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RESEARCH PAPER

Bi-level model reduction for coupled problems

Application to a 3D wing

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Abstract In this work a methodology is proposed for the optimization of coupled problems, and applied to a 3D flexible wing. First, a computational fluid dynamics code coupled with a structural model is run to obtain the pressures and displacements for different wing geometries controlled by the design variables. Secondly, the data are reduced by Proper Orthogonal Decomposition (POD), allowing to expand any field as a linear combination of specific modes; finally, a surrogate model based on Moving Least Squares (MLS) is built to express the POD coefficients directly as functions of the design variables. After the validation of this bi-level model reduction strategy, the approximate models are used for the multidisciplinary optimization of the wing. The proposed method leads to a reduction of the weight by 6.6%, and the verification of the solution with the accurate numerical solvers confirms that the solution is feasible.

Keywords Multidisciplinary optimization (MDO) · Model reduction · Proper orthogonal decomposition (POD) · Moving least squares (MLS) · 3D wing · Fluid–structure coupling

1 Introduction

In recent years, more and more applications in structural engineering take different disciplines into consideration. Indeed, in various cases, the structural analysis is tightly coupled to one or more disciplines (typically fluid and/or thermal). On the other hand, the trend nowadays is to use independent and dedicated computational codes for each discipline.

In this context, the aim of multidisciplinary analysis (= MDA) is to develop mathematical and numerical methods in order to guarantee the coherence of the physical variables involved. Several approaches have been proposed in the literature to reach this goal, as the fixed-point method (Alexandrov and Lewis 2000), the minimization of the discrepancy between the coupling variables from the different disciplines (Tedford and Martins 2006) and the CASCADE method (Hulme and Bloebaum 1999).

However, beside the specific features of each MDA technique, all of them require the models to exchange information interactively, which has greatly hindered the systematic use of MDA in industrial applications. Indeed, since the responses (e.g. the mass, the maximum von Mises stress, the displacements at given nodes) are generally post-processed results computed

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by commercial software, the actual coupling of models is a tedious task:

- a large amount of data has to be exchanged;
- the variables are usually provided in different file formats, with possibly non-coincident geometries and incompatible meshes;
- the codes may be installed on separate platforms, with different hardware configurations (single PC / workstation clusters / supercomputers, 32-bit / 64-bit), operating systems and software versions;
- licensing issues may also bring limitations to the code interfacing.

The problem becomes even more critical when the coupled models are integrated in an optimization task, because a high number of calls to the simulations have to be executed before reaching an optimal solution, therefore leading to unacceptable CPU time and memory costs if no adapted strategy is followed (Cramer et al. 1994; Alexandrov and Lewis 1999). To address this issue, a series of specific multidisciplinary optimization (MDO) strategies have been proposed in the literature (Tedford and Martins 2006), attempting to separate as much as possible the computations of the different models. Examples of these methods are Collaborative optimization (CO) (Braun and Kroo 1997), Concurrent subspace optimization (CSSO) (Tedford and Martins 2006), Bi-level integrated system synthesis (BLISS) (Sobieszczanski et al. 1998; Agte 2005), Multidisciplinary Optimisation and Robust Design Approaches applied to Concurrent Engineering (MORDACE) (Giassi et al. 2004) and Disciplinary Interaction Variable Elimination (DIVE) (Masmoudi and Parte 2006).

Nevertheless, whatever strategy is used, the analysis and optimization of coupled systems still face a major challenge related to the interconnections between disciplines. Indeed, in order to decrease the size of the data to be transmitted from one model (e.g. a temperature field from a thermal simulation) to another model (e.g. a structural finite element computation with material properties depending on the temperature), the development of adapted model reduction strategies is mandatory. Once the coupling variables (generally the responses of the models, as the displacements in a structural calculation) have been reduced, the path is open to build efficient surrogate models of the whole disciplines.

Therefore, in this paper, an original bi-level model reduction technique is proposed. Based upon a set of numerical simulations, the proposed method allows for reducing the data exchanges between disciplines, and use this information to build surrogate models of the coupled problems. Initially proposed in the case of a 2D wing demonstrator (Filomeno Coelho et al. 2008), the method is extended and applied here to the multidisciplinary analysis and optimization of a 3D flexible wing (where the aerodynamics and the mechanical resistance are coupled).

The paper is organized as follows: after the problem presentation and formulation (Section 2), the description of the bi-level model reduction method is detailed (Section 3). Then, the example of a 3D flexible wing is presented (Section 4), and used to illustrate the benefits of the model reduction strategy for the multidisciplinary analysis and optimization of the wing (Section 5). The last section draws the conclusions of the study and discusses research prospects (Section 6).

