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Reliability-based structural optimization with probability and convex set hybrid models

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Abstract For structural systems exhibiting both probabilistic and bounded uncertainties, it may be suitable to describe these uncertainties with probability and convex set models respectively in the design optimization problem. Based on the probabilistic and multi-ellipsoid convex set hybrid model, this paper presents a mathematical definition of reliability index for measuring the safety of structures in presence of parameter or load uncertainties. The optimization problem incorporating such reliability constraints is then mathematically formulated. By using the performance measure approach, the optimization problem is reformulated into a more tractable one. Moreover, the nested double-loop optimization problem is transformed into an approximate single-loop minimization problem by considering the optimality conditions and linearization of the limit-state function, which further facilitates efficient solution of the design problem. Numerical examples demonstrate the validity of the proposed formulation as well as the efficiency of the presented numerical techniques.

Keywords Structural optimization · Reliability · Probability · Convex set · Hybrid model

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1 Introduction

Along with the ever increasing computational power, the past two decades has seen a rapid development of structural optimization in both theories and industrial applications. In particular, the design optimization problem incorporating various uncertainties has been intensively studied (Schuëller and Jensen 2008). Among other non-deterministic optimal design formulations, the reliability-based design optimization provides an effective tool for seeking the best designs against structural failures in presence of system variations. Basically, the uncertainty models employed by a typical structural reliability analysis can be classified into two categories: the probabilistic model and non-probabilistic models. As the most mature uncertainty model, the probabilistic model describes the stochastic parameters and structural responses with random fields or discrete random variables that have certain statistical distribution characteristics. The probabilistic model has been successfully used in many real-life engineering applications for structural reliability-based design optimization (RBDO) (Frangopol and Corotis 1996; Papadrakakis and Lagaros 2002) as well as robust design optimization (Doltsinis and Kang 2004; Beyer and Sendhoff 2007). In practical applications, the probabilistic distribution type and the corresponding statistical parameters of inputs are usually extracted from a sufficient amount of measured data or assumed on the basis of engineering experiences.

A meaningful probabilistic reliability analysis relies on availability of precise description of the statistical characteristics, particularly, the tail distribution of the random inputs. However, these data cannot be accurately extracted in some circumstances where only limited number of samples are available. Construction of appropriate probability model using insufficient data or incomplete knowledge of the uncertainty has been a classical problem in the probability theory. A conventional treatment is to model the concerned uncertainty with a probabilistic distribution that is closest to be uniform by using the principle of maximum entropy (Jaynes 1957; Soize 2001).

As illustrated by Elishakoff (1995, 1999), in particular cases a small error in constructing the probabilistic density function for input quantities may give rise to misleading prediction of the probabilistic reliability. This means that the traditional probabilistic approaches might be questionable to deal with those problems involving incomplete information or inherently non-probabilistic uncertainties. Consequently, non-traditional uncertainty models, such as the fuzzy possibility model (Mourelatos and Zhou 2005; Du et al. 2006a), the fuzzy/statistical hybrid model (Möller and Beer 2004; Buckley 2005; Du et al. 2006b; Du and Choi 2008) and the convex model (Ben-Haim and Elishakoff 1990; Elseifi et al. 1999), have also been developed as alternative models for describing uncertainty with incomplete statistical information. The interested readers are referred to review papers by e.g. Moens and Vandepitte (2005), Möller and Beer (2008).

In many circumstances, the bounds of uncertainty, compared with the precise probability distribution data, are more easily available. This is particularly the case for the geometry uncertainty arising as a result of manufacturing errors, which are subject to the tolerance band control. The set theory-based convex models, especially the ellipsoidal model, provide an attractive framework for treating those so-called *uncertain-but-bounded* variations of a structural system. Two characteristics make the ellipsoidal convex model more appealing when no sufficient data on the inner distributions of system variations are available. Firstly, only bound information of the uncertainties is needed for construction of a convex model. Secondly, the convex model presents a mathematically differentiable bound description of the inputs and thus facilitates use of common optimization algorithms in seeking the minimum of a function value. A number of studies on the non-probabilistic optimization designs using convex sets have been presented in literatures (Qiu and Elishakoff 1998; Pantelides and Ganzerli 1998; Au et al. 2003; Jiang et al. 2007; Luo et al. 2009).

In some typical engineering problems, some of the concerned uncertainties are probabilistic variables with precise-enough probability distribution information, while others are only uncertain-but-bounded due to their inherent characteristic or lack of sufficient sample data. Therefore, it is desirable to select the best suitable model to quantify these different types of uncertainties. Early attempts have been made by e.g. Elishakoff and Colombi (1993) to accommodate both probabilistic and convex models into structural reliability analysis and design. Recently, the function approximation technique (Penmetsa and Grandhi 2002; Adduri and Penmetsa 2007) and the probability bounds (p-box) approach (Karanki et al. 2009) have been suggested for estimating the lower and upper bounds of the structural reliability when both random and interval parameters are concerned. Similar problems have also been studied by Hall and Lawry (2004), Berleant et al. (2005), Utkin and Kozine (2005), Kreinovich et al. (2006) and Qiu et al. (2008). Moreover, Du et al. (2005) extended the conventional RBDO method to structural design problems where a part of the uncertainties appear as interval variables. In their study, a procedure for seeking the worst-case combination of the interval variables is embedded into the probabilistic reliability analysis. As the literature survey shows, the existing studies mainly focus on solving the combination of random variables and interval variables. Basically, the interval set can be regarded as the simplest specific case of set-value based convex models and it does not account for the dependencies among the bounded uncertainties. It is therefore meaningful to fully explore the reliability-based design optimization formulation using the mixed description of probability and more general convex set model, e.g. the multi-ellipsoid convex model.

This paper aims to provide a mathematical statement and associated numerical techniques to incorporate simultaneously stochastic randomness and general uncertain-but-bounded quantities into the design optimization problem. To achieve this goal, a mathematical definition of structural reliability index based on probabilistic model and multi-ellipsoid convex set is first proposed. Then, a nested optimization model for reliability-based structural design problems with constraints on such reliability indices is presented. To improve the convergence of the overall solution, the performance measure approach (PMA) is employed. Further, an iteration scheme based on linear approximation of the limit-state function and the optimality conditions is proposed for converting the nested double-loop problem into an approximate single-loop one. This approach is employed to solve the present optimization problem and its performance is compared with that of the direct nested double-loop method in aspect of efficiency. Finally, to demonstrate the applicability of the proposed model and the efficiency of the numerical techniques, three pure mathematical or engineering design examples are presented.

2 General concept of multi-ellipsoid convex model

Several frequently used convex models are the envelope-bound convex model, the instantaneous energy-bound convex model, the cumulative energy-bound convex model and the ellipsoid convex model (Ben-Haim and Elishakoff 1990). In this study, the multi-ellipsoid convex model (Luo et al. 2009) is applied for the uncertain-but-bounded uncertainty description. This model and the corresponding normalization procedures are described as follows.

Assume the uncertain-but-bounded variables are collected in the vector $\mathbf{Y} \in R^n$. The conventional single-ellipsoid convex model assumes that all the possible value of uncertainties are bounded by a multidimensional (hyper-) ellipsoid, which is expressed by:

$$\mathbf{Y} \in \mathrm{E}\left(\mathbf{W}_{\mathrm{y}}, \varepsilon\right) = \left\{\mathbf{Y} : \left(\mathbf{Y} - \hat{\mathbf{Y}}\right)^{\mathrm{T}} \mathbf{W}_{\mathrm{y}} \left(\mathbf{Y} - \hat{\mathbf{Y}}\right) \leq \varepsilon^2\right\}, \quad (1)$$

where $\hat{\mathbf{Y}}$ is the nominal value vector of $\mathbf{Y}$, $\mathbf{W}_{\mathrm{y}}$ is the characteristic matrix of the ellipsoid, and it is a symmetric positive-definite real matrix defining the orientation and aspect ratio of the ellipsoid, and ε is a real number defining the magnitude of the parameter variability.

