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RESEARCH PAPER

# An optimization approach for unconfined seepage problem with semipermeable conditions

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**Abstract** In this paper we study the unconfined potential steady flow through a porous media with semipermeable bottom. We propose a model that leads to a free boundary-value-problem with complementarities conditions on the bottom. The shape of a part of the domain boundary, called free boundary, is one of the unknown of the problem. The pressure of the flow as well as the flow velocity on the another part of the boundary, that is a one way permeable bottom, are also unknowns and satisfy a complementarity condition. We present the numerical implementation of the model based on an optimization approach. Performing a boundary-element discretization we get a nonlinear mathematical programming problem with complementarities conditions. To solve it we use Herskovits's interior point algorithm. Numerical examples are presented.

**Keywords** Free boundary problem · Shape optimization · Boundary elements method · Mathematical programming · Complementarity condition · Dam problem

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## 1 Introduction

The phenomenon of unconfined steady flow through porous media belongs to the category of free boundary value problems. A part of the boundary is *a priori* unknown and can be found as a component of the solution (Friedman 1982; Polubarinova-Kochinab 1962). Let $\mathcal{R}$ be an open and, for the sake of convenience, rectangular domain with base length $\ell$ occupied by the porous media, $h_1$ and $h_2$ the fluid piezometric levels on the left and on the right sides of $\mathcal{R}$ respectively and $h_1 \geq h_2$, see Fig. 1. The classical formulation looks for the location of the phreatic surface (water table) $\Gamma_\lambda$ and their associated seepage surface $\Gamma_\sigma$ as well as for the flow velocity potential $u$ into the domain $\Omega$ with the frontier $\Gamma = \Gamma_\lambda \cup \Gamma_\sigma \cup \Gamma_\circ \cup \Gamma_1 \cup \Gamma_2$. The fluid is assumed to be ideal, the dam is homogeneous and isotropic with the permeability coefficient $k = 1$, fluid specific weight $\gamma = 1$ and assume that the external pressure is equal to zero.

The classical unconfined seepage flow problem was widely analyzed from the theoretical and the numerical points of view. The analytical solution can be obtained with the theory of analytical functions for linear ordinary differential equations (Polubarinova-Kochinab 1962). In the iterative approach proposed in Liggett (1997), one of the two conditions defined at the free boundary is chosen to solve, at each iteration, the direct value problem. Guessing an initial approximation, the location of the free boundary is adjusted at each iteration to make the other boundary condition hold, then the direct problem is re-solved, etc. The method of transformation known as "Baiocchi's transformation" (Baiocchi and Capelo 1984) consists in changing the problem variables to transform the free

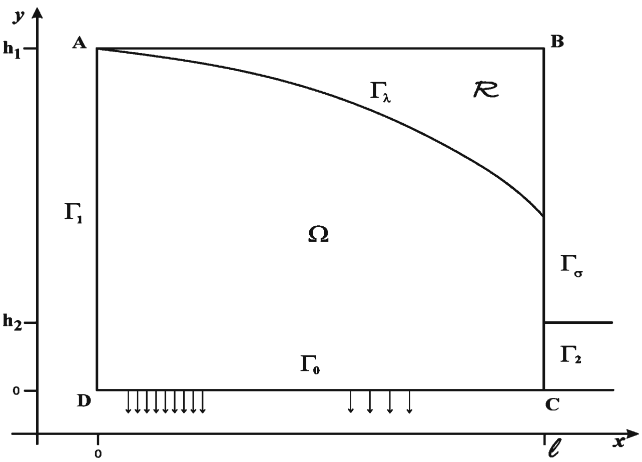


**Fig. 1** Seepage problem geometry

boundary domain into a fixed domain. The problem on this new domain becomes a variational inequality. The optimization method consists in the interpretation of the free boundary as an optimal boundary and applying mathematical programming techniques (Leontiev and Huacasi 2001). The seepage problem can be considered also as a "codimentional-two free boundary problem" (Howison et al. 1997), in which the only geometrical unknowns are the "free points" which mark the points at which the free boundary meets the top of the dam. Employing a penalization technique, known as extended pressure method, the seepage interval can be identified through Signorini conditions imposed at the part of the boundary of a fixed (extended) domain where appearance of seepage is expected (Zheng et al. 2009).

All these methods were proposed to solve the classical seepage problem, i.e. the problem without any evaporation or infiltration effect on the boundary. Some unconfined steady flow problems with a *a piori* prescribed evaporation zone were considered by Jensen (1980) and Pozzi (1974). We mention also a problem of unconfined flow in porous media with possible fluid discharge (evaporation) through the water table due to tree roots suction, called "forest impact problem" (Leontiev et al. 2004). The location of the water table under the forest suction effect, the flow characteristics as well as the region of contact of the aquifer with the tree roots are the unknowns of this problem.

