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AUTHORS' REPLY

Exact truss topology optimization for external loads and friction forces

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Abstract In replying to a valuable Discussion by Mariano Vázquez Espi, the authors show that the problem of friction forces in general can be handled by the Prager-Rozvany layout theory, and the optimal Michell layout does not always correspond to the maximum value of the static friction force. Moreover, it is explained that discontinuities in the specific cost function can be accommodated by an extended version of the Prager-Shield optimality criteria, which was already demonstrated in the second author's first (1976) book.

Keywords Topology optimization · Michell trusses · Support costs · Prager-Rozvany layout theory · Discontinuous cost functions

1 Introduction

This text was prompted by a Discussion by Vázquez Espi (2013), subsequently called the 'Writer', who is to be commended for putting forward highly constructive ideas about a research paper by the authors (Rozvany and Sokół 2012).

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It will be explained subsequently that discontinuous cost functions in general, and reaction forces consisting partially of friction forces in particular, can be handled by the layout theory of Prager and Rozvany (1977), or an extension (Rozvany 1974, 1976) of the Prager and Shield (1967) conditions.

The Writer has also suggested (Vázquez Espi 2013) the correct optimal region layout for a generalization of the much-cited 'MBB-beam' benchmark example (Lewiński et al. 1994) to non-vertical reaction forces, as will be explained below.

2 Truss with friction forces at the supports

In his Discussion, the Writer considered a Michell-type truss (Michell 1904) with a central vertical point load, and two supports having horizontal reaction components consisting of friction forces. These he denoted by X, subject to $X \leq \mu Y$, where μ is the static friction coefficient and Y is the vertical reaction component. It is known that a solid body resting on a horizontal surface does not start sliding if the above inequality is satisfied. According to the Writer, this example was mentioned originally by Cox (1965).

Some exact optimal truss topologies for the above problem, considering various μ-values, are shown in Fig. 1. Very similar solutions were presented by the Writer in his Discussion.

For $\mu = 0$, the exact analytical solution, shown in Fig. 1a or b, has been originally derived by Lewiński et al. (1994).

For $\mu = 1$, the well-known analytical solution, also reviewed recently (Rozvany 2011, Fig. 4d), is shown in Fig. 1d. It consists of two R-regions (see e. g., Rozvany et al. 1995 for optimal regions). The interesting aspect of this solution is that it has been derived for two hinge supports, which would allow any arbitrary horizontal reactions. The optimality of this topology implies that it has a lower volume than a truss topology for any other horizontal reaction, which means that it is optimal also for $\mu > 1$. If we had, for example $\mu = 1.5$, then the horizontal friction force could be $X = 1.5Y$, but the optimal topology would only use up part of this, with the optimal horizontal reaction value of $X = Y$.

The above result implies that the optimal truss topology does not necessarily use up the highest possible value of the static friction force.

Naturally, the Writer is correct that for $\mu < 1$ the solution in Fig. 1d is not feasible, and the solution makes use of the highest feasible value of the friction force. The latter observation can be concluded from the fact that in Table 1 below the optimum volume monotonically decreases with increasing friction coefficients, provided that $\mu < 1$. These solutions are of the type shown in Fig. 1c, which consists of a circular fan CDE, a both-way curved Hencky-net DEGF and a straight member AF. This most interesting feature was pointed out by the Writer. As μ tends to zero, the length of the bar AF converges to zero (see Fig. 1a). If μ tends to 1, then the region DEGF disappears, only one bar (i.e. CD) of the circular fan CDE will have a non-zero cross-sectional area, and this bar becomes collinear with the bar AF.

Table 1 A comparison of numerical solutions by the writer and the first author and the analytical solution derived in Section 3 (percentage errors in brackets)

$\bar{V} = V\sigma_P/PL$

μ	Numerical writer	Numerical first author	Analytical
0.00	1.435 (0.14 %)	1.4333 (0.025 %)	1.432959
0.25	1.310 (0.12 %)	1.3087 (0.023 %)	1.308394
0.50	1.190 (0.08 %)	1.1892 (0.016 %)	1.189034
0.75	1.084 (0.02 %)	1.0838 (0.009 %)	1.083750
1.00	1.000 (0.00 %)	1.0000 (0.000 %)	1.000000

The First Author has derived accurate numerical solutions by his method (Sokół 2011), using half a billion potential truss elements for the problems in Fig. 1 of the Writer (Vázquez Espi 2013). The results by the Writer and those by the First Author show a good agreement. This is shown in Table 1 (here σ_P is the permissible stress).

It can be seen that there is a three-digit agreement between the solutions of the Writer and those of the First Author, and an at least four digit agreement between the (rounded) analytical solutions and the numerical solutions by the First Author. The numerically and analytically derived optimal topologies for the above cases are shown in Fig. 2.