The vector $\mathbf{Y}$ can be transformed into a dimensionless vector $\boldsymbol{\delta} \in R^n$ and the components of the vectors $\boldsymbol{\delta}$ and $\mathbf{Y}$ are related by:

$$\delta_i = \frac{Y_i - \hat{Y}_i}{\hat{Y}_i} \quad (i = 1, 2, \ldots, n), \quad (2)$$

where $\hat{Y}_i$ denotes the nominal value of the i-th uncertain-but-bounded variable. Thus, the ellipsoid set can be further transformed into its dimensionless form:

$$\boldsymbol{\delta} \in \mathbf{E}\left(\mathbf{W}, \varepsilon\right) = \left\{\boldsymbol{\delta} : \boldsymbol{\delta}^{\mathrm{T}} \mathbf{W} \boldsymbol{\delta} \leq \varepsilon^2\right\}, \quad (3)$$

where $\mathbf{W}$ is the dimensionless characteristic matrix.

In some structural design problems, it might be suitable to divide all the uncertain-but-bounded variables into groups and assume that variations of parameters in different groups are uncorrelated. In such circumstances, the multi-ellipsoid convex model is competent for the description of these uncertainties (Luo et al. 2009). Supposing the uncertain-but-bounded variables are divided into N_{E} groups denoted by:

$$\mathbf{Y}^{\mathrm{T}} = \left\{\mathbf{Y}_1^{\mathrm{T}}, \mathbf{Y}_2^{\mathrm{T}}, \ldots, \mathbf{Y}_{N_{\mathrm{E}}}^{\mathrm{T}}\right\}, \quad (4)$$

the multi-ellipsoid convex model treats each group of uncertainties $\mathbf{Y}_i \in R^{n_i}$ $(i = 1,2,\ldots,N_{\mathrm{E}})$ with an individual ellipsoid convex set, respectively, as:

$$\boldsymbol{\delta} \in \mathbf{E} = \left\{\boldsymbol{\delta} : \boldsymbol{\delta}_i^{\mathrm{T}} \mathbf{W}_i \boldsymbol{\delta}_i \leq \varepsilon_i^2 \quad (i = 1, 2, \ldots, N_{\mathrm{E}})\right\}, \quad (5)$$

where $\boldsymbol{\delta}_i$ is the dimensionless vector of $\mathbf{Y}_i$, $\mathbf{W}_i$ is the dimensionless characteristic matrix of the i-th ellipsoid, $\varepsilon_i (i = 1, 2, \ldots, N_{\mathrm{E}})$ are real numbers, N_{E} is the total number of groups of bounded uncertainties, n_i is the number of bounded uncertainties in the i-th group and $\sum_{i=1}^{N_{\mathrm{E}}} n_i = n$.

The multi-ellipsoid convex model provides a unified approach that accommodates coexisting conventional ellipsoid sets and interval sets in the representation of bounded uncertainties. In particular, if each group consists of only one uncertain parameter, the multi-ellipsoid set in (5) degenerates into a multi-dimensional interval set (hyper-box model). On the other hand, the single-ellipsoid model represents another degenerated case of the multi-ellipsoid model, where all the bounded uncertainties are correlated.

3 Reliability definition under hybrid model of probability and convex sets

3.1 Pure probabilistic description

In the conventional probabilistic framework (Melchers 1999), the uncertainties are modelled as random variables with certain distribution characteristics. Let $\mathbf{X} = \{X_1, X_2, \ldots, X_m\}^{\mathrm{T}}$ denotes the vector of random variables, the structural failure probability can be given as:

$$P_{\mathrm{f}} = \Pr\left[G\left(\mathbf{X}\right) \leq 0\right] = \int \cdots \int_{G(\mathbf{X}) \leq 0} p_{\mathrm{X}}\left(\mathbf{x}\right) dx_1 \ldots dx_m \quad (6)$$

where $\Pr[\cdot]$ denotes the probability, $G(\mathbf{X})$ is a system performance function and $G(\mathbf{X}) \leq 0$ defines the failure event, $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}^{\mathrm{T}}$ represents the realization of $\mathbf{X}$, $p_{\mathrm{x}}(\mathbf{x})$ is the joint probability density function, which is usually approximated using measured data sets of the system parameters. The accuracy of the approximation for $p_{\mathrm{x}}(\mathbf{x})$ is limited by the total number of available samples. For assessing the multi-variate integral in (6), many available techniques, such as the Monte Carlo simulation and the familiar first-order reliability method (FORM) (Hasofer and Lind 1974), can be implemented.

3.2 Probability and convex set hybrid model description

In practice, engineering structures may exhibit both probabilistic uncertainties and bounded uncertainties. In such a case, the uncertain variables involved in the design problem can be classified into random variables and uncertain-but-bounded variables described by multi-ellipsoid convex sets. They are respectively denoted by $\mathbf{X} = \{X_1, X_2, \ldots, X_m\}^T$ and $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_n\}^T$ (or $\mathbf{Y} = \left\{\mathbf{Y}_1^{\mathrm{T}}, \mathbf{Y}_2^{\mathrm{T}}, \ldots, \mathbf{Y}_{N_{\mathrm{E}}}^{\mathrm{T}}\right\}^{\mathrm{T}}$ in terms of N_{E} vectors for grouped bounded variables), which are expressed using above-mentioned notations as:

$$\mathbf{X} \sim p_{\mathrm{X}}\left(\mathbf{x}\right), \quad (7)$$

$$\mathbf{Y} \in \mathbf{E} = \left\{\boldsymbol{\delta} : \boldsymbol{\delta}_i^{\mathrm{T}} \mathbf{W}_i \boldsymbol{\delta}_i \leq \varepsilon_i^2, \quad i = 1, 2, \ldots, N_{\mathrm{E}}\right\}, \quad (8)$$

where $p_X(\mathbf{x})$ is the joint probability density function for the random variables $\mathbf{X}$, $\mathbf{E}$ is the multi-ellipsoid convex set defining the variation range of $\mathbf{Y}$.

3.2.1 *Normalization of probabilistic random variables*

A normal random variable X can be transformed into a standard normal random variable u by:

$$u = \frac{X - \bar{X}}{\sigma}, \tag{9}$$

where $\bar{X}$ and σ are the mean value and the standard deviation of X, respectively.

In general cases, the random variables $\mathbf{X}$ can be transformed to a set of uncorrelated normal random variables via the Rosenblatt transformation (Rosenblatt 1952):

$$\begin{aligned}\mathbf{u} &= \{u_1, u_2, ..., u_m\}^{\mathrm{T}} \\ &= \Big\{\Phi^{-1}\left[H_1(x_1)\right], \Phi^{-1}\left[H_2(x_2\,|x_1)\right], \ldots, \\ &\qquad \Phi^{-1}\left[H_m(x_m\,|x_1, x_2, \ldots, x_{m-1})\right]\Big\}^{\mathrm{T}},\end{aligned} \tag{10}$$

where $H_j(x_j|x_1, x_2, \ldots, x_{j-1})$ is the marginal cumulative distribution function and $\Phi^{-1}(\cdot)$ is the inverse standard normal cumulative distribution function. In this way, all the random variables are transformed into statistically independent standard normal ones in the normalized space (or $\mathbf{u}$-space).

3.2.2 *Normalization of uncertain-but-bounded uncertainties*

For the case of convex set modelling, it is convenient to convert the original uncertainties into dimensionless normalized variables with a linearization transformation. This procedure is elaborated in what follows.

Firstly, the following eigenvalue problems need to be solved:

$$\mathbf{Q}_i^{\mathrm{T}}\mathbf{W}_i\mathbf{Q}_i = \mathbf{\Lambda}_i \quad (i = 1, 2, \ldots, N_{\mathrm{E}}), \tag{11}$$

where $\mathbf{Q}_i$ is the orthogonal matrix comprising the normalized eigenvectors, $\mathbf{\Lambda}_i$ is a diagonal matrix containing the eigenvalues of $\mathbf{W}_i$.

