RESEARCH PAPER

A semi-Lagrangian level set method for structural optimization

Mingdong Zhou · Michael Yu Wang

Abstract In level set based structural optimization, semi-Lagrange method has an advantage to allow for a large time step without the limitation of Courant–Friedrichs–Lewy (CFL) condition for numerical stability. In this paper, a line search algorithm and a sensitivity modulation scheme are introduced for the semi-Lagrange method. The line search attempts to adaptively determine an appropriate time step in each iteration of optimization. With consideration of some practical characteristics of the topology optimization process, incorporating the line search into semi-Lagrange optimization method can yield fewer design iterations and thus improve the overall computational efficiency. The sensitivity modulation is inspired from the conjugate gradient method in finite-dimensions, and provides an alternative to the standard steepest descent search in level set based optimization. Two benchmark examples are presented to compare the sensitivity modulation and the steepest descent techniques with and without the line search respectively.

Keywords Structural optimization · Level set method · Line search · Sensitivity modulation · Semi-Lagrange method

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1 Introduction

Among various numerical methods for structural shape and topology optimization, the level set based approach excels in smooth structural boundary representation, compared with the material based approaches. Popular material based techniques include the homogenization method (Bendsoe and Kikuchi 1988), the method of Solid Isotropic Material with Penalization (Zhou and Rozvany 1991; Bendsøe and Sigmund 1999), and the Evolutionary Structural Optimization (Xie and Steven 1993). The level set method, as introduced in Osher and Sethian (1988), enables numerical computations of a time-dependent evolution of a free-form shape, which is represented implicitly over a Cartesian grid in a higher dimensional space. The pioneer works of incorporating this method into structural optimization can be found in Sethian and Wiegmann (2000), Allaire et al. (2004) and Wang et al. (2003). For a comprehensive review of structural optimization, the readers may refer to excellent surveys (Haftkaa and Grandhib 1986; Eshenauer and Olhoff 2001; Guo and Cheng 2010).

For a material based optimization method, design variables are explicitly defined using the element densities or properties. Thus many approximation algorithms, like the optimality criteria method and the method of moving asymptotes, are workable and efficient (Bendsoe and Sigmund 2003). But in the level set based optimization, the implicit design model is updated by solving a Hamilton–Jacobi type partial differential equation. It is well known that for the traditional upwind scheme (Osher and Fedkiw 2002), the time step is strictly constrained by the CFL condition (Courant et al. 1967), which limits the moving front to propagate not more than one grid length in a single time step. Therefore, the optimization process usually takes hundreds of iterations to converge, even for simple problems.

To circumvent this limitation, a number of techniques have been proposed, including the fast marching method (Sethian 1996), AOS scheme (Lu et al. 1991; Weickert et al. 1998; Luo et al. 2008), radial basis function based parameterization method (Luo et al. 2007), and the semi-Lagrange method (Strain 1999; Xia et al. 2006).

Structural optimization with semi-Lagrange method (Bargteil et al. 2006; Strain 1999; Xia et al. 2006) has great potential for fast convergence. It is unconditionally stable and free of CFL condition, with a larger time step allowed than that of the upwind scheme. The theoretical framework of semi-Lagrange method for contouring moving interface was given by Strain in his seminal articles (Strain 1999, 2000, 2001). In this method, the level set equation is solved by evaluating an explicit semi-Lagrangian formula through a backward path tracing. Numerical accuracy is the only factor that restricts the upper limit of the time step rather than the CFL condition (Ritchie 1986). Early applications, like that in (Xia et al. 2006), use semi-Lagrange method with a fixed time step for level set based structural optimization. In such case, although the optimization process is accelerated without considering the restriction of CFL condition, to find a proper time step still requires tedious and repetitive testing for different practical problems. Moreover, the fixed time step scheme lacks flexibility and theoretical validity.

In this paper, a line search scheme is proposed for the semi-Lagrange method to allow adaptive time steps in optimization. By leveraging the stable property of semi-Lagrange method, this scheme automatically spurs a sufficient descent of the objective for each design iteration. Although performing a line search inevitably incurs additional computational cost, the scheme has a mechanism to avoid unnecessary searches by taking into account of some practical characteristics of shape and topology optimization. To better illustrate the significance of the line search in level set based structural optimization, a new Sensitivity Modulation (SM) scheme is proposed in this paper for comparative study, which can be regarded as an alternative to the standard steepest descent method. From the viewpoint of an optimization algorithm, the Conjugate Gradient (CG) method is generally known of better performance than the Steepest Descent (SD) method for finite dimensional problems (Nocedal and Wright 2006). Various nonlinear CG techniques have been well studied in last decades (Dai and Yuan 1999; Fletcher and Reeves 1964; Fletcher 1987; Hestenes and Stiefel 1952; Liu and Storey 1991; Polyak 1969; Zhou et al. 2011). Directly inspired from the CG method in finite dimensions, the scheme is devised in a heuristic manner with a similar format of the traditional CG method. Numerical experiments reveal that, by incorporating the line search, the overall design efficiency of using SD and SM can improve substantially. Moreover, the proposed sensitivity modulation scheme behaves as effectively as the steepest descent method with using line search.

This paper is organized as follows. The basic theory of the level set based structural optimization and the semi-Lagrange method is reviewed briefly in Section 2. Detailed algorithms of the line search and sensitivity modulation are elaborated in Section 3. Numerical experiments of two benchmark examples are presented and discussed in Section 4. Finally, conclusion and future work are stated in Section 5.

2 Semi-Lagrangian level set method

2.1 Level set based structural optimization

In the level set framework, a Lipschitz-continuous implicit function $\Phi(\mathbf{x}(t), t)$ is used to represent the design domain D, which includes the admissible shape Ω of underlying structure. The domain contains two regions: the structural interior Ω and the void $D \backslash \Omega$. Let Γ be the structural boundary:

$$\Gamma = \{\mathbf{x}(t) : \Phi(\mathbf{x}, t) = 0, \mathbf{x} \in \Omega\}. \tag{1}$$

Thus, the level set equation can be written as:

$$\frac{\partial \Phi}{\partial t} + v_n \cdot |\nabla \Phi| = 0, \quad \Phi(\mathbf{x}, 0) = \Phi_0(\mathbf{x}), \tag{2}$$

where $\Phi_0(\mathbf{x})$ defines the initial model, and v_n is the magnitude of the velocity of shape transformation along the local unit normal outward to the structural boundary (Allaire et al. 2004; Wang et al. 2003).

The level set method has been applied to numerous topics of structural optimization, such as mean compliance, stress and frequency response problems. In this paper, only the linearly elastic problem of the compliance minimization with volume constraint is concerned to illustrate the effectiveness of the proposed algorithms. Extensions to other objectives and constraints are straightforward.

The classical formulation of mean compliance minimization with volume constraint is defined as:

$$\min_{\Phi} \quad J(\mathbf{u}, \Phi) = \int_\Omega \frac{1}{2} \mathbf{E} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) \, d\Omega, \tag{3}$$

$$\text{s.t.} \quad G(\Phi) = \int_\Omega d\Omega - V_0 \le 0, \tag{4}$$

$$a(\mathbf{u}, \mathbf{w}) = l(\mathbf{w}) \text{ for all } \mathbf{w} \in U, \tag{5}$$

where U denotes the space of kinematically admissible displacement field with predefined displacement at Dirichlet

boundary, $\mathbf{u}$ the elastic displacement under external loads, ε the strain tensor, $\mathbf{E}$ the elasticity tensor, and V_0 the allowable volume of design. Equation (5) corresponds to the linear elastic equilibrium, where a and l denote the energy bilinear and the load linear variational form respectively:

$$a(\mathbf{u}, \mathbf{w}) = \int_{\Omega} \mathbf{E}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{w}) \, d\Omega, \tag{6}$$

$$l(\mathbf{w}) = \int_{\Omega} (f \cdot \mathbf{w}) \, d\Omega + \int_{\Gamma} (h \cdot \mathbf{w}) \, d\Omega, \tag{7}$$

where f is the body force, and h the traction force on the Neumann boundary.

