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INDUSTRIAL APPLICATION

Optimal design of rectangular RC sections for ultimate bending strength

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Abstract A minimum cost problem for ultimate strength in bending of rectangular reinforced concrete sections is investigated. The design variables are section depth and steel reinforcement areas. State equations are those of equilibrium with compression depth as state variable. The Kuhn-Tucker optimality conditions are solved analytically and formulas for nondimensional design and state variables are obtained in four cases: Two singly-reinforced solutions with either maximum allowable depth or smaller; Two doubly-reinforced with maximum allowable depth and either maximum compression depth or smaller. Each of the solutions is optimal in a region of the plane 'nondimensional bending moment'–'cost-effectiveness ratio of concrete to steel'. The formulas are for an arbitrary concrete constitutive law with tension cut-off and are specialized for the parabola-rectangle law of Eurocode 2.

Keywords Reinforced concrete · Section design · Reinforcement optimization · Bending moment · Eurocode 2

1 Introduction

The sizing of reinforced concrete (RC) beam and skeletal structures often falls to the design of rectangular sections for ultimate strength. For some bending moment (the case considered in this paper), the designer must specify the section dimensions (if not specified already by the architectural design) and the reinforcement area, or areas if a doubly-reinforced section is required. Construction codes and related literature are available for this task, in the guise of simple formulas, tables and abacuses. The designer, however, is left with an infinity of feasible designs with little help in the choice of the most economical, other than general guidelines and his own experience, and may have to compute and compare a large number of them. This situation becomes more acute when there is a rapid variation of raw materials costs, which has been the case of steel bar worldwide in recent years, and the designs must adapt accordingly for economy sake. A key point of the design codes is the constitutive law of concrete, such as parabola-rectangle, bilinear, etc., and simplifications such as the rectangular stress block. The same code may—as is the case of Eurocode 2 (EC2 2001)—allow for different concrete laws.

A number of researchers have investigated the optimal design of concrete sections in bending without resorting to computer implementation of numerical methods. Al-Salloum and Siddiqi (1994) obtain close-form solutions for optimal depth and steel ratio of a singly-reinforced rectangular RC section in terms of material costs (including formwork) and strength ratios. Constraints are flexural strength and bounds on reinforcement area of the American Concrete Institute (ACI) code and a maximum allowable section depth. Five combinations of active side constraints are analyzed. Samman and Erbatur (1995) present an iterative procedure for economical design of RC beams compliant with the ACI code with various support conditions where section depth and steel ratios are determined. Equilibrium equations with rectangular stress block for singly-reinforced

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rectangular sections are used. The effect of steel and concrete costs is shown in plots of optimum steel ratios against costs ratio. Ceranic and Frier (2000) present formulas for the optimal depth and steel reinforcement for both singly- and doubly-reinforced RC rectangular sections. Analysis is by means of limit state with rectangular stress block and the objective function is the cost of materials. Design curves for materials cost and strength parameters are shown. Barros et al. (2005) investigate the cost optimization of rectangular RC sections using the non-linear MC90 equation, showing that, for this constitutive relation, the maximum strain in concrete lies between the strain for peak stress and the ultimate strain. Explicit formulas for optimal section depth, traction armature area and, in the doubly-reinforced case, compression/traction armature ratio, are presented. In a parallel development (Barros et al. 2011) investigate the minimal cost problem of rectangular sections in bending using the parabola-rectangle law of EC2 (1991). The objective function is raw materials cost and the design variables are section depth and steel reinforcement areas; the state variables are traction armature strain and maximum strain in concrete. With the constraint on steel strain of the 1991 version of the code 7 solutions are obtained: 1 doubly-reinforced and 2 singly-reinforced with concrete rupture, analogous to those presented here; 4 additional singly-reinforced solutions with steel rupture (2) and simultaneous rupture of steel and concrete (2). The equivalent of the optimality map shown here in Fig. 2 is partitioned in 6 regions.

Other investigators addressed the optimal design of RC beams and frames, including structural analysis in the formulation. Kanagasundaram and Karihaloo (1990) treat the optimal design of beams, simply supported and multi-span, and columns as a non-linear programming problem. Objective function is the cost of materials and formwork and the constraints are strength at critical sections, serviceability and other, conforming to Australian Standard AS3600-1988. Design variables are section depth and width and steel reinforcement, either uniform or different for each span. Non-linear minimization algorithms, developed for structural optimization, are used to exemplify the formulation. Adamu and Karihaloo (1995) state the minimum cost design of RC frames as a discrete optimization of structures composed of a number of beam and column elements. The cost is the sum of concrete, steel and formwork costs and the design variables are the cross-sectional parameters and steel ratios of beams, columns and elements. Design constraints include bending and shear strength of beams and columns and limits on nodal deflections. The stiffness method is used in the analysis of the real and adjoint structures in an iterative optimization procedure. Adamu et al. (1994) apply continuum-type optimality criteria (COC) to the design of RC beams. Design variables considered are width, depth and traction reinforcement steel ratio; these are functions of the longitudinal coordinate. Constraints are flexural and shear strength, maximum deflection and side constraints. Calculus of variations on an augmented Lagrangian functional yields the COC optimality conditions. An iterative procedure is used to obtain discrete numerical approximations of optimal solutions. Genetic algorithms have been used for the optimization of concrete structures, e.g., Lepš and Šejnoha (2003) demonstrate the technique with RC beams. Objective function is the cost of raw materials and behavior constraints, conforming to EC2, are flexural and shear strength and deflection criteria. Mindlin beam is used, discretized by finite elements. Design variables are cross-sectional dimensions of the beam and steel reinforcements over multiple beam spans.

In this paper a minimal cost problem of a rectangular section in simple bending is investigated where the objective function is the cost of raw materials and the design variables are section depth, s.t. an upper side constraint, and steel reinforcement areas, traction and compression. The fundamental problem statement is an augmented Lagrangian function. The state constraint equations are those of equilibrium—force and moment—written for an arbitrary concrete law with tension cut-off. The state variables are maximum concrete strain and compression depth. The Kuhn-Tucker (K-T) necessary optimality conditions (see, e.g., Bazaraa et al. 1993) are solved with the aid of a symbolic processor (CAS). Only optimal solutions with concrete rupture are accounted for; in this case the concrete constitutive law appears as two constants in the equilibrium equations and it is found that the optimal strain is indeed the ultimate for concrete laws satisfying a condition on these constants. Active or inactive side constraints (on section depth and compression armature area) are combined in 4 cases to predefine primal and/or dual variables and the remaining optimality conditions are used to complete the solution, cf Haug and Arora (1979).

For the sake of simplicity, the perfectly plastic model is used for steel reinforcements, as a first approach; this avoids the added complication of checking for the yield condition of the armatures. In this case three solutions are obtained: (1) A singly-reinforced section with an optimal depth smaller than the maximum allowable; (2) A singly-reinforced section with the maximum depth allowable and steel reinforcement as required by equilibrium; (3) A doubly-reinforced section with maximum depth and an optimal distribution of armature steel. The 4th case of calculated depth with double reinforcement does not have a feasible solution.

Elastic-perfectly plastic armatures are accounted for in Section 7 in the guise of a side constraint on compression depth, added to the augmented Lagrangian. It is found the three solutions previously obtained are also solutions in this case, under revised conditions. An additional doubly-reinforced solution (4) with the new constraint active and maximum section depth is obtained.

Each of these solutions has a domain of optimality in the plane 'nondimensional bending moment'–'cost-effectiveness ratio of concrete to steel'. These domains are disjoint, except that those of solutions 3 and 4 (mostly) overlap.

2 Stress resultants in RC section

Conventions regarding the RC section are represented in Fig. 1. Section breadth is denoted by b and effective depth by d. The maximum design depth is $\bar{d}$, i.e., $d \leq \bar{d}$. Let η and r be adimensionalised (w.r.t. the maximum depth) effective height and concrete cover, respectively:

$$\eta = \frac{d}{\bar{d}} \leq 1$$

$$r = \frac{a}{\bar{d}} \geq 0 \tag{1}$$

Depth in compression x is adimensionalised w.r.t. section depth:

$$\gamma = \frac{x}{d} \tag{2}$$

Armature areas are denoted by A and A', traction and compression, respectively. Steel reinforcement ratios w.r.t. the maximum effective section area are defined by (compression armature area is 0 for singly-reinforced sections):

$$\rho = \frac{A}{b\bar{d}} > 0$$

$$\rho' = \frac{A'}{b\bar{d}} \geq 0 \tag{3}$$

Compressive strain in concrete ϵ for simple bending is proportional to distance to the mean axis y (↑) (Fig. 1), $\epsilon = \frac{y}{R}$, where $R > 0$ is the radius of curvature of the unstrained longitudinal fibres. Let $\epsilon_c = \frac{x}{R}$ denote the maximum strain in concrete. Using the latter identity to eliminate R on the former, the strain equation is obtained:

$$\epsilon = \frac{y}{x}\epsilon_c$$

Extension of traction steel reinforcement is $\epsilon_s = \frac{d-x}{R}$, in nondimensional form

$$\epsilon_s = \frac{1-\gamma}{\gamma}\epsilon_c$$

and positiveness condition $\epsilon_s > 0$ sets the bounds on nondimensional compression depth:

$$0 < \gamma < 1$$

Compressive strain in the upper armature is $\epsilon_s' = \frac{x-a}{R}$, or

$$\epsilon_s' = \frac{\gamma - \frac{r}{\eta}}{\gamma}\epsilon_c$$

and for positiveness, $\epsilon_s' > 0$:

$$r < \gamma\eta$$

This condition is regarded as a limitation on concrete cover r, to be checked in the doubly-reinforced section case, and as such will not be included in the optimization constraints (r is not a design variable).