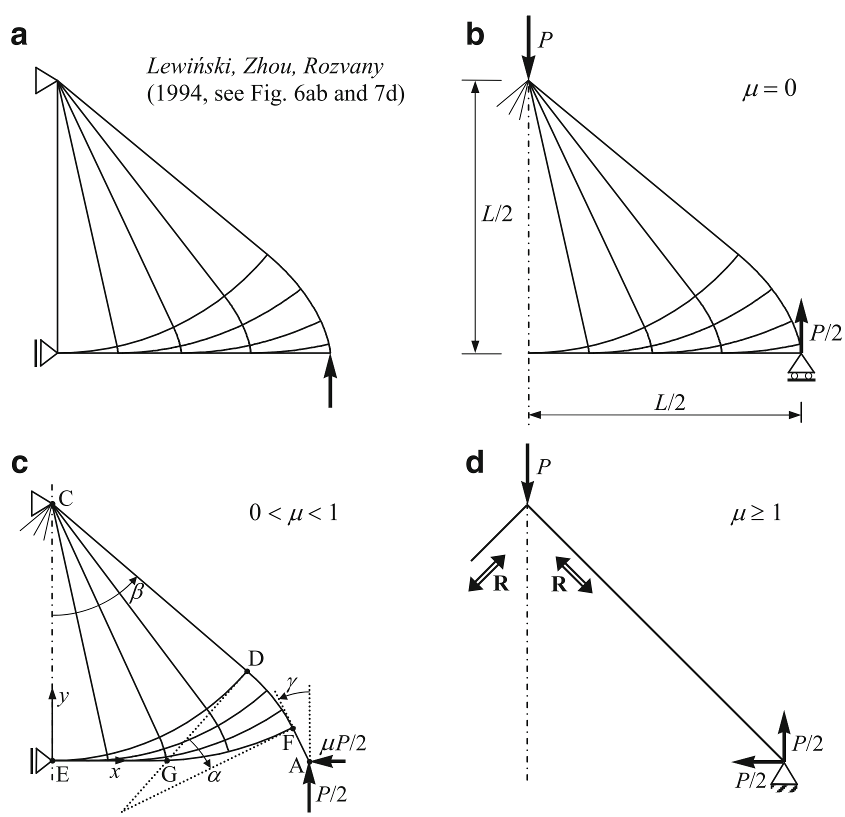

Fig. 1 Optimal layouts for selected friction coefficients

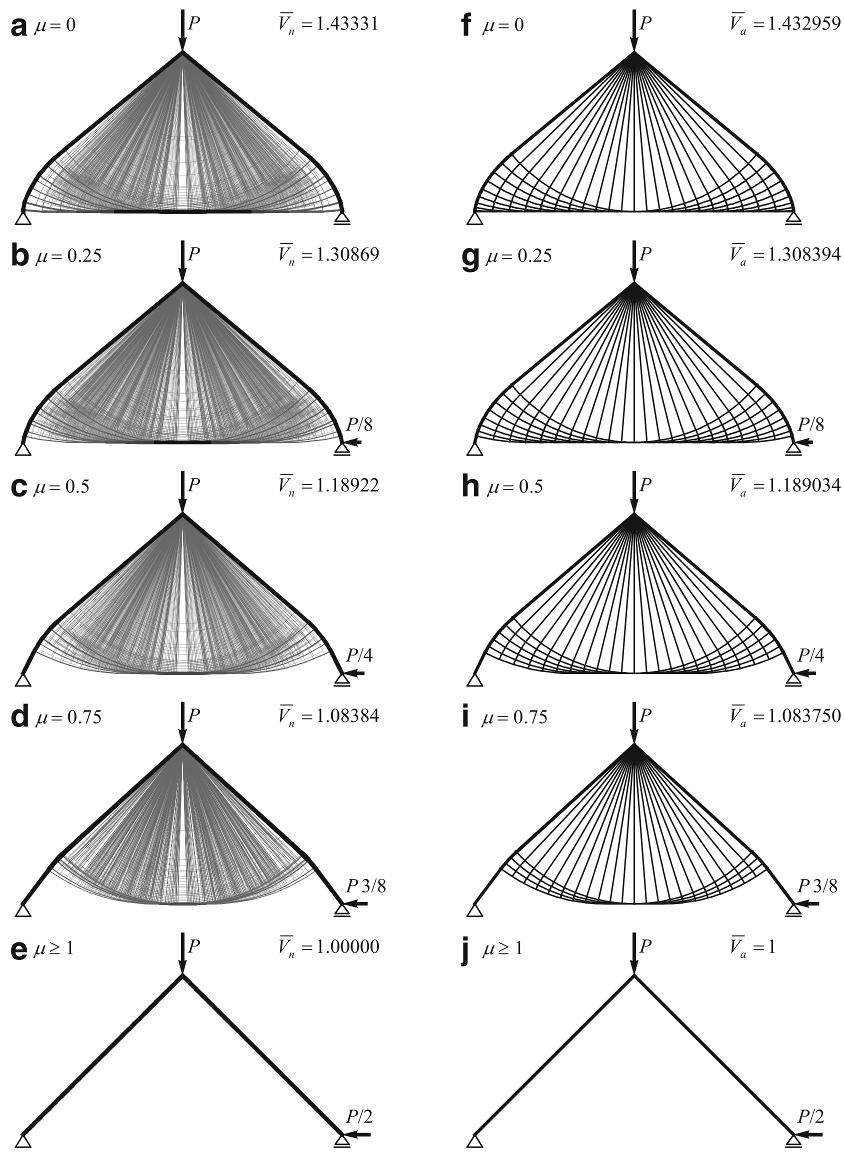

Fig. 2 Numerical (**a**–**e**) and analytical (**f**–**j**) solutions for various friction coefficients

3 Details of the analytical derivation of the exact solutions in Fig. 1 and Table 1

The general topology of the problem investigated, see Fig. 1c, consists of a truss-like continuum CEGFD and the straight bar AF. The solution can be derived by adjusting the angles α and β to properly link these two substructures.

The geometry and displacement field of the most important region EGFD were derived by Lewiński et al. (1994). In particular, using (194) and (195) from this paper we can define the coordinates of point F as

$$\begin{aligned} x^{\mathrm{F}} &= x(\alpha,\beta) = \frac{L}{2}\,(\cos(\beta-\alpha)\,[F_1(\beta,\alpha) - F_3(\alpha,\beta)] \\ &\quad + \sin(\beta-\alpha)\,[F_2(\beta,\alpha) - F_2(\alpha,\beta)]) \\ y^{\mathrm{F}} &= y(\alpha,\beta) = \frac{L}{2}\,(-\cos(\beta-\alpha)\,[F_2(\beta,\alpha) - F_2(\alpha,\beta)] \\ &\quad + \sin(\beta-\alpha)\,[F_1(\beta,\alpha) - F_3(\alpha,\beta)]) \end{aligned} \tag{1}$$

The functions F_1, F_2 and F_3 in (1) and G_1 and G_0 in (8), were defined in the above paper (Lewiński et al. 1994).

Note that the external chord AFDC has to pass smoothly through the point F, hence

$$\beta - \alpha = \gamma \tag{2}$$

where γ is the angle of inclination of bar AF (measured from vertical axis y, see Fig. 1c) and defined as

$$\gamma = \arctan \mu \tag{3}$$

Moreover, the tangent to the curve DF at point F has to pass through the point A, hence