A common technique to solve the constrained optimization problem is to construct an augmented objective functional as:

$$L = J + \lambda \left(\int_{\Omega} d\Omega - V_0 \right), \tag{8}$$

where $\lambda > 0$ is the Lagrange multiplier. The shape gradient of the general objective L is determined effectively as:

$$\dot{L}(\Omega) = \int_{\Gamma} G v_n d\Gamma, \tag{9}$$

where G is known as the shape gradient density of the problem. The SD method simply takes $v_n = -G$, and this choice guarantees $L(\Omega) \leq 0$. This is a common strategy for the level set method in structural optimization facilitated with sensitivity analysis of the optimization problem (Wang and Wang 2004).

For simplicity, traction-free boundary and body-force free conditions are assumed in this paper. Thus, the steepest descent direction of optimization search is found by setting the normal velocity v_n as (Allaire et al. 2004; Wang et al. 2003):

$$v_n = -\left(\lambda - \frac{1}{2} \mathbf{E}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) \right), \tag{10}$$

with λ updated by the following rule, for iteration number i:

$$\lambda_{i+1} = \max \left\{ 0, \lambda_i + \mu \left(\int_{\Omega} d\Omega - V_0 \right) \right\}, \tag{11}$$

where μ is a penalty parameter. Consequently, the optimal design can be obtained by solving the Hamilton–Jacobi (2) with (10) and (11).

2.2 Semi-Lagrange method

To be self-contained, the basic theory of semi-Lagrange method for hyperbolic partial differential equation is reviewed briefly in this section. Readers may refer to relevant literature, such as Bargteil et al. (2006), Staniforth and Cote (1991), Strain (1999, 2000, 2001) and Xia et al. (2006), for more details. Semi-Lagrange method leverages the fact that (2) propagates the solution Φ along the characteristic curve $\mathbf{x} = \mathbf{c}(t)$ by:

$$\dot{\mathbf{c}}(t) = \mathbf{v}(\mathbf{c}(t), t). \tag{12}$$

Evaluation of $\Phi(\mathbf{x}_{i+1}, t_i + \Delta t)$ amounts to tracing the characteristic back in time from $\mathbf{x}_{i+1} = \mathbf{c}(t_i + \Delta t)$ to the previous point $\mathbf{x}_i = \mathbf{c}(t_i)$, and then setting $\Phi(\mathbf{x}, t_{i+1}) = \Phi(\mathbf{c}(t_i), t_i)$. Different interpolation schemes (Staniforth and Cote 1991; Xia et al. 2006) are pertinent to compute $\Phi(\mathbf{x}_i, t_i)$ even if $\mathbf{x}_i$ might be off the grid.

In this paper, the first order Courant–Isaacson–Rees formula (Courant et al. 1952) is adopted to approximate the characteristic curve through any spatial point $\mathbf{x}$ via a straight line with the speed $\mathbf{v}(\mathbf{x}, t - \Delta t)$ of the previous time step:

$$\mathbf{c}(t) \approx \mathbf{x} - \Delta t \cdot \mathbf{v}(\mathbf{x}, t - \Delta t). \tag{13}$$

Thus, the Hamilton–Jacobi equation can be readily solved as:

$$\Phi(\mathbf{x}, t_{i+1}) = \Phi(\mathbf{x} - \Delta t \cdot \mathbf{v}(\mathbf{x}, t_i), t_i). \tag{14}$$

The explicit semi-Lagrange method is unconditionally stable, and it allows for a much larger time step compared to that of the upwind scheme restricted by CFL condition. Despite the relaxation, it requires repetitive testing to determine a proper time step for a practical application (Xia et al. 2006). To alleviate this problem, a line search algorithm is proposed here to determine an appropriate time step for each iteration, instead of using a fixed setting throughout the optimization process. It will be seen that the advantages of semi-Lagrange method can be fully realized together with the line search for the level set based structural optimization.

3 Line search and sensitivity modulation

3.1 A line search algorithm

A line search technique generally refers to two mainstreams: exact and soft line search (Nocedal and Wright 2006). Given

a predetermined descent direction, the exact line search locates the exact minimum along the direction, while a soft search only leads to a result satisfying some liberal criteria but with a cheaper computational cost. For the level set based structural optimization, performing exact line search is not necessary due to two considerations. Firstly, each line search requires a finite element analysis for evaluating the structural performance, which is computationally expensive especially for high-accuracy analysis. Secondly, the time step cannot be set arbitrarily large due to numerical accuracy and stability concerns in solving the partial differential equation, even using semi-Lagrange method. Therefore, a soft line search strategy is preferred for semi-Lagrange method.

The infinite-dimensional version of the line search algorithm is listed in Table 1 in pseudo code. It is a direct adoption of Armijo–Golden's rule (Burger 2003), which is well known for its effectiveness. The basic rationale of this algorithm lies in the idea of sufficient descent in each search. A bisection technique is adopted here to find an appropriate time step to locate the target state Φ_k^* meeting the following condition:

$$L\left(\Phi_k^0\right) + \beta_2 \cdot \dot{L}\left(\Phi_k^0\right) \cdot \Delta t \le L\left(\Phi_k^*\right) \le L\left(\Phi_k^0\right)$$
$$+ \beta_1 \cdot \dot{L}\left(\Phi_k^0\right) \cdot \Delta t, \tag{15}$$

where Φ_k^0 is the initial level set model at iteration k. β_1 and β_2 denote two thresholds for checking whether the objective reduced within the current time step Δt meets the expectation, and, by default, $0 < \beta_1 < \beta_2 < 1.0$. $T_{\min}$ and $T_{\max}$ represent the lower and upper limit of time step respectively. It is quite common that the optimal time step Δt may not be found within a few steps, or it even lies outside the interval between $T_{\min}$ and $T_{\max}$. Thus, for each iteration, a maximum number of search M is set to limit unnecessary computation.

Practical experience of using the level set method reveals that the remarkable shape and topological changes occur mostly at the beginning of the optimization process. It is in this phase that the design process finds its way to the

Table 1 Pseudo code of the line search algorithm

Algorithm 1. Line Search

Input: current level set model Φ_k^0; maximum time step allowed by CFL condition t_{CFL}; lower and upper limits of time step $T_{\min}$ and $T_{\max}$; maximum number M of line search for each iteration.