The design compressive ultimate strength of concrete is denoted by $f_{cd} > 0$ and the ultimate design stress of steel (either traction or compression) by $f_{yd} > 0$. The strength ratio of steel to concrete is the parameter:

$$\beta = \frac{f_{yd}}{f_{cd}} > 0 \tag{4}$$

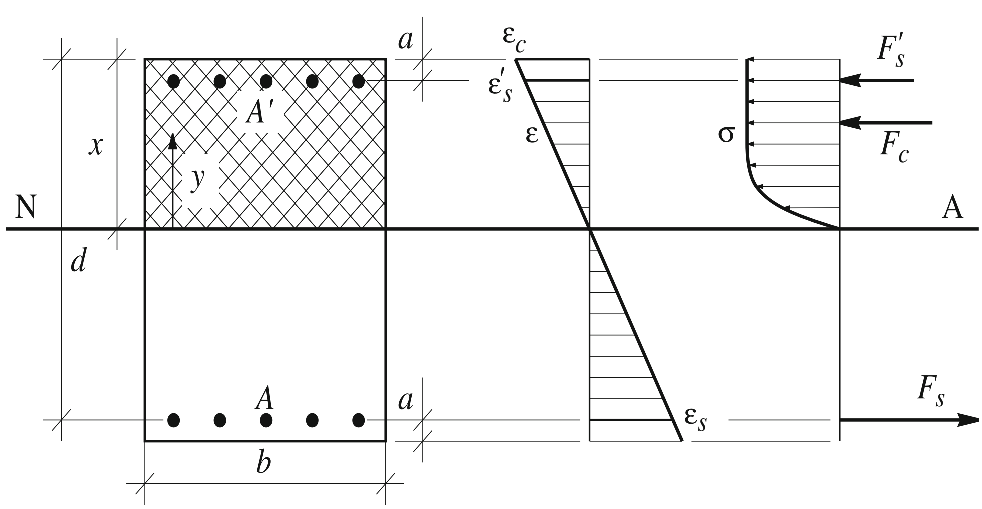

Fig. 1 Strain, stress and forces on RC section

The response function of concrete is denoted by $\sigma_c(\cdot)$, taken to be a [positive] compression for positive argument, with tension cut-off, and a nondimensional response function is defined:

$$\tilde{\sigma}_c(\cdot) = \frac{\sigma_c(\cdot)}{f_{cd}} \tag{5}$$

The resultant axial force—which will be set to 0—is the vector sum ($\rightarrow$) of resultant forces in concrete and armatures, traction and compression, (Fig. 1):

$$N_{\text{res}} = -F_c + F_s - F_s'$$

The stress resultant in concrete is the stress integral in the compression region

$$F_c = b \int_0^x \sigma(y) dy$$

and by change of integration variable using the strain equation:

$$F_c = bx \frac{\int_0^{\epsilon_c} \sigma_c(\epsilon) d\epsilon}{\epsilon_c}$$

The force resultants in steel reinforcements are:

$$F_s = A f_{yd}$$
$$F_s' = A' f_{yd}$$

A nondimensional force resultant is defined

$$\tilde{N} = \frac{N_{\text{res}}}{b\bar{d} f_{cd}}$$

and the formula for it is:

$$\tilde{N} = -\gamma\eta \frac{\int_0^{\epsilon_c} \tilde{\sigma}_c(\epsilon) d\epsilon}{\epsilon_c} + \rho\beta - \rho'\beta \tag{6}$$

The resultant bending moment ($\curvearrowleft$), evaluated at the lower armature, is the sum of concrete and upper armature moments:

$$M_{\text{res}} = M_c + M_s'$$

The moment of compressive stresses in concrete is

$$M_c = b \int_0^x \sigma(y) \cdot (y + d - x) \, dy$$

and changing the integration variable as before:

$$M_c = bx^2 \frac{\int_0^{\epsilon_c} \sigma_c(\epsilon) \cdot \epsilon d\epsilon}{\epsilon_c^2} + bx(d - x) \frac{\int_0^{\epsilon_c} \sigma_c(\epsilon) d\epsilon}{\epsilon_c}$$

The moment of compression armature is

$$M_s' = F_s'(d - a)$$

The bending moment, adimensionalised w.r.t. the maximum section dimensions, is

$$\tilde{M} = \frac{M_{\text{res}}}{b\bar{d}^2 f_{cd}}$$

and is evaluated by:

$$\tilde{M} = \gamma^2\eta^2 \frac{\int_0^{\epsilon_c} \tilde{\sigma}_c(\epsilon) \cdot \epsilon d\epsilon}{\epsilon_c^2} + (1 - \gamma)\gamma\eta^2 \frac{\int_0^{\epsilon_c} \tilde{\sigma}_c(\epsilon) d\epsilon}{\epsilon_c} + \rho'\beta(\eta - r) \tag{7}$$

3 Optimal design problem

Let C_s and C_c be specific cost factors (economical, environmental or other) of raw materials steel bar and concrete, respectively. The cost ratio of steel to concrete is the parameter

$$\alpha = \frac{C_s}{C_c} > 0 \tag{8}$$

It is introduced at this point the quotient of cost and strength ratios of raw materials

$$\xi = \frac{\alpha}{\beta} > 0 \tag{9}$$

which is found to arise naturally in the solution of the optimization problem in the following. It can be interpreted as either the efficiency, or cost-effectiveness, ratio of concrete to steel

$$\xi = \frac{\frac{f_{cd}}{C_c}}{\frac{f_{yd}}{C_s}}$$

where by cost-effectiveness is understood the strength-to-cost ratio of a raw material; or as the quotient of steel and concrete cost-to-strength ratios:

$$\xi = \frac{\frac{C_s}{f_{yd}}}{\frac{C_c}{f_{cd}}}$$

The beam cost per unit length is

$$C = C_s(A + A') + C_c b(d + a)$$

a nondimensional cost is defined

$$\tilde{C} = \frac{C}{b\bar{d}C_c}$$

and it is evaluated as the function of nondimensional variables and parameters:

$$\tilde{C} = \alpha\left(\rho + \rho'\right) + \eta + r \tag{10}$$

To account for the replacement of concrete area by steel reinforcement the numerical value of $\alpha - 1$ can be used instead of α.

The optimal design problem is the cost minimization w.r.t. the design variables

$$\tilde{C} \underset{\eta,\,\rho,\,\rho'}{\longrightarrow} \min$$

nondimensional depth (η) and armature areas, traction (ρ) and compression (ρ'), s.t.:

(1) State equations of equilibrium

$$\tilde{N} = 0$$

$$\tilde{M} = M$$

where $\tilde{N}$ (6) and $\tilde{M}$ (7) are functions of the design variables and $M > 0$ is the nondimensional specified bending moment.

(2) Side constraints on state

$$\epsilon_c \leq \epsilon_{cu}$$

where $\epsilon_{cu} > 0$ is the ultimate compressive strain in concrete, and design variables:

$$\eta \leq 1$$

$$\rho' \geq 0$$

(3) Inclusion, or domain, constraints:

$$\epsilon_c > 0$$

$$\gamma > 0$$

$$\gamma < 1$$

$$\eta > 0$$

$$\rho > 0$$

An augmented Lagrangian for the constrained optimization problem is defined

$$L(\epsilon_c, \gamma, \eta, \rho, \rho', \lambda, \mu, \nu, \pi, \kappa) = \tag{11}$$

$$= \tilde{C} + \lambda\tilde{N} + \mu\left(\tilde{M} - M\right) + \nu\left(\epsilon_c - \epsilon_{cu}\right) + \pi\left(\eta - 1\right) - \kappa\rho'$$

a function of state variables (ϵ_c, γ), design variables (η, ρ, ρ') and Lagrange multipliers (λ, μ, ν, π, κ). The saddle-point optimality conditions, on primal and dual variables, are

$$\begin{cases} L \underset{\epsilon_c>0,\,0<\gamma<1,\,\eta>0,\,\rho>0,\,\rho'}{\longrightarrow} \min \\ L \underset{\lambda,\,\mu,\,\nu\geq 0,\,\pi\geq 0,\,\kappa\geq 0}{\longrightarrow} \max \end{cases}$$