In this paper we consider the model of unconfined steady flow through a porous media with semipermeable bottom. We impose Signorini boundary conditions in order to ensure one way permeability of the bottom. Moreover, the bottom becomes permeable for the flow when the pressure of the flow reaches a certain value. So, the permeable region of the bottom is unknown *a priori*. An equivalent formulation of the problem in terms of the quasivariational inequality can be given and the existence and uniqueness of the solution can be proved (Piermatei Filho 2006).

We reformulate this complementarity problem as a bilinear mathematical program. This approach interprets the free boundary is an optimal boundary. An interior point algorithm for non linear optimization (Herskovits 1998) is employed to solve this problem. Numerical results for test problems with different permeability properties of the bottom are presented.

## 2 Problem formulation

The classical seepage model does not suppose any evaporation or infiltration effects on the water table $\Gamma_\lambda$. The dam bottom $\Gamma_\circ$ is considered impermeable. The model can be formulated as a free boundary problem:

**Problem 1** Find the potential $u(x, y)$ and the decreasing function $\psi(x)$ that defines the location of the water table $\Gamma_\lambda$, satisfying

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = h_1 & \text{on} \quad \Gamma_1, \\ u = h_2 & \text{on} \quad \Gamma_2, \\ u = y & \text{on} \quad \Gamma_\sigma \cup \Gamma_\lambda, \\ q = 0 & \text{on} \quad \Gamma_\circ \cup \Gamma_\lambda, \end{cases}$$

where $q \equiv \partial u/\partial n$ and $n$ is the outward normal to $\Gamma_\circ \cup \Gamma_\lambda$.

In the unknown part $\Gamma_\lambda$ of the boundary, the function $u(x, y)$ has to fulfill two boundary conditions, called free boundary conditions, $u = y$ and $q = 0$. Problem 1 admits a unique solution pair $\{\psi, u\}$, where $\psi(x)$ is smooth and $u \in H^1(\Omega) \cap C^\circ(\overline{\Omega})$, Kinderlehrer and Stampacchia (1980).

Performing the Baiocchi transformation

$$w(x, y) = \int_y^{\psi(x)} (u(x, t) - t)dt,$$

a variational inequality equivalent to Problem 1 can be obtained (Baiocchi and Capelo 1984):

$$w \in \mathcal{K}, \quad \forall v \in \mathcal{K},$$

$$\int_{\mathcal{R}} (w_x(v-w)_x + w_y(v-w)_y)dxdy \geq -\int_{\mathcal{R}} (v-w)dxdy.$$

Here $\mathcal{K} = \{v \in H^1(\mathcal{R}) \mid v \geq 0 \text{ in } \mathcal{R} \text{ and } v = g \text{ on } \partial\mathcal{R}\}$, the function $g$ depends on $h_1, h_2$ and $\ell$, and the subscript $x$ (or $y$) denotes the derivative with respect to $x$ (or $y$). From the solution $w$ of this inequality, the velocity potential is defined as $u = y - w_y$ and $\Gamma_\lambda$ is determined as the curve that separates the areas with $w = 0$ and $w > 0$. To obtain this variational formulation it is necessary to know the discharge across any vertical section of the aquifer, $Q(x) \equiv -\int_0^{\psi(x)} u_x(x,t)dt$. The Dupuit formula $Q(x) = \left(h_1^2 - h_2^2\right)/2\ell$ is used in the classical case.

In our model, since we suppose that the bottom of the dam is one way permeable, we put at $\Gamma_\circ$ the Signorini conditions:

$$q \le 0,\ u \le p_\circ,\ q(u - p_\circ) = 0.$$

The unilateral character of these conditions guarantees that only one side of the bottom is permeable and only outgoing flux is possible. Moreover, the bottom becomes permeable for the flow when $u = p_\circ$. If $u < p_\circ$ the flux through the bottom is equal to zero. We note that the region of permeability of the bottom is unknown *a priori* and is a part of the solution. This model can be formulated in the following way:

**Problem 2** Let be $p_\circ \ge 0$. Find the potential $u(x, y)$ and the decreasing function $\psi(x)$ that defines the location of the water table $\Gamma_\lambda$, satisfying

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = h_1 & \text{on} \quad \Gamma_1, \\ u = h_2 & \text{on} \quad \Gamma_2, \\ u = y & \text{on} \quad \Gamma_\sigma \cup \Gamma_\lambda, \\ q = 0 & \text{on} \quad \Gamma_\lambda, \\ q \le 0,\ u \le p_\circ,\ q(u - p_\circ) = 0 & \text{on} \quad \Gamma_\circ. \end{cases}$$