1: Set $\varepsilon > 0$, $\alpha > 0$, $t_1 = T_{\min}$, $t_2 = T_{\max}$, $0 < \beta_1 < \beta_2 < 1.0$, $i = 1$;

2: **if** (conditions (16)–(18) are all satisfied)

$$\Delta t = t_{CFL};$$

3: **else**

4:     **while** $i \le M$

$$\Delta t = \frac{1}{2}(t_1 + t_2);$$

5:         Compute the new level set model Φ_k^i using semi-Lagrange method;

6:         Evaluate objective function $L(\Phi_k^i)$;

7:         **if** ($L(\Phi_k^i) - L(\Phi_k^0) \le \beta_2 \cdot \dot{L}(\Phi_k^0) \cdot \Delta t$)

$$t_1 = \Delta t;$$

8:         **else if** ($L(\Phi_k^i) - L(\Phi_k^0) \ge \beta_1 \cdot \dot{L}(\Phi_k^0) \cdot \Delta t$)

$$t_2 = \Delta t;$$

9:         **else**   **break**;

10:         **end**

11:         $i++$;

12:     **end**

13: **end**

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near-optimum region. Once the region has been found, the overall topology usually has little change, but a relatively long oscillation process follows afterwards which deals with local geometry adjustment as well as convergence. This two-phase phenomenon is basically due to the augmented Lagrange multiplier strategy, and it is also affected by parameter setting, such as the penalty parameter. Therefore, performing a line search seems necessary to guide the optimization at the beginning, when there is little information known except the search direction. For the later phase, as the proper topology and general shape have been determined, improving the accuracy can reduce oscillation, and thus lead to a quick and smooth convergence. Therefore, a line search is not needed for the later stage.

For these reasons, a criterion is set with the following three conditions:

1) $\left| \dfrac{V - V_0}{V_0} \right| \leq \varepsilon,$ (16)

2) $\left| \dfrac{L\left(\Phi_k\right) - L\left(\Phi_{k-1}\right)}{L\left(\Phi_{k-1}\right)} \right| \leq \alpha,$ (17)

3) $\left| \dfrac{L\left(\Phi_{k-1}\right) - L\left(\Phi_{k-2}\right)}{L\left(\Phi_{k-2}\right)} \right| \leq \alpha,$ (18)

where V is the structural volume at current iteration, V_0 the volume constraint,and and α the thresholds defined by the

user. In each iteration, if all the three conditions are satisfied, it implies that the optimization process is about to converge. Then the time step is simply set to be the maximum allowed by the CFL condition. Otherwise, the line search is performed.

Note that the above criterion is restricted to the optimization problem with volume constraint. For other different problem formulations, a proper relaxation of the convergence criterion may be served as a similar criterion to terminate the line search.

With the proposed line search scheme, although additional computation cost arises for each search, the overall convergence efficiency can be enhanced with much fewer iterations. A reasonable final design can also be obtained effectively, as shown in the numerical tests presented in Section 4.

3.2 A sensitivity modulation scheme

In this section, a sensitivity modulation scheme is proposed for semi-Lagrangian level set based structural optimization. The scheme is directly inspired from the idea of CG method in finite dimensions, and thus is formulated similar to the CG method.

In the algorithm of steepest descent of semi-Lagrange method, it simply takes $v_n = -G$ as current search direction, without considering the information of either the structural

Table 2 Sensitivity modulation algorithm

Algorithm 2. Quasi-conjugate gradient
Input: level set model Φ_k of iteration k ; resetting condition number n_p .
1: Start with $k = 0$, set $\gamma > 0$, and initialize $v_{n_0} = -G_0$;
2: Determine the time step Δt using line search method in direction of v_{n_k} ;
3: Compute the new shape using Semi-Lagrange method: $$\Phi(\mathbf{x},t_{k+1}) = \Phi(\mathbf{x} - \Delta t \cdot v_{n_k}(\mathbf{x},t_k) \cdot \mathbf{n}, t_k) ;$$
4: Find the new direction of search by: $$v_{n_{k+1}} = -G_{k+1} + \beta_k v_{n_k}, \qquad \beta_k = \dfrac{\int_{\Gamma_{k+1}} G_{k+1} \cdot G_{k+1} d\Gamma}{\int_{\Gamma_k} G_k \cdot G_k d\Gamma}$$
5: If any of following cases happens, reset $\beta_k = 0$
1) $\int_{\Gamma_k} G_k \cdot G_k d\Gamma < \gamma$
2) $L(\Phi_{k+1}) - L(\Phi_k) \geq 0$
3) $(k \bmod n_p) = 1$
6: $k++$, and then repeat steps 2 - 5 until convergence

configurations or the search directions at previous steps. A strong motivation for devising a modulated sensitivity is essentially to incorporate the earlier updating information into the current search.

Table 2 lists the detailed algorithm of the sensitivity modulation. Starting with an initial design $\Phi_0(k = 0)$ and a normal velocity of steepest descent $v_{n_0} = -G_0$, the line search is first evoked to find a proper time step. After the updated design Φ_{k+1} with the corresponding shape gradient density G_{k+1}, a modulated search direction is defined by:

$$v_{n_{k+1}} = -G_{k+1} + \beta_k v_{n_k}, \tag{19}$$

$$\beta_k = \frac{\int_{\Gamma_{k+1}} G_{k+1} \cdot G_{k+1} d\Gamma}{\int_{\Gamma_k} G_k \cdot G_k d\Gamma}, \tag{20}$$

where Γ_k denotes the structural boundary at iteration k. Note that β_k is taken in a similar form to that of the finite dimensional problem (Fletcher 1987). Here, for the infinite dimensional problem at hand, the gradient vector norm is replaced analogously with the integration over the boundary.

Nonetheless, the newly generated direction from (19) may lose significance for optimization after a few iterations, due to the inexact soft line search and the infinite dimensional nature of the problem. As a remedy, it is advisable to restart with the steepest descent search to ensure meaningful computation and efficiency. The following three resetting conditions are suggested:

1) $\displaystyle\int_{\Gamma_k} G_k \cdot G_k d\Gamma < \gamma,$ $\tag{21}$

2) $L(\Phi_{k+1}) - L(\Phi_k) \geq 0,$ $\tag{22}$

3) $(k \mod n_p) = 1,$ $\tag{23}$

where γ is a small positive number and n_p is a pre-defined number of steps. If any one of the conditions occurs, then the search direction will be reset to the steepest descent direction. However, a proper selection of n_p depends on applications, and no universal optimal setting may exist.

4 Numerical examples

In this section, two benchmark examples of structural optimization are studied using the schemes of SD and SM with and without line search respectively. An "ersatz material" approach (Allaire et al. 2004; Wang et al. 2003) is adopted for finite element analysis. Without loss of generality, a fixed time step of $\Delta t = t_{CFL}$ is used for the traditional cases without using the line search, so that the numerical accuracy and stability are guaranteed. Re-initialization is performed at the end of each design iteration after the level set model has been updated. All the examples are carried out on a PC with 4 GB RAM and Intel Core2Quad CPU of 2.66 GHz speed.

4.1 Cantilever beam

The first classical example, a cantilever beam, is considered with the design domain and initial design shown in Fig. 1. The structure is discretized with a mesh of 80×40 elements. The volume ratio of the final design is constrained to be 60 % of the design domain.

In order to better demonstrate the effectiveness of the proposed algorithms, the following parameters are consistent for all the four experiments: $L = 200$, $H = 100$, $F = 10$, $\lambda = 0.3$, $T_{\min} = 0.8 \times t_{CFL}$, $T_{\max} = 6 \times t_{CFL}$, $\beta_1 = 0.1$, $\beta_2 = 0.9$, $E = 1$, $E_o = le - 3$, $= 5\%$, $\alpha = 1\%$, $\gamma = 0.1$, where E_o denotes the Young's modulus of the weak material phase.