and the K-T necessary optimality conditions, in differential form, are the set of equations

$$\begin{cases} \dfrac{\partial L}{\partial \epsilon_c} = \lambda\dfrac{\partial \tilde{N}}{\partial \epsilon_c} + \mu\dfrac{\partial \tilde{M}}{\partial \epsilon_c} + \nu = 0 \\ \dfrac{\partial L}{\partial \gamma} = \lambda\dfrac{\partial \tilde{N}}{\partial \gamma} + \mu\dfrac{\partial \tilde{M}}{\partial \gamma} = 0 \\ \dfrac{\partial L}{\partial \eta} = 1 + \lambda\dfrac{\partial \tilde{N}}{\partial \eta} + \mu\dfrac{\partial \tilde{M}}{\partial \eta} + \pi = 0 \\ \dfrac{\partial L}{\partial \rho} = \alpha + \lambda\dfrac{\partial \tilde{N}}{\partial \rho} = 0 \\ \dfrac{\partial L}{\partial \rho'} = \alpha + \lambda\dfrac{\partial \tilde{N}}{\partial \rho'} + \mu\dfrac{\partial \tilde{M}}{\partial \rho'} - \kappa = 0 \\ \dfrac{\partial L}{\partial \lambda} = \tilde{N} = 0 \\ \dfrac{\partial L}{\partial \mu} = \tilde{M} - M = 0 \end{cases} \tag{12}$$

and the constraint conditions, positivity of Lagrange multipliers and complementarity conditions:

$$\begin{cases} \dfrac{\partial L}{\partial \nu} = \epsilon_c - \epsilon_{cu} \leq 0, & \nu \geq 0, & \nu\dfrac{\partial L}{\partial \nu} = 0 \\ \dfrac{\partial L}{\partial \pi} = \eta - 1 \leq 0, & \pi \geq 0, & \pi\dfrac{\partial L}{\partial \pi} = 0 \\ \dfrac{\partial L}{\partial \kappa} = -\rho' \leq 0, & \kappa \geq 0, & \kappa\dfrac{\partial L}{\partial \kappa} = 0 \end{cases} \tag{13}$$

4 Simplified optimality conditions

Only solutions with concrete rupture

$$\epsilon_c = \epsilon_{cu} \tag{14}$$

will be considered here. The condition under which this assumption is valid will be obtained from the analysis of Lagrange multiplier ν.

Under this assumption the strain functions in $\tilde{N}$ (6) and $\tilde{M}$ (7) become the constants:

$$C_1 = \frac{\int_0^{\epsilon_{cu}} \tilde{\sigma}_c(\epsilon)d\epsilon}{\epsilon_{cu}} > 0$$

$$C_2 = \frac{\int_0^{\epsilon_{cu}} \tilde{\sigma}_c(\epsilon)\cdot\epsilon d\epsilon}{\epsilon_{cu}^2} > 0 \tag{15}$$

The physical meaning of C_1 is the mean stress $\bar{\sigma}_c$ in the stress–strain curve of concrete and

$$\frac{C_2}{C_1} = \frac{\bar{\epsilon}_c}{\epsilon_{cu}}$$

is the ratio of the mean to the maximum strains. By the mean theorem

$$\frac{C_2}{C_1} < 1 \tag{16}$$

and, therefore, $C_1 > C_2$, which will be used recurrently.

It will be further assumed—as a restriction to concrete laws for which the results presented here are applicable—that

$$\frac{C_2}{C_1} \geq \frac{1}{2} \tag{17}$$

from which the following inequality ensues, recorded here for later reference:

$$\frac{1}{2}\frac{C_1}{C_1 - C_2} \geq 1 \tag{18}$$

Under the latter assumption the bending moment (7) is a monotonically increasing function of γ for $\gamma < 1$

$$\frac{\partial \tilde{M}}{\partial \gamma} = \eta^2 C_1 \left(1 - \frac{\gamma}{\frac{1}{2}\frac{C_1}{C_1 - C_2}}\right) > 0$$

hence it is a sufficient condition for root uniqueness of $\gamma \in]0, 1[$ in the moment equilibrium equation.

The following identities are needed for the K-T optimality conditions (12) (note that $\tilde{\sigma}_c(\epsilon_{cu}) = 1$):

$$\left.\frac{\partial}{\partial \epsilon_c}\frac{\int_0^{\epsilon_c}\tilde{\sigma}_c(\epsilon)d\epsilon}{\epsilon_c}\right|_{\epsilon_c=\epsilon_{cu}} = \frac{1 - C_1}{\epsilon_{cu}}$$

$$\left.\frac{\partial}{\partial \epsilon_c}\frac{\int_0^{\epsilon_c}\tilde{\sigma}_c(\epsilon)\cdot\epsilon d\epsilon}{\epsilon_c^2}\right|_{\epsilon_c=\epsilon_{cu}} = \frac{1 - 2C_2}{\epsilon_{cu}}$$

For solutions with $\epsilon_c = \epsilon_{cu}$ the optimality conditions are, the simplified K-T equations (12)

$$\begin{cases} \gamma\eta\left(\mu\eta\left(1-\gamma\right)-\lambda\right)\dfrac{1-C_1}{\epsilon_{cu}}+\mu\gamma^2\eta^2\dfrac{1-2C_2}{\epsilon_{cu}}+\nu=0 & \text{(a)} \\ -\lambda\eta C_1+\mu\eta^2\left(C_1+2\gamma\left(C_2-C_1\right)\right)=0 & \text{(b)} \\ 1-\lambda\gamma C_1+2\mu\gamma\eta\left(1-\gamma\right)C_1+2\mu\gamma^2\eta C_2+\mu\rho'\beta+\pi=0 & \text{(c)} \\ \alpha+\lambda\beta=0 & \text{(d)} \\ \alpha-\lambda\beta+\mu\beta\left(\eta-r\right)-\kappa=0 & \text{(e)} \\ -\gamma\eta C_1+\rho\beta-\rho'\beta=0 & \text{(f)} \\ \gamma^2\eta^2 C_2+\left(1-\gamma\right)\gamma\eta^2 C_1+\rho'\beta\left(\eta-r\right)=M & \text{(g)} \end{cases} \tag{19}$$

and conditions (13)

$$\begin{cases} \nu \geq 0 \\ \eta - 1 \leq 0, \quad \pi \geq 0, \quad \pi\left(\eta - 1\right) = 0 \\ -\rho' \leq 0, \quad \kappa \geq 0, \quad \kappa\rho' = 0 \end{cases} \tag{20}$$

plus the inclusion constraints:

$$\begin{cases} \gamma > 0 \\ \gamma < 1 \\ \eta > 0 \\ \rho > 0 \end{cases} \tag{21}$$

There are 9 variables to determine: 1 state variable (γ); 3 design variables (η, ρ, ρ'); 5 Lagrange multipliers (λ, μ, ν, π, κ). Owing to it's simplicity, the second and third sets of conditions in (20) predefine 2 variables—either a design variable (η, ρ') or the Lagrange multiplier for the corresponding side constraint (π, κ). Thus, there are 4 cases—free or constrained height, singly- or doubly-reinforced section—to be analyzed, using the 7 core equations (19) to determine the remaining variables. This will be done in the next section. Detailed analysis of the solutions obtained will be postponed to the following section.

5 Solutions of the optimality conditions

System of equations (18) is nonlinear. However, it has a distinct sparse nature, lending itself to solution by successive elimination. It will be seen that it can be solved uniquely for each of the cases mentioned above, except one for which it will be shown no solution exists. The first five equations are solved independently of case. Firstly, equations (19d), (19b) and (19a) are solved for Lagrange multipliers λ, μ and ν; secondly, the equilibrium equations (19f) and (19g) are solved for ρ and γ; finally, the 4 cases are analyzed, using equations (19c) and (19e) to determine the design variables η and ρ', if free, or the Lagrange multipliers π and κ, if the side constraints are saturated, in 3 cases and in 1 is shown not to be feasible.

5.1 Lagrange multipliers λ, μ and ν

From equation (19d) is obtained

$$\lambda = -\xi$$

and from equation (19b):

$$\mu = -\frac{\xi}{\eta}\frac{C_1}{C_1 - 2\gamma\left(C_1 - C_2\right)}$$

Replacing λ and μ on equation (19a) and solving for ν:

$$\nu = \frac{\xi\gamma^2\eta}{\epsilon_{cu}}\frac{2\left(C_1 - C_2\right) - C_1^2}{C_1 - 2\gamma\left(C_1 - C_2\right)}$$

The algebraic in the denominator is positive, $C_1 - 2\gamma(C_1 - C_2) > 0$ is equivalent to

$$\gamma < \frac{1}{2}\frac{C_1}{C_1 - C_2}$$

which is always true on account of inequality (18). Positivity of the Lagrange multiplier ($\nu \geq 0$) requires that the numerator be positive, $2(C_1 - C_2) - C_1^2 \geq 0$, from which the condition on constants C_1 and C_2 is derived:

$$\frac{C_2}{C_1} \leq 1 - \frac{C_1}{2} \tag{22}$$

This is the condition for the optimal solutions (all of them) to have concrete strained to the maximum, $\epsilon_c = \epsilon_{cu}$. It is seen this is so for some concrete laws—those for which

$$\frac{\bar{\epsilon}_c}{\epsilon_{cu}} \leq 1 - \frac{\bar{\sigma}_c}{2}$$

In particular, on account of (17), the average stress in concrete $\bar{\sigma}_c$ can not be larger than 1. This may occur when the peak stress is greater than the ultimate one, and the optimal strain will be smaller than the ultimate, as would be expected.