Then, by defining a dimensionless vector $\mathbf{v}_i$ as:

$$\mathbf{v}_i = \left(1/\varepsilon_i\right)\mathbf{\Lambda}_i^{1/2}\mathbf{Q}_i^{\mathrm{T}}\boldsymbol{\delta}_i \quad (i = 1, 2, \ldots, N_{\mathrm{E}}) \tag{12}$$

and substituting (12) into (8), the original multi-ellipsoid convex set is rewritten as:

$$\mathbf{E} = \left\{\mathbf{v}_i : \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i \le 1, \quad i = 1, 2, \ldots, N_{\mathrm{E}}\right\}, \tag{13}$$

where $\mathbf{v}_i$ is the normalized vector of the i-th group of uncertain-but-bounded variables. It can be seen that the normalized convex set forms multiple spheres of unit radius.

3.3 Definition of the reliability measure under hybrid model description

For assessment of the structural reliability, a quantitative definition of the failure state is needed. We assume a desired performance of a structure is represented by $G(\mathbf{X}, \mathbf{Y}) > 0$. After the uncertainty normalization procedures given by (10) and (12), the performance function $G(\mathbf{X}, \mathbf{Y})$ is symbolically transformed into its normalized form, which is expressed by $g(\mathbf{u}, \mathbf{v})$.

In the conventional probability model, the limit state $g = 0$ defines a unique surface which is called the limit-state surface. However, when the coexistence of random and uncertain-but-bounded variables is encountered, there are a cluster of limit-state surfaces $g(\mathbf{u}, \mathbf{v}) = 0$, which form a critical region in the $\mathbf{u}$-space for all possible values of $\mathbf{v} \in \mathbf{E}$. As shown in Fig. 1, the whole $\mathbf{u}$-space Ω is thus divided into a safe region $\left(\Omega_{\mathrm{s}} = \left\{\mathbf{u} : \min_{\mathbf{v}\in\mathbf{E}} g(\mathbf{u}, \mathbf{v}) > 0\right\}\right)$, a critical region ($\Omega_{\mathrm{c}} = \{\mathbf{u} : \exists\mathbf{v} \in \mathbf{E}, g(\mathbf{u}, \mathbf{v}) = 0\}$) and a failure region ($\Omega_{\mathrm{f}} = \Omega \backslash (\Omega_{\mathrm{s}} \cup \Omega_{\mathrm{c}})$). If multiple failure modes (each represented by a limit-state equation $g_i(\mathbf{u}, \mathbf{v}) = 0$ for $j = 1, 2, \ldots, N_F$) are considered, the safe region of the problem becomes $\Omega_{\mathrm{s}} = \Omega_{\mathrm{s},1} \cap \Omega_{\mathrm{s},2} \cap \ldots \cap \Omega_{\mathrm{s},N_{\mathrm{F}}}$. Here N_{F} is the total number of failure modes and $\Omega_{s,j}$ is the safe region for the j-th failure mode.

It is recalled that the structural reliability is defined as the probability that the structure performs its desired functionality under given random system variations. By extending this concept, we define the reliability under probability and convex set hybrid model as: the least probability that the structural behaviour satisfies the design requirements for all the possible values of the bounded uncertainties. This reliability measure provides a conservative assessment on the

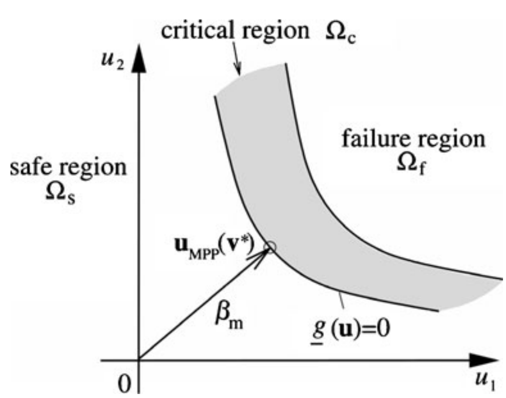

Fig. 1 Schematic representation of the reliability measure under hybrid model in $\mathbf{u}$-space

probability that a structure performs as designed in presence of bounded variations in addition to conventional randomness. Mathematically, by replacing the limit-state function in the conventional reliability definition, we can express the structural reliability under the hybrid model as:

$$P_{\mathrm{m}} = \Pr\left\{\underline{g}\,(\mathbf{u}) > 0\right\}, \tag{14}$$

where $\underline{g}\,(\mathbf{u}) = \min_{\mathbf{v}:\mathbf{v}\in\mathbf{E}} g\,(\mathbf{u}, \mathbf{v})$ and $\underline{g}\,(\mathbf{u}) = 0$ indicates the failure of the structure. The subscript $\langle \mathrm{m}\rangle$ stands for the hybrid model.

For a given limit-state surface $g(\mathbf{u},\mathbf{v}) = 0$, we define the most probable failure point (MPP) $\mathbf{u}_{\mathrm{MPP}}\,(\mathbf{v})$ for a certain combination of bounded variables $\mathbf{v}$ as the solution of $\min_{\mathbf{u}:g(\mathbf{u},\mathbf{v})=0} \mathbf{u}^{\mathrm{T}}\mathbf{u}$. As analogues to the probabilistic reliability index, we further define a quantified reliability index β_{m} as:

$$\beta_{\mathrm{m}} = \mathrm{sgn}\,(g\,(\mathbf{0}, \mathbf{v}^{*})) \cdot \min_{\mathbf{v}:\mathbf{v}\in\mathbf{E}} \sqrt{\{\mathbf{u}_{\mathrm{MPP}}\,(\mathbf{v})\}^{\mathrm{T}}\,\mathbf{u}_{\mathrm{MPP}}\,(\mathbf{v})} \tag{15}$$

where $\mathbf{v}^{*}$, the solution of the constrained minimization problem, is referred to as the *worst-case point* (i.e. worst-case realization of the bounded uncertainties) in this paper since it corresponds to the largest failure probability for all $\mathbf{v} \in \mathbf{E}$. The signum function sgn$(\cdot)$ is included here to yield a negative value when $g(\mathbf{0}, \mathbf{v}^{*}) < 0$, which indicates that the design violates the design requirement for the mean values of the probabilistic uncertainties under the worst-case combination of the bounded parameters.

Substituting the definition of $\mathbf{u}_{\mathrm{MPP}}(\mathbf{v})$ into (15), it yields:

$$\beta_{\mathrm{m}} = \mathrm{sgn}\,(g\,(\mathbf{0}, \mathbf{v}^{*})) \cdot \min_{\mathbf{v}:\mathbf{v}\in\mathbf{E}} \left(\min_{\mathbf{u}:g(\mathbf{u},\mathbf{v})=0} \sqrt{\mathbf{u}^{\mathrm{T}}\mathbf{u}} \right) \tag{16}$$

With a deeper insight into problem (16), we can find that this bi-level optimization problem is equivalent to the following single-level optimization problem:

$$\begin{aligned} &\min_{\mathbf{u},\mathbf{v}} \quad \mathbf{u}^{\mathrm{T}}\mathbf{u} \\ &\text{s.t.} \quad g\,(\mathbf{u}, \mathbf{v}) = 0 \\ &\qquad \mathbf{v}_i^{T}\mathbf{v}_i \le 1 \quad (i = 1, 2, \ldots, N_{\mathrm{E}}) \end{aligned} \tag{17}$$

From geometrical point of view, the reliability index β_{m} is the shortest distance from the origin to the critical region in $\mathbf{u}$-space, as shown in Fig. 1. Obviously, if all the uncertainties can be provided with precise probabilistic distributions, the proposed reliability index degenerates into a conventional probability reliability index.