4.1.1 Steepest descent

First, the SD method is used for the cases of with and without line search respectively. Figures 2 and 3 illustrate the results of optimization for the respective case. The corresponding convergence curves are depicted in Fig. 4a and b. In Fig. 4, the left vertical marking line in each sub-figure indicates the near-optimal region of the first phase of the optimization process, where the volume constraint is satisfied, and the optimal topology and shape are generally determined. The second vertical marking line indicates that the final optimum design has been obtained. Key indicators

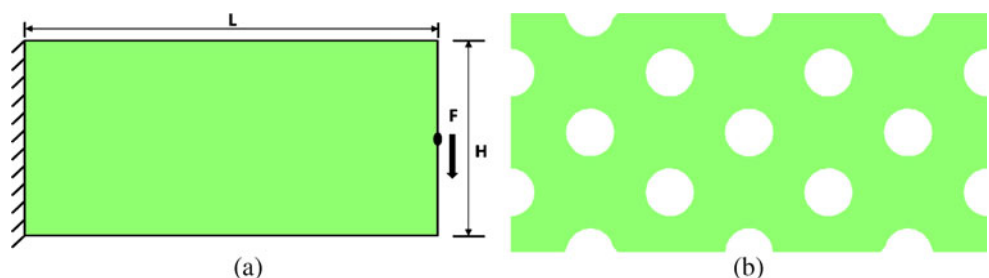**Fig. 1** Cantilever beam: (**a**) design domain; (**b**) initial design

(a)   (b)

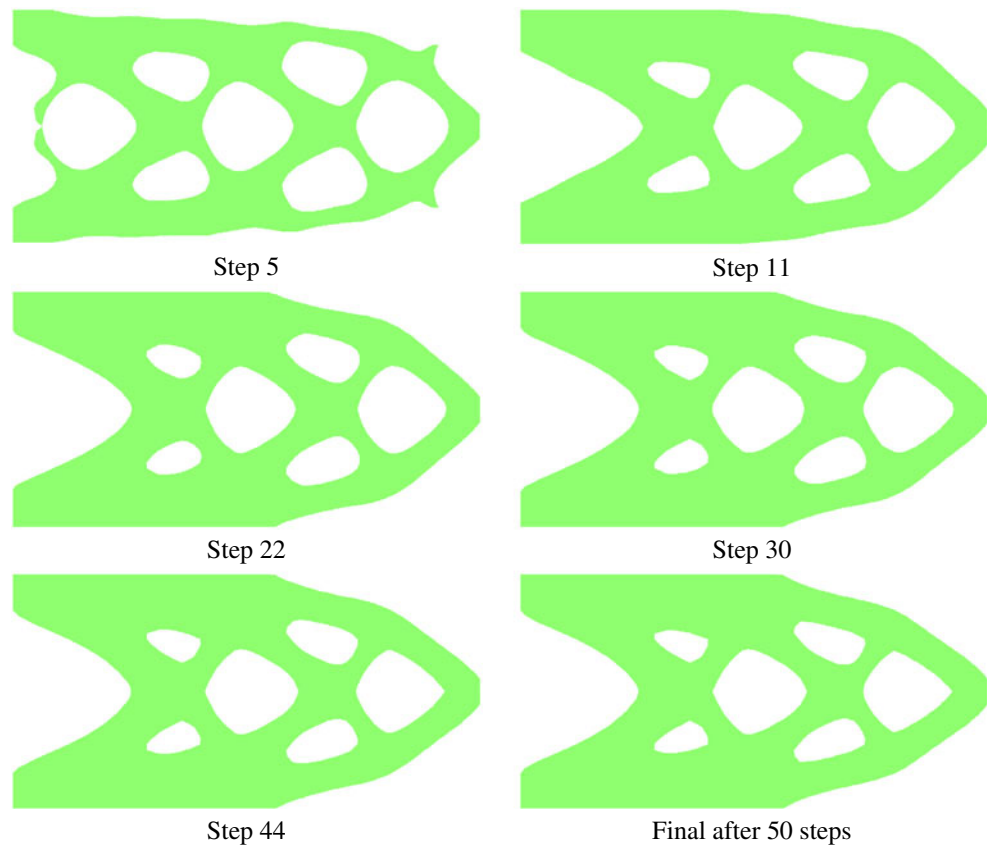**Fig. 2** Cantilever beam: steepest descent with line search

