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BRIEF NOTE

A note on the derivation of global stress constraints

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Abstract The purpose of this brief note is to derive the KS global constraint function. The first derivation based on the maximum entropy theory elaborates the statistical significance of KS function. The second one points out the relationship between KS and the *p*-norm global function. The properties of these two global functions validate these substitutions are reasonable.

Keywords Topology optimization · Stress constraints · Global constraints · KS function · Shannon's entropy · Maximum entropy principle

1 Introduction

In topology design of structures, the optimum distribution of material is sought in a given design domain. Topology optimization has been developed to efficiently deal with compliance formulations. This compact type model can be solved conveniently, although it has a few shortcomings in the mathematical and engineering meaning.

In most engineering situations, pointwise failure criteria need to be considered, usually based on the stress tensor. There are two main questions in topology optimization with stress constraints. One is the singularity phenomenon, which has been proposed by Kirsch (1990). This phenomenon arises from the relaxation of primary discrete 0–1 formulation. Once the element becomes zero, the permissible stress in the corresponding stress constraints jumps suddenly from a finite value to infinity. This difficulty has been dealt with by the smooth envelope functions (SEF's) method (Rozvany and Sobieszczanski-Sobieski 1992; Rozvany et al. 1995; Rozvany 1996) or the ε-relaxation technique (Cheng and Guo 1997; Duysinx and Bendsoe 1998; Duysinx and Sigmund 1998). About singular topologies, Rozvany (2001) has provided a comprehensive review. It was shown in this that the SEF method proposed by Rozvany and Sobieszczanski-Sobieski (1992) and the epsilon-relaxation method give very similar results in handling singular topologies. The other question is the large number of local constraints in the discretized finite elements. A large scale optimization problem created by these local constraints and the large number of design variables, however, is still a challenge to the optimization algorithm. In the following sections, we'll endeavor to overcome this obstacle.

Some different techniques have been proposed in an attempt to deal with the large number of local constraints. For example, Pereira et al. (2004) used the augmented Lagrangian method to the optimization model with the local material failure constraints. But most of researches prefer transforming the local constraints to a global function, such as Yang and Chen (1996), Duysinx and Sigmund (1998), París et al. (2009). The optimization problem with the global function becomes easier to solve since the number of constraints is drastically reduced.

Nowadays, *p*-norm and KS function (Kreisselmeier and Steinhauser 1979) are two common global

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functions. They both can tend to the maximum local stress constraint along with the change of the parameter. KS function is also called aggregate function by Li (1992) as its' global properties. And the smooth KS function is very suitable for the optimization algorithms. But we can't find any derivative process of the KS function in the references of the structural optimization. We don't know where it comes from and what it means. In this paper, we give two different derivations by means of the Jeynes' maximum entropy principle (Jaynes 1957) and exponential transaction of p-norm. Furthermore, the significance of KS global function and the relationship between these two functions are also elaborated, while the rationality of these substitutions is validated.

2 Topology optimization with stress constraints

Topology optimization is defined as finding the material distribution in the given domain so that an objective function is minimized, while a set of constraints are satisfied. The objective maybe minimum weight, maximum fundamental frequency or the other wanted properties. Generally, the discrete variables are replaced by the continuous ones, and Solid Isotropic Microstructure with Penalty (SIMP) scheme is used to show the stiffness or Young's Module attenuation of the medium density (Bendsoe 1989; Rozvany et al. 1992). SIMP was also applied to stress design by Rozvany et al. (1992), who assumed that the permissible stress value is proportional to the varying material density, but penalized intermediate densities. We use this idea in the stress constraint of this paper. So the common stress constraint can be formulated as

$$\sigma_{\mathrm{VM}}(\rho_e) \le \rho_e^{\eta}\sigma_l \quad e = 1, \cdots, Ne$$

$$0 < \rho_{\min} \le \rho_e \le 1 \tag{1}$$

where ρ_e is the material density at element e, $\sigma_{\mathrm{VM}}(\rho_e)$ is the von Mises stress at element e, σ_l is the material yielding stress, Ne is the number of finite elements in the discretized domain and $\rho_{\min}$ is the prescribed lower bound of the material density. Based on experience, we use the penalization factor $\eta = 3$.

