the above sample points s^0 and the corresponding responses.
4) In the kth iteration of the RBDO, select new sample points x^k using the sampling criterion IBS_1 in (19), and then evaluate the responses of the constraint functions $g_j(x^k)$, $j=1,\cdots,N$ at the selected points s^k. Add these new sample points x^k to the sample set s^k, then reconstruct the Kriging models $\widehat{g}_j(x)$, $j=1,\cdots,N$ for constraint functions

Table 1 Summary of the optimization results for example 2

Methods	Design variables	Samples	Relative error (%)	β_1	β_2	β_3
SORA	(5.8545, 3.4236)	294	/	Inf.	3.0247	2.9937
LHS	(5.8530, 3.2904)	50	1.3319	Inf.	2.5869	3.2266
CBS	(5.8586, 3.4228)	33	0.0418	Inf.	3.0251	2.9839
IBS	(5.8545, 3.4236)	21	0.0003	Inf.	3.0252	2.9946

based on all the existing sample points s^k and the corresponding responses.

5) Conduct optimization using the above constructed Kriging models $g_j(x^k)$, $j=1,\cdots,N$.

6) Select new sample points x^k using the sampling criterion IBS_2 in (22), and then evaluate the responses of the constraint functions $g_j(x^k)$, $j=1,\cdots,N$ at the selected points s^k. Add these new sample points x^k to the sample set s^k, then reconstruct the Kriging models $\widehat{g}_j(x)$, $j=1,\cdots,N$ for constraint functions based on all the existing sample points s^k and the corresponding responses.

7) Conduct reliability analysis using the above constructed Kriging models $g_j(x^k)$, $j=1,\cdots,N$. This paper assumes that the original performance functions are implicit, such as expensive physical test and complex finite element simulation. The searching of the IMPP is based on the cheap Kriging models. When compared with the implicit performance function calls, the computational effort to find the MPP is very cheap.

8) If the design (d^{k+1}, μ^{k+1}) converges, then the whole process will stop, $(d^*, \mu^*)_{RBDO} = (d^{k+1}, \mu^{k+1})$; otherwise, $k=k+1$, and the process will go back to step (4).

4 Application

In order to verify the accuracy and efficiency of the proposed IBS method, five examples are tested and compared to analytical method, Latin Hypercube sampling (LHS) method and constraint boundary sampling (CBS) method. The SORA is chosen as the RBDO method which directly calls the true performance functions and can yield a relatively accurate design. The optimal results μ^* obtained from Kriging-model-based RBDO methods are assessed through the relative error $\|\mu^* - \mu_A^*\| / \|UB - LB\|$, μ_A^* is the optimal design obtained from SORA, LB and UB are the lower and upper boundaries of the design variables respectively. The results of the mathematical examples are also evaluated by MCS with ten million sample size. These examples are tested in the Matlab, and the tool "fmincon" is used as the optimizer.

4.1 Numerical example 1

The model of this example is the same as the (20), however, its design variables become random variables $x_i \sim N(\mu_i, 0.05)$, $i=1,2$, $\beta=2.0$, and the initial design point is located at (0.5, 0.5). This example is used to compare the behaviors of the proposed IBS method and the local adaptive sampling (LAS) method (Chen et al. 2014).

The comparison results are shown in Fig. 7. After initialization with 18 uniform sample points, sequential sampling is conducted using the LAS and IBS methods separately. The LAS method chooses 15 sample points, as seen in Fig. 7 (a), most of these samples are located in the middle feasible region, and only the local optimum is obtained. The IBS method selected 19 sample points, and with the advantage of the IBS_1 (I_2), most of the sample points are located in the right upper feasible region, where the optimum is located. So, the proposed IBS method is more applicable than the LAS method.