5.2 Equilibrium equations

Zero force resultant equation (19f) determines the traction armature

$$\rho = \frac{1}{\beta}\left(\gamma\eta C_1 + \rho'\beta\right)$$

and moment equilibrium equation (19g) is used to obtain the nondimensional compression depth:

$$\gamma = \frac{1}{2}\frac{C_1}{C_1 - C_2} - \frac{\sqrt{C_1^2\eta^2 - 4(C_1 - C_2)(M - \rho'\beta(\eta - r))}}{2(C_1 - C_2)\eta}$$

Equation (19g) has a second root (with a plus sign instead of minus for the second term) which is inadmissible ($\gamma > 1$) on account of inequality (18). With the solving strategy followed, the equilibrium equations were solved independently from the remaining—as they would have in an analysis context.

5.3 Singly-reinforced section, free height: Solution 1

In this case one design variable and one Lagrange multiplier are predefined:

$$\rho' = 0$$

$$\pi = 0$$

Equation (19e) is solved for Lagrange multiplier κ

$$\kappa = 2\xi\beta - \frac{\xi\beta C_1(\eta - r)}{\sqrt{C_1^2\eta^2 - 4M(C_1 - C_2)}}$$

and equation (19c) for design variable η:

$$\eta = \frac{\xi C_1^2 + 2(C_1 - C_2)}{\sqrt{\xi C_1^2 + C_1 - C_2}}\frac{\sqrt{M}}{C_1}$$

Equation (19c) has a second negative solution (it is the symmetric of a square root), inadmissible—it violates inclusion constraint $\eta > 0$.

5.4 Singly-reinforced section, maximum height: Solution 2

In this case two design variables are predefined (the third, ρ, was determined by equilibrium equation (19f)):

$$\rho' = 0$$

$$\eta = 1$$

Equation (19e) is solved for κ, obtaining the same result as in solution 1 (with $\eta = 1$), and equation (19c) for Lagrange multiplier π:

$$\pi = \frac{\xi C_1^3 - \left(\xi C_1^2 + 2(C_1 - C_2)\right)\sqrt{C_1^2 - 4M(C_1 - C_2)}}{2(C_1 - C_2)\sqrt{C_1^2 - 4M(C_1 - C_2)}}$$

5.5 Doubly-reinforced section, maximum height: Solution 3

In this case one Lagrange multiplier and one design variable are predefined:

$$\kappa = 0$$

$$\eta = 1$$

Equation (19e) is solved for compression steel reinforcement ratio

$$\rho' = \frac{1}{\beta}\frac{16M(C_1 - C_2) - C_1^2\left(3 + 2r - r^2\right)}{16(C_1 - C_2)(1 - r)}$$

and equation (19c) for Lagrange multiplier π:

$$\pi = \frac{16\xi M(C_1 - C_2) - 4\xi C_1^2 r^2 + \left(\xi C_1^2 - 8(C_1 - C_2)\right)(1 - r)^2}{8(C_1 - C_2)(1 - r)^2}$$

5.6 Doubly-reinforced section, free height: No solution

In this case two Lagrange multipliers are predefined

$$\kappa = 0$$

$$\pi = 0$$

and two design variables (η, ρ') are to be determined, using equations (19c) and (19e).

Replacing λ and μ, as obtained by elimination in equations (19d) and (19b), in equations (19c) and (19e), solving both for γ and equating the results and, finally, solving for η the result is:

$$\eta = \frac{4\xi\,(C_1 - C_2)\,\rho'\beta + \left(\xi C_1^2 + 2\,(C_1 - C_2)\right) r}{2\,(C_1 - C_2) - \xi C_1^2}$$

Noting that the numerator is positive, for positivity of the denominator the cost-effectiveness ratio

$$\xi < 2\frac{C_1 - C_2}{C_1^2}$$

is outside the range of admissibility that will be set on the next section.

6 Analysis of optimal solutions

6.1 Singly-reinforced section, free height: Solution 1

The full solution, obtained by back substitution in the results of the last section, is:

$$\gamma = \frac{C_1}{\xi C_1^2 + 2\,(C_1 - C_2)}$$

$$\eta = \frac{\xi C_1^2 + 2\,(C_1 - C_2)}{\sqrt{\xi C_1^2 + C_1 - C_2}} \frac{\sqrt{M}}{C_1}$$

$$\rho = \frac{1}{\beta} \frac{C_1\sqrt{M}}{\sqrt{\xi C_1^2 + C_1 - C_2}}$$

$$\rho' = 0$$

$$\lambda = -\xi$$

$$\mu = -\frac{\sqrt{\xi C_1^2 + C_1 - C_2}}{C_1\sqrt{M}}$$

$$\nu = \frac{\sqrt{M}}{\epsilon_{cu}} \frac{2\,(C_1 - C_2) - C_1^2}{C_1^2\sqrt{\xi C_1^2 + C_1 - C_2}}$$

$$\pi = 0$$

$$\kappa = \beta\frac{\xi C_1^2 - 2\,(C_1 - C_2)}{C_1^2} + \beta r \frac{\sqrt{\xi C_1^2 + C_1 - C_2}}{C_1\sqrt{M}} \qquad (23)$$

Compliance to domain constraints $\rho > 0$, $\eta > 0$ and $\gamma > 0$ can be confirmed directly by inspection of the formulas above. The condition $\gamma < 1$ sets the lower bound on the cost-effectiveness ratio:

$$\xi > \frac{2C_2 - C_1}{C_1^2} \qquad (24)$$

Positiveness condition on Lagrange multiplier $\kappa \geq 0$ can, equivalently, be stated as a restriction on concrete cover

$$r \geq -\frac{\xi C_1^2 - 2\,(C_1 - C_2)}{\sqrt{\xi C_1^2 + C_1 - C_2}} \frac{\sqrt{M}}{C_1}$$

and for this to be an arbitrary positive number ($r \geq 0$) it suffices that the r.h.s. be non-positive, which will be the case if

$$\xi \geq 2\frac{C_1 - C_2}{C_1^2} \qquad (25)$$

and this will be considered here as a lower bound on the cost-effectiveness ratio. If

$$\frac{C_2}{C_1} < \frac{3}{4}$$

the latter bound is larger than the required by condition $\gamma < 1$ (24) and supersedes it.

The side constraint on nondimensional height $\eta \leq 1$ sets the upper bound on bending moment:

$$M \leq C_1^2 \frac{\xi C_1^2 + C_1 - C_2}{\left(\xi C_1^2 + 2\,(C_1 - C_2)\right)^2} \qquad (26)$$

The bending moment in solution 1, adimensionalised w.r.t. the actual section depth,

$$M_1 = \frac{M_{\text{res}}}{bd^2 f_{cd}}$$

relates to the specified (adimensionalised w.r.t. the maximum depth) $M_1 = \frac{M}{\eta^2}$ and is found to be a function of the cost-effectiveness ratio only

$$M_1 = C_1^2 \frac{\xi C_1^2 + C_1 - C_2}{\left(\xi C_1^2 + 2\,(C_1 - C_2)\right)^2} \qquad (27)$$

and equal to the upper bound (26). The traction reinforcement ratio adimensionalised w.r.t. the actual section area

$$\rho_1 = \frac{A}{bd}$$

is the function of two design variables $\rho_1 = \frac{\rho}{\eta}$ and is also a function of the efficiency ratio (and not of the specified bending moment):

$$\rho_1 = \frac{1}{\beta} \frac{C_1^2}{\xi C_1^2 + 2\,(C_1 - C_2)} \qquad (28)$$