Because of the continuation of the discrete variables, singularity phenomenon appears in the topology optimization with stress constraints. It results in the impossibility for the optimization algorithm to create or to remove materials during the optimization process. The ε-relaxation technique proposed by Cheng and Guo (1997) is used to circumvent this difficulty. For the continuum type topology optimization problem, Duysinx and Sigmund (1998) developed a more perfect relaxation form

$$\left.\begin{aligned} & g_e = \frac{\sigma_{\mathrm{VM},e}}{\rho_e^{\eta}\sigma_l} + \varepsilon - \frac{\varepsilon}{\rho_e} \le 1 \\ & 1 \ge \rho_e \ge \varepsilon^2 \quad e = 1, \cdots, Ne \end{aligned}\right\} \tag{2}$$

where $\sigma_{\mathrm{VM},e}$ is the abbreviation of $\sigma_{\mathrm{VM}}(\rho_e)$, $\varepsilon > 0$ is the small relaxation factor.

The relaxation form (2) can be rewritten as the equivalent maximum form

$$\left.\begin{aligned} & g_{\max} = \max_{e=1,\cdots,Ne}\left(g_e = \frac{\sigma_{\mathrm{VM},e}}{\rho_e^{\eta}\sigma_l} + \varepsilon - \frac{\varepsilon}{\rho_e}\right) \le 1 \\ & \qquad 1 \ge \rho_e \ge \varepsilon^2, \quad e = 1, \cdots, Ne \end{aligned}\right\} \tag{3}$$

or the relaxed surrogate form

$$\left.\begin{aligned} & g_s = \sum_{e=1}^{Ne} \lambda_e g_e \le 1 \\ & \lambda \in \Lambda = \left\{\lambda_e \ge 0, \ \sum_{e=1}^{Ne} \lambda_e = 1\right\} \\ & g_e = \frac{\sigma_{\mathrm{VM},e}}{\rho_e^{\eta}\sigma_l} + \varepsilon - \frac{\varepsilon}{\rho_e} \\ & 1 \ge \rho_e \ge \varepsilon^2 \\ & e = 1, \cdots, Ne \end{aligned}\right\} \tag{4}$$

This is because $g_s \le g_{\max}$ when $\lambda \in \Lambda$. The factor λ_e is called surrogate multipliers.

3 Derivation of KS function (aggregate function) by maximum entropy principle

In information theory, Shannon's entropy is a measure of the uncertainty associated with a random variable. The formulation of Shannon's entropy is

$$H(\lambda) = -\sum_{i=1}^{m} \lambda_i \ln(\lambda_i)$$

$$\lambda_i \ge 0, \quad \sum_{i=1}^{m} \lambda_i = 1 \tag{5}$$

Comparing with the surrogate constraint form (4) and the maximum constraint form (3), we can see

$$g_s = \sum_{e=1}^{Ne} \lambda_e g_e \le g_{\max} \tag{6}$$

So g_s must be maximized to make (6) become an equality, that is, it solves a maximization problem:

$$\max_{\lambda\in\Lambda}\ g_s = \sum_{e=1}^{Ne} \lambda_e g_e \tag{7}$$

Unfortunately, this linear optimization problem rarely has an explicit solution λ^* for the design variables. In search of a smooth solution, we regard the surrogate multiplier λ_e as the probability of $g_s = g_{\max}$. That is to say, every local stress constraint maybe the maximum one, and the probability is denoted by λ_e. Furthermore, the maximum entropy principle can provide a means to obtain least-biased statistical inference when insufficient information is available. In order to get fair result for every local constraint, the formulation of maximum entropy is used

$$\max_{\lambda\in\Lambda}\ H(\lambda) = -\sum_{e=1}^{Ne} \lambda_e \ln(\lambda_e) \tag{8}$$

If satisfaction of the objects (7) and (8) are demanded at the same time, we can construct a composite maximization problem with the weighting coefficient p

$$\max_{\lambda\in\Lambda}\ g_p = g_s + H(\lambda)/p \tag{9}$$

Shannon's entropy $H(\lambda)$ is always nonnegative, so the effect of the entropy term on the solution of (9) will diminish as p approaches infinity. By a simple calculation, we can get an analytical solution

$$\lambda_e = \exp\left[p g_e\right] \Big/ \sum_{e=1}^{Ne} \exp\left[p g_e\right] \tag{10}$$

Substituting (10) for λ_e into g_p eliminates λ and yields a function

$$g_p = \frac{1}{p} \ln \left\{ \sum_{e=1}^{Ne} \exp\left[p g_e\right] \right\} \tag{11}$$

This is KS global constraint function, and we often call it aggregate function.