Employing the Baiocchi transformation, a variational formulation equivalent to Problem 2 can be obtained (Piermatei Filho 2006):

$$w \in \mathcal{K}, \quad \int_{\mathcal{R}} \nabla w \cdot \nabla(v - w) \ge -\int_{\mathcal{R}} (v - w), \qquad \forall\, v \in \mathcal{K},$$

where $\mathcal{K} \subset H^1(\mathcal{R})$ is a non empty subset:

$$\mathcal{K} = \big\{ v \in H^1(\mathcal{R}) \mid v \ge 0 \ \text{in}\ \mathcal{R},\ v = \mathcal{G}\ \text{on}\ \partial\mathcal{R}\ \text{and}$$
$$v_y + p_\circ \ge 0,\ (v_y + p_\circ)v_{xx} = 0 \text{ on } \Gamma_\circ \big\},$$

with $\mathcal{G}(x, y) \in W^{2,\infty}(\mathcal{R})$:

$$\mathcal{G}(x,y) = \begin{cases} \dfrac{1}{2}(h_1 - y)^2 + \dfrac{[k(x) - k(0)]}{2[k(\ell) - k(0)]}\left[(h_2 - y)^2 - (h_1 - y)^2\right], \\ \hfill 0 \le y \le h_2, \\ \dfrac{1}{2}(h_1 - y)^2 - \dfrac{[k(x) - k(0)]}{2[k(\ell) - k(0)]}(h_1 - y)^2, \\ \hfill h_2 \le y \le h_1. \end{cases}$$

and

$$k(x) = \frac{1}{2}h_1^2 - \frac{x}{\ell}\left(\frac{h_1^2 - h_2^2}{2} - \int_0^\ell (\ell - t)q(t,0)dt\right) - \int_0^x (x - t)q(t,0)dt.$$

By the definition of the function $k(x)$, the subset $\mathcal{K}$ depends implicitly on the flow through the bottom. This flow is unknown *a priori* and is defined by the function $w$. Hence, the variational formulation of our problem is a quasivariational inequality. To prove the existence of a solution, we construct a family of variational inequalities and indicate a sequence of its solutions that converges to the solution of our quasivariational inequality. Under some reasonable assumptions the solution of this quasivariational inequality is unique (Piermatei Filho 2006).

## 3 The numerical algorithm

Since in our problem the constraints are imposed on the boundary, it is quite suitable to look for a numerical solution by means of the direct boundary element method (BEM). The unknown boundary values are the potential and the flux, which are considered as primary variables in the BEM. Then, their values are obtained directly. As a consequence, the BEM yields higher accuracy as compared with the finite element method. Boundary variational inequality formulations for potential problems with Signorini boundary conditions were studied by many authors (Han 1990; Simunovic and Saigal 1992; Spann 1993). Usually, these kind of conditions are applied for contact problems in solid mechanics (Eck et al. 1999; Eck and Wendland 2003). Some variational principles of the BEM for contact problems in elasticity were obtained by Polizzotto (1993), for the direct determination of the unknown boundary quantities, considering the relative displacements in a contact region as independent variables. Other authors developed variational formulations of the BEM for contact problems based on the use of Green's functions (Alliney et al. 1990). In general, all these approaches lead to a solution of linear complementarity problems. Depending on the method, linear complementarity problems have different properties (non symmetric matrix (Alliney et al. 1990), symmetric sign-definite matrix (Polizzotto 1993), etc.).

We apply a technique recently developed to solve boundary value problems with the Signorini conditions (Leontiev et al. 2002) and a one way permeable thin obstacle situated inside of the flow domain (Leontiev and

Khludnev 2006). Performing the boundary elements discretization we obtain a mixed linear complementarity problem.

In the two-dimensional case for Problem 2 the flux and potential satisfy the following integral equation on the boundary $\Gamma$:

$$\frac{1}{2}u(\xi) + \int_{\Gamma} q^*(\xi,\chi)u(\chi)d\Gamma = \int_{\Gamma} u^*(\xi,\chi)q(\chi)d\Gamma, \quad (1)$$

where $\chi = (x, y) \in \Gamma$, $u^*(\xi,\chi)$ is the fundamental solution of the Laplace equation, $q^*(\xi,\chi)$ its normal derivative, and $\xi \in \Gamma$ is the collocation point (Brebbia et al. 1984).

In the present approach, we make first a boundary element discretization based on the (1) and then introduce the semipermeability conditions for the discrete model.