6.2 Singly-reinforced section, maximum height: Solution 2

The complete solution is:

$$\gamma = \frac{1}{2}\frac{C_1 - \sqrt{C_1^2 - 4M(C_1 - C_2)}}{C_1 - C_2}$$
$$\eta = 1$$
$$\rho = \frac{1}{\beta}\frac{C_1}{2}\frac{C_1 - \sqrt{C_1^2 - 4M(C_1 - C_2)}}{C_1 - C_2}$$
$$\rho' = 0$$
$$\lambda = -\xi$$
$$\mu = -\frac{\xi C_1}{\sqrt{C_1^2 - 4M(C_1 - C_2)}}$$
$$\nu = \frac{\xi}{\epsilon_{cu}}\frac{2(C_1 - C_2) - C_1^2}{2(C_1 - C_2)^2} \times \frac{C_1^2 - 2M(C_1 - C_2) - C_1\sqrt{C_1^2 - 4M(C_1 - C_2)}}{\sqrt{C_1^2 - 4M(C_1 - C_2)}}$$
$$\pi = \frac{\xi C_1^3 - \left(\xi C_1^2 + 2(C_1 - C_2)\right)\sqrt{C_1^2 - 4M(C_1 - C_2)}}{2(C_1 - C_2)\sqrt{C_1^2 - 4M(C_1 - C_2)}}$$
$$\kappa = 2\xi\beta - \frac{\xi\beta C_1(1 - r)}{\sqrt{C_1^2 - 4M(C_1 - C_2)}} \tag{29}$$

Inclusion constraints $\rho > 0$ and $\gamma > 0$ are readily verified to hold for all $M > 0$. Constraint $\gamma < 1$ sets an upper bound on bending moment for validity of this solution:

$$M < C_2 \tag{30}$$

It is straightforward to conclude that, under the latter condition, positivity of the radicand $C_1^2 - 4M(C_1 - C_2) > 0$ in (29) holds in the entire range of constants C_1 and C_2, i.e., $\frac{C_2}{C_1} \in [\frac{1}{2}, 1[$.

Positivity of Lagrange multiplier $\pi \geq 0$ sets the lower bound on bending moment

$$M \geq C_1^2 \frac{\xi C_1^2 + C_1 - C_2}{\left(\xi C_1^2 + 2(C_1 - C_2)\right)^2} \tag{31}$$

which is the upper bound (26) of solution 1, derived from condition $\eta \leq 1$, complementary to $\pi \geq 0$.

Lagrange multiplier positiveness condition $\kappa \geq 0$ sets an upper bound on bending moment:

$$M \leq \frac{C_1^2}{16}\frac{4 - (1 - r)^2}{C_1 - C_2} \tag{32}$$

The lower bound (31) is a monotonically decreasing function of ξ (the first derivative is negative for positive ξ), hence it can not be larger than the value at the lower bound of ξ (25)

$$C_1^2 \frac{\xi C_1^2 + C_1 - C_2}{\left(\xi C_1^2 + 2(C_1 - C_2)\right)^2} \leq \frac{3}{16}\frac{C_1^2}{C_1 - C_2}$$

and equality holds only at this point, whereas the upper bound (32)

$$\frac{C_1^2}{16}\frac{4 - (1 - r)^2}{C_1 - C_2} \geq \frac{3}{16}\frac{C_1^2}{C_1 - C_2}$$

hence the two bounds are compatible and intersect at the lower bound of ξ (25) for $r = 0$. The upper bound imposed by $\kappa \geq 0$ (32) supersedes (30) if:

$$r < \frac{3C_1 - 4C_2}{C_1}$$

6.3 Doubly-reinforced section, maximum height: Solution 3

The full solution is:

$$\gamma = \frac{C_1(1 + r)}{4(C_1 - C_2)}$$
$$\eta = 1$$
$$\rho = \frac{1}{\beta}\frac{16M(C_1 - C_2) + C_1^2\left(1 - 2r - 3r^2\right)}{16(C_1 - C_2)(1 - r)}$$
$$\rho' = \frac{1}{\beta}\frac{16M(C_1 - C_2) - C_1^2\left(3 + 2r - r^2\right)}{16(C_1 - C_2)(1 - r)}$$
$$\lambda = -\xi$$
$$\mu = -\frac{2\xi}{1 - r}$$
$$\nu = \frac{\xi}{\epsilon_{cu}}\frac{C_1\left(2(C_1 - C_2) - C_1^2\right)(1 + r)^2}{8(C_1 - C_2)^2(1 - r)}$$
$$\pi = \frac{16\xi M(C_1 - C_2) - 4\xi C_1^2 r^2 + \left(\xi C_1^2 - 8(C_1 - C_2)\right)(1 - r)^2}{8(C_1 - C_2)(1 - r)^2}$$
$$\kappa = 0 \tag{33}$$

Positivity of the compression armature $\rho' \geq 0$ sets the lower bound on bending moment

$$M \geq \frac{C_1^2}{16}\frac{4 - (1 - r)^2}{C_1 - C_2} \tag{34}$$

which is the upper bound (32) of solution 2, set by the complementary condition $\kappa \geq 0$.

Positivity of Lagrange multiplier $\pi \geq 0$ imposes the condition

$$M \geq \frac{8(C_1 - C_2)(1-r)^2 - \xi C_1^2 \left(1 - 2r - 3r^2\right)}{16\xi (C_1 - C_2)}$$

and it ensues from the former because the r.h.s. of the latter is smaller than the former's if

$$\xi \geq 2\frac{C_1 - C_2}{C_1^2}\frac{1-r}{1+r}$$

which is true, considering the lower bound on ξ (25).

From the positiveness condition $\rho > 0$ is obtained

$$M > \frac{C_1^2}{16}\frac{3r^2 + 2r - 1}{C_1 - C_2}$$

which always holds because, comparing with (34), $4 - (1-r)^2 > 3r^2 + 2r - 1$ for all $r \in [0, 1[$.

The condition $\gamma < 1$ imposes the constraint on concrete cover

$$r < \frac{3C_1 - 4C_2}{C_1} \tag{35}$$

and for the r.h.s. to be a positive number it is necessary that constants C_1 and C_2 satisfy

$$\frac{C_2}{C_1} < \frac{3}{4} \tag{36}$$

otherwise the doubly-reinforced solution is unfeasible.

The condition $r < \gamma$ simplifies to

$$r < \frac{C_1}{3C_1 - 4C_2}$$

and the r.h.s. is not smaller than 1 for $\frac{C_2}{C_1} \in [\frac{1}{2}, \frac{3}{4}[$, hence it is trivially verified.

7 Elastic-perfectly plastic armatures

In this section previous results are extended in order to account for the yield of armatures by adding conditions $\epsilon_s \geq \epsilon_{sy}$ and $\epsilon_s' \geq \epsilon_{sy}$ where $\epsilon_{sy} > 0$ is the design yield strain of steel.

For the traction armature the yield condition (with concrete compressed to the maximum)

$$\epsilon_s = \frac{1-\gamma}{\gamma}\epsilon_{cu} \geq \epsilon_{sy}$$

sets an upper bound on nondimensional compression depth

$$\gamma \leq \bar{\gamma} \tag{37}$$

where

$$\bar{\gamma} = \frac{\epsilon_{cu}}{\epsilon_{sy} + \epsilon_{cu}} < 1 \tag{38}$$

For the compression armature

$$\epsilon_s' = \frac{\gamma - \frac{r}{\eta}}{\gamma}\epsilon_{cu} \geq \epsilon_{sy}$$

a condition on concrete cover r is obtained

$$r \leq \gamma\eta\frac{\epsilon_{cu} - \epsilon_{sy}}{\epsilon_{cu}}$$

to be checked separately, and not included in the optimization constraints, as before.

With the additional side constraint on γ (37) augmented Lagrangian (11) is replaced by

$$\begin{aligned} L(\epsilon_c, \gamma, \eta, \rho, \rho', \lambda, \mu, \nu, \pi, \kappa, \tau) \\ = \tilde{C} + \lambda\tilde{N} + \mu\left(\tilde{M} - M\right) + \nu(\epsilon_c - \epsilon_{cu}) \\ + \pi(\eta - 1) - \kappa\rho' + \tau(\gamma - \bar{\gamma}) \end{aligned} \tag{39}$$

where τ is the additional Lagrange multiplier, and simplified optimality conditions (19a) and (19b) are replaced by (note that they coincide if $\tau = 0$):

$$\begin{cases} \gamma\eta(\mu\eta(1-\gamma) - \lambda)\dfrac{1 - C_1}{\epsilon_{cu}} + \mu\gamma^2\eta^2\dfrac{1 - 2C_2}{\epsilon_{cu}} + \nu - \tau\dfrac{\epsilon_{sy}}{(\epsilon_{sy} + \epsilon_{cu})^2} = 0 & \text{(a)} \\ -\lambda\eta C_1 + \mu\eta^2(C_1 + 2\gamma(C_2 - C_1)) + \tau = 0 & \text{(b)} \end{cases} \tag{40}$$

The feasible region of the optimization problem with additional side constraint (37) is a subset of the one in the previous sections and the objective function is the same. Hence, solutions 1, 2 and 3 of the latter problem are also solutions of the former (with Lagrange multiplier $\tau = 0$) if they satisfy (37). This will be seen to be the case, under additional conditions. It is not, therefore, necessary to investigate the 4 additional cases with constraint (37) active. It will also be seen that solution 3 verifies (37) only under a limiting condition; if this condition is not met, another doubly-reinforced solution needs to be sought. For the latter case a fourth solution will be obtained, with constraint (37) active. The additional K-T optimality conditions in the case where the constraint is active are:

$$\begin{cases} \gamma = \bar{\gamma} \\ \tau \geq 0 \end{cases} \tag{41}$$

7.1 Single reinforcement, free height: Solution 1

With side constraint $\gamma \leq \bar{\gamma}$ instead of condition $\gamma < 1$ the lower bound (24) is replaced by:

$$\xi \geq \frac{C_1 - 2\bar{\gamma}\,(C_1 - C_2)}{\bar{\gamma} C_1^2} \tag{42}$$

Lower bound (25) supersedes (42) if

$$\bar{\gamma} \geq \frac{1}{4} \frac{C_1}{C_1 - C_2}$$

otherwise (42) shall be taken as the lower bound on the cost-effectiveness ratio ξ, instead of (25).