4 Relationship with the *p*-norm function and properties of global functions

If p is a real number, $p \geq 1$, the p-norm of vector $x = \{x_1, x_2, \ldots, x_n\}$ is defined by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{12}$$

When p tends to infinity, p-norm has the property

$$\|\mathbf{x}\|_p \geq \max_i\ |x_i|$$
$$\lim_{p\to\infty}\ \|\mathbf{x}\|_p = \max_i\ |x_i| = \|\mathbf{x}\|_\infty \tag{13}$$

Compared with the maximum stress constraint form (3), we can get the p-norm global constraint function

$$\left[\sum_{e=1}^{Ne} (\max\{0, g_e\})^p \right]^{1/p} \leq 1 \tag{14}$$

Since g_e maybe negative, max $\{0, g_e\}$ is used as the component of p-norm.

Now, we construct a new function $\boldsymbol{\Psi}$ with the exponent of every local constraint component

$$\psi_e = \exp(g_e)\,,\ e = 1, \cdots, Ne \tag{15}$$

Since any component of $\boldsymbol{\Psi}$ is positive, the p-norm of $\boldsymbol{\Psi}$ is

$$\|\boldsymbol{\Psi}\|_p = \left\{ \sum_{e=1}^{Ne} [\psi_e]^p \right\}^{1/p} = \left\{ \sum_{e=1}^{Ne} \exp\left[p g_e\right]^p \right\}^{1/p} \tag{16}$$

By taking a logarithmic operation on both sides of (16), we immediately get

$$\ln \|\boldsymbol{\Psi}\|_p = \frac{1}{p} \ln \left\{ \sum_{e=1}^{Ne} \exp\left[p g_e\right] \right\} = g_p \tag{17}$$

From the above-mentioned formulations, KS function can be derived from exponential transaction of p-norm. According to the property of p-norm, we have

$$\lim_{p\to\infty}\ g_p = \lim_{p\to\infty}\ \ln \|\boldsymbol{\Psi}\|_p = \ln\left[\max_e\ \psi_e\right]$$
$$= \max_e\ \ln[\psi_e] = g_{\max} \tag{18}$$

The following inequality is evident

$$g_p - g_{\max} = \ln \|\Psi\|_p - \ln\{\exp(g_{\max})\}$$
$$= \frac{1}{p} \ln \sum_{e=1}^{Ne} \exp\left\{p\left[g_e - g_{\max}\right]\right\} \geq 0 \tag{19}$$

Then if $g_p \leq 1$, we can get $g_{\max} \leq 1$ immediately. So when we use KS global function in the constraint, the property (19) can ensure the maximum form (3) is strictly followed. In a similar way, based on the property (13), the p-norm global function can keep (3) valid.

The stress constraint with KS global function can be written as

$$\left.\begin{array}{l} g_p = \dfrac{1}{p} \ln \left\{ \displaystyle\sum_{e=1}^{Ne} \exp\left[p g_e \right] \right\} \leq 1 \\ g_e = \dfrac{\sigma_{\mathrm{VM},e}}{\rho_e^{\eta} \sigma_l} + \varepsilon - \dfrac{\varepsilon}{\rho_e} \\ 1 \geq \rho_e \geq \varepsilon^2 \\ e = 1, \cdots, Ne \end{array} \right\} \tag{20}$$

Based on the property (18), this formulation is transformed into the maximum form (3) as p approaches infinity. So it's equivalent to the local constraints formulation (2). Moreover, maximum entropy principle can ensure this global constraint formulation is fair to every local constraint.

5 Conclusions

In this paper, two different derivations of the global KS stress constraints function are proposed. The first one is based on the maximum entropy principle. Since this principle can provide a least-biased statistical inference with the known information, the local constraints are distributed fairly in KS global constraint function. The second one applies the exponential transaction of p-norm. This derivation points out the relationship between KS and the p-norm global constraint function. Furthermore, both two functions are bigger than the maximum local one. This property leads that the feasible domain is smaller than the primary local constraints model. And they are equivalent to the primary local constraints, as p approaches infinity. So these two substitutions are reasonable. Comparing these two global functions, we can see the smooth KS function is more adaptive for the optimization algorithms.

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