Let $N$ be the number of (geometrical) nodes and elements $\Gamma_i$ such that $\Gamma = \sum_{j=1}^{N} \Gamma_j$. Assuming that the flux and the potential are approximated by constant functions for each $\Gamma_j$, $j = 1, ..., N$, we perform the following discretization of the integral equation:

$$\frac{1}{2}u_i + \sum_{j=1}^{N}\left(\int_{\Gamma_j} q_i^* d\Gamma_j\right) u_j = \sum_{j=1}^{N}\left(\int_{\Gamma_j} u_i^* d\Gamma_j\right) q_j,$$

where $i = 1, ..., N$, $u_i = u(\xi_i)$, $u_i^* = u^*(\xi_i,\chi)$, $q_i^* = q^*(\xi_i,\chi)$, $\xi_i \in \Gamma_i$ and $u_j = u(\chi)$, $q_j = q(\chi)$, $\chi \in \Gamma_j$, $j = 1, ..., N$. Using the notations $H_{ij} = \int_{\Gamma_j} q_i^* d\Gamma_j$ for $i \neq j$, $H_{ii} = 0.5 + \int_{\Gamma_i} q_i^* d\Gamma_i$ and $G_{ij} = \int_{\Gamma_j} u_i^* d\Gamma_j$, we can write this equation in the matrix form:

$$Hu = Gq,$$

where $H, G \in \mathbf{R}^{N\times N}$ and $u, q \in \mathbf{R}^N$.

Let $(x_i, y_i)$ be the coordinates of the geometrical nodes, $i = 1, ..., N$, and $x_{N+1} = x_1$, $y_{N+1} = y_1$. Then, we can obtain explicit formulas for the elements of $G$ and $H$:

When $i \neq j$:

$$G_{ij} = -\sum_{k=1}^{4} \frac{1}{2}\omega_k \left(a_x^2 + a_y^2\right)^{1/2} \ln \times \left((x_c - a_x\gamma_k - b_x)^2 + (y_c - a_y\gamma_k - b_y)^2\right), \quad (2)$$

$$H_{ij} = -\sum_{k=1}^{4} \frac{\omega_k\left(a_y(a_x\gamma_k + b_x - x_c) - a_x(a_y\gamma_k + b_y - y_c)\right)}{\left(x_c - a_x\gamma_k - b_x\right)^2 + \left(y_c - a_y\gamma_k - b_y\right)^2}, \quad (3)$$

and when $i = j$:

$$G_{ii} = 2\left(a_x^2 + a_y^2\right)\left(1 - \ln\left(a_x^2 + a_y^2\right)^{1/2}\right), \quad (4)$$

$$H_{ii} = \pi, \quad (5)$$

where $a_x = 0.5(x_{j+1} - x_j)$, $b_x = 0.5(x_{j+1} + x_j)$, $a_y = 0.5(y_{j+1} - y_j)$, $b_y = 0.5(y_{j+1} + y_j)$, $x_c = 0.5(x_i + x_{i+1})$, $y_c = 0.5(y_i + y_{i+1})$, and $\gamma_k$, $\omega_k$ are the abscissa and weight of the Gauss quadrature.

Let $n, m, l, k$ and $r$ be the numbers of the boundary elements located at the segments $\Gamma_\sigma$, $\Gamma_\lambda$, $\Gamma_1$, $\Gamma_\circ$ and $\Gamma_2$, respectively, see Fig. 2, and $K := n + m$, $L := K + l$, $M := L + k$, $N := M + r$, $P := N + n$, $R := P + m$ and $S := R + k - 1$.

We consider as independent variables $n - 1$ y-coordinates of the seepage surface nodes:

$$X_1, ..., X_{n-1},$$

$m$ y-coordinates of the free boundary nodes:

$$X_n, ..., X_{K-1},$$

the flux at the boundary elements of $\Gamma_1$:

$$X_K, ..., X_{L-1},$$

the potential at the boundary elements of $\Gamma_\circ$:

$$X_L, ..., X_{M-1},$$

the flux at the boundary elements of $\Gamma_2$, $\Gamma_\sigma$, $\Gamma_\lambda$ and $\Gamma_\circ$:

$X_M, ..., X_{N-1}$, $X_N, ..., X_{P-1}$, $X_P, ..., X_{R-1}$ and $X_R, ..., X_S$,

respectively.

