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BRIEF NOTE

Sensitivity filtering from a continuum mechanics perspective

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Abstract In topology optimization filtering is a popular approach for preventing numerical instabilities. This short note shows that the well-known sensitivity filtering technique, that prevents checkerboards and ensures mesh-independent designs in density-based topology optimization, is equivalent to minimizing compliance for nonlocal elasticity problems known from continuum mechanics. Hence, the note resolves the long-standing quest for finding an explanation and physical motivation for the sensitivity filter.

Keywords Topology optimization · Regularization · Filtering

1 Introduction

The numerical topology optimization approach in its most basic form solves the problem of minimizing compliance by distributing a fixed amount of material in a design domain. Usually, the problem is solved on a fixed finite element grid where the densities of each finite element or node constitute the design variables (Bendsøe and Sigmund 2004). The simple formulation suffers from instabilities like checkerboarding (Díaz and Sigmund 1995; Jog and Haber 1996) and mesh-dependencies (Sigmund and Petersson 1998). A wide range of formulations that ensure mesh independence and eliminate such instabilities have been proposed and include: sensitivity filtering (Sigmund 1997; Lazarov and Sigmund 2011), gradient control (Petersson and Sigmund 1998; Borrvall 2001), density filtering (Bruns and Tortorelli 2001; Bourdin 2001), regularized penalization (Borrvall and Petersson 2001), perimeter control (Ambrosio and Buttazzo 1993; Haber et al. 1996) and lately projection filtering (Guest et al. 2004; Sigmund 2007; Xu et al. 2010) and robust optimization approaches (Sigmund 2009; Wang et al. 2011; Schevenels et al. 2011). Sigmund (2009) provides an extensive review and comparison of different filtering approaches. Of the approaches mentioned above, the so far most popular regularization technique has been the sensitivity filter which is very simple to implement and has been extensively used, a.o. in public domain codes (Sigmund 2001; Andreassen et al. 2011; Tcherniak and Sigmund 2001; Aage et al. 2012) as well as commercial codes.

Sensitivity filtering was originally implemented as a heuristic modification of the strain energy densities inspired by image processing techniques (Sigmund 1997). Later, it was interpreted as filtering of sensitivities times the local density (Sigmund and Petersson 1998). In order to mitigate the dependency of the results on the mesh density, element compliance sensitivity is computed as a weighted average over elements within a fixed size neighborhood. Despite its widespread successes in academia and commercial codes, sensitivity filtering has suffered under its

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predicate "heuristic" and "numerically inconsistent" as the filtered sensitivities cannot be derived directly from the objective function, i.e. compliance. Many researchers have tried to find the objective function that corresponds to the modified sensitivities but without success.

The "suffering" of the sensitivity filter from its apparent lag of mathematical rigor in the field of topology optimization is actually somewhat surprising since many other engineering, physics and even mathematics disciplines, use similar concepts, mostly in the form of stabilization techniques. Examples are pressure stabilization techniques in fluid mechanics (Brooks and Hughes 1982; Tezduyar et al. 1992), nonlocal bone-growth rules in bio-mechanics (Mullender et al. 1994; Huiskes 2000), velocity extension approaches in level-set methods (Adalsteinsson and Sethian 1995; Allaire et al. 2004; De Gournay 2006), regularization of shape flow (Charpiat et al. 2007) and parameter estimation (Bukshtynov et al. 2011), nonlocal optical responses (Boardman 1982) and nonlocal elasticity (Toupin 1962; Eringen and Edelen 1972) and plasticity (Leblond et al. 1994; Tvergaard and Needleman 1995) as reviewed in Peerlings et al. (2001). Concerning the continuum mechanics applications, nonlocal processes are introduced to prevent physically unrealistic localizations of damage, plasticity, stress singularities and other effects. In practise the nonlocality is implemented by filtering operators or Helmholtz approaches just as done for filtering operations in topology optimization (Allaire 2007; Lazarov and Sigmund 2011; Andreassen et al. 2011).