2 Problem presentation and formulation

2.1 Formal description of the problem for 2 coupled models

When the input and output variables of 2 models depend on each other, the analysis of the coupling has to be tackled. For 2 models M_1 and M_2, the following options are available (cf. Fig. 1):

- *strongly-coupled models*: the coupling variables and functions are formulated in a unique system:

$$\begin{cases} v_1 = M_1(v_2) \\ v_2 = M_2(v_1) \end{cases}; \tag{1}$$

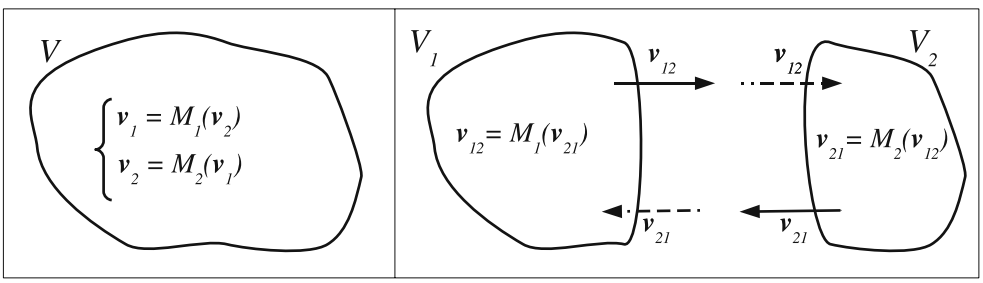

Fig. 1 Models M_1 and M_2: strongly-coupled (*left*) and loosely-coupled (*right*) models

– *loosely-coupled models*: the models are treated separately, and interact only by means of the coupling variables. In Fig. 1 (right), the variable v_{ij} denotes the output variables of model M_i used as inputs in M_j. The coherence of the coupling variables implies that the set of coupling variables (v_{12}, v_{21}) is a fixed point, expressing formally that the following conditions hold:

$$\begin{aligned} v_{12} &= M_1(M_2(v_{12})); \\ v_{21} &= M_2(M_1(v_{21})). \end{aligned} \tag{2}$$

Whereas the first approach is more satisfactory from a mathematical point of view, it is generally not practical because the models consist in commercial software for which an intrusive approach is not permitted. Therefore, the use of loosely-coupled models should be considered. However, three important features must be analyzed:

– while the coupling variables are usually defined on a common interface $S_{interface}$ (e.g. the external surface of a wing in an aeroelastic problem), these variables are generally expressed on different node locations, and possibly on non-coincident geometries. Therefore, an additional operator I_{ij} must be used to interpolate the data on a common reference grid (cf. Fig. 2);
– a high amount of data has to be exchanged between models, which can dramatically slow down the analysis of the coupled system. To address this issue, efficient model reduction strategies are mandatory;
– since the final responses of the models consist in a small set of post-processed (scalar) values (e.g. the maximum von Mises stress in the structure), the use of surrogate models is often a well-adapted option to consider, especially within an optimization task.

The next section discusses the use of reduced and approximate models for the analysis and optimization of coupled systems.

2.2 Reduction and approximation methods for MDO

Due to the high number of simulations required by the analysis and optimization of structures involved in a coupled system, several reduction methods have been proposed in the literature. In most cases, one or several disciplines are fully replaced by a meta-model based upon polynomial response surfaces or kriging; for example, in Simpson et al. (1998), an aerospike nozzle is optimized according to aerodynamics and structure, with a significant reduction of the simulation analysis and gradient calls thanks to the approximate models.

The need for surrogate models becomes crucial in MDO multi-level techniques, which work by reformulation of the MDO problem in different levels of optimization. Most of these techniques divide the optimization task into 2 levels: an upper level managing the global and coupling variables, and a lower one dealing with each discipline separately. For example, the BLISS method uses surrogate models (based on disciplines already optimized at a local level), by means of polynomial response surfaces (Kodiyalam et al. 1999) or kriging (Agte 2005), in order to be optimized eventually at the global level. Other meta-models, based on Moving Least Squares techniques, have been used in collaborative optimization (Zadeh et al. 2005), in order to construct approximations of the multidisciplinary constraints, and more precisely to identify the feasible region (a trust region algorithm being used to make the actual search inside the admissible region).