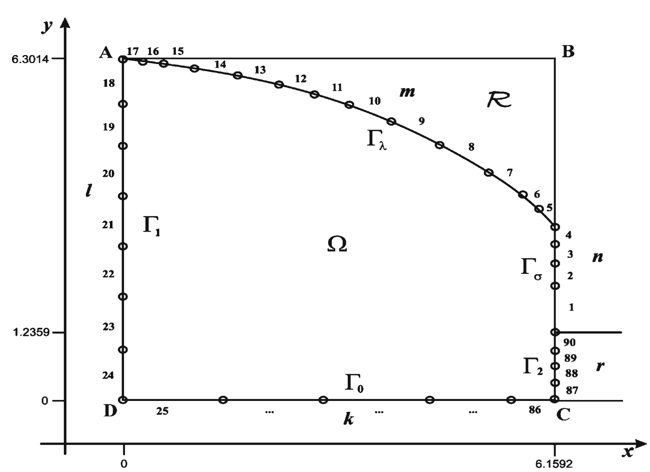


**Fig. 2** The BEM discretization

Let be

$$X = (X_1 ... X_S),$$
$$U = (u_1 ... u_l, X_L ... X_{M-1}, u_{l+k-1} ... u_N),$$
$$Q = (X_K ... X_{L-1}, X_M ... X_S),$$

where the values of the potential $u$ on the segment $\Gamma_\sigma \cap \Gamma_\lambda$ are defined using the boundary conditions of the problem

$$U_{l+k+r+i} = 0.5(X_{i-1} + X_i), \qquad i = 2, ..., K-1,$$

$$U_{l+k+r+1} = 0.5(h_2 + X_1), \qquad U_N = 0.5(h_1 + X_{K-1}).$$

as well as the remaining values of $u$ and $q$.

For the complementarity conditions we have:

$$X_{L+i} \leq p_\circ, \quad X_{R+i} \leq 0, \quad (X_{L+i} - p_\circ) \cdot X_{R+i} = 0, \quad i = 0, .., k-1.$$

It follows from (2–5) that $H$ and $G$ are functions of $X$, more precisely, of the $y$-coordinates of the free boundary and seepage surface nodes:

$$H(X) \equiv H(X_1, ..., X_{n+m-1}),$$

$$G(X) \equiv G(X_1, ..., X_{n+m-1}).$$

Then, the discrete problem can be formulated as the following mixed linear complementarity problem:

$$\begin{cases} H(X)u - G(X)q = 0, \\ X_{L+i} \leq p_0, \quad X_{R+i} \leq 0, \quad i = 0, .., k-1, \\ (X_{L+i} - p_\circ) \cdot X_{R+i} = 0, \quad i = 0, .., k-1, \\ X_i = 0, \quad i = P, ..., R-1, \\ U_{l+k+r+i} = 0,5(X_{i-1} + X_i), \quad i = 2, ..., K-1, \\ U_{l+k+r+1} = 0,5(h_2 + X_1), \quad U_N = 0,5(h_1 + X_{K-1}), \\ h_2 \leq X_i \leq h_1, \quad i = n, ..., n+m. \end{cases}$$

Different techniques can be employed to solve linear complementarity problems. We mention Lemke's method (Bazaraa and Shetty 1979), gradient projection, quasi-Newton and conjugate gradient projection algorithms (Xiao et al. 1999), the decomposition-coordination techniques (Spann 1993), and some heuristic iterative procedures with trial and error. Another approach to treat the linear complementarity problem is their reformulation as optimization problems (Friedlander et al. 1995). The most convenient way is to exploit a bilinear mathematical program (Cottle et al. 1992). We consider the equivalent bilinear mathematical program:

$$\begin{cases} \min_X F(X) \\ H(X)u - G(X)q = 0, \\ X_{L+i} \leq p_\circ, \quad X_{R+i} \leq 0, \quad i = 0, .., k-1, \\ (X_{L+i} - p_\circ) \cdot X_{R+i} = 0, \quad i = 0, .., k-1, \\ U_{l+r+k+i} = 0.5(X_{i-1} + X_i), \quad i = 2, ..., K-1, \\ U_{l+r+k+1} = 0.5(h_2 + X_1), \quad U_N = 0.5(h_1 + X_{K-1}), \\ h_2 \leq X_i \leq h_1, \quad i = n, ..., n+m. \end{cases}$$

with the objective functional:

$$F(X) = \sum_{i=P}^{R-1} X_i^2.$$

To find a solution of this problem, we use the interior point algorithm for nonlinear mathematical programming, FAIPA (Herskovits 1998). This algorithm solves the Karush-Kuhn-Tucker conditions without need to employ any penalization functions or to solve quadratic programming sub-problems. Some of the advantages of FAIPA in the case of inequality constrains is that all the iterated points are feasible and the objective function is reduced at each iteration. In the case of equality constraints a feasible initial point is not required. Even if the complementarity conditions are non-convex, the iterates given by FAIPA belong to a convex region defined by the inequality constraints. In consequence, we have global convergence. The search along an arc improves theoretical and numerical convergence. This algorithm is widely used to solve engineering optimization problems (Herskovits et al. 2005; Canelas et al. 2008a, b; Herskovits and Mazorche 2008).