In this short note we show how an optimization problem solved by the sensitivity filtering approach for the standard compliance minimization problem is equal to minimizing the compliance for a nonlocal elasticity problem as introduced by Eringen (1983), Ru and Aifantis (1993), Gutkin and Aifantis (1999), Gutkin (2000), Aifantis (2003) and Askes et al. (2008).

The note is composed as follows: in Section 2 we review the nonlocal and staggered elasticity approach by Aifantis and co-workers and extend it to filtering of strain energy densities; in Section 3 we review the continuum formulation of the sensitivity filtering approach to topology optimization, and in Section 4 we identify the commonalities of the two approaches and discuss implications of this observation.

2 Nonlocal elasticity theory

As presented by Ru and Aifantis (1993), Gutkin and Aifantis (1999), Gutkin (2000), Aifantis (2003) and Askes et al. (2008), a simple constitutive law for nonlocal elasticity theory can be written as

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - l^2\varepsilon_{kl,mm}\right) \tag{1}$$

where σ is the Cauchy stress, C the constitutive tensor, ε the linear engineering strain, l a material length scale parameter, and indices following a comma denote spatial derivatives. The parameter l represents the size of microstructural features like grain, inclusion sizes or interparticle distances and removes strain or stress singularities. Ignoring volume forces the static equilibrium equations are given by

$$\sigma_{ij,j} = 0 \tag{2}$$

and the strain-displacement relationships are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{3}$$

Combining (1)–(3) results in a fourth order equation

$$C_{ijkl}\left(u_{k,jl} + u_{l,jk} - l^2(u_{k,jl} + u_{l,jk})_{,mm}\right) = 0. \tag{4}$$

By the Ru–Aifantis theorem (Ru and Aifantis 1993) the fourth order equation can be simplified to two staggered second order problems: the standard elasticity problem

$$C_{ijkl}\left(u^c_{k,jl} + u^c_{l,jk}\right) = 0 \tag{5}$$

and the Helmholtz equation

$$u^g_i - l^2 u^g_{i,jj} = u^c_i, \tag{6}$$

where superscript ^{c} indicates classical (local) elasticity and superscript ^{g} indicates gradient enhanced (nonlocal) elasticity. Hence, (5) corresponds to the standard linear elasticity problem which then is followed by a separate filtering (smoothing) of the individual components of the displacement vector. Here, the filtered (nonlocal) displacement u^g resulting from the solution of (6) is identical to the solution u of the fourth order problem in (4). Solution of (5) and (6) is much simpler than directly solving (4), partly due to the lower order (i.e. standard elements can be used) and partly because the solution of (6) is computationally inexpensive and simple, once the solution of (5) has been computed.

The displacement-based filtering in (6) can be translated to the same type of expression in terms of strains (Gutkin and Aifantis 1999; Gutkin 2000) by differentiation

$$\varepsilon^g_{ij} - l^2\varepsilon^g_{ij,mm} = \varepsilon^c_{ij}, \tag{7}$$

which relates the nonlocal strains ε^g to the local strains ε^c.

Next, Askes et al. (2008) assume that both nonlocal and local stresses follow Hooke's law and hence they get

$$\sigma^g_{ij} = C_{ijkl}\varepsilon^g_{kl} \text{ and } \sigma^c_{ij} = C_{ijkl}\varepsilon^c_{kl} \tag{8}$$

which entails that (7) can, by premultiplication with the constitutive tensor C (Gutkin and Aifantis 1999; Gutkin 2000; Aifantis 2003), be rewritten to

$$\sigma_{ij}^{g} - l^2 \sigma_{ij,mm}^{g} = \sigma_{ij}^{c} \tag{9}$$

At this stage, we extend the theory by Aifantis and coworkers and multiply the stress smoothing expression (9) with the local strains ε^c on both sides of the equality sign to get