7.2 Single reinforcement, maximum height: Solution 2

Side constraint $\gamma \leq \bar{\gamma}$ imposes an upper bound on bending moment not larger than (30):

$$M \leq \bar{\gamma} C_1 - \bar{\gamma}^2\,(C_1 - C_2) < C_2 \tag{43}$$

The upper bound on bending moment (32) is smaller than (43) if concrete cover verifies

$$r \leq \frac{4\bar{\gamma}\,(C_1 - C_2) - C_1}{C_1} \tag{44}$$

otherwise (43) shall be taken as the upper bound for solution 2. Inequality (44) can be rewritten as:

$$r \leq \frac{3C_1 - 4C_2}{C_1} \frac{\epsilon_{cu}}{\epsilon_{sy} + \epsilon_{cu}} - \frac{\epsilon_{sy}}{\epsilon_{sy} + \epsilon_{cu}}$$

The lower bound of bending moment (31) equals the upper bound (43) at the lower bound (42) of the cost effectiveness ratio ξ, hence the two bounds (31) and (43) are consistent because the r.h.s. of (31) is a monotonically decreasing function of ξ.

7.3 Double reinforcement, maximum height: Solution 3

Side constraint $\gamma \leq \bar{\gamma}$ is found to set condition (44) on concrete cover r and, if this condition is met, upper bound (32) of solution 2 and lower bound (34) of solution 3 separate the singly- and doubly-reinforced solutions. For (44) to allow for positive concrete cover r at all it is necessary that

$$\bar{\gamma} \geq \frac{1}{4} \frac{C_1}{C_1 - C_2}$$

or, equivalently,

$$\frac{\epsilon_{sy}}{\epsilon_{cu}} \leq \frac{3C_1 - 4C_2}{C_1}$$

and the r.h.s. of the latter has to be strictly positive, which requires condition (36). If condition (44) is not met then solution 3 is unfeasible. This will be the case, in particular, if concrete and steel strains do not satisfy the latter inequality.

The yield condition of the compression armature sets the upper bound on concrete cover

$$r \leq \frac{C_1\left(\epsilon_{cu} - \epsilon_{sy}\right)}{(3C_1 - 4C_2)\,\epsilon_{cu} + C_1 \epsilon_{sy}}$$

and this bound is superseded by (44) if

$$\frac{\epsilon_{sy}}{\epsilon_{cu}} \geq -\frac{3C_1 - 4C_2}{C_1}$$

which is always true because the r.h.s. is negative.

7.4 Double reinforcement, maximum height, maximum compression depth: Solution 4

The complete solution for the system of equations (40) and (19c)–(19g) in this case ($\kappa = 0$, $\eta = 1$, $\gamma = \bar{\gamma}$) is (reinforcement areas are determined by the equilibrium equations (19g) and (19f) alone):

$$\gamma = \bar{\gamma}$$
$$\eta = 1$$
$$\rho = \frac{1}{\beta} \frac{M + \bar{\gamma}^2\,(C_1 - C_2) - \bar{\gamma} C_1 r}{1 - r}$$
$$\rho' = \frac{1}{\beta} \frac{M + \bar{\gamma}^2\,(C_1 - C_2) - \bar{\gamma} C_1}{1 - r}$$
$$\lambda = -\xi$$
$$\mu = -\frac{2\xi}{1 - r}$$
$$\nu = \frac{\xi}{\epsilon_{cu}} \frac{\bar{\gamma}}{1 - r} \left(4\bar{\gamma}^2\,(C_1 - C_2) - \bar{\gamma} C_1\,(3 + r) + 1 + r\right)$$

$$\pi = \frac{2\xi M - 2\bar{\gamma}^2 \xi (C_1 - C_2)(1-2r) + \bar{\gamma}\xi C_1 \left(1 - 2r - r^2\right) - (1-r)^2}{(1-r)^2}$$

$$\kappa = 0$$

$$\tau = \xi \frac{C_1 (1+r) - 4\bar{\gamma} (C_1 - C_2)}{1-r} \tag{45}$$

From the Lagrange multiplier positiveness condition $\tau \geq 0$ the condition on concrete cover r is obtained

$$r \geq \frac{4\bar{\gamma} (C_1 - C_2) - C_1}{C_1} \tag{46}$$

and comparing with (44) it is found this is the solution when solution 3 is unfeasible.

Positivity of compression armature $\rho' \geq 0$ sets the lower bound on the bending moment

$$M \geq \bar{\gamma} C_1 - \bar{\gamma}^2 (C_1 - C_2) \tag{47}$$

which is the upper bound (43) of solution 2. When (46) holds, (43) and (47) separate the singly- and doubly-reinforced solutions 2 and 4.

Comparing the formulas for both armatures it is immediately recognized that $\rho > \rho'$ $(r < 1)$ hence $\rho > 0$.

Lagrange multiplier positivity $\nu \geq 0$ can be restated as a condition on concrete cover r (property $C_1 \leq 1$ is used)

$$r \geq \frac{3\bar{\gamma} C_1 - 4\bar{\gamma}^2 (C_1 - C_2) - 1}{1 - \bar{\gamma} C_1}$$

and comparing the r.h.s. of this inequality with that of (46) it is found that the condition that the latter be greater than the former is equivalent to (22), hence the Lagrange multiplier is positive.

Positivity of Lagrange multiplier $\pi \geq 0$ sets the lower bound on the bending moment

$$M \geq \frac{(1-r)^2}{2\xi} + \bar{\gamma}^2 (C_1 - C_2)(1-2r) - \bar{\gamma} C_1 \frac{1 - 2r - r^2}{2}$$

comparing with bound (47) it is found the latter is larger than the former if

$$\xi \geq \frac{1-r}{\bar{\gamma}\left(3C_1 + C_1 r - 4\bar{\gamma} (C_1 - C_2)\right)}$$

and comparing this bound on the cost-effectiveness ratio with (42) the latter supersedes the former for all concrete cover r satisfying (46), hence (47), (42) and (46) imply $\pi \geq 0$, and no additional condition is required.

The yield condition of compression armature simplifies to

$$r \leq \frac{\epsilon_{cu} - \epsilon_{sy}}{\epsilon_{sy} + \epsilon_{cu}} \tag{48}$$

and, for solution 4 to be feasible, it is necessary for concrete cover r to lie between the lower and upper bounds (46) and (48), respectively. The two bounds are consistent for $\frac{C_2}{C_1} \geq \frac{1}{2}$ (17).

8 Specialization to parabola-rectangle law

The parabola-rectangle concrete response function specified by EC2 (2001) is used:

$$\tilde{\sigma}_c(\cdot) = \begin{cases} 1 - \left(1 - \frac{\epsilon}{\epsilon_{c2}}\right)^n & \text{if} \quad 0 \leq \epsilon \leq \epsilon_{c2} \\ 1 & \text{if} \quad \epsilon_{c2} \leq \epsilon \leq \epsilon_{cu} \end{cases}$$

For this law concrete constants C_1 and C_2 evaluate:

$$C_1 = 1 - \frac{\epsilon_{c2}}{(n+1)\,\epsilon_{cu}}$$

$$C_2 = \frac{1}{2} - \frac{\epsilon_{c2}^2}{\left(2 + 3n + n^2\right)\epsilon_{cu}^2}$$

These constants are listed in Table 1 for the fourteen concrete classes of EC2.