4 Reliability-based design optimization under hybrid models

A structural optimization problem aims to seek the best design that satisfies certain structural behaviour requirements. Under the hybrid model, the structural behaviours can be represented by the performance functions $g_j\,(\mathbf{d},\mathbf{u},\mathbf{v})$ $(j = 1, 2, \ldots, N_{\mathrm{F}})$, which are functions of the design variables $\mathbf{d}$, the normalized probabilistic variables $\mathbf{u}$ and the normalized uncertain-but-bounded variables $\mathbf{v}$. It should be noted that the design variables $\mathbf{d}$ can also be the mean (or nominal) values of geometrical dimensions and material properties when their variations are modelled as stochastic (or bounded) uncertainties. In the present paper, analogously as the stochastic optimization under pure probabilistic framework (see e.g. Enevoldsen and Sørensen 1994; Vietor 1997), the reliability-based design optimization problem under the probability and convex set hybrid model is mathematically formulated as:

$$\begin{aligned} &\min_{\mathbf{d}} \quad f\,(\mathbf{d}) \\ &\text{s.t.} \quad \beta_{\mathrm{m}}\left[g_j\,(\mathbf{d}, \mathbf{u}, \mathbf{v})\right] \ge \underline{\beta}_{\mathrm{m},j} \quad (j = 1, 2, \ldots, N_{\mathrm{F}}) \\ &\qquad \mathbf{d}_{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}_{\mathrm{U}} \end{aligned} \tag{18}$$

where $f(\mathbf{d})$ is the objective function to be minimized; $\beta_{\mathrm{m}}[g_j\ (\mathbf{d},\ \mathbf{u},\ \mathbf{v})]$ is the reliability index (see (15)) corresponding to the j-th performance function; $\underline{\beta}_{\mathrm{m},j}$ is the prescribed target value of the reliability index; $\mathbf{d}_{\mathrm{L}}$ and $\mathbf{d}_{\mathrm{U}}$ are the lower and upper bounds of the design variables, respectively.

The above reliability-based design problem is a double-loop optimization one. The outer-loop aims to minimize the cost function of the structural design, while the inner-loop represents the reliability analysis. For improving the computational efficiency in solving this problem, the popular performance measure approach (PMA), which was proposed by Tu et al. (1999) and enhanced by e.g. Youn et al. (2005), is employed in this paper. Researches showed that the performance measure approach is inherently robust and superior in aspects of both computational efficiency and numerical stability (Lee et al. 2002). In contrast to the direct method relying on the comparison between the current reliability index value and the prescribed target value, the performance measure approach checks the satisfaction of a reliability constraint according to the target performance value. Based on a similar idea, we transform the design problem (18) into an equivalent optimization problem expressed by:

$$\begin{aligned} &\min_{\mathbf{d}} \quad f\,(\mathbf{d}) \\ &\text{s.t.} \quad \alpha_j\,(\mathbf{d}) \ge 0 \qquad (j = 1, 2, \ldots, N_{\mathrm{F}}) \\ &\qquad \mathbf{d}_{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}_{\mathrm{U}} \end{aligned} \tag{19}$$

where $\alpha_j(\mathbf{d})$ is the target performance value corresponding to the prescribed target reliability index $\underline{\beta}_{\mathrm{m},j}$ for the j-th failure mode and it is expressed by:

$$\alpha_j\,(\mathbf{d}) = \min_{\mathbf{u}:\mathbf{u}^{\mathrm{T}}\mathbf{u}=\underline{\beta}_{\mathrm{m},j}^{2}} \underline{g}_j\,(\mathbf{d},\mathbf{u}) \tag{20}$$

where $\underline{g}_j(\mathbf{d}, \mathbf{u})$ is mathematically stated as:

$$\underline{g}_j(\mathbf{d},\mathbf{u}) = \min_{\mathbf{v}:\mathbf{v}\in\mathbf{E}} g_j(\mathbf{d}, \mathbf{u},\mathbf{v}) \tag{21}$$

Here, problems (20) and (21) can be combined into a single-level problem:

$$\begin{aligned} \alpha_j(\mathbf{d}) = \min_{\mathbf{u},\mathbf{v}} \quad & g_j(\mathbf{d},\mathbf{u}, \mathbf{v}) \\ \text{s.t.} \quad & \mathbf{u}^{\mathrm{T}}\mathbf{u} = \underline{\beta}^2_{\mathrm{m},j} \\ & \mathbf{v}_i^T\mathbf{v}_i \le 1 \qquad (i = 1, 2, \ldots, N_{\mathrm{E}}) \end{aligned} \tag{22}$$

Note that the reliability-based optimization formulation expressed by (19) and (22) reduces to the conventional RBDO in cases where only probabilistic random variables are concerned. Problem (22) is more general than the hybrid random/interval model-based formulation proposed by Du et al. (2005), since the interval model considered therein can be regarded as a specific instance of the multi-ellipsoid model used in this study. Moreover, this problem also differs from the PMA (Tu et al. 1999) for the conventional RBDO or the mentioned formulation considering random/interval hybrid uncertainties in aspects of constraint conditions. Here, the target performance value $\alpha_j(\mathbf{d})$ is to be found by minimizing the performance function on a hyper-sphere surface in $\mathbf{u}$-space under multiple quadratic inequality constraints defined in the sub-spaces of bounded uncertainties. This nature also necessitates special handling techniques, which will be described in the following section.

5 Solution strategy

The structural optimization problem incorporating reliability constraints under hybrid model of probabilistic randomness and convex models presents a challenging problem with nested optimization. While a direct nested double-loop approach is applicable, we proposed a single-loop approach based on linearization of the limit-state function and the optimality conditions in order to reduce the computational cost.

5.1 Nested double-loop approach

In the reliability-based optimization problem under the hybrid model, the inner-loop of the target performance evaluation is embedded in the outer-loop for the overall design optimization. A direct nested double-loop procedure, as schematically depicted in Fig. 2, can be resorted for solving the nested problem (19) and (22). Herein, the mathematical programming package CFSQP (Lawrence et al. 1997), which is an implementation of the Sequential Quadratic Programming (SQP) algorithm, is used to solve the problem.

Though the nested double-loop approach is applicable, it still requires prohibitively lengthy calculations. Since the inner-loop (evaluation of the target performance) needs many evaluations of limit-state functions, and each iteration of the outer-loop optimization consists of an execution of the inner-loop minimization, the total number of limit-state function evaluations is usually very high.

5.2 Linearization-based approach

Various techniques have been proposed to decouple the nested optimization problem involved in the conventional RBDO. In a sequential optimization strategy (Royset et al. 2001; Du and Chen 2004; Cheng et al. 2006), the inner-loop and the outer-loop are treated sequentially and thus the optimum design is obtained by solving a sequence of sub-programming problems. Moreover, a decoupled single-loop strategy (Chen et al. 1997; Kuschel and Rackwitz 2000; Liang et al. 2007; Minguez and Castillo 2009), which allows the solution to be infeasible before convergence and to satisfy the constraints only at the final

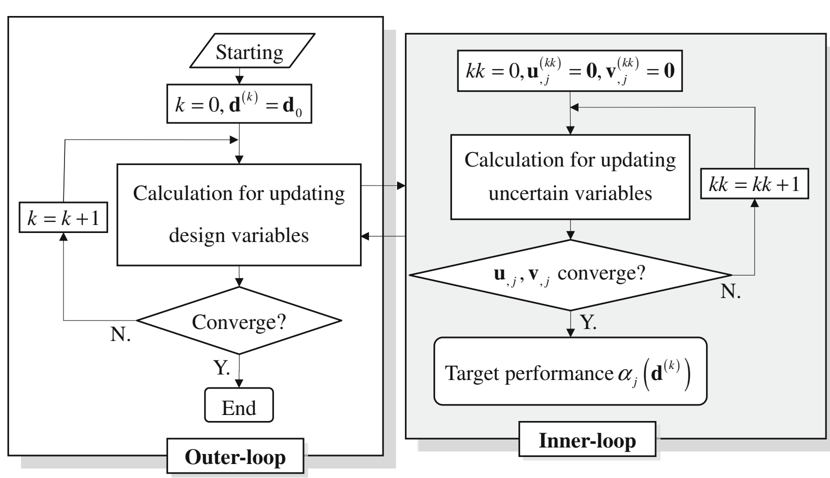

Fig. 2 Flow chart of the direct nested double-loop approach

optimum, has also been developed to address the computational challenges. Numerical investigations suggested that these methods could considerably improve the efficiency in solving the RBDO problem. In this paper, based on a similar decoupling strategy as the above-mentioned single-loop approach, a linearization-based method is presented to solve the proposed reliability-based optimization problem under the hybrid uncertainty model.