$$\varepsilon_{ij}^{c}\sigma_{ij}^{g} - l^2 \varepsilon_{ij}^{c}\sigma_{ij,mm}^{g} = \varepsilon_{ij}^{c}\sigma_{ij}^{c}, \tag{10}$$

which can be written in a simpler form as

$$\tilde{w} - l^2 \tilde{w}_{,mm} = w^c, \tag{11}$$

where $w^c = C_{ijkl}\varepsilon_{ij}^{c}\varepsilon_{kl}^{c}$ is the local (classical) strain energy density and $\tilde{w}$ is a nonlocal (filtered) strain energy density. Hence, a smoothed strain energy density distribution can, by use of nonlocal elasticity theory, be obtained as a post-processing step (11) to the standard linear elasticity problem (5). We remark here that the multiplication with the local strains in (10) is chosen as opposed to a multiplication with its non-local counterpart since the latter would entail a complex, non-linear (double) filtering operation on the left hand side of the equation.

3 Sensitivity filtering

For the simple compliance minimization problem the objective function can be written in continuous form as

$$\Phi = \int_\Omega C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} dV = \int_\Omega w dV, \tag{12}$$

where the strain energy density is defined as $w = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$. Assuming that the constitutive tensor depends on the density as $C_{ijkl} = \rho^p C_{ijkl}^0$ (according to the SIMP interpolation scheme (Bendsøe 1989; Zhou and Rozvany 1991)) and making use of the standard adjoint analysis technique, we can find the gradients of the objective function as

$$\begin{aligned}\Phi_{,\rho} &= -\int_\Omega C_{ijkl,\rho}\varepsilon_{ij}\varepsilon_{kl} dV \\ &= -p\int_\Omega \frac{C_{ijkl}}{\rho}\varepsilon_{ij}\varepsilon_{kl} dV = -p\int_\Omega \frac{w}{\rho} dV.\end{aligned} \tag{13}$$

The sensitivity filter works by modifying the gradients as

$$\begin{aligned}\widehat{\Phi(\mathbf{x})_{,\rho}} &= \frac{\int_\Omega H(\mathbf{y},\mathbf{x})\rho(\mathbf{y})\Phi(\mathbf{y})_{,\rho} dV}{\rho(\mathbf{x})\int_\Omega H(\mathbf{y},\mathbf{x}) dV} \\ &= \frac{\int_\Omega H(\mathbf{y},\mathbf{x}) w(\mathbf{y}) dV}{\rho \int_\Omega H(\mathbf{y},\mathbf{x}) dV},\end{aligned} \tag{14}$$

where $H(\mathbf{y},\mathbf{x})$ is a weighting function usually defined as an inverse linear distance function with $H(\mathbf{y},\mathbf{x}) = \max[0, \mathrm{dist}(\mathbf{y},\mathbf{x})]$, where $\mathrm{dist}(\mathbf{y},\mathbf{x})$ is the distance between points $\mathbf{x}$ and $\mathbf{y}$.

As suggested in Lazarov and Sigmund (2011) (and many other places) the sensitivity filtering operation (14) can be substituted with a PDE or Helmholtz filtering approach

$$\hat{w} - r^2 \hat{w}_{,mm} = \overline{w}, \tag{15}$$

where $\overline{w} = \rho\Phi_{,\rho} = -pw$. The filtered sensitivities are subsequently found from $\widehat{\Phi_{,\rho}} = \hat{w}/\rho$. Among other advantages the Helmholtz approach can be implemented very efficiently for use in parallel computing and makes it simpler and more stringent to filter complex domains with small void details like cracks and sharp corners (Lazarov and Sigmund 2011).

4 Discussion

Since the right hand sides of the filtering operations for nonlocal elasticity (11) and sensitivity filtering (15) are the same (except for a negative constant $-p$), the filtered strain energy densities are equal as well ($-p\tilde{w} = \hat{w}$). Furthermore, since the constitutive tensor C inside $\tilde{w}$ is a local property according to (8), the derivative of the nonlocal strain energy measure $\tilde{w}$ is simply $\tilde{w}_{,rho} = \frac{-p\tilde{w}}{\rho}$. This means that one can define a compliance minimization problem based on nonlocal elasticity theory and by mathematically stringent differentiation obtain the sensitivities as provided by the sensitivity filter! Hence, we conclude that the sensitivity filtering technique in topology optimization can be interpreted as an optimization problem based on a nonlocal elasticity approach.