## 4 Numerical results

As an example, we describe the numerical results for the test problem with $h_1 = 6.3014$, $h_2 = 1.2359$ and $\ell = 6.1592$. The discretization includes 90 boundary elements ($n = 4$, $m = 13$, $l = 7$, $r = 62$, $k = 4$), see Fig. 2. The $y$-coordinates of the nodes 1-3 at the seepage interval $\Gamma_\sigma$ and the $y$-coordinates of the nodes 4-16, that define location of the water table $\Gamma_\lambda$, are the variables of the problem. The coordinates of the rest of the nodes are fixed. The mathematical program has 168 variables, 152 nonlinear equality constraints and 137 "box" constraints. We adopt the algorithm stopping criterion with precision 10E−6, (Herskovits 1998). With different initial data points, the convergence of the algorithm was obtained in no more than 30 iterations. The algorithm converges to the same solution with absolute error less that 10E−6 for pressure, flux and the $y$-coordinates

**Table 1** Seepage interval height ($S.P.$) and bottom permeability interval length ($\ell_\circ$) coordinates for different values of the $p_\circ$

| $p_\circ$ | $S.P.$ | $\ell_\circ$ |
|---|---|---|
| 6.5 | 2.5129 | 0.0 |
| 6.0 | 2.4934 | 0.2 |
| 5.5 | 2.4661 | 0.8 |
| 5.0 | 2.4074 | 1.3 |
| 4.5 | 2.3423 | 1.8 |
| 4.0 | 2.2490 | 2.3 |
| 3.5 | 2.1361 | 2.8 |
| 3.0 | 1.9691 | 3.4 |
| 2.5 | 1.8034 | 3.9 |
| 2.0 | 1.5878 | 4.4 |
| 1.7 | 1.4482 | 4.9 |
| 1.6 | 1.4103 | 5.0 |

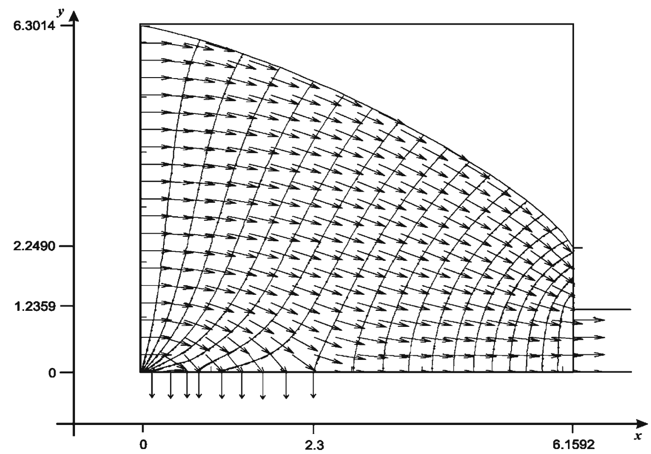


**Fig. 4** Case $p_\circ = 4.0$. The bottom is permeable at the interval $(0.0 - 2.3)$

of the water table and the seepage interval. FAIPA proved to be very strong for this kind of applications, solving all the test problems very efficiently with the same set of parameters. The boundary elements mesh at $\Gamma_\circ$ has the uniform step equal to 0.1, except for the last element. Thus, the location of the non zero flux interval at the bottom (permeable part of the bottom) is defined with absolute error less than 0.2. We perform the tests for the following values of $p_\circ$: 6.5, 6.0, 5.5, 5.0, 4.5, 4.0, 3.5, 3.0, 2.5, 2.0, 1.7 and 1.6.

The computed values of the seepage interval height and the bottom permeability interval length are given in Table 1. These results show the monotone decreasing of the water table and, at the same time, the enlargement of the permeability interval at the bottom with decreasing of $p_\circ$. For $p_\circ = 6.5$ the bottom is impermeable and we have the solution of the classical seepage problem, see Fig. 3. Our numerical results in this case are consistent with the numerical results obtained in the paper by Leontiev and Huacasi (2001). In the cited paper the