Table 2 Summary of the optimization results for example 3

| Methods	Design variables	Samples	Relative error (100 %)	β_1
Anal.	(2.5078, 4.3986, 5.4951, 5.0963, 3.2037, 4.3986, 6.4925, 3.2037, 4.5979, 2.0115, 5.4951, 3.7014, 3.0048, 6.3927, 3.8010, 4.3986)	323	/	2.9925
LHS	(2.5425, 4.4427, 5.5422, 5.1425, 3.2422, 4.4429, 6.5425, 3.2424, 4.6427, 2.0423, 5.5419, 3.7424, 3.0424, 6.4420, 3.8425, 4.4420)	300	1.7094	3.5624
CBS	(2.5226, 4.4234, 5.5228, 5.1233, 3.2232, 4.4228, 6.5231, 3.2230, 4.6232, 2.0232, 5.5229, 3.7231, 3.0234, 6.4231, 3.8231, 4.4233)	250	0.9484	3.2983
IBS	(2.5082, 4.3990, 5.4944, 5.0959, 3.2043, 4.3978, 6.4922, 3.2047, 4.5975, 2.0106, 5.4956, 3.7013, 3.0055, 6.3921, 3.8016, 4.3988)	219	0.0235	2.9924
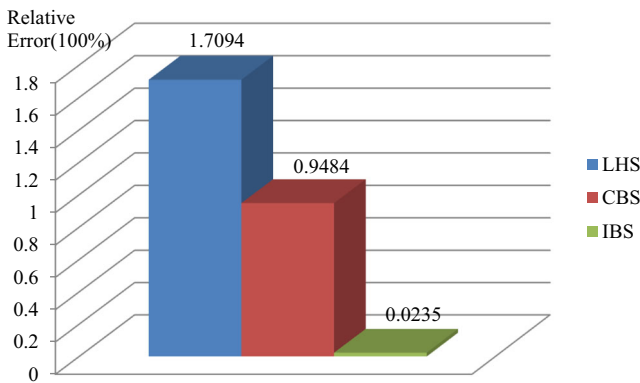
Fig. 11 Relative errors of the LHS、CBS、IBS methods for example 3

here, lines are the predicted boundaries by Kriging; shaded area is the feasible region; triangle and fork denote the sample points from LAS and IBS methods respectively.

4.2 Numerical example 2

This is a non-linear mathematical problem (Youn and Choi 2004). There are two random design variables X_1, X_2, and three probabilistic constraints g_1, g_2, g_3. No deterministic design variable or random parameter exists.

$$
\begin{aligned}
&\text{find } \ \boldsymbol{\mu} = [\mu_1, \mu_2]^{\text{T}} \\
&\min \ f(\boldsymbol{\mu}) = 10 - \mu_1 + \mu_2 \\
&s.t. \ \ prob\left[g_j(\boldsymbol{X}) > 0\right] \leq \Phi\left(-\beta_1^t\right) \quad j = 1, 2 \\
&g_1(\boldsymbol{X}) = 1 - \frac{X_1^2 X_2}{20} \\
&g_2(\boldsymbol{X}) = 1 - \frac{(X_1 + X_2 - 5)^2}{30} - \frac{(X_1 - X_2 - 12)^2}{120} \\
&g_3(\boldsymbol{X}) = 1 - \frac{80}{(X_1^2 + 8X_2 + 5)} \\
&0.0 \leq \mu_i \leq 10.0, \quad X_i \sim N(\mu_i, 0.3^2), \quad i = 1, 2 \\
&\beta_1^t = \beta_2^t = \beta_3^t = 3.0, \quad \boldsymbol{\mu}^{(0)} = [5.00, 5.00]^{\text{T}}
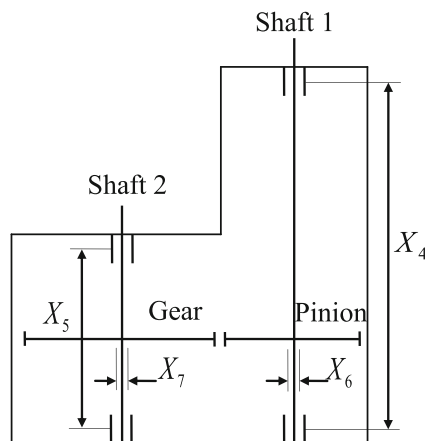\end{aligned} \tag{25}
$$

Fig. 12 A speed reducer

Table 3 Summary of the optimization results for the speed reducer

Methods	Design variables	Samples	Relative error (100 %)
Anal.	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3652, 5.3017)	1218	/
LHS	(3.5767, 0.7000, 17.0000, 7.3013, 7.7541, 3.3669, 5.3014)	80	0.2212
CBS	(3.5769, 0.7000, 17.0000, 7.3013, 7.7544, 3.3662, 5.3016)	77	0.1741
IBS	(3.5765, 0.7000, 17.0000, 7.3000, 7.7541, 3.3653, 5.3017)	50	0.0121

The objective function value decreases down and to the right in design region. The third constraint g_3 $(X)=0$ is highly nonlinear. The optimal design $\boldsymbol{\mu}^{opt}$ for RBDO is located at (5.8545, 3.4236) as seen in Fig. 8 (a). The shaded area is the feasible region. The circle around the optimal design is the β-circle.

here, dashed lines are the true constraint boundaries; solid lines are the predicted boundaries by Kriging; shaded area is the feasible region and square denotes the sample point from LHS method.