In continuum mechanics the concept of nonlocal elasticity was introduced to include size-effects for microstructured media, e.g. to provide strain or stress smoothing effects when small features are present. For example, nonlocal elasticity causes smaller eigenfrequencies and buckling loads and larger deflections for carbon nanotubes (Reddy and Pang 2008). Likewise, sensitivity filtering can be interpreted as having a softening effect on small structural details hence making them uneconomical in the optimization process. For compliance minimization, the smoothing of strain energies for small structural details results in soft elements which then are eliminated during the optimization process.

Furthermore, the Helmholtz filtering process is "volume preserving" (at least inside the modeling domain), which means that the integral of the strain energy density remains the same before and after the filtering process, i.e. $\int_\Omega w dV = \int_\Omega \hat{w} dV$. This entails that from a continuum mechanics perspective, the filtering operation does not change the objective function (12).

It is also interesting to note that the present "strain energy density interpretation" of the sensitivity filter is a return to the original thoughts of the first author as discussed in Sigmund (1997). Here, the filter was defined in terms of the strain energy density for a compliant mechanism design problem. This also indicates that the filtering concept is more general and holds beyond the simple compliance problem. For example, the strain used for premultiplication of the stress filter in (10) to yield the strain energy density filter (11), can be substituted with the strain field associated with the adjoint problem for the output port (c.f. Sigmund 1997). Hence, the concept given here also holds for compliant mechanism design problems and probably others as well.

The discussions sofar were based on mechanical principles. However, if one is just interested in descend directions the explanation of sensitivity filtering may also be viewed from another perspective. That the filtering of the sensitivities does not change the objective function but only the descend direction is well-known and much used in mathematics and image processing communities (Charpiat et al. 2007; De Gournay 2006; Allaire 2007; Bukshtynov et al. 2011). With certain assumptions on the filter operator which for example are satisfied by the so-called Laplace-Beltrami operator $\mathcal{L}(w) = v - l^2 \Delta w$ (which corresponds to the left hand side of (15)), e.g. Charpiat et al. (2007) proves that the filtered sensitivities still provide a descend direction for the original objective function. In general, a gradient can be altered considerably without compromising the convergence (although the optimization path is altered). As stated in this literature, filtering of gradients may promote convergence of some length scales over others and thereby speed up convergence.

With the derivations given in this paper, we have shown that sensitivity filtering can be rigorously derived from continuum mechanics and nonlocal elasticity concepts. We have shown that for the minimum compliance problem the sensitivity filtering concept corresponds to the minimization of strain energy using a nonlocal elasticity formulation, which is generally accepted in continuum mechanics. From this perspective sensitivity filtering is neither "heuristic" nor "inconsistent". Our arguments hold for simple compliance minimization problems, however, nothing prevents us in substituting the strain multiplier used in extending (10) to (11) with a strain associated with adjoint displacements (for e.g. mechanism design), hence adapting the arguments to other mechanical objectives than simple compliance. The arguments should also hold for other physical problems involving multiple length scales and localization phenomena. The latter statement is further supported if sensitivity filtering is seen in the light of simple algebraic gradient modifications that maintain descend properties as discussed above.

Finally, we add a remark on the comparison of sensitivity and density filtering methods. With the findings of this paper we have provided a continuum mechanics motivation for the sensitivity filter. Interestingly, the same motivation is not available for the density filter (Bruns and Tortorelli 2001; Bourdin 2001). Although mathematically rigorous, we can at present, from a continuum mechanics perspective, not provide a physical justification for letting the local material properties be a function of the neighboring material properties as is the case for the density filter. This aspect, however, is still open for discussions and we hope that our paper will spur further interest and developments in this area.