Beside general meta-models where no knowledge about the physics involved is introduced, another category of reduction methods attempt to take benefit of the information contained in the models. The Proper Orthogonal Decomposition (POD), well-spread among

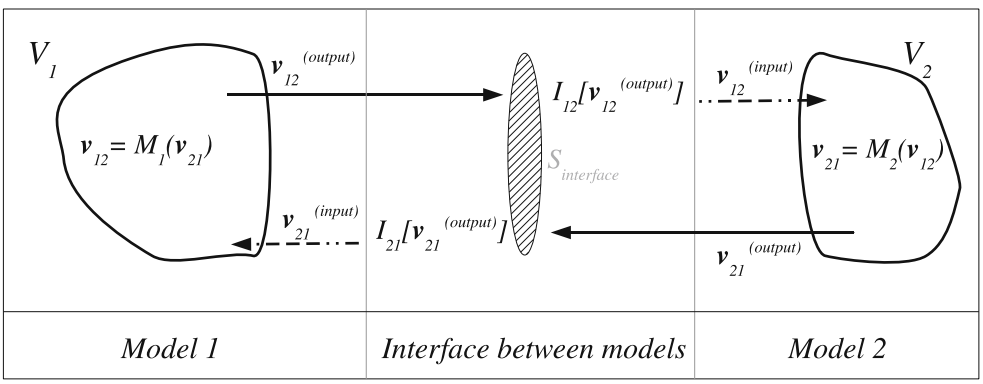

Fig. 2 Interface schematic description for two coupled models M_1 and M_2

the fluid dynamics community (Berkooz et al. 1993), is part of this family of techniques, and has been successfully incorporated in BLISS for airfoil design applications (LeGresley and Alonso 2004; LeGresley 2005).

In this paper, taking benefit of approximation and reduction methodologies, the original contribution consists in a combination of both techniques to build inexpensive but reliable surrogate models for fluid and structure models.

3 Bi-level model reduction technique

3.1 Level 1: POD to reduce the interconnections between disciplines

As mentioned in Section 2, one of the major issues in loosely-coupled systems lies in the management of the interconnections between disciplines: an efficient and practical method should allow each discipline to work with a minimum quantity of data to exchange with the other disciplines.

Therefore, a suitable reduction technique should be applied. In other words, instead of transferring a large set of data (e.g. the whole field of pressure on the wing), a limited number of (scalar) numbers should be sent. In this work, the POD or Proper Orthogonal Decomposition, also known as the Karhunen-Loève expansion (KLE) or Principal Component Analysis (PCA), is selected for this purpose (Berkooz et al. 1993).

The POD is widely used in computational fluid dynamics to decompose flows (for example, to expand a velocity vector evolving with time). In structural analysis, the method has also been adapted in various cases. For example, in Schenk et al. (2005), POD helps calculating for a moderate cost the non-stationary response of large, nonlinear finite element systems under stochastic loading. In interactive design and manufacturing, the analysis in real time of nonlinear mechanical model by finite element method (FEM) is also largely accelerated by POD reduction techniques (Dulong et al. 2007).

In this work, a parameter-dependent POD is proposed to decompose the coupling variables (Bui-Thanh et al. 2003; Newman 1996). For the test application presented in this paper, the pressure and displacements on the wing surface are considered. For the sake of clarity, the method is described only for the fluid discipline, but the same reasoning holds for the structure.

Practically, for a variable p (e.g. a pressure field), starting from the values of p_i (called snapshots) obtained by accurate numerical simulations for a representative set of designs, the idea is to build a linear basis to represent any vector $\tilde{p}$, this basis being guaranteed by construction to be optimal to describe a given sample set of observations:

$$p_{POD} = \bar{p} + \sum_{i=1}^{M} \alpha_i \phi_i^p, \tag{3}$$

where:

- $\bar{p}$ is the average vector;
- the ϕ_i^p are the POD basis vectors;
- the α_i are scalar coefficients of the linear expansion.
- M is the number of snapshots.

The construction of the POD basis is divided in 3 parts. First, the deviation matrix P is built by storing the pressure vectors of all snapshots, and subtracting the mean vector $\bar{p}$ to each row of P:

$$P = \begin{bmatrix} p_1^{(1)} - \bar{p}_1 & p_1^{(2)} - \bar{p}_1 & \cdots & p_1^{(M)} - \bar{p}_1 \\ p_2^{(1)} - \bar{p}_2 & p_2^{(2)} - \bar{p}_2 & \cdots & p_2^{(M)} - \bar{p}_2 \\ \vdots & \vdots & \vdots & \vdots \\ p_N^{(1)} - \bar{p_N} & p_N^{(2)} - \bar{p_N} & \cdots & p_N^{(M)} - \bar{p_N} \end{bmatrix}. \tag{4}$$