$$x^{\mathrm{F}} + \mu\, y^{\mathrm{F}} = L/2 \tag{4}$$

Using the above compatibility equation together with (1)–(3), one can prove that the optimal angle α has to satisfy the following transcendental equation:

$$\cos\gamma - F_1(\alpha+\gamma,\alpha) + F_3(\alpha,\alpha+\gamma) = 0 \tag{5}$$

It can be shown that the solution to this equation is unique (the left-hand side is monotonically decreasing function with respect to α).

Now, using dual formulation, we can derive the volume of the optimal structure as the work of external loads over the adjoint displacements. The member compression force in the bar AF is known and given by

$$N = \frac{P}{2\cos\gamma} \tag{6}$$

This force works along the length $L_{\mathrm{AF}} = \sqrt{(L/2 - x^{\mathrm{F}})^2 + (y^{\mathrm{F}})^2}$, implying by (1)

$$L_{\mathrm{AF}} = \frac{L}{2}\,(F_2(\alpha,\ \alpha+\gamma) - F_2(\alpha+\gamma,\ \alpha) + \tan\gamma\ [F_1(\alpha+\gamma,\ \alpha) - F_3(\alpha,\ \alpha+\gamma)]) \tag{7}$$

and on the corresponding displacement of point F (see (203) in the paper by Lewiński et al. 1994) which is defined by

$$u^{\mathrm{F}} = \frac{L}{2}\,[G_0(\alpha,\ \alpha+\gamma) + 2\gamma\, G_1(\alpha,\ \alpha+\gamma) + 2F_2(\alpha,\ \alpha+\gamma)] \tag{8}$$

Finally, the volume of the whole structure can be written as

$$V = 2\frac{N}{\sigma_{\mathrm{P}}}\left(u^{\mathrm{F}} + L_{\mathrm{AF}}\right) \tag{9}$$

or

$$V = \frac{P\,L}{2\,\sigma_P}\frac{1}{\cos\gamma}\,(3F_2(\alpha,\ \alpha+\gamma) - F_2(\alpha+\gamma,\ \alpha) + G_0(\alpha,\ \alpha+\gamma) + 2\gamma\, G_1(\alpha,\ \alpha+\gamma) + \sin\gamma) \tag{10}$$

where α is solution of (5).