The Latin hypercube sampling method is shown in Fig. 8 (b). 50 points are evenly selected within the design region. However, many sample points are located out of the feasible region. The limit state constraints 1 and 3 are not accurate.

The CBS method adopts grid sampling with nine points, 3-level full factorial design, as initial sampling. It applies 33 sample points to approximate the three constraints as seen in Fig. 9 (a). Most of the sample points are selected on the limit state constraints within the feasible region. The boundaries of the feasible region are well fitted. However, only a few sample points are located in the vicinity of the optimal design μ^{opt}, which means that most of the samples are not well explored in improving the accuracy of the results.

The proposed IBS method also adopts grid sampling with nine points as initial sampling. And it uses 21 sample points to
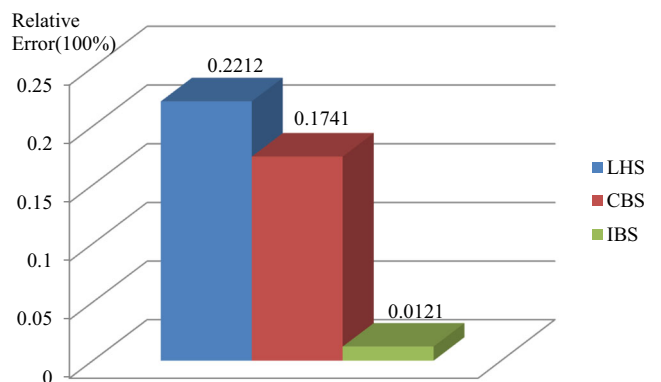

Fig. 13 Relative errors of the LHS、CBS、IBS methods for the speed reducer

Table 4 Summary of the MCS results for the speed reducer

Methods	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}
Anal.	Inf.	Inf.	Inf.	Inf.	3.0000	3.0023	Inf.	3.0002	Inf.	Inf.	2.9991
LHS	Inf.	Inf.	Inf.	Inf.	3.3297	2.9513	Inf.	3.0106	Inf.	Inf.	3.0288
CBS	Inf.	Inf.	Inf.	Inf.	3.2029	2.9835	Inf.	3.0172	Inf.	Inf.	3.0447
IBS	Inf.	Inf.	Inf.	Inf.	3.0196	3.0037	Inf.	3.0003	Inf.	Inf.	2.9984

approximate the three constraints as seen in Fig. 9 (b). Most of the sample points are located on the important parts of the limit state constraint boundaries. The region in the vicinity of the optimal design μ^{opt} is accurately fitted, and this will ensure the accuracy of the RBDO results.

here, dashed lines are the true constraint boundaries; solid lines are the predicted boundaries by Kriging; shaded area is the feasible region; triangle and fork denote the sample points from CBS and IBS methods respectively.

The comparison results for example 1 are shown in Table 1. It can be seen that analytical method is not efficient. The CBS method is more efficient than LHS. The number of constraint function calls in IBS is equal to 21. So IBS is the very efficient.

For the second constraint, the reliability indexes in IBS (β_2=3.0252) and CBS (β_2=3.0251) are almost equal to each other. However, for the third constraint, the reliability index in IBS (β_3=2.9946) is closer to the target value ($\beta_3{}^t$=3.0) than that in CBS (β_3=2.9839). The relative error in IBS also has the smallest value 0.0003 % as seen in Fig. 10. Therefore, the proposed IBS method is very accurate.

In this example, the reliability indexes for the optimal design are (Inf., 3.0247, 2.9937), β_3=2.9946 is samller than the target value $\beta_3{}^t$=3.0, that is because the MPP-based RBDO method has a common issue that it will have some error for highly nonlinear problems. This paper is based on the MPP-based RBDO method, although it is roboust, but it may have some error.