It can be readily shown that these constants satisfy all constraints that have arisen here, without resorting to a case-by-case analysis; it suffices to note that $\epsilon_{cu} \geq \epsilon_{c2}$ and $n > 1$ for all classes. Solving inequality $\frac{C_2}{C_1} \geq \frac{1}{2}$ (17) for ϵ_{cu}

$$\epsilon_{cu} \geq \frac{2}{2+n} \epsilon_{c2}$$

Table 1 Constants C_1 and C_2 for parabola-rectangle law (EC2 2001)

Parameter	C12/15–C50/60	C55/67	C60/75	C70/85	C80/95	C90/105
ϵ_{c2}	$\frac{20}{10{,}000}$	$\frac{22}{10{,}000}$	$\frac{23}{10{,}000}$	$\frac{24}{10{,}000}$	$\frac{25}{10{,}000}$	$\frac{26}{10{,}000}$
ϵ_{cu}	$\frac{35}{10{,}000}$	$\frac{31}{10{,}000}$	$\frac{29}{10{,}000}$	$\frac{27}{10{,}000}$	$\frac{26}{10{,}000}$	$\frac{26}{10{,}000}$
n	$\frac{200}{100}$	$\frac{175}{100}$	$\frac{160}{100}$	$\frac{145}{100}$	$\frac{140}{100}$	$\frac{140}{100}$
C_1	$\frac{17}{21}$	$\frac{23}{31}$	$\frac{262}{377}$	$\frac{281}{441}$	$\frac{187}{312}$	$\frac{7}{12}$
C_2	$\frac{139}{294}$	$\frac{13{,}007}{28{,}830}$	$\frac{42{,}586}{98{,}397}$	$\frac{222{,}661}{547{,}722}$	$\frac{53{,}327}{137{,}904}$	$\frac{77}{204}$

Table 2 Nondimensional compression depth γ and height η, mechanical reinforcement ratios $\rho\beta$ (traction) and $\rho'\beta$ (compression) for C12/15–C50/60

Variable	Solution 1	Solution 2	Solution 3	Solution 4
γ	$\frac{357}{289\xi+297}$	$\frac{119-7\sqrt{289-594M}}{99}$	$\frac{119(1+r)}{198}$	$\bar{\gamma}$
η	$\frac{(289\xi+297)\sqrt{2M}}{17\sqrt{578\xi+297}}$	1	1	1
$\rho\beta$	$\frac{17\sqrt{2M}}{\sqrt{578\xi+297}}$	$\frac{289-17\sqrt{289-594M}}{297}$	$\frac{2376M+289\left(1-2r-3r^2\right)}{2376(1-r)}$	$\frac{294M+99\bar{\gamma}^2-238\bar{\gamma}r}{294(1-r)}$
$\rho'\beta$	0	0	$\frac{2376M-289\left(3+2r-r^2\right)}{2376(1-r)}$	$\frac{294M+99\bar{\gamma}^2-238\bar{\gamma}}{294(1-r)}$

which is always true. Solving inequality $\frac{C_2}{C_1} \leq 1 - \frac{C_1}{2}$ (22) for the same variable an equally self-evident result is obtained:

$$\epsilon_{cu} > \frac{\epsilon_{c2}}{1+n}$$

On account of the previous inequality (22), noting that in this case $C_1 > \frac{1}{2}$ because

$$\epsilon_{cu} > \frac{2}{1+n}\epsilon_{c2}$$

and, therefore, $\frac{C_2}{C_1} < \frac{3}{4}$ (36).

In Table 2 are listed the primal variables for C12/15–C50/60 concrete classes. Optimality regions of solutions 1, 2, 3 and 4 in the plane 'cost-effectiveness ratio'–'nondimensional bending moment' (ξ–M) are shown in Fig. 2, for the same concrete classes, and a three-dimensional representation of the cost function is shown in Fig. 3.

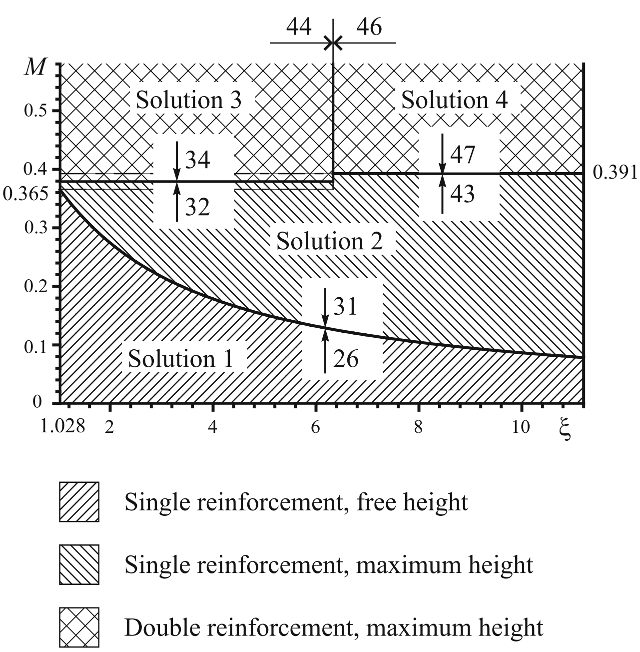

Fig. 2 Optimality regions for EC2 C12/15–C50/60 concrete classes. The numbering refers to the inequalities bounding the solutions. Axis markings are for S400 steel (E=200GPa). For S600 solution 3 is unfeasible. M is the nondimensional bending moment and ξ is the cost-effectiveness ratio of concrete to steel

9 Examples

9.1 Example 1

Optimal solutions for $\xi = 3$ and $\xi = 6$ with increasing specified bending moment M are compared. The parabola-rectangle law of EC2 (2001) for classes C12/15–C50/60 is used. The nondimensional concrete cover used is $r = 0.05$. Nondimensional height η and total mechanical reinforcement ratio $\left(\rho + \rho'\right)\beta$ (see Table 2) are shown in Fig. 4. Differences are observed up to the bound $M \lesssim 0.217$ of solution 1 with $\xi = 3$. For the comparatively smaller cost-effectiveness ratio of concrete to steel $\xi = 3$ less concrete and more steel reinforcement are used than for $\xi = 6$; conversely, for the comparatively higher cost-to-strength ratio of steel to concrete $\xi = 6$ less armature and a larger section are used. The bound of solution 1, up to which the section is downsized, for the higher cost-effectiveness ratio of concrete to steel $\xi = 6$ is $M \lesssim 0.132$, lower than for $\xi = 3$. The upper bound for solution 2 (in both cases) is $M \lesssim 0.377$; for $M \gtrsim 0.377$ the optimal solution is doubly-reinforced (solution 3).

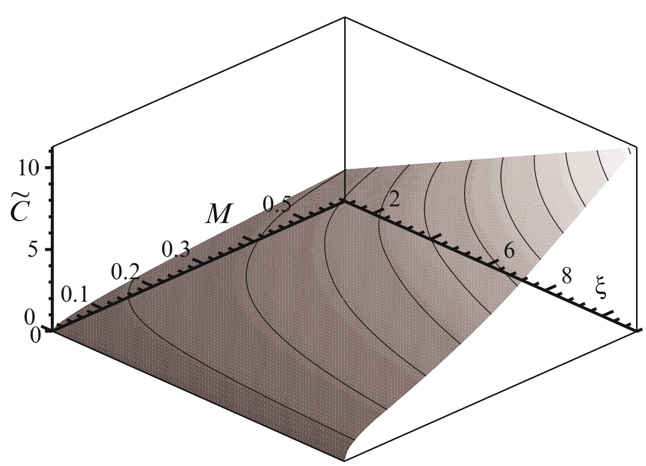

Fig. 3 Cost contour curves for EC2 C12/15–C50/60 concrete classes (solutions 1-2-3, $r = 0$)

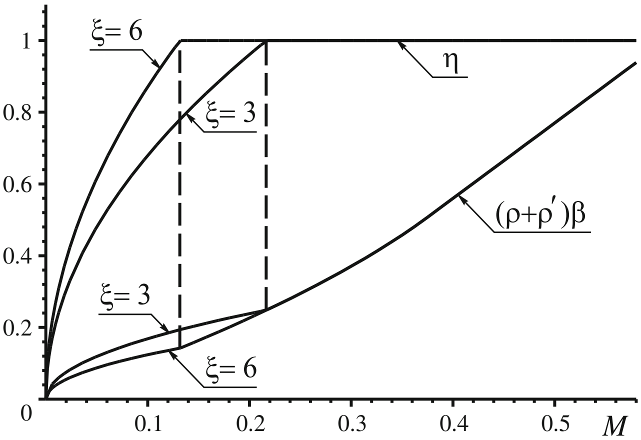

Fig. 4 Section 9.1: Optimal nondimensional heigth η and total mechanical reinforcement ratio $(\rho + \rho')\,\beta$ vs bending moment M for $\xi = 3$ and $\xi = 6$

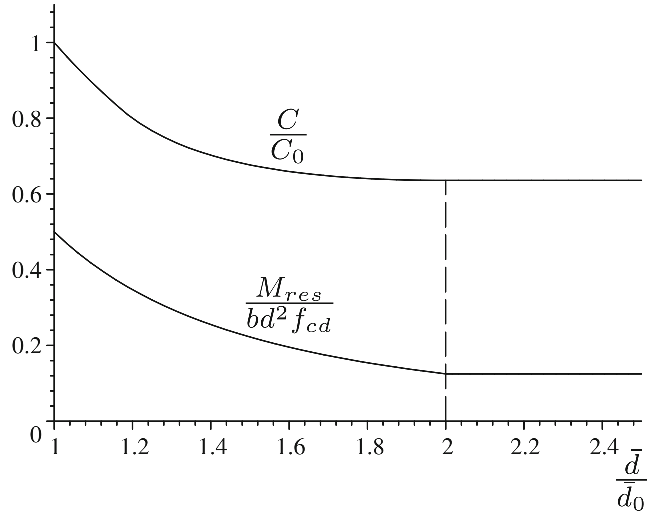

Fig. 6 Section 9.2: Section cost $\frac{C}{C_0}$ and nondimensional bending moment $\frac{M_{\mathrm{res}}}{bd^2 f_{cd}}$ vs maximum allowable depth $\frac{\bar{d}}{\bar{d}_0}$