Then, the covariance matrix C is calculated:

$$C = P.P^T. \tag{5}$$

Finally, the POD modes ϕ_i are obtained by extracting the eigenvectors of C. To estimate the quality of the POD reduction, an energy criterion error is defined with respect to the eigenvalues λ_i of C:

$$\varepsilon_{energy}(m) = 1 - \frac{\sum_{i=1}^{m} \lambda_i}{\sum_{i=1}^{M} \lambda_i}. \tag{6}$$

Any vector can be projected on each POD basis vector ϕ_i, and the corresponding scalar coefficient α_i represents the relative importance of the ith eigenmode on the reconstruction of the total field. Therefore, the actual reconstruction error $\varepsilon_{rec}(m)$ caused by neglecting the contribution due to the last $(M - m)$ eigenmodes can be evaluated by:

$$\varepsilon_{rec}(m) = 1 - \frac{\|\bar{p} + \sum_{i=1}^{m} \alpha_i \phi_i\|}{\|\bar{p} + \sum_{i=1}^{M} \alpha_i \phi_i\|}. \tag{7}$$

3.2 Level 2: Surrogate models

Once the behaviour of a coupling variable has been reduced by the POD technique, the m scalar values contain enough information to build an approximation of the whole field. Therefore, the main idea proposed in this paper is to build a surrogate surface for *each* of the scalar POD coefficients retained, defined with

respect to the design variables: in other words, the model reduction is performed at two stages:

- first, a POD reduction of the coupling variables (pressures and displacements) is performed, hence leading to a database of POD coefficients for a representative set of designs;
- secondly, response surfaces are built to link each POD coefficient with the design variables.

Figure 3 illustrates the workflow to build the reduced models, combining a general approximator to the POD procedure. First, a series of numerical simulations are executed for a representative set of designs $x_{DOE}^{(i)}$, each simulation consisting in a resolution of the coupled system, also called multidisciplinary analysis (MDA). Then, the variables of interest are decomposed by POD, and a limited number m of the scalar POD coefficients $\alpha_j^{(i)}$ are stored. For any vector p, the POD coefficients are simply calculated by projection on the POD mode:

$$\alpha_i =< p - \bar{p}, \phi_i >, \tag{8}$$

where $<,>$ is the scalar product between vectors. The same is done for the displacements in the three directions.

Finally, from these POD coefficients, surrogate response surfaces are built with respect to the design variables.

The approximation techniques used for making the response surfaces are described in the next section.

3.3 Approximation techniques

To carry out the surrogate response surfaces, two approximation techniques are used:

- the Polynomial Response Surface Method (Giunta et al. 1998);
- the Moving Least Squares method, also called Diffuse Approximation (Nayroles et al. 1992).

The polynomial response surface method (PRSM) is one of the most widespread general approximation techniques, and its main features are detailed in Giunta et al. (1998). However, although the surrogate surfaces built by PRSM generally give a good trend of the global behaviour of the response, they might be inadequate to capture local phenomena. Therefore, the use of the Moving Least Squares (MLS) technique is also investigated.

The MLS approximation of a function $\alpha(x)$ can be written as follows (Breitkopf et al. 2002):

$$\alpha(x) \approx \alpha_{app}(x) = b^T(x)\, a(x), \tag{9}$$

where:

$$b(x) = [1 \quad x_1 \quad x_2 \quad \ldots \quad x_1^2 \quad x_1 x_2 \ldots]. \tag{10}$$

The difference with the classical least squares method lies in the fact that the coefficients $a(x)$ are not constant over the domain, but depend upon the value of x. In practice, these coefficients minimize the functional $J_x(a)$ defined by:

$$J_x(a) = \frac{1}{2} \sum_{i=1}^{M} w_i(x_i, x) \left(b^T(x_i) a - u_i \right)^2, \tag{11}$$

where,

- $b(x)$ is the polynomial;
- x_i are the sample points;
- w_i are the weights depending on some distance (here: the Euclidean norm) between x_i and x.

Moreover, the terms composing the basis vector $b(x)$ can be chosen according to the problem, dropping for instance the crossed terms $x_i x_j$ in a quadratic surface, or adding other terms (e.g. sinusoidal, exponential). Beside the flexibility authorized for the definition of the basis, the inherently nonlinear behaviour of the MLS surface enables to capture the possibly irregular shape of the response to approximate.