Step 5

Step 11

Step 22

Step 30

Step 44

Final after 50 steps

Fig. 3 Cantilever beam: steepest descent without line search

Step 10

Step 20

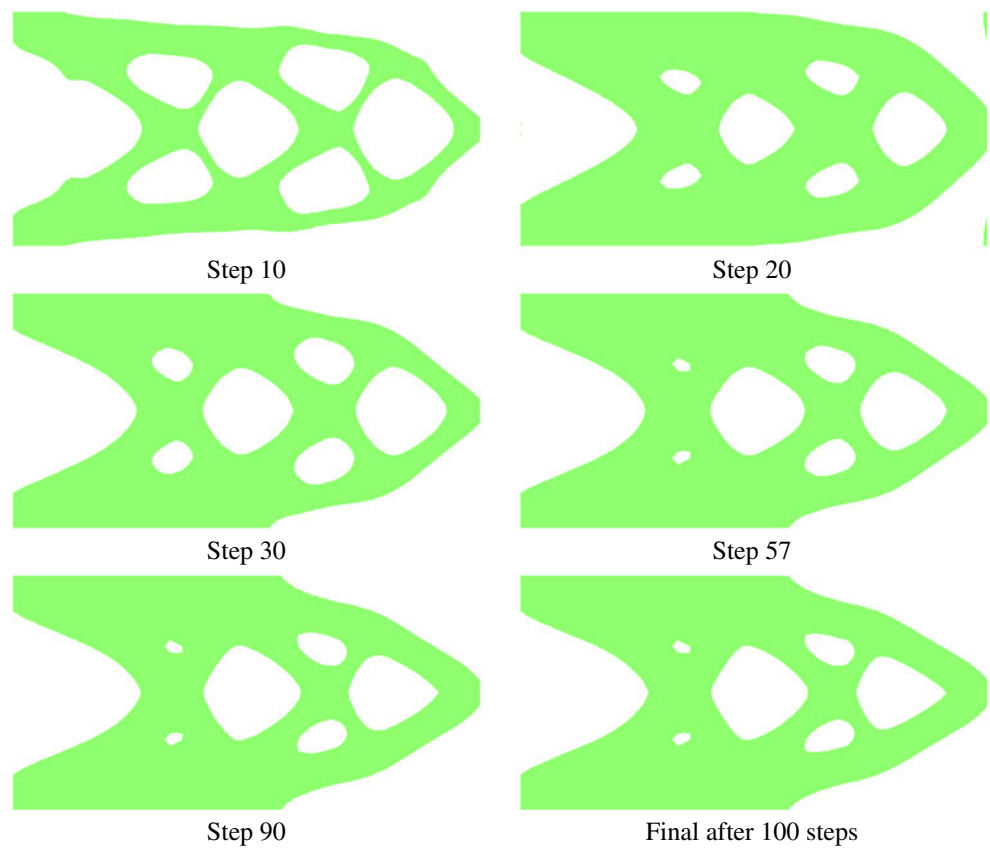Step 30

Step 57

Step 90

Final after 100 steps

Fig. 4 Convergence curves of
the cantilever beam example:
(**a**) SD with line search;
(**b**) SD without line search;
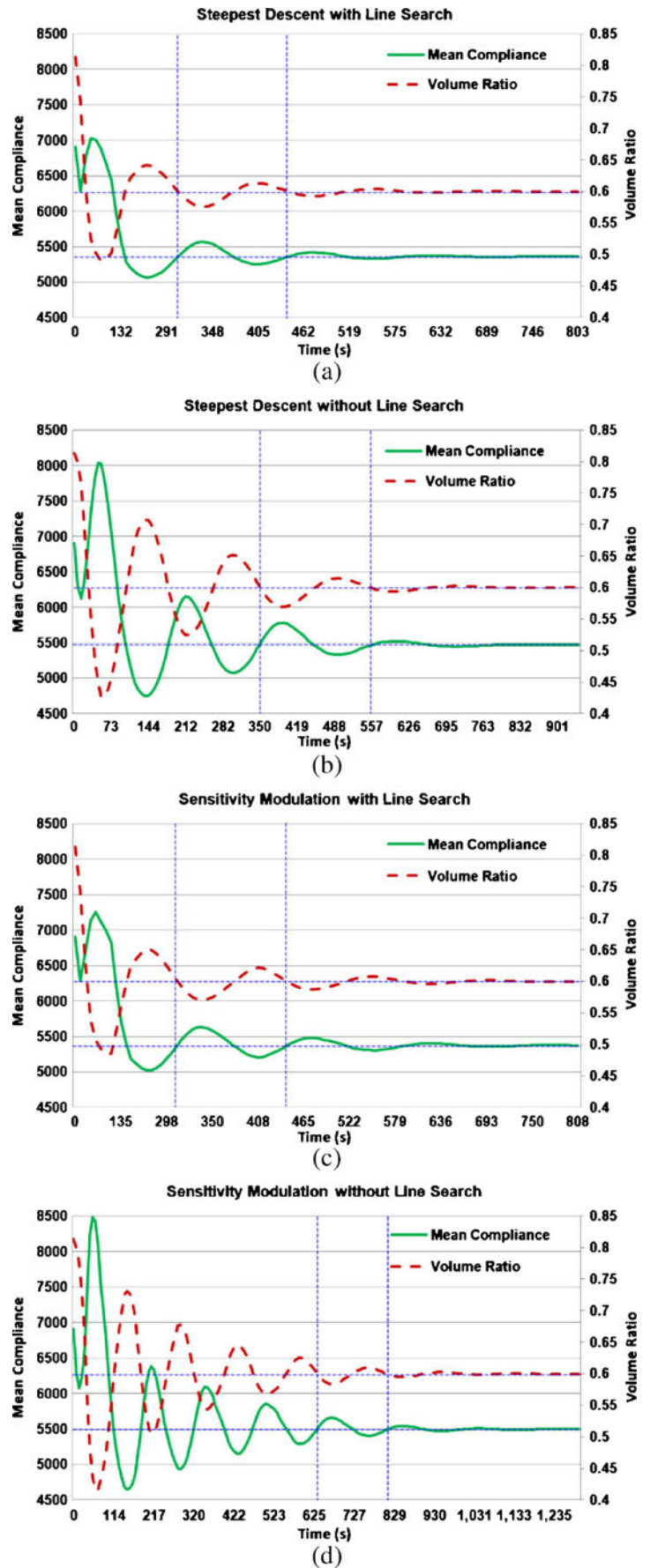(**c**) SM with line search;
(**d**) SM without line search

Table 3 Results of the cantilever beam example for different search algorithms

	Final mean compliance	No. of steps to near-optimum	Time to near-optimum (s)	No. of steps to converge	Time to converge (s)
SD with line search	5363	22	303	44	443
SD	5466	57	356	90	563
SM with line search	5373	22	310	43	439
SM	5500	93	632	119	808

of the optimization processes are listed in Table 3 for the cantilever beam example. Note that for structural optimization, in order to find a global optimum, it relies on convex problem formulation and proper optimization techniques, which is beyond the scope of this paper. Thus, only the local optimum is considered here.

As shown in Fig. 2, with the line search, major topological and shape changes occur at the first 10 steps. The line search is performed only for about 20 steps at the beginning before conditions of (16)–(18) are satisfied. Subsequently, a smooth and quick convergence is observed. For this example, the SD method with line search finds the near-optimal region within 22 steps of 303 s. Convergence is obtained within 44 steps of 443 s. Comparatively, for the case without the line search, as shown in Fig. 3, it takes 57 steps of 356 s

to find the near-optimal region, and 90 steps of 563 s to converge. The final designs are also slightly different for the two cases, with the line search scheme resulting in a lower mean compliance in the final design, thus a better design.

With the line search, each iteration generally takes longer execution time due to additional finite element analysis, but the total time to reach the near-optimal region and the final convergence is reduced by about 15 % and 21 % respectively, compared to that without the line search, as shown in Table 3.

4.1.2 Sensitivity modulation

The SM is also applied to the same cantilever beam example with and without using line search. Figures 5 and 6

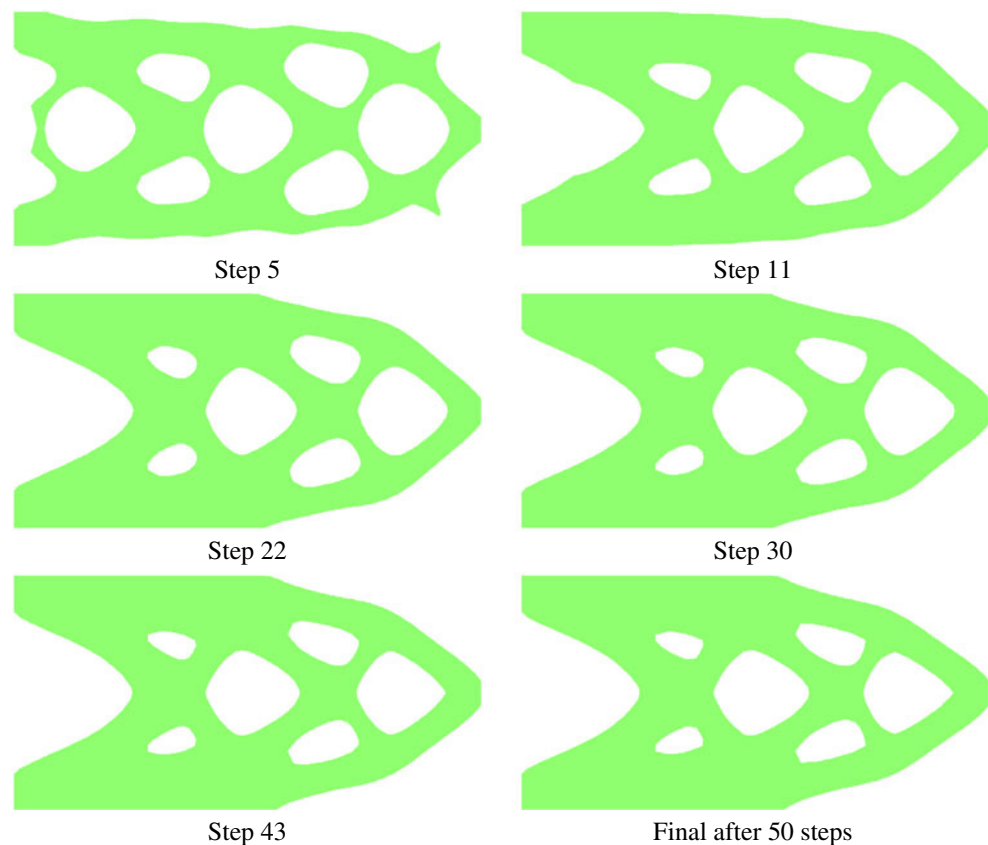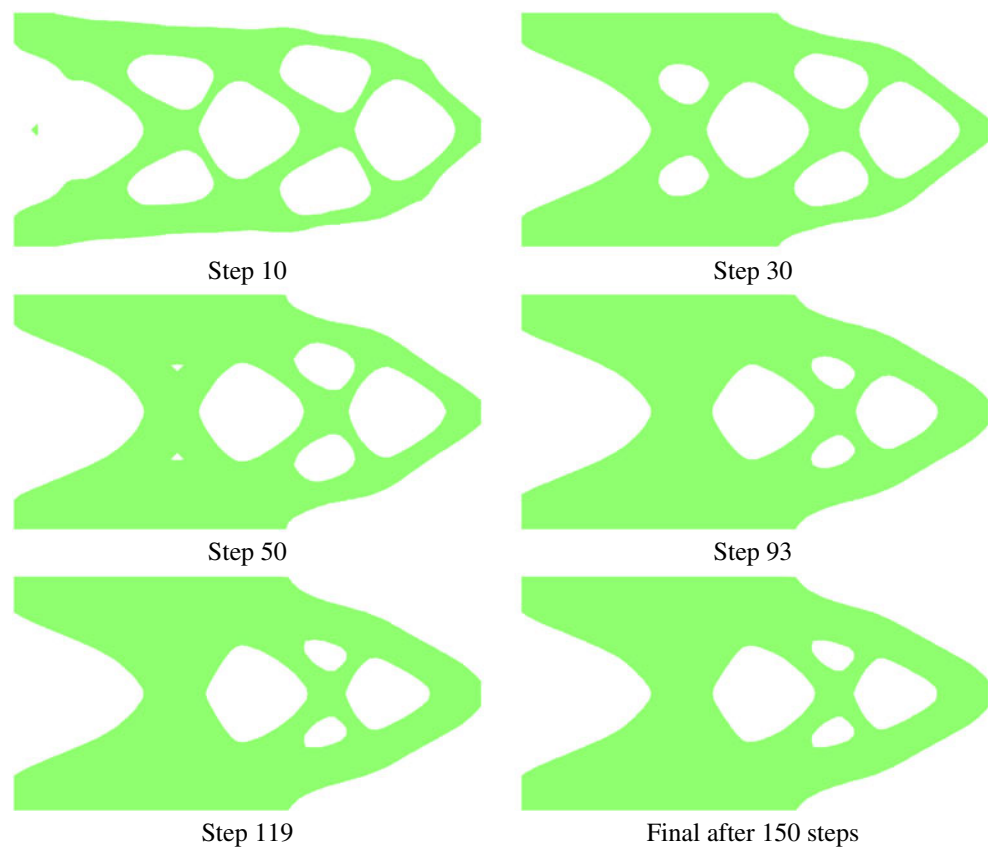**Fig. 5** Cantilever beam: sensitivity modulation with line search