In most practical circumstances, the variability of the uncertain-but-bounded variables is relatively small or moderate. Therefore, it is reasonable to assume that the mechanical performance functions are monotonic with respect to these quantities within their variation bounds (see e.g. Neumaier and Pownuk 2007). In virtue of this, the inequality constraints in (22) should be active at the optimum. Thus, an iteration scheme for solving the optimum $\left(\mathbf{u}^*_{,j}, \mathbf{v}^*_{,j}\right)$ is derived based on the optimality conditions in the following.

Denoting the approximate solution of (22) in the k-th iteration by $\left(\mathbf{u}^{(k)}_{,j}, \mathbf{v}^{(k)}_{,j}\right)$ $\left(j = 1, 2, \ldots, N_{\mathrm{F}}\right)$, by approximating the limit-state function g_j with the first-order Taylor expansion about $\left(\mathbf{u}^{(k)}_{,j}, \mathbf{v}^{(k)}_{,j}\right)$, we rewrite (22) as:

$$\alpha_j\left(\mathbf{d}^{(k)}\right) = \min_{\mathbf{u},\mathbf{v}} \left\{ g_j\left(\mathbf{d}^{(\mathrm{k})}, \mathbf{u}^{(\mathrm{k})}_{,\mathrm{j}}, \mathbf{v}^{(\mathrm{k})}_{,\mathrm{j}}\right) + \mathbf{G}^{\mathrm{T}}_{\mathbf{u}^{(k)}_{,j}}\left(\mathbf{u} - \mathbf{u}^{(k)}_{,j}\right) + \mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}_{,j}}\left(\mathbf{v} - \mathbf{v}^{(k)}_{,j}\right) \right\}$$
$$\text{s.t.} \quad \mathbf{u}^{\mathrm{T}}\mathbf{u} = \underline{\beta}^2_{\mathrm{m},j}$$
$$\mathbf{v}^T_i \mathbf{v}_i = 1 \qquad (i = 1, 2, ..., N_{\mathrm{E}}) \tag{23}$$

where $\mathbf{d}^{(k)}$ denotes the k-th (current) iteration design variables, the subscript $\langle, j\rangle$ stands for the j-th performance function. $\mathbf{G}_{\mathbf{u}^{(k)}_{,j}}$ and $\mathbf{G}_{\mathbf{v}^{(k)}_{,j}}$ are partial derivatives of the performance function, which are expressed by:

$$\mathbf{G}_{\mathbf{u}^{(k)}_{,j}} = \left.\frac{\partial g_j}{\partial \mathbf{u}}\right|_{\mathbf{d}^{(k)}, \mathbf{u}^{(k)}_{,j}, \mathbf{v}^{(k)}_{,j}}, \quad \mathbf{G}_{\mathbf{v}^{(k)}_{,j}} = \left.\frac{\partial g_j}{\partial \mathbf{v}}\right|_{\mathbf{d}^{(k)}, \mathbf{u}^{(k)}_{,j}, \mathbf{v}^{(k)}_{,j}}, \tag{24}$$

Both derivatives can be either analytically obtained or evaluated by using classical approaches such as the semi-analytical method. For design problems involving a large number of uncertain variables, the adjoint variable scheme is preferably used in view of its computational efficiency.

Before derivation of the iteration scheme for solving the above constrained minimization problem, a Lagrangian function is first constructed:

$$L = g_j\left(\mathbf{d}^{(k)}, \mathbf{u}^{(k)}_{,j}, \mathbf{v}^{(k)}_{,j}\right) + \mathbf{G}^{\mathrm{T}}_{\mathbf{u}^{(k)}_{,j}}\left(\mathbf{u} - \mathbf{u}^{(k)}_{,j}\right) + \mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}_{,j}}\left(\mathbf{v} - \mathbf{v}^{(k)}_{,j}\right) + \kappa\left(\mathbf{u}^{\mathrm{T}}\mathbf{u} - \underline{\beta}^2_{\mathrm{m},j}\right) + \sum_{i=1}^{N_{\mathrm{E}}} \lambda_i\left(\mathbf{v}^T_i \mathbf{v}_i - 1\right), \tag{25}$$

where κ and λ_i are the Lagrangian multipliers for the corresponding constraints.

Applying the Karush–Kuhn–Tucker optimality condition at $\left(\mathbf{u}^*_{,j}, \mathbf{v}^*_{,j}\right)$, we have:

$$\begin{cases} \left.\dfrac{\partial L}{\partial \mathbf{u}}\right|_{\mathbf{u}^*_{,j}} = \mathbf{G}_{\mathbf{u}^{(k)}_{,j}} + 2\kappa \mathbf{u}^*_{,j} = \mathbf{0}, \\ \left.\dfrac{\partial L}{\partial \mathbf{v}_i}\right|_{\mathbf{v}^*_{i,j}} = \mathbf{G}_{\mathbf{v}^{(k)}_{i,j}} + 2\lambda_i \mathbf{v}^*_{i,j} = \mathbf{0}, \\ \left(\mathbf{u}^*_{,j}\right)^{\mathrm{T}} \mathbf{u}^*_{,j} - \underline{\beta}^2_{\mathrm{m},j} = 0, \\ \left(\mathbf{v}^*_{i,j}\right)^{\mathrm{T}} \mathbf{v}^*_{i,j} - 1 = 0 \quad (i = 1, 2, \ldots, N_{\mathrm{E}}). \end{cases} \tag{26}$$

After some manipulations, the equations in (26) lead to:

$$\begin{cases} \mathbf{u}^*_{,j} = -\dfrac{\underline{\beta}_{\mathrm{m},j} \mathbf{G}_{\mathbf{u}^{(k)}_{,j}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{u}^{(k)}_{,j}} \mathbf{G}_{\mathbf{u}^{(k)}_{,j}}}}, \\ \mathbf{v}^*_{i,j} = -\dfrac{\mathbf{G}_{\mathbf{v}^{(k)}_{i,j}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}_{i,j}} \mathbf{G}_{\mathbf{v}^{(k)}_{i,j}}}} \quad (i = 1, 2, \ldots, N_{\mathrm{E}}). \end{cases} \tag{27}$$

Thus, a heuristic scheme for updating $\left(\mathbf{u}^*_{,j}, \mathbf{v}^*_{,j}\right)$ corresponding to the j−th reliability constraint would be:

$$\left(\mathbf{u}^{(k+1)}_{,j}, \mathbf{v}^{(k+1)}_{,j}\right) = \left(\mathbf{u}^{(k+1)}_{,j}, \mathbf{v}^{(k+1)}_{1,j}, \ldots, \mathbf{v}^{(k+1)}_{N_{\mathrm{E}},j}\right) = -\left(\frac{\underline{\beta}_{\mathrm{m},j} \mathbf{G}_{\mathbf{u}^{(k)}_{,j}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{u}^{(k)}_{,j}} \mathbf{G}_{\mathbf{u}^{(k)}_{,j}}}}, \frac{\mathbf{G}_{\mathbf{v}^{(k)}_{1,j}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}_{1,j}} \mathbf{G}_{\mathbf{v}^{(k)}_{1,j}}}}, \ldots, \frac{\mathbf{G}_{\mathbf{v}^{(k)}_{N_{\mathrm{E}},j}}}{\sqrt{\mathbf{G}^{\mathrm{T}}_{\mathbf{v}^{(k)}_{N_{\mathrm{E}},j}} \mathbf{G}_{\mathbf{v}^{(k)}_{N_{\mathrm{E}},j}}}}\right). \tag{28}$$

where the superscript k and $k+1$ denote the iteration step of the overall design optimization of the structure.