4.3 Numerical example 3

This example is a multi-dimensional problem. There are sixteen random design variables X_i, i=1,...,16, which are statistically independent and have normal distributions. The

constraint function is highly nonlinear (Shan et al. 2009). This problem can be expressed as follows:

$$find \ \boldsymbol{\mu} = [\mu_1, \cdots, \mu_{16}]^{\mathrm{T}}$$

$$min \ f(\boldsymbol{\mu}) = \sum_i^{16} (\mu_i - c_i)^2$$

$$s.t. \ prob \ [g_1(\boldsymbol{X}) > 0] \geq \Phi(\beta^t)$$

$$g_1(\boldsymbol{X}) = \sum_i^{16} X_i \left(\tau + \ln \left(\frac{X_i}{X_1 + \cdots + X_{16}} \right) \right)$$

$$X_i \sim N(\mu_i, 0.3^2), \quad 1e-6 \leq \mu_i \leq 10, \quad i = 1, \cdots, 16$$

$$\boldsymbol{c} = \begin{bmatrix} 2.20, & 4.10, & 5.20, & 4.80, & 2.90, & 4.10, & 6.20, & 2.90, \\ & 4.30, & 1.70, & 5.20, & 3.40, & 2.70, & 6.10, & 3.50, & 4.10 \end{bmatrix}$$

$$\tau = -17.164, \quad \beta_1{}^t = 3.0, \quad \boldsymbol{\mu}^{(0)} = [4.0, \cdots, 4.0]^{\mathrm{T}}$$

$$(26)$$

The comparison results for example 2 are shown in Table 2. Analytical method is not efficient. CBS is more efficient than the LHS. The total number of function calls in the proposed IBS method is equal to 219. So IBS is very efficient.

LHS is not accurate, for its relative error is the largest one as seen in Fig. 11, and the reliability index evaluated by MCS is 3.5624 which is far away from the target value 3.0. The optimal design from the IBS method is very close to the result of the analytical method; the relative error is 0.0235 %; also the reliability index in IBS is closer to the target value 3.0 than the result of CBS. So the proposed IBS method is very accurate.

Fig. 14 Box girder design
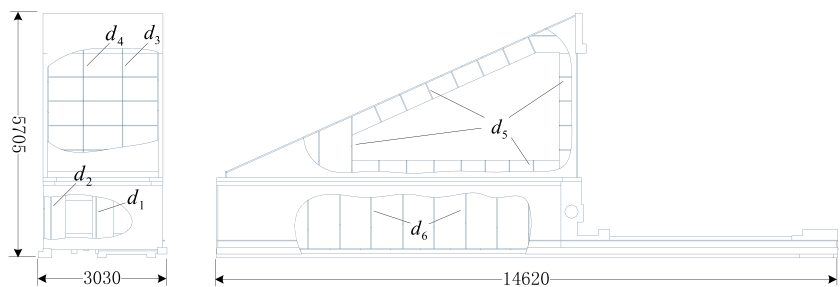
4.4 A speed reducer

A speed reducer shown in Fig. 12 is used to rotate the engine and propeller with efficient velocity in light plane (Ju and Lee 2008). This problem has seven random variables and 11 probabilistic constraints. The objective function is to minimize the weight, and probabilistic constraints are related to physical quantities such as bending stress, contact stress, longitudinal displacement, stress of the shaft, and geometry constraints. The random design variables are gear width (X_1), gear module (X_2), the number of pinion teeth (X_3), distance between bearings (X_4, X_5), and diameter of each shaft (X_6, X_7)

The description of the RBDO model of the speed reducer is shown as follows:

$$find \ \ \boldsymbol{\mu} = [\mu_1, \cdots, \mu_7]^{\mathrm{T}}$$
$$\min \ f(\boldsymbol{\mu}) = 0.7854\mu_1\mu_2^2\left(3.3333\mu_3^2 + 14.9334\mu_3 - 43.0934\right)$$
$$\qquad -1.508\mu_1\left(\mu_6^2 + \mu_7^2\right) + 7.477\left(\mu_6^3 + \mu_7^3\right)$$
$$\qquad + 0.7854\left(\mu_4\mu_6^2 + \mu_5\mu_7^2\right)$$
$$s.t. \quad prob\left[g_j(\boldsymbol{X}) > 0\right] \leq \Phi\left(-\beta_j^t\right), \quad j = 1, \cdots, 11$$
$$where \ \ g_1(\boldsymbol{X}) = \frac{27}{X_1X_2^2X_3} - 1, \ \ g_2(\boldsymbol{X}) = \frac{397.5}{X1X_2^2X_3^2} - 1,$$
$$g_3(\boldsymbol{X}) = \frac{1.93X_4^3}{X_2X_3X_6^4} - 1, \ \ g_4(\boldsymbol{X}) = \frac{1.93X_5^3}{X_2X_3X_7^4} - 1,$$
$$g_5(\boldsymbol{X}) = \frac{\sqrt{\left(745X_4/(X_2X_3)\right)^2 + 16.9 \times 10^6}}{0.1X_6^3} - 1100,$$
$$g_6(\boldsymbol{X}) = \frac{\sqrt{\left(745X_5/(X_2X_3)\right)^2 + 157.5 \times 10^6}}{0.1X_7^3} - 850,$$
$$g_7(\boldsymbol{X}) = X_2X_3 - 40, \ \ g_8(\boldsymbol{X}) = 5 - \frac{X_1}{X_2},$$
$$g_9(\boldsymbol{X}) = \frac{X_1}{X_2} - 12, \ \ g_{10}(\boldsymbol{X}) = \frac{1.5X_6 + 1.9}{X_4} - 1,$$
$$g_{11}(\boldsymbol{X}) = \frac{1.1X_7 + 1.9}{X_5} - 1,$$
$$\beta_1^t = \cdots = \beta_{11}^t = 3.0,$$
$$2.6 \leq X_1 \leq 3.6, \quad 0.7 \leq X_2 \leq 0.8, \quad 17 \leq X_3 \leq 28,$$
$$7.3 \leq X_4 \leq 8.3, \quad 7.3 \leq X_5 \leq 8.3,$$
$$2.9 \leq X_6 \leq 3.9, \quad 5.0 \leq X_7 \leq 5.5,$$
$$X_i \sim N\left(\mu_i, 0.005^2\right), \qquad i = 1, \cdots, 7,$$
$$\boldsymbol{\mu}^{(0)} = [3.2, \ 0.75, \ 23.0, \ 8.0, \ 8.0, \ 3.6, \ 5.0]^{\mathrm{T}}$$

(27)

The comparison results for the speed reducer are shown in Table 3. The number of constraint function calls in analytical method is 1218, it is not efficient. CBS is more efficient than the LHS. The number of function calls in the proposed IBS method is equal to 50. So IBS is very efficient.

The optimal design from the IBS method is almost identical with the result of the analytical method; the relative error is 0.0121 % as seen in Fig. 13. Table 4 shows the MCS results.
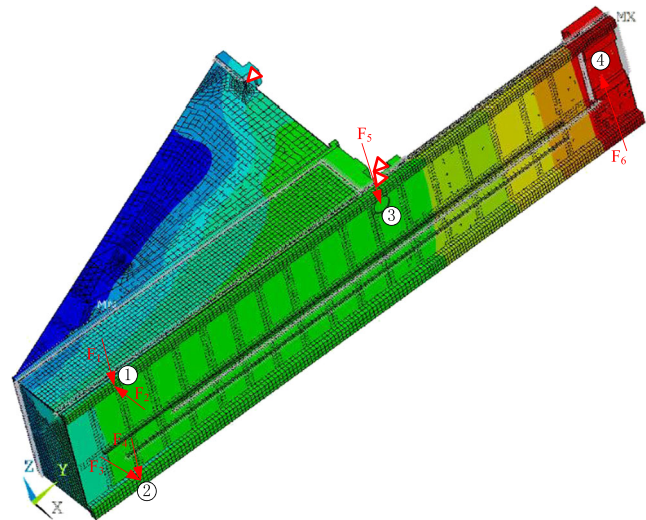

Fig. 15 Force analysis of the box girder

In IBS, the reliability indexes for constraints 5, 6, 8 and 11 are closer to the target value ($\beta_i^t = 3.0$, $i = 1, \ldots, 11$) than that in CBS. Therefore the proposed IBS method is very accurate during the Kriging-model-based RBDO methods.

4.5 Box girder design

As shown in Fig. 14, it is a large scale box girder; the total length is $14,620\,mm$. The target of the box girder design is to reduce its total weight. There are six random design variables d_i, $i = 1, \ldots, 6$, which denote the thicknesses of the stiffeners at different parts of the box girder. The main loads applied to the box girder are F_1, F_2, F_3, F_4, F_5 and F_6, as seen in Fig. 15. There are eight probabilistic constraints $g_i(\boldsymbol{X})$, $i = 1, \cdots, 8$, which denotes the displacements along x and z axes at 1, 2, 3 and 4 places.