Step 5

Step 11

Step 22

Step 30

Step 43

Final after 50 steps

Step 10

Step 30

Step 50

Step 93

Step 119

Final after 150 steps

show the optimization processes of each case. The corresponding convergence history are depicted in Fig. 4c and d respectively.

For the design process shown in Fig. 5 with the line search, it takes 22 steps of 310 s to find the near-optimal region, and 43 steps of 439 s to converge. Comparing to that using the standard SD method with line search as shown in Fig. 2, although different intermediate designs are observed, the final optimal designs are similar, which demonstrate the effectiveness of the proposed SM scheme with line search.

On the other hand, if no line search is used for the SM scheme, the optimization process is highly oscillatory with a much larger number of iterations to converge (Fig. 4d).

Specifically, it requires 93 steps of 632 s to find the near-optimal region, and 119 steps of 808 s to converge. This scheme produces the largest mean compliance in the final design among all the numerical schemes examined. The corresponding final structure, as shown in Fig. 6, has less geometric complexity than the benchmark result (Allaire et al. 2004; Xia et al. 2006). In the implementation, $n_p = 4$ is adopted. If a larger value is chosen, invalid results, such as thin-wall or broken structure, may be obtained.

4.2 Bridge-type structure

The second example studied is a bridge-type structure with the design domain and initial design shown in Fig. 7. A
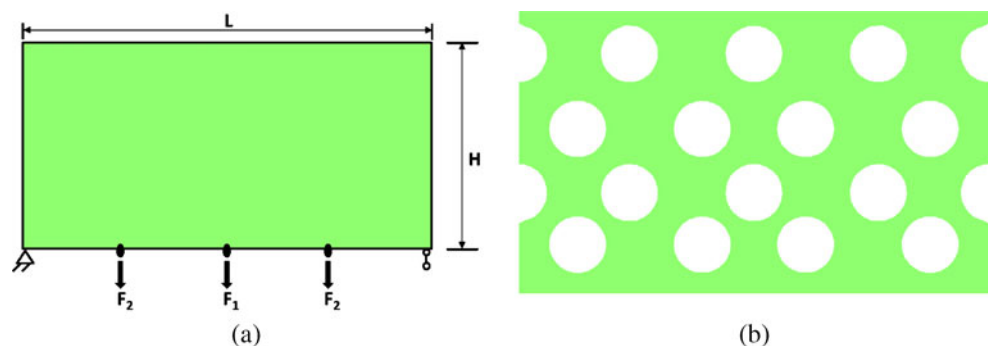
(a)

(b)

Fig. 8 Convergence curves of
the bridge-type structure
example: (**a**) SD with line
search; (**b**) SD without line
search; (**c**) SM with line search;
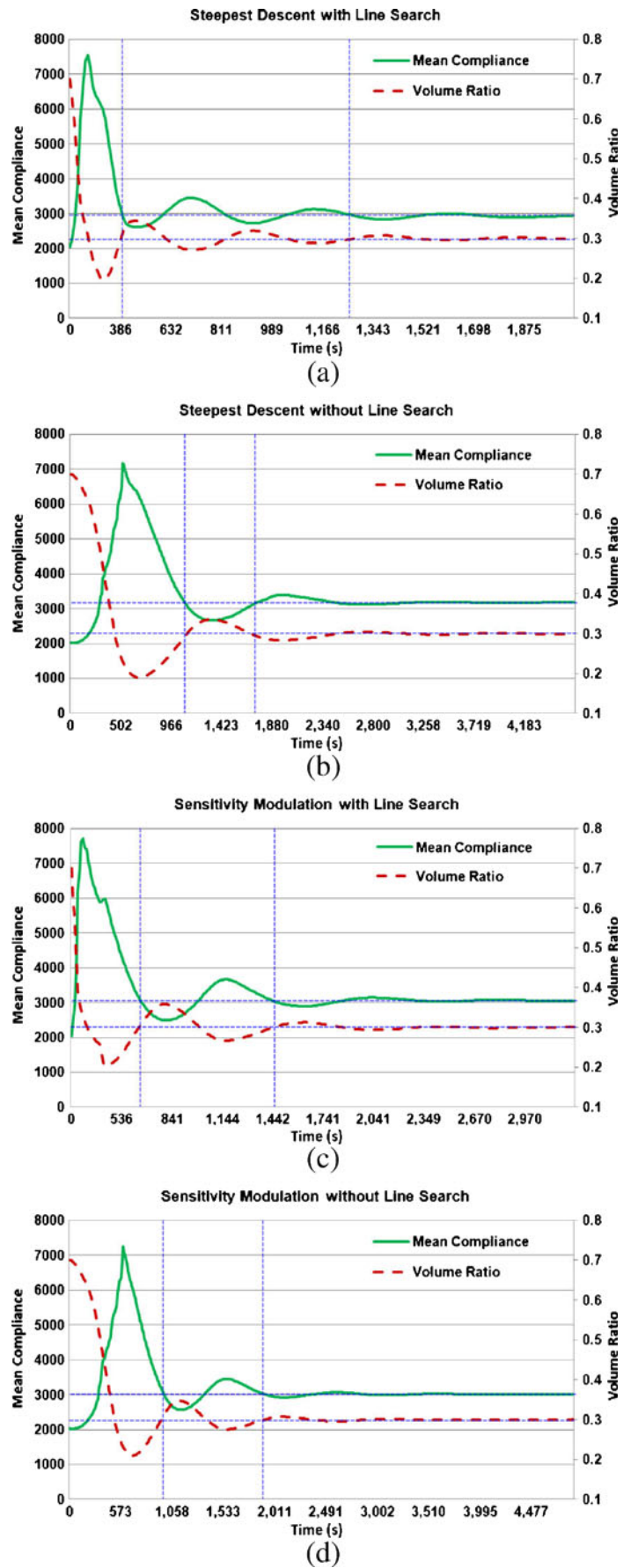(**d**) SM without line search