Using the iteration scheme given in (28), $\mathbf{u}^*_{,j}$ and $\mathbf{v}^*_{,j}$ can be explicitly updated in each step of the structural optimization. Then the optimum structural design is searched in the design space by a gradient-based optimizer. This process continues until the objective function converges and all the target performance constraints are satisfied. The flowchart of the optimization process using the proposed linearization-based approach is given in Fig. 3. With this procedure, the computational efficiency should be much higher than that of the nested double-loop approach because the expensive iteration procedures for solving the inner-loop minimization problems are avoided.

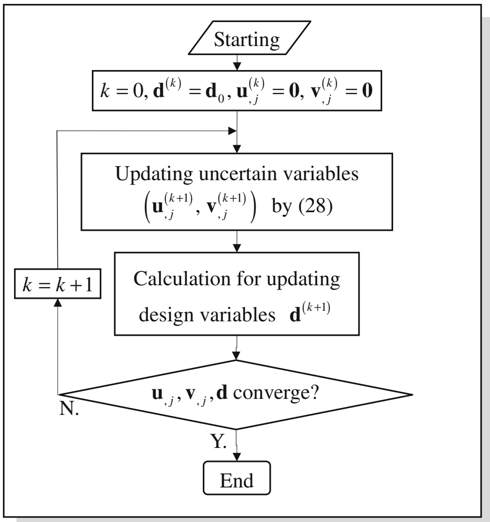

Fig. 3 Flow chart of the optimization process using the linearization-based approach

However, it should be noted that the suggested linearization-based approach relies on the assumption of the local monotonicity of the performance function with respect to the uncertainty variables. In addition, the global optimality cannot be guaranteed if the performance function is non-convex or not smooth. In such a case, the nested double-loop approach enhanced with special techniques such as multiple initial guesses and sequential convex approximations can be resorted to.

6 Numerical examples

In this section, three numerical examples are presented to demonstrate the validity and efficiency of the proposed method. The optimization package CFSQP is employed for solving all the minimization problems, where the stop criterion of iterations is: the relative difference between the objective function values of two adjacent iterations is less than 10^{-4}, or the number of iteration steps exceeds 500.

6.1 Minimization of a mathematical function under reliability constraints

The first example considers minimization of an explicit function under reliability constraints. Two random variables (denoted by $\mathbf{X} = X_1, X_2\}^{\mathrm{T}}$) and two uncertain parameters (denoted by $\mathbf{Y} = \{Y_1, Y_2\}^{\mathrm{T}}$) bounded by an ellipsoid model are taken into account in the problem. Let X_1 be a Gaussian distributed random variable and X_2 be a log-normally distributed random variable. The optimization problem is expressed as:

$$\begin{aligned}
\min_{\mathbf{d}} \quad & f(\mathbf{d}) = (d_1+3)^2 + (d_2+3)^2 \\
\text{s.t.} \quad & \beta_{\mathrm{m}}\left[G_1(\mathbf{X},\mathbf{Y}) \geq 0\right] \geq \beta_{\mathrm{m}}, \\
& \beta_{\mathrm{m}}\left[G_2(\mathbf{X},\mathbf{Y}) \geq 0\right] \geq \beta_{\mathrm{m}}, \\
& 0.01 \leq d_1 \leq 10,\ 0.01 \leq d_2 \leq 10,
\end{aligned} \tag{29}$$

in which:

$$\begin{aligned}
G_1(\mathbf{X},\mathbf{Y}) &= X_1(X_2+Y_1) - Y_2, \\
G_2(\mathbf{X},\mathbf{Y}) &= X_1 - (X_2+Y_1)^2 Y_2,
\end{aligned} \tag{30}$$

where the design variables are $\mathbf{d} = \{d_1, d_2\}^T$, with d_1 and d_2 representing the mean value of X_1 and the median of X_2, respectively. The coefficients of variation (COV) for X_1 and X_2 are both 0.1. The variation ranges of Y_1 and Y_2 are given by $\mathbf{Y} = \{Y_1, Y_2\}^{\mathrm{T}} \in \mathrm{E} = \left\{\mathbf{Y}\middle|\left(\mathbf{Y}-\hat{\mathbf{Y}}\right)^{\mathrm{T}}\mathbf{W}_{\mathrm{y}}\left(\mathbf{Y}-\hat{\mathbf{Y}}\right) \leq 0.5^2\right\}$, where their nominal values are $\hat{\mathbf{Y}} = \left\{\hat{Y}_1, \hat{Y}_2\right\}^{\mathrm{T}} = \{0.25, 2\}^{\mathrm{T}}$ and the characteristic matrix is $\mathbf{W}_{\mathrm{y}} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

Table 1 Solutions for the mathematic example

	Linearization-based approach	Nested double-loop approach
Objective	76.0129	76.0129
Optimal design (d_1, d_2)	(4.9025, 0.6828)	(4.9025, 0.6828)
Nominal value $(\bar{X}_1, \bar{X}_2, \hat{Y}_1, \hat{Y}_2)$	(4.9025, 0.6828, 0.25, 2)	(4.9025, 0.6828, 0.25, 2)
$\left(X_1^*, X_2^*, Y_1^*, Y_2^*\right)$ for G_1	(3.7068, 0.5736, 0.0300, 2.2374)	(3.7080, 0.5734, 0.0300, 2.2368)
$\left(X_1^*, X_2^*, Y_1^*, Y_2^*\right)$ for G_2	(3.8715, 0.8452, 0.4661, 2.2515)	(3.8693, 0.8449, 0.4662, 2.2510)
Number of iterations for outer-loop	21	27
Average number of iterations for each inner-loop solution	1	10
Total number of performance function evaluations	42	540

The target reliability index of the constraints is $\underline{\beta}_{\mathrm{m}} = 3.0$, which means the failure probability of the structure must be less than 0.135%. After the uncertainty normalization, problem (29) can be reformulated by using the PMA approach as:

$$\begin{aligned} \min_{\mathbf{d}} \quad & f(\mathbf{d}) = (d_1 + 3)^2 + (d_2 + 3)^2 \\ \text{s.t.} \quad & \alpha_1(\mathbf{d}) \geq 0, \\ & \alpha_2(\mathbf{d}) \geq 0, \\ & 0.01 \leq d_1 \leq 10, \quad 0.01 \leq d_2 \leq 10, \end{aligned} \tag{31}$$

in which:

$$\begin{aligned} \alpha_j(\mathbf{d}) = \min_{\mathbf{u},\mathbf{v}} \quad & g_j(\mathbf{d}, \mathbf{u}, \mathbf{v}) \\ \text{s.t.} \quad & \mathbf{u}^{\mathrm{T}}\mathbf{u} = 3.0^2, \\ & \mathbf{v}^{\mathrm{T}}\mathbf{v} \leq 1, \end{aligned} \tag{32}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the normalized vectors of $\mathbf{X}$ and $\mathbf{Y}$, $g_j(\cdot)$ $(j = 1, 2)$ are the normalized performance functions of $G_j(\cdot)$ $(j = 1, 2)$.

The initial values of the design variables are set to be $d_1^{(0)} = d_2^{(0)} = 5$. The proposed linearization-based approach results in the identical optimal solutions as the nested double-loop approach, as listed in Table 1. However, the former approach is much more efficient. This is due to the fact that it requires only one limit-state function evaluation for each inner-loop solution. For further testing of the convergence behaviour of the linearization-based approach, three different initial guesses of design variables $\left(d_1^{(0)}, d_2^{(0)}\right) = (1, 1)$, $\left(d_1^{(0)}, d_2^{(0)}\right) = (3, 3)$ and $\left(d_1^{(0)}, d_2^{(0)}\right) = (8, 8)$ are also fed into the optimizer. From the iteration histories shown in Fig. 4, it can be seen that the iterations converge to the same optimum, though the number of iterations relies on the initial design point.