4 Answering the query about discontinuous cost functions

The Writer is asking an interesting question, how the Prager and Shield (1967) condition and the Prager and Rozvany (1977) layout theory can be applied to discontinuous cost functions. This problem was discussed already in the author's first book (Rozvany 1976, pp. 85–87), and even earlier in a paper (Rozvany 1974). It will be explained in the context of discrete value problems involving beams (and grillages), but the same method can be extended to other classes of problems.

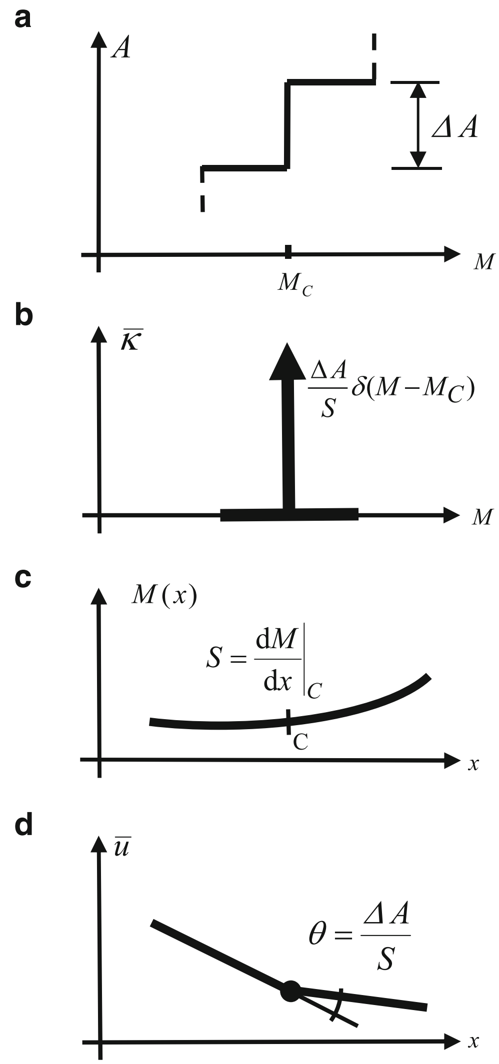

Fig. 3 Extension of the layout theory to discontinuous cost functions illustrated with beam and grillage optimization: **a** Discintinuity in the cost function, **b** corresponding impulse in the adjoint strain (i.e. curvature) field, **c** Strain field (moment diagram), **d** concentrated rotation in the adjoint beam (i.e. adjoint displacement field) (after Rozvany 1974, 1976)

Figure 3a shows part of a functional relation between the cross-sectional area of a beam (it can be in a grillage), and the bending moment in it. This is a discrete value problem, with a finite number of discrete cross sections available for our design. It can be seen that there is a discontinuity of ΔA in the cost function at $M = M_C$. It is shown in the above publications, that this discontinuity causes in the adjoint strain field (here: curvature field $\bar{\kappa}$) an impulse or Dirac delta 'function' (actually generalized function), whose magnitude depends on (a) the magnitude (ΔA) of the discontinuity and (b) the first derivative (here denoted by S for slope) of the beam moment M with respect to the longitudinal spatial coordinate x (see Figs. 3b and c).

The impulse in the adjoint curvature causes a concentrated rotation in the 'adjoint beam' having the adjoint displacements $\bar{u}$ (Fig. 3d). For examples the reader is referred to the publications cited above.

Returning to the problem of Michell trusses with friction forces, the horizontal component X of the support force can take on the range of values $0 \leq X \leq \mu Y$. The Writer correctly suggests that one way of formulating the upper constraint is making the cost of the horizontal reaction zero for $X \leq \mu Y$, and infinity for $X > \mu Y$. This makes the cost function for the horizontal reaction component discontinuous, which prompted the Writer to ask the above mentioned question about suitability of the Prager and Rozvany (1977) layout theory for problems with discontinuous cost functions.

It was explained above that the layout theory can handle discontinuous cost functions, but the Authors believe that the problems discussed in Section 1 can be formulated in a simpler fashion, for example by imposing the constraint $0 \leq X \leq \mu Y$ on our volume minimization problem.

Finally, the Authors agree with the Writer that the vertical reaction components do not influence the volume of the truss, because these components are invariable and are given uniquely by equilibrium.

5 Concluding remarks

The Authors would like to thank the Writer of the Discussion (Vázquez Espi 2013) for extremely constructive and useful ideas and questions. They have also resulted in a new class of optimal truss topologies, which constitute a generalization of the much cited 'MBB-beam' solutions (Lewiński et al. 1994).

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