9.2 Example 2

The purpose of this example is to analyze the influence of maximum depth $\bar{d}$ on optimal solutions. Parabola-rectangle law for EC2 concrete classes C12/15–C50/60 is used. Suppose that the nondimensional bending moment is $M_0 = 0.5$ for some specified bending moment M_{res} and initial design $\bar{d}_0$, hence a doubly-reinforced section is required (solution 3), and that the section cost is C_0. The parameters $\xi = 6.42$ and $r = 0$ are used. With increasing $\bar{d}$ the nondimensional bending moment M decreases and it is found that the boundary between solutions 3 and 2 is for $\frac{\bar{d}}{\bar{d}_0} = 1.17$ and between solutions 2 and 1 for $\frac{\bar{d}}{\bar{d}_0} = 2.00$. In Fig. 5 are plotted the nondimensional optimal depth $\frac{d}{\bar{d}_0}$ and the mechanical reinforcement ratio $\frac{A+A'}{bd}\beta$, adimensionalised w.r.t. the optimal section area. Section cost ratio $\frac{C}{C_0}$ and bending moment $\frac{M_{\mathrm{res}}}{bd^2 f_{cd}}$ are shown in Fig. 6. It is seen the optimal depth increases and armature decreases, with decreasing cost, until solution 1 is reached and then all variables level off. The most economical section (singly-reinforced) is reached for twice the initial maximum depth with 63.6% of the initial cost and bending moment $M = 0.125$.

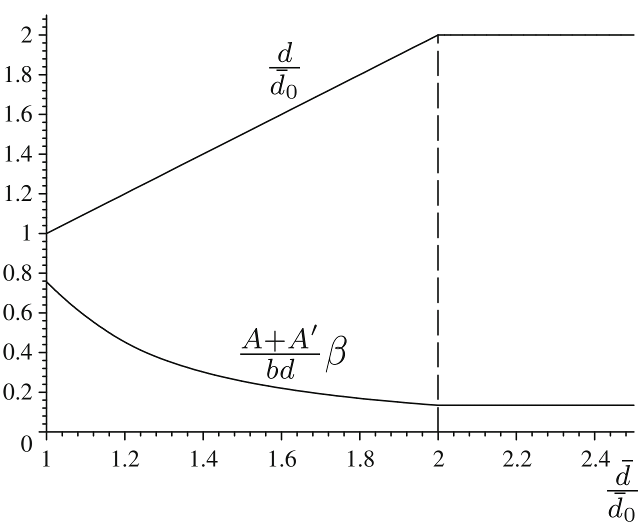

Fig. 5 Section 9.2: Section depth $\frac{d}{\bar{d}_0}$ and mechanical reinforcement ratio $\frac{A+A'}{bd}\beta$ vs maximum allowable depth $\frac{\bar{d}}{\bar{d}_0}$

9.3 Example 3

This is a counter-example, where results derived in this paper are not applicable. Adjusted for conventions used here, the MC90 nondimensional concrete response function is (see Table 3 for material constants description and numerical values):

$$\tilde{\sigma}_c(\cdot) = \frac{1 + \frac{E_{cc1}-2}{\epsilon_{c1}}\epsilon_{cu}}{\frac{E_{cc1}}{\epsilon_{c1}}\epsilon_{cu} - \frac{\epsilon_{cu}^2}{\epsilon_{c1}^2}} \, \frac{\frac{E_{cc1}}{\epsilon_{c1}}\epsilon - \frac{\epsilon^2}{\epsilon_{c1}^2}}{1 + \frac{E_{cc1}-2}{\epsilon_{c1}}\epsilon}$$

It can be readily verified that constants C_1 and C_2 listed in Table 3 for 4 concrete classes violate condition (22)—for C16/20 and C25/30 no computation is even required as, on account of (17), C_1 can not be larger than 1. Therefore, basic assumption (14), that concrete is strained to the maximum (ϵ_{cu}), does not hold for this constitutive law. In Barros et al. (2005) is demonstrated that, at the maximum bending

Table 3 Section 9.3: Mechanical properties of EC2 concrete classes C16/20–C50/60 for model code MC90

Parameter	C16/20	C25/30	C40/50	C50/60
ϵ_{c1}	0.001875	0.002069	0.002324	0.002465
ϵ_{cu}	0.0035	0.0035	0.0035	0.0035
E_{cc1}	3.5431	2.7762	2.2366	2.0059
C_1	1.032 5	1.011 8	0.947 99	0.906 89
C_2	0.565 14	0.568 16	0.550 52	0.536 61

ϵ_{c1}: strain at peak stress; ϵ_{cu}: ultimate strain; E_{cc1}: conventional elasticity modulus; C_1 and C_2: material constants defined in (15)

moment allowable, maximum concrete strain lies between the strain for peak stress (ϵ_{c1}) and the ultimate strain (ϵ_{cu}).

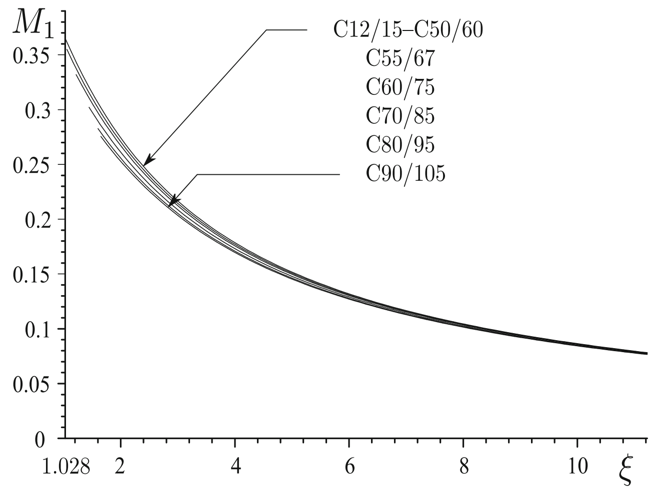

Fig. 7 Optimal bending moment M_1 (27) vs cost-effectiveness ratio ξ (9) for EC2 C12/15–C90/105 concrete classes

10 Conclusions

Solutions of the K-T optimality conditions were obtained for a generic concrete constitutive law and conditions to determine wether they are applicable to a specific law were established. The parabola-rectangle law of EC2 (2001) is one case of applicability for all concrete classes, as shown in Section 8, and MC90 of non-applicability (Section 9.3).

The four solutions obtained—(23), (29), (33) and (45)—were analyzed for the range of parameters where each is optimal. This is illustrated summarily in Fig. 2. It is noteworthy that all formulas are sensibly short.

The investigation presented here confirms practical experience—an interesting result in it's own right—and adds guidelines (and mathematical formulas) for the cost optimization of rectangular RC sections. Known basic designs are shown to be cost-optimal under certain conditions, but not always. The singly reinforced section with predefined height, for some given bending moment (solution 2), for instance, may be optimal for some value of the cost-effectiveness ratio but not for another (illustrated in Section 9.1): the optimal may be a section with reduced height (solution 1). Or the doubly-reinforced section with maximum compression depth, as allowed by the minimum strain in traction armature (solution 4): it will not be optimal if there is a feasible solution with less than maximum compression depth (solution 3), for which an optimal distribution of armatures (traction and compression) can be obtained. Solutions 3 and 4 are unaffected by cost factors because the amount of concrete is fixed and the optimal solution is simply the one using less steel. It was shown that there is not an optimal doubly-reinforced section if the depth is allowed to vary freely.

The argument is made that, compared to solution 1, the other solutions are sub-optimal (illustrated in Section 9.2). Indeed, all other solutions have an additional active constraint—maximum section depth—and can not, therefore, be more economical than solution 1; if the constraint is removed this will be the only solution. Moreover, the bending moment in solution 1 adimentionalised w.r.t. the actual section depth M_1 (27) is a function of the cost-effectiveness ratio alone. This can be used as a straightforward design recommendation, leaving the designer with one degree of freedom less: section depth, breadth, or other. This optimal nondimensional bending moment is plotted in Fig. 7 for the parabola-rectangle law of EC2 (2001). Armature is given by ρ_1 (28), also a decreasing function of the cost-effectiveness ratio; with increasing cost-effectiveness of concrete (or decreasing of steel) a larger section with less armature is more economical.

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