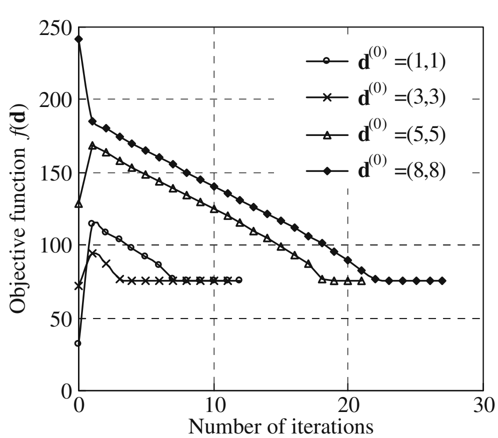

Fig. 4 Iteration histories of the optimization with different initial design points

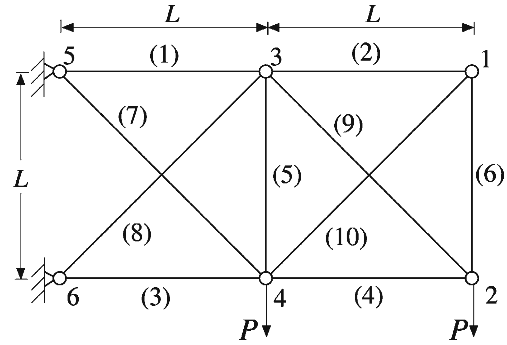

Fig. 5 The ten-bar truss structure

6.2 Reliability-based optimization of a ten-bar truss structure

Figure 5 shows a frequently studied planar ten-bar truss structure, which is to be optimized for minimum weight under non-deterministic conditions in this study. The horizontal and vertical bar members have a length of $L = 360$. The mass density of the material is $\rho = 0.1$. Two external loads P are applied to node 2 and node 4. A constraint $U \leq 2.0$ is imposed on the vertical displacement of node 2. The bar cross-sectional areas A_i $(i = 1, 2, \ldots, 10)$ and the Young's modulus E of the material are Gaussian normal random variables, whereas the external load P is an uncertain-but-bounded variable. The uncertainty properties of these parameters are summarized in Table 2.

The mean values of the member section areas $\bar{A}_i$ $(i = 1, 2, \ldots, 10)$ are taken as design variables, with lower bound limits $\underline{d}_i = 0.1$ and initial values $d_i^{(0)} = 40.0$ $(i = 1, 2, \ldots, 10)$. The target reliability index is set as $\underline{\beta}_{\mathrm{m}} = 3.0$.

The obtained optimal design by the present method based on the hybrid model is given in the second column

Table 2 Uncertainty properties for the ten-bar truss structure

Uncertainty	Cross-sectional areas $A_i (i = 1,2,\ldots,10)$	Young's modulus E	External load P
Distribution	Normal distribution	Normal distribution	Uncertain-but-bounded
Nominal value	$\bar{A}_i$	10^7	10^5
COV	0.05	0.05	–
Variation range	–	–	15%

Table 3 Optimal solutions using different approaches

Member number	Optimal cross-sectional area $\bar{A}_i$			
	Proposed method using hybrid model	Deterministic optimization	Conventional RBDO	Worst-case scenario approach
1	42.91	31.37	39.23	49.83
2	0.10	0.10	0.10	0.10
3	29.32	21.48	26.81	34.16
4	21.01	15.46	19.23	24.57
5	0.10	0.10	0.10	0.10
6	0.10	0.10	0.10	0.10
7	3.38	2.83	3.21	4.19
8	30.81	22.56	28.18	35.81
9	29.94	21.86	27.36	34.75
10	0.10	0.10	0.10	0.10
Total weight	6638.0	4880.4	6076.9	7729.8
β_m	3.00	<0	1.50	5.33

of Table 3. Therein, a reliability index $\beta_\mathrm{m} = 3.00$ is achieved. The iteration history plotted in Fig. 6 shows a steady decrease of the objective function as well as a stable convergence during the optimization process.

For comparison's purpose, the deterministic optimization based on nominal values, the reliability-based optimization in the pure probabilistic framework (Conventional RBDO) and the worst-case scenario approach using pure non-probabilistic description were also run. The well-known PMA approach (Tu et al. 1999) is employed for solving the conventional RBDO problem, where all the uncertainties are assumed to have a Gaussian normal distribution with the coefficient of variation being 0.05 and the probabilistic reliability is required to be 3.0. In the worst-case scenario approach, all the uncertain parameters are described by interval variables. Therein, the variation ranges of A_i ($i = 1,2,\ldots,10$) and E about their nominal values are assumed to be 15%, i.e. three times that of the corresponding coefficient of variation.

The optimal solutions using the above three methods are also listed in Table 3. For these optimal designs, the corresponding reliability indices evaluated with the hybrid model are given in the last row of the table. The deterministic optimization presents a design with the least structural weight, though the reliability requirement is not satisfied. The conventional RBDO solution has a reliability index of $\beta_\mathrm{m} = 1.50$ and therefore also violates the reliability constraint. The design obtained by the worst-case scenario approach

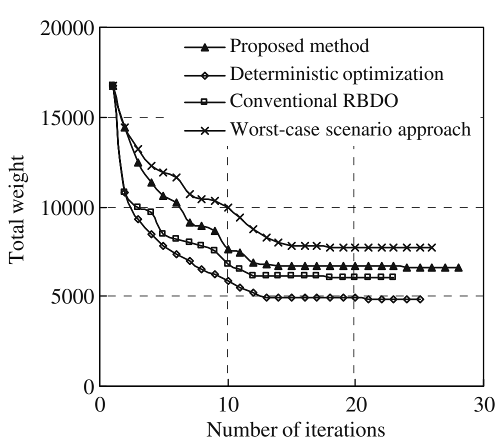

Fig. 6 Iteration histories

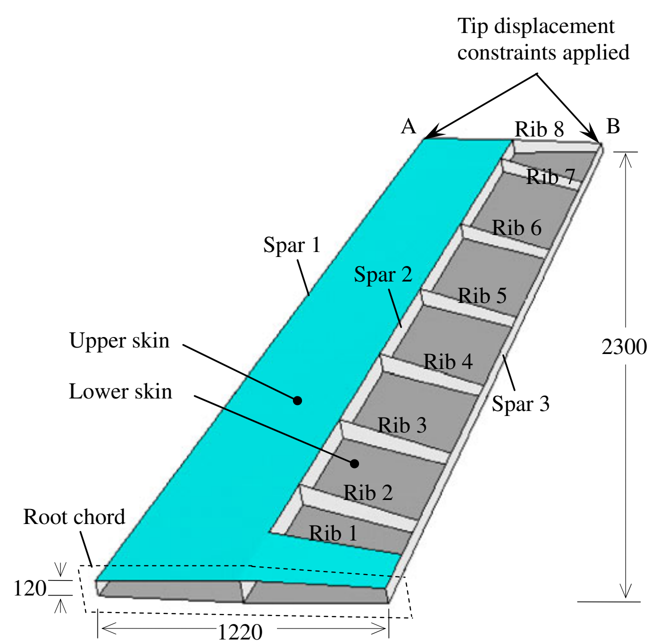

Fig. 7 The wing box structure (unit: millimeters)

Table 4 Uncertainty properties for the wing box structure

Uncertainty	Thicknesses of spars, ribs and skins (mm) $t_i\,(i=1,2,\ldots,13)$	Young's moduli (GPa) (E_1, E_2)	External loads (kN) (P_1, P_2, P_3)
Distribution	Normal distribution	Normal distribution	Uncertain-but-bounded
Nominal value	Design variables d_i	(72.42, 68.98)	(146.23, 64.44, 17.26)
COV	0.02	0.03	–
Convex model	–	–	$[\delta_{P1}\ \delta_{P2}\ \delta_{P3}]\begin{bmatrix}1 & 0 & 0\\ 0 & 2.78 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}\delta_{P1}\\ \delta_{P2}\\ \delta_{P3}\end{bmatrix} \le 0.1^2$

demands for the most volume of material. With a corresponding reliability index $\beta_{\mathrm{m}} = 5.33$, it turns out to be an over-conservative design. It is also noted that there are considerable differences between the optimal solution by the present method and those by the conventional RBDO or the worst-case scenario approach. This implies that it may be not appropriate to treat probabilistic uncertainties with non-probabilistic models, and vice versa. The importance of the present method based on the hybrid model is thus demonstrated.