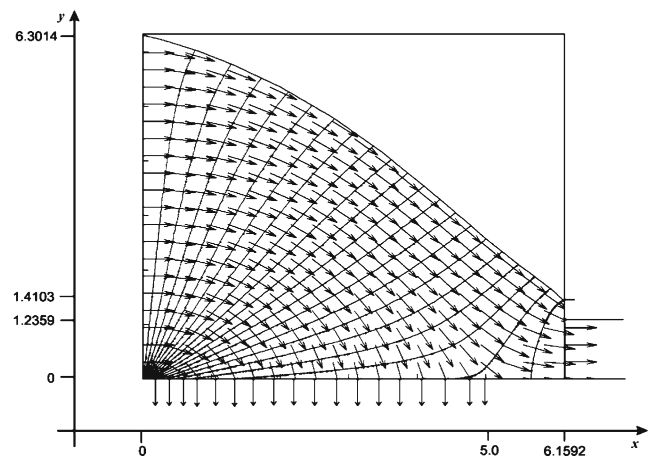


**Fig. 5** Case $p_\circ = 1.6$. The bottom is permeable at the interval $(0.0 - 5.0)$

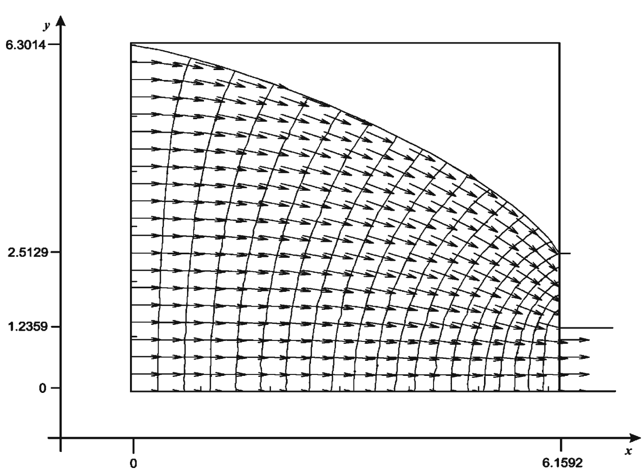


**Fig. 3** Case $p_\circ = 6.5$. The bottom is impermeable (classical seepage problem)

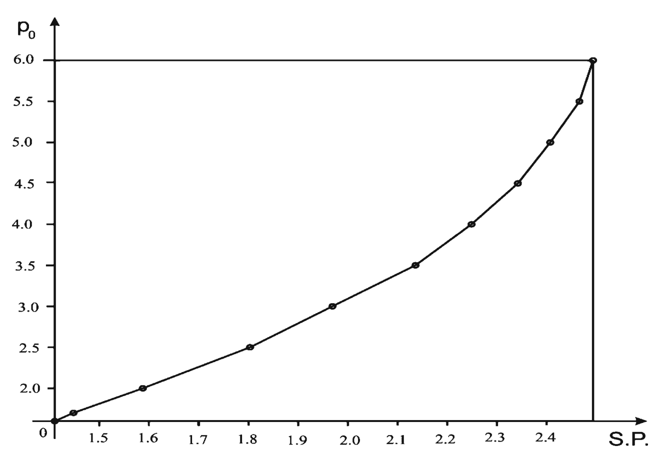


**Fig. 6** Seepage interval height ($S.P.$) versus value of $p_\circ$

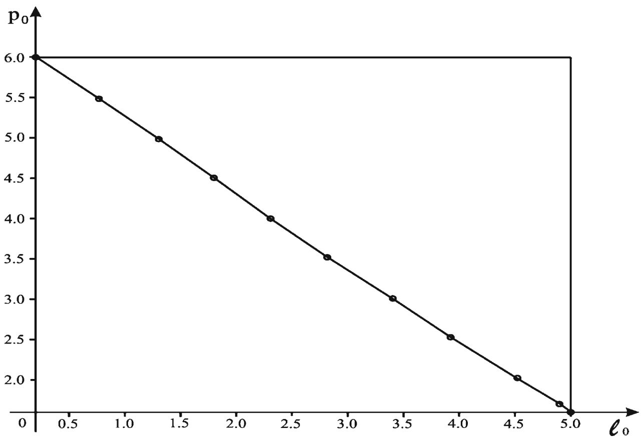


**Fig. 7** Permeability interval length ($\ell_\circ$) versus value of $p_\circ$

numerical results are compared with an analytical solution with very good compatibility. Figure 4 presents the numerical result for $p_\circ = 4.0$. In this case the permeable part of the bottom is the interval $(0.0 - 2.3)$. The numerical results for $p_\circ = 1.6$ are given in Fig. 5. The obtained permeability interval is $(0.0 - 5.0)$. Figures 6 and 7 show the seepage interval height $(S.P.)$ and the bottom permeability interval length ($\ell_\circ$) versus the value of $p_\circ$, respectively. These relations make possible an approximate prediction of the location of the seepage interval and the length of the permeable part of the bottom for a given value of $p_\circ$ in the case of the direct problem, or the values of $p_\circ$ and $\ell_\circ$ observing the height of the seepage interval for the inverse problem.