Table 4 Results of the bridge-type structure example for different search algorithms

	Final mean compliance	No. of steps to near-optimum	Time to near-optimum (s)	No. of steps to converge	Time to converge (s)
SD with line search	2914	22	410	88	1041
SD	3184	112	1076	180	1697
SM with line search	3042	40	638	121	1441
SM	3010	92	972	193	1935

mesh of 80×48 elements is used for both the structural analysis and the level set calculation. The final design is constrained to occupy 30 % volume of the design domain. Meanwhile, the parameters used for each experiment are as follows: $L = 200$, $H = 120$, $F_1 = 10$, $F_2 = 3$, $\lambda = 0.5$, $T_{min} = 0.8 \times t_{CFL}$, $T_{max} = 6 \times t_{CFL}$, $\beta_1 = 0.1$, $\beta_2 = 0.9$, $E = 1$, $E_o = 1e - 3$, $= 5$ %, $\alpha = 1$ %, $\gamma = 0.1$.

4.2.1 Steepest descent

The bridge-type structure is firstly optimized using the SD method with and without line search. The optimization processes are illustrated in Figs. 9 and 10 respectively, and the corresponding convergence curves are depicted in Fig. 8a

and b. Key indicators of the optimization processes are listed in Table 4 for the example.

As shown in Fig. 9, major topological and shape change occur during first 22 steps of 410 s. After that, a process of local geometrical adjustment is observed, followed with a convergence in 88 steps of 1041 s. Comparatively, if no line search is employed as shown in Fig. 10, it takes 112 steps of 1076 s to find the near-optimal region, and 180 steps of 1697 s to converge.

Similar to the previous example, the line search scheme results in a better structural design of lower mean compliance. Meanwhile, as shown in Table 4, the total execution time to reach the near-optimal region and the final convergence is reduced by about 62 % and 39 % respectively.

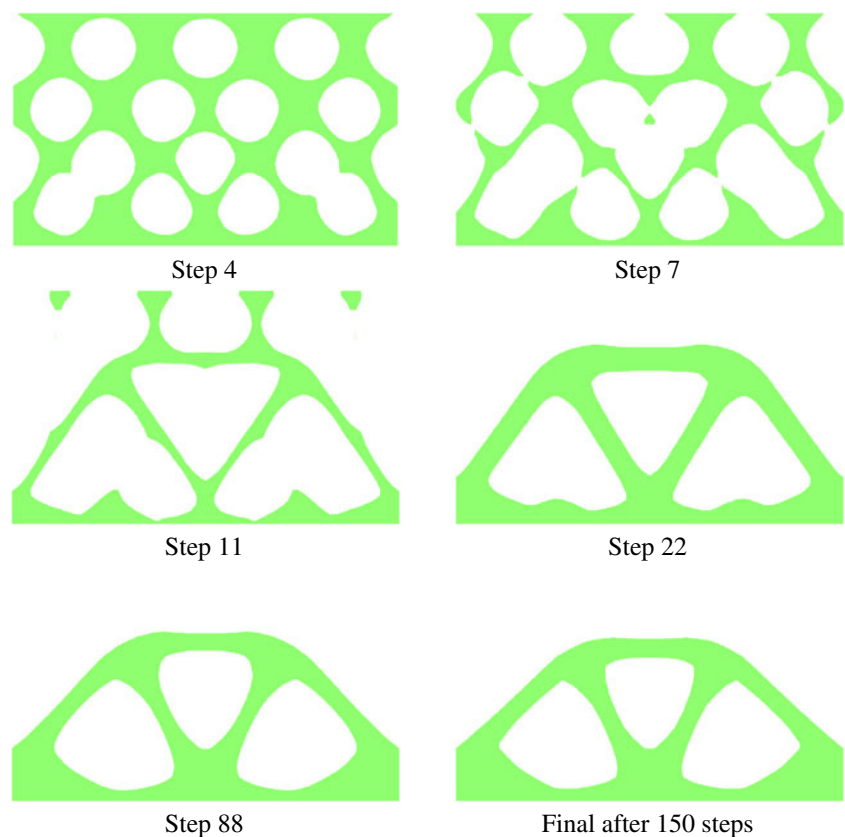**Fig. 9** Bridge-type structure: steepest descent with line search