Though all the computations are based on the finite element model, an explicit expression of the displacement constraint would be useful for gaining an insight into the nonlinearity of this problem. Using the node equilibrium equations and the compatibility relations, the member axial forces N_i (i = 1,2,...,10) can be easily obtained as (Au et al. 2003):

$$\begin{aligned}
&N_1 = P - \frac{\sqrt{2}}{2}N_8, && N_2 = -\frac{\sqrt{2}}{2}N_{10},\\
&N_3 = -3P - \frac{\sqrt{2}}{2}N_8, && N_4 = -P - \frac{\sqrt{2}}{2}N_{10}\\
&N_5 = -P - \frac{\sqrt{2}}{2}\left(N_8 + N_{10}\right), && N_6 = -\frac{\sqrt{2}}{2}N_{10},\\
&N_7 = 2\sqrt{2}P + N_8, && N_8 = \frac{a_{12}\Delta_{2P} - a_{22}\Delta_{1P}}{a_{11}a_{22} - a_{12}a_{21}},\\
&N_9 = \sqrt{2}P + N_{10}, && N_{10} = \frac{a_{21}\Delta_{1P} - a_{11}\Delta_{2P}}{a_{11}a_{22} - a_{12}a_{21}}.
\end{aligned} \tag{33}$$

Table 5 Optimal solutions for the wing box structure

Component	Initial design (mm)	Optimal thickness (mm)	
		Proposed method using hybrid model	Deterministic optimization
Spar 1	25.4	4.59	4.53
Spar 2	25.4	15.74	14.02
Spar 3	25.4	10.56	9.60
Rib 1	25.4	2.54	2.54
Rib 2	25.4	3.01	2.64
Rib 3	25.4	2.54	2.54
Rib 4	25.4	2.54	2.54
Rib 5	25.4	2.54	2.54
Rib 6	25.4	2.54	2.54
Rib 7	25.4	2.54	2.54
Rib 8	25.4	2.54	2.54
Upper skin	25.4	32.85	28.71
Lower skin	25.4	32.69	28.56
Weight (kg)	346.79	329.16	289.12
Reliability index β_{m} (for constraint $U_{\mathrm{A}} \le 25.4$ mm)	−0.91	3.00	−1.22
Reliability index β_{m} (for constraint $U_{\mathrm{B}} \le 25.4$ mm)	−2.77	3.00	−1.21

where a_{ij} ($i = 1, 2$; $j = 1, 2$) and Δ_{iP} ($i = 1, 2$) are defined as:

$$a_{11} = \frac{L}{2E}\left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right),$$

$$a_{12} = a_{21} = \frac{L}{2EA_5},$$

$$a_{22} = \frac{L}{2E}\left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right),$$

$$\Delta_{1P} = \frac{\sqrt{2}PL}{2E}\left(-\frac{1}{A_1} + \frac{3}{A_3} + \frac{1}{A_5} + \frac{4\sqrt{2}}{A_7}\right),$$

$$\Delta_{2P} = \frac{\sqrt{2}PL}{2E}\left(\frac{1}{A_4} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_9}\right). \tag{34}$$

In a similar way, the virtual member forces N_i^0 ($i = 1, 2, \ldots, 10$) can be also obtained by applying a unit virtual load vertically at node 2.

Following the virtual load method, one may find the analytic expression for the vertical displacement of node 2, which is:

$$U = \left(\sum_{i=1}^{6} \frac{N_i^0 N_i}{A_i} + \sqrt{2}\sum_{i=7}^{10} \frac{N_i^0 N_i}{A_i}\right)\frac{L}{E}. \tag{35}$$

In addition, it can be easily checked that a design problem formulated with the explicit displacement constraint will yield the same results as presented above.

6.3 Reliability-based optimization of a wing box structure

In this example, we considered optimal design of an aircraft wing box. The simplified structural model has three spars, eight ribs, an upper skin and a lower skin, as shown in Fig. 7. The finite element model consists of 64 membrane elements for the skins and 55 shell elements for the ribs and spars. The wing box is fixed at the root chord. The aerodynamic lifting forces are represented by three static loads (P_1, P_2 and P_3), which are applied along the three spars. The spars and ribs are made of an aluminium alloy with the Young's modulus $E_1 = 72.42$ GPa, the Poisson's ratio $\mu_1 = 0.3$, and the mass density $\rho_1 = 2.768 \times 10^3 \text{kg}/\text{m}^3$. The skins are made from a boron/epoxy composite material with the Young's modulus $E_2 = 68.98$ GPa, the Poisson's ratio $\mu_2 = 0.21$, and the mass density $\rho_2 = 2.007 \times 10^3 \text{kg}/\text{m}^3$.

The Young's moduli and the thicknesses of spars, ribs and skins are assumed to be normally distributed random variables. A total number of 15 probabilistic variables are considered, including two for the Young's moduli E_1 and E_2, and 13 for the thicknesses of the spars, ribs and skins. In addition, the applied loads are bounded uncertainties and they are described by a three-dimensional ellipsoid model. The uncertainty properties are given in Table 4.

Two constraints, $U_\text{A} \le 25.4$ mm and $U_\text{B} \le 25.4$ mm, are imposed on the vertical tip displacements. The nominal thicknesses of all the spars, ribs and skins are taken as design variables d_i ($i = 1,2,\ldots,13$), with the lower bounds limit of 2.54 mm and the initial values of 25.4 mm. The design objective is to minimize the total structural weight while achieving a target reliability index $\underline{\beta}_\text{m} = 3.0$ for the displacement constraints.

The optimal results obtained by the proposed reliability-based optimization formulation as well as the deterministic optimization are listed in Table 5 for comparison's purpose. When the uncertainties are considered, both the initial design and the deterministic optimal design result in a severe violation of the reliability constraints. On the contrary, in the final reliability-based design obtained by the present method, the reliability requirements of the tip displacement constraints are all met. Again, the results showed the effectiveness of the present method based on the hybrid model. This example seems to confirm that the present optimization method is capable of treating medium-scale engineering problems.

7 Conclusions

While the probabilistic randomness is a natural model for the stochastic parameter scatters exhibited by a structure, the multi-ellipsoid convex model provides an appealing non-probabilistic description method for uncertain-but-bounded variations arising from different sources. This paper explores the reliability-based optimization design of non-deterministic structures with both stochastic and uncertain-but-bounded variations. Based on the probability and convex set hybrid model, the reliability index is defined as the shortest distance from the origin to the critical region in **u**-space. Then, the reliability-based optimization with constraints on such reliability indices is formulated as a nested optimization problem. By employing the performance measure approach, the original optimization problem is reformulated into an inherently more robust and numerically tractable one, in which the outer-loop aims to minimize the cost function while the inner-loop evaluates the target performance value. Two approaches, namely the nested double-loop approach and the linearization-based approach, are employed to solve the optimization problem. The numerical examples confirm that both approaches are applicable but the linearization-based approach is more efficient since it avoids expensive iterations for inner-loop solution. The present optimization method is proven to be effective in ensuring the structural reliability requirements in presence of probabilistic and bounded uncertainties. In addition, the comparison of numerical examples also reveals that it may be not appropriate to treat the inherently probabilistic variations with non-probabilistic models, and vice versa. This again implies the importance of the present study.

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