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References

Aage N, Jørgensen M, Andreasen C, Sigmund O (2012, submitted) Interactive topology optimization on hand-held devices

Adalsteinsson D, Sethian J (1995) A fast level set method for propagating interfaces. J Comput Phys 118(2):269–277. doi:10.1006/jcph.1995.1098

Aifantis E (2003) Update on a class of gradient theories. Mech Mater 35(3–6):259–280. doi:10.1016/S0167-6636(02)00278-8

Allaire G (2007) Conception optimale de structures. Springer, Berlin, Heidelberg. doi:10.1007/978-3-540-36856-4

Allaire G, Jouve F, Toader AM (2004) Structural optimization using sensitivity analysis and a level-set method. J Comput Phys 194(1):363–393

Ambrosio L, Buttazzo G (1993) An optimal design problem with perimeter penalization. Calc Var 1:55–69

Andreassen E, Clausen A, Schevenels M, Lazarov B, Sigmund O (2011) Efficient topology optimization in matlab using 88 lines of code. Struct Multidisc Optim 43:1–16. doi:10.1007/s00158-010-0594-7. MATLAB code available online at www.topopt.dtu.dk

Askes H, Morata I, Aifantis E (2008) Finite element analysis with staggered gradient elasticity. Comput Struct 86(11–12):1266–1279. doi:10.1016/j.compstruc.2007.11.002

Bendsøe MP (1989) Optimal shape design as a material distribution problem. Struct Optim 1:193–202

Bendsøe MP, Sigmund O (2004) Topology optimization - theory, methods and applications. Springer Verlag, Berlin, Heidelberg

Boardman A (1982) Electromagnetic surface modes. Wiley, Chichester

Borrvall T (2001) Topology optimization of elastic continua using restriction. Arch Comput Methods Eng 8(4):351–385

Borrvall T, Petersson J (2001) Topology optimization using regularized intermediate density control. Comput Methods Appl Mech Eng 190:4911–4928

Bourdin B (2001) Filters in topology optimization. Int J Numer Methods Eng 50(9):2143–2158

Brooks A, Hughes T (1982) Streamline upwind/petrov-galerkin formulations for convection dominated flows with particular emphasis on the incompressible navier-stokes equations. Comput Methods Appl Mech Eng 32(1–3):199–259

Bruns TE, Tortorelli DA (2001) Topology optimization of non-linear elastic structures and compliant mechanisms. Comput Methods Appl Mech Eng 190(26–27):3443–3459

Bukshtynov V, Volkov O, Protas B (2011) On optimal reconstruction of constitutive relations. Phys D Nonlinear Phenom 240(16):1228–1244. doi:10.1016/j.physd.2011.04.006

Charpiat G, Maurel P, Pons JP, Keriven R, Faugeras O (2007) Generalized gradients: Priors on minimization flows. Int J Comput Vis 73(3):325–344. doi:10.1007/s11263-006-9966-2

De Gournay F (2006) Velocity extension for the level-set method and multiple eigenvalues in shape optimization. SIAM J Control Optim 45(1):343–367. doi:10.1137/050624108

Díaz AR, Sigmund O (1995) Checkerboard patterns in layout optimization. Struct Optim 10(1):40–45

Eringen A (1983) On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J Appl Phys 54:4703. doi:10.1063/1.332803

Eringen A, Edelen D (1972) On nonlocal elasticity. Int J Eng Sci 10(3):233–248. doi:10.1016/0020-7225(72)90039-0

Guest J, Prevost J, Belytschko T (2004) Achieving minimum length scale in topology optimization using nodal design variables and projection functions. Int J Numer Methods Eng 61(2):238–254

Gutkin M (2000) Nanoscopics of dislocations and disclinations in gradient elasticity. Rev Adv Mater Sci 1(1):27–60