These probabilistic constraints are implicit and need computer simulations, each finite element simulation will take about 2 min, the total computational cost is unaffordable when
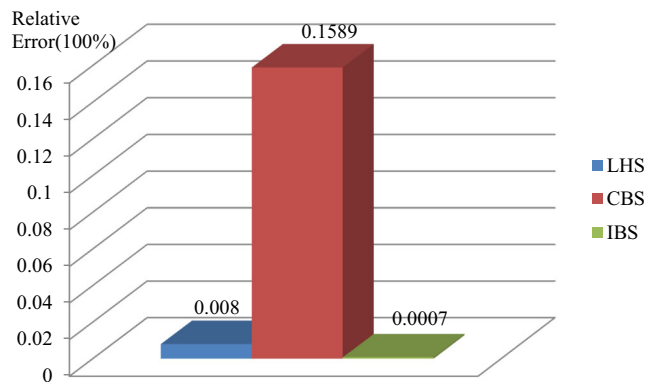

Fig. 16 Relative errors of the LHS、CBS、IBS methods for the box girder design

Table 5 Summary of the optimization results for the box girder design

Methods	Design variables	Samples	Relative error (100 %)
Anal.	(16.0000, 22.0000, 16.0000, 16.0000, 16.0000, 20.2776)	4736	/
LHS	(16.0000, 22.0000, 16.0000, 16.0000, 16.0000, 20.2811)	350	0.0080
CBS	(16.0000, 22.0000, 16.0000, 16.0000, 16.0000, 20.3472)	250	0.1589
IBS	(16.0000, 22.0000, 16.0000, 16.0000, 16.0000, 20.2776)	193	0.0007

these computer simulations are directly applied in RBDO. The optimization problem is defined as below:

$$find \quad (d_1, d_2, d_3, d_4, d_5, d_6)$$
$$\min : \quad total\ weight$$
$$s.t. \quad prob\Big(g_j(\boldsymbol{X}) < 1.059\Big) \geq \varPhi(\beta^t) \quad j = 1, \cdots, 8 \quad (28)$$
$$X_i \sim N\big(d_i,\ 0.2^2\big), \quad \beta_i^t = 3.0$$
$$16.00 \leq d_i \leq 22.00 \quad i = 1, \cdots, 6$$

The comparison results for the box girder design are shown in Table 5. The number of function calls in analytical method is 4,736, which is not efficient. Though the CBS method uses a smaller number of function calls than LHS, it cannot yield an accurate result and its relative error is 0.1589 % as seen in Fig. 16. In the proposed IBS method, the number of function calls is 193, which means the computational cost is reduced significantly and IBS is very efficient. The optimal design obtained from IBS is almost identical with the result of the analytical method; the relative error is 0.0007 %. Therefore, IBS is also very accurate.

5 Conclusion

In this paper, the importance boundary sampling method is proposed to improve the efficiency of Kriging-model-based RBDO method. Rather than fitting all the parts of the limit state constraint boundaries precisely within the whole design region, the proposed IBS method mainly selects sample points on these important constraint boundary parts. In the RBDO process, two importance coefficients are proposed to identify the critical parts of the boundaries. Also, in IBS, the sampling and optimization processes are conducted alternatively so that the optimization results can be used in the following sampling process to make the selection of the sample points more appropriate.

Several examples are tested to verify the accuracy and efficiency of the proposed method. Through the examples it can be seen that the proposed IBS method is very efficient. By using the importance coefficients, the IBS identifies the critical parts of the limit state constraint boundaries, and then chooses sample points mainly on these important boundary parts. Most of sample points could give useful information to the optimization process, and the Kriging models in the

vicinity of the optimal design are well fitted. The results from IBS are almost identical with that of analytical method, which verifies that the proposed IBS is very accurate. Also, the IBS method uses the smallest number of constraint function calls, which confirms that the computational cost is reduced significantly and the proposed IBS method is very efficient.

However, there are still many challenges need to be overcome to improve the accuracy of kriging model. For example, selecting a small number of initial sample points can lead to loss of information in certain regions of the design space, while selecting a large number of initial sample points might lead to more computational times. The initial samples will also decide whether the global solution can be found always. So it is a significant research to find an efficient method for deciding the initial raining set size. In the future work, we will study this issue to make the IBS method more applicable to RBDO problems.

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