## 5 Conclusions

The optimization approach was employed to study the unconfined seepage problem with one way permeable bottom. The BEM avoids the need of mesh generation for the domain discretization of the problems with free boundary required by the FEM. The one way permeability of the bottom turns the discrete problem into a mixed linear complementarity program. Transforming this problem to the bilinear mathematical program we solve it using an interior point algorithm. Numerical results allows to observe relations between bottom permeability parameter, length of the permeable part of the bottom and height of the seepage interval.

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## References

Alliney S, Tralli A, Alessandri C (1990) Boundary variational formulation and numerical solution techniques for unilateral contact problems. Comput Mech 6:247–257

Baiocchi C, Capelo A (1984) Variational and quasivariational inequalities. Applications to free boundary problems. Wiley, Chichester

Bazaraa MS, Shetty CM (1979) Nonlinear programming. Theory and algorithms. Wiley, New York

Brebbia CA, Telles JCF, Wrobel LC (1984) Boundary elements techniques—theory and applications in engineering. Springer, Berlin

Canelas A, Herskovits J, Telles J (2008a) Shape optimization using the boundary element method and a SAND interior point algorithm for constrained optimization. Comput Struct 86:1517–1526

Canelas A, Roche JR, Herskovits J (2008b) The inverse electromagnetic shaping problem. Struct Multidiscipl Optim. doi:10.1007/s00158-008-0285-9

Cottle RW, Pang JS, Stone RE (1992) The linear complementarity problem. Academic, New York

Eck C, Steinbach O, Wendland WL (1999) A symmetric boundary element method for contact problems with friction. Math Comput Simul 50:43–61

Eck C, Wendland WL (2003) A residual-based error estimator for BEM-descretizations of contact problems. Numer Math 95:253–282

Friedlander A, Martínez JM, Santos SA (1995) Solution of linear complementarity problems using minimization with simple bounds. J Glob Optim 6:253–267

Friedman A (1982) Variational principles and free-boundary problems. Wiley, New York

Jensen R (1980) Fluid flow through a porous media. In: Free boundary problems, proceedings of a seminar held in Pavia, September–October 1979. Istituto Nazionale di Alta Matematica Francesco Severi, Roma, pp 133–149

Han H (1990) A direct boundary element method for Signorini problems. Math Comput 5:115–128

Herskovits J (1998) Feasible direction interior-point technique for nonlinear optimization. J Optim Theory Appl 99:121–146

Herskovits J, Mazorche SR (2008) A feasible directions algorithm for nonlinear complementarity problems and applications in mechanics. Struct Multidiscipl Optim. doi:10.1007/s0015800802525

Herskovits J, Mota Soares CM, Goulart E, Mappa PC (2005) Mathematical programming models and algorithms for engineering design optimization. Comput Methods Appl Mech Eng 194:3244–3268

Howison SD, Morgan JD, Ockendon JR (1997) A class of codimension-two free boundary problems. SIAM Rev 39:221–235

Kinderlehrer D, Stampacchia G (1980) An introduction to variational inequalities and their applications. Academic, New York

Leontiev A, Huacasi W (2001) Mathematical programming approach for unconfined seepage problem. Int J Eng Anal Bound Elem 25:49–56

Leontiev A, Khludnev A (2006) A problem of flow through semipermeable obstacle. Sib J Numer Math 9:173–188

Leontiev A, Busse RS, Huacasi W (2004) A mathematical approach to a forest-impact problem. J Eng Math 48:27–42

Leontiev A, Huacasi W, Herskovits J (2002) An optimization technique for the solution of Signorini problem using boundary elements method. Struct Multidiscipl Optim 24:72–77

Liggett JA (1997) Location of the free surface in porous media. J Hydraul Div ASCE 103:353–365
Piermatei Filho O (2006) Modelo de infiltração em barragem com fundo semi-permeável. Tese de doutorado, UFRJ - Universidade Federal do Rio de Janeiro, Brasil
Polizzotto C (1993) Variational boundary-integral-equation approach to unilateral contact problems in elasticity. Comput Mech 13:100–115
Polubarinova-Kochinab PYa (1962) Theory of ground water movement. Princeton University Press, Princeton
Pozzi G (1974) On a free-boundary problem arising from fluid flow through a porous medium in the presence of evaporation. Boll UMI 4:416–440
Simunovic S, Saigal S (1992) Frictionless contact with B.E.M. using quadratic programming. J Eng Mech ASCE 118: 1876–1891
Spann W (1993) On the boundary-element method for the Signorini problem of the laplacian. Numer Math 65:337–356
Xiao JR, Lok TS, Xu SZ (1999) Boundary elements linear complementary equation solution to polygonial unilaterally simply supported plates. Int J Eng Anal Bound Elem 23: 297–305
Zheng H, Dai HC, Liu DF (2009) A variational inequality formulation for unconfined seepage problem in porous media. Appl Math Model 33:437–450