Gutkin M, Aifantis E (1999) Dislocations in the theory of gradient elasticity. Scr Mater 40(5):559–566. doi:10.1016/S1359-6462(98)00424-2

Haber RB, Jog CS, Bendsøe MP (1996) A new approach to variable-topology shape design using a constraint on the perimeter. Struct Optim 11(1):1–11

Huiskes R (2000) If bone is the answer, then what is the question? J Anat 197:145–156

Jog CS, Haber RB (1996) Stability of finite element models for distributed-parameter optimization and topology design. Comput Methods Appl Mech Eng 130(3–4):203–226

Lazarov B, Sigmund O (2011) Filters in topology optimization as a solution to Helmholtz type differential equation. Int J Numer Methods Eng 86(6):765–781. doi:10.1002/nme.3072

Leblond JB, Perrin G, Deveaux J (1994) Bifurcation effects in ductile metals with damage delocalization. J Appl Mech 61:236–242

Mullender MG, Huiskes R, Weinans H (1994) A physiological approach to the simulation of bone remodelling as a self-organizational controll process. J Biomech 11:1389–1394

Peerlings R, Geers M, de Borst R, Brekelmans W (2001) A critical comparison of nonlocal and gradient-enhanced softening continua. Int J Solids Struct 38(44–45):7723–7746. doi:10.1016/S0020-7683(01)00087-7

Petersson J, Sigmund O (1998) Slope constrained topology optimization. Int J Numer Methods Eng 41(8):1417–1434

Reddy J, Pang S (2008) Nonlocal continuum theories of beams for the analysis of carbon nanotubes. J Appl Phys 103(2):023,511. doi:10.1063/1.2833431

Ru C, Aifantis E (1993) A simple approach to solve boundary-value problems in gradient elasticity. Acta Mech 101:59–68. doi:10.1007/BF01175597

Schevenels M, Lazarov B, Sigmund O (2011) Robust topology optimization accounting for spatially varying manufacturing errors. Comput Methods Appl Mech Eng 200(49–52):3613–3627. doi:10.1016/j.cma.2011.08.006

Sigmund O (1997) On the design of compliant mechanisms using topology optimization. Mechan Struct Mach 25(4):493–524

Sigmund O (2001) A 99 line topology optimization code written in MATLAB. Struct Multidisc Optim 21:120–127. doi:10.1007/s001580050176. MATLAB code available online at www.topopt.dtu.dk

Sigmund O (2007) Morphology-based black and white filters for topology optimization. Struct Multidisc Optim 33(4–5):401–424

Sigmund O (2009) Manufacturing tolerant topology optimization. Acta Mech Sin 25(2):227–239. doi:10.1007/s10409-009-0240-z

Sigmund O, Petersson J (1998) Numerical instabilities in topology optimization: A survey on procedures dealing with checkerboards, mesh-dependencies and local minima. Struct Optim 16(1):68–75

Tcherniak D, Sigmund O (2001) A web-based topology optimization program. Struct Multidisc Optim 22(3):179–187

Tezduyar T, MS, Ray S, Shih R (1992) Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements. Comput Methods Appl Mech Eng 95:221–242

Toupin RA (1962) Elastic materials with couple-stresses. Arch Ration Mech Anal 11:385–414. doi:10.1007/BF00253945

Tvergaard V, Needleman A (1995) Effects of nonlocal damage in porous plastic solids. Int J Solids Struct 32(8–9):1063–1077. doi:10.1016/0020-7683(94)00185-Y

Wang F, Lazarov B, Sigmund O (2011) On projection methods, convergence and robust formulations in topology optimization. Struct Multidisc Optim 43(6):767–784. doi:10.1007/s00158-010-0602-y

Xu S, Cai Y, Cheng G (2010) Volume preserving nonlinear density filter based on heaviside funtions. Struct Multidisc Optim 41:495–505

Zhou M, Rozvany GIN (1991) The COC algorithm, part II: Topological, geometry and generalized shape optimization. Comput Methods Appl Mech Eng 89(1–3):309–336