Step 4

Step 7

Step 11

Step 22

Step 88

Final after 150 steps

Fig. 10 Bridge-type structure: steepest descent without line search

Step 33

Step 44

Step 56

Step 112

Step 180

Final after 300 steps

Fig. 11 Bridge-type structure: sensitivity modulation with line search

Step 5

Step 10

Step 20

Step 40

Step 121

Final after 200 steps
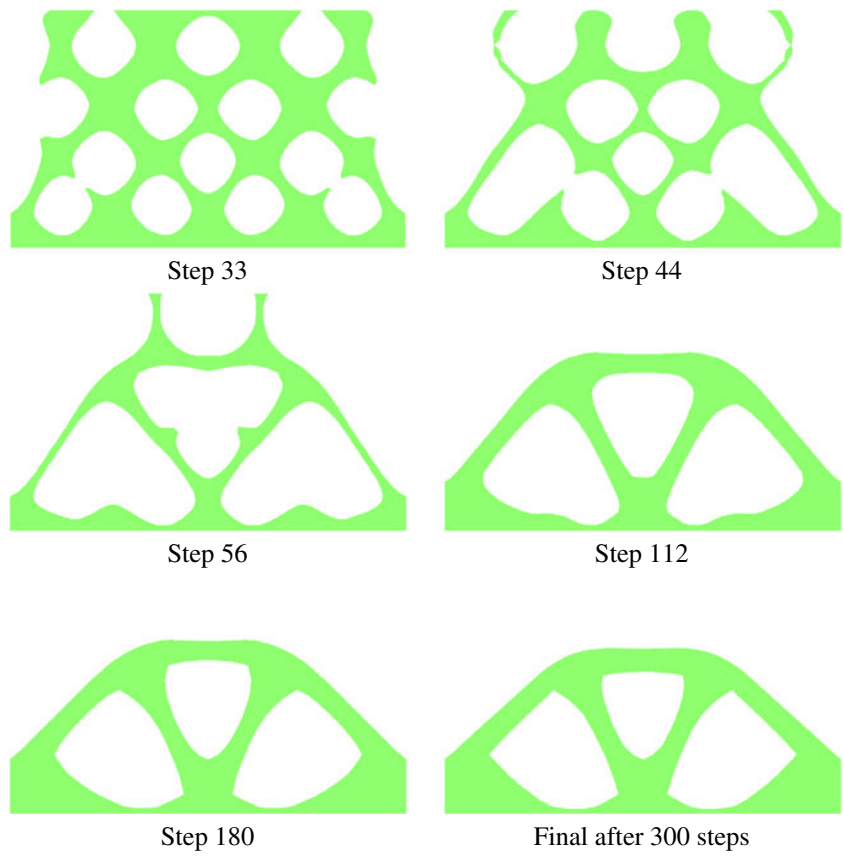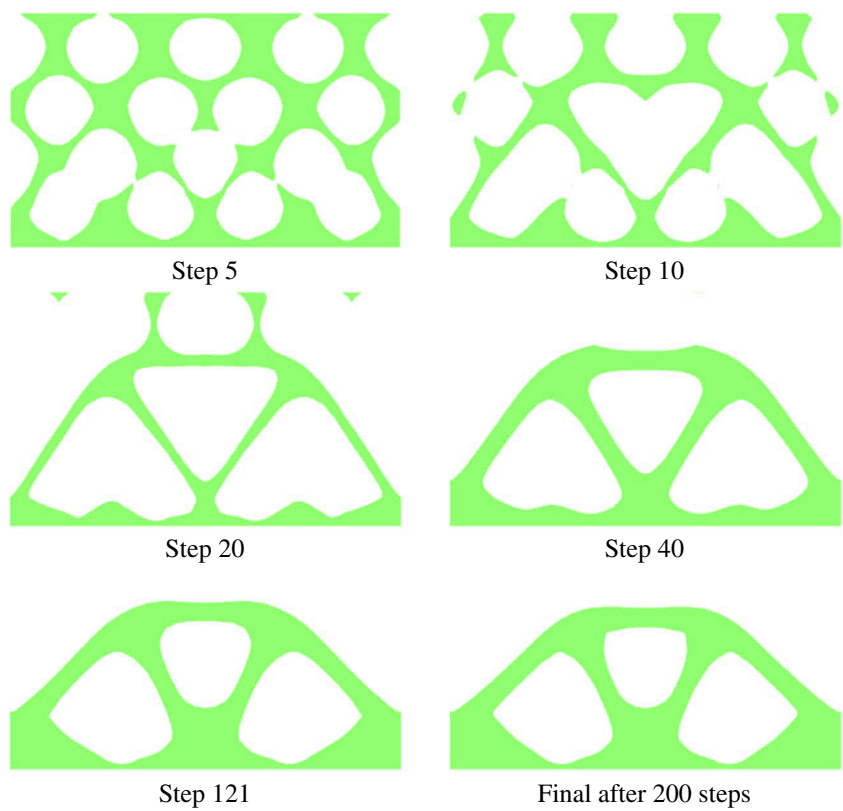
4.2.2 Sensitivity modulation

Figures 11 and 12 illustrate the optimization processes of the bridge-type structure using the SM with and without line search respectively. Figure 8c and d depict the corresponding design history.

As shown in Table 4, only 40 steps of 638 s are needed to reach the near-optimum region, and 121 steps of 1441 s are needed to obtain the optimal design. On the other hand, the SM without line search takes 92 steps of 972 s and 193 steps of 1935 s to achieve the same level of convergence respectively. Thus, line search scheme is more efficient for the SM method.

Interestingly, as shown in Figs. 9, 10, 11 and 12, although the design processes vary for different techniques used, all the final structural configurations have the same topology. However, as shown in Table 4, the SD with line search scheme is the most effective and efficient combination, which results in the lowest mean compliance of the final design and the shortest convergence time among all the test cases. The SM without line search behaves as effectively as the pure SD method with a lower mean compliance value of the final design but a longer convergence time. Note that, every optimization technique has its strength and weakness under certain circumstances. As a numerical technique, the proposed SM scheme may serve as an alternative to the standard SD method. In this implementation, $n_p = 7$ is adopted.

5 Conclusion

In this paper, a simple and workable line search algorithm is presented to enhance the efficiency of semi-Lagrangian level set based structural optimization. It allows for an adaptive time step to achieve sufficient descent of the objective for each design iteration. A strategy is given for its implementation to consider practical characteristics of shape and topology optimization. The strategy makes the line search computation worthwhile. For comparative study, a sensitivity modulation scheme is also presented as an alternative of the steepest descent search.

Numerical tests of two benchmark examples reveal that both the number of design steps and overall computational cost can be reduced substantially by incorporating the proposed line search scheme into semi-Lagrangian level set based structural optimization, compared to the conventional scheme. Meanwhile, both SD and SM can produce reasonable optimal designs effectively with the line search.

However, the numerical experiments show that the SM scheme may not behave as effectively as the steepest descent method in some cases without the line search. In other

Fig. 12 Bridge-type structure: sensitivity modulation without line search

Step 33

Step 44

Step 56

Step 92

Step 193

Final after 300 steps
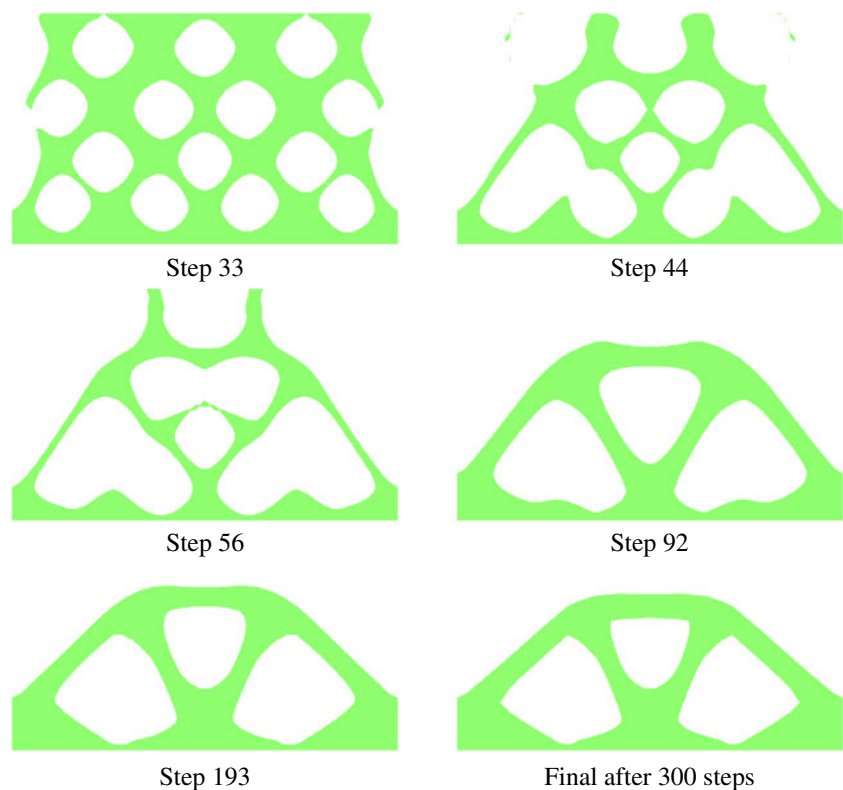
words, by altering the search direction along from the steepest descent direction to the modulated direction, it may not contribute to a significant reduction in computational effort. Employing a line search is important and indispensible to put the sensitivity modulation into effect. Note that, the proposed sensitivity modulation scheme is limited to level set based structural optimization, not a direct extension of CG method for general infinite dimensional problems. Its mathematical property will be further explored.

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