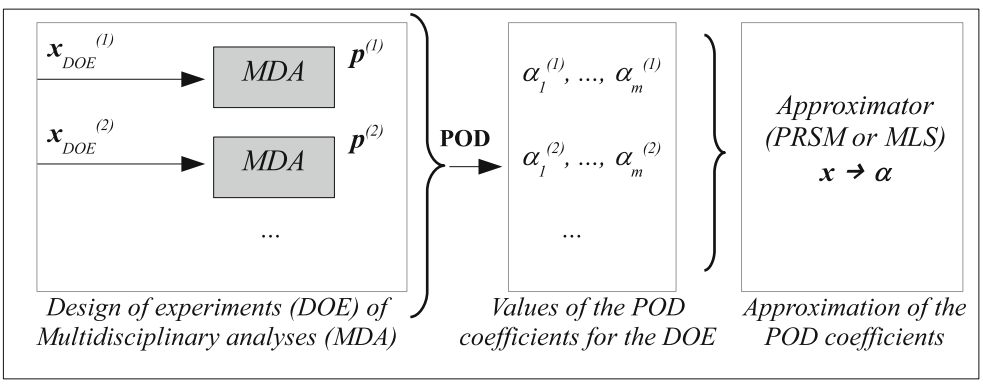

Fig. 3 Model reduction workflow

In this work, the weights w_i are given by:

$$w_i = w_{ref}\left(\frac{dist(x_i, x)}{r}\right), \tag{12}$$

where w_{ref} is a C^1 piecewise cubic spline expressed by:

$$w_{ref}(s) = \begin{cases} 1 - 3s^2 + 2s^3 & \text{if } 0 \le s \le 1, \\ 0 & \text{if } s \ge 1, \end{cases} \tag{13}$$

and r is a radius defining the (local) influence zone, defined here such as it covers the k closest points to x (k being selected such as $k = n_p + n_{dim}$, where n_p is number of terms in the polynomial basis vector p and n_{dim} is the dimension of the inputs).

The surrogate surface established by this technique will usually not pass through all sample points; to force the approximation to interpolate the points, the weights can be specifically scaled to fit the data points, as explained in detail in Breitkopf et al. (2002).

Now that the theoretical description of the bi-level model reduction has been exposed, the next section presents a 3D flexible wing example that will be used eventually to validate the bi-level strategy.

4 Description of a 3D wing example for transonic flow

4.1 Formulation of the optimization problem

In the purpose of investigating the bi-level model reduction method on a realistic example, the original test case of a 3D wing (from a business jet) has been developed. The goal is to minimize the weight of the wing in transonic operating conditions (Mach number $M = 0.8$, incidence angle $\gamma_{inc} = 3°$), with constraints related to the aerodynamics (lift) and the structure (mechanical resistance).

The multidisciplinary optimization of the 3D wing is formulated as follows:

$$\min_X f_{obj} \tag{14}$$

$$\text{subject to: } \begin{array}{l} C_L \ge C_L^0, \\ \sigma_{max} \le \sigma_{max}^0, \end{array} \tag{15}$$

where:

- $X \in [X_{lb}, X_{ub}]$ are the (bounded) design variables, controlling the shape of the wing section (see Section 4.2);
- f_{obj} is the mass of the structure;
- C_L is the lift coefficient;
- σ_{max} is the maximum von Mises stress over the wing structure;
- C_L^0 and σ_{max}^0 are the values obtained for the reference wing defined in Section 4.2.

The next sections describe the design variables (Section 4.2), as well as the numerical models required to calculate the objective and constraints (Section 4.3 and Section 4.4).

4.2 The parameterization

The geometry of the wing is built from a NACA64A010 airfoil section at the hub submitted to a homothetic extrusion from hub to tip in order to keep the same maximum thickness along the span.

In order to perform the shape optimization of the wing, new geometries must be generated automatically. In this study, a Free Form Deformation (FFD) parameterization is used. This technique was initially developed in the computer graphics field, but its flexibility and ease of implementation have made it a very practical tool for aerodynamic optimization (Samareh 2004; Duvigneau 2006). The principle of FFD consists in parameterizing the 3D space directly, instead of the (possibly complex) surface itself; then, a modification of some control points defining the 3D space will automatically transform the coordinates of all points inside the domain.

In the general 3D formulation, the geometry to be parameterized is contained in a 3D hexahedral lattice, where a local system of relative coordinates $(\xi,\eta,\theta) \in [0, 1] \times [0, 1] \times [0, 1]$ is defined. To deform the lattice, a displacement Δq is applied to each point q inside the lattice, computed as a third-order Bézier tensor product:

$$\Delta q = \sum_{i=1}^{n_i}\sum_{j=1}^{n_j}\sum_{k=1}^{n_k} B_i^{n_i}(s_q) B_j^{n_j}(t_q) B_k^{n_k}(u_q) \Delta c_{ijk}, \tag{16}$$

where the $B_p^n(t)$ are the n-order Bernstein polynomials given by:

$$B_p^n(t) = C_n^p t^p (1-t)^{(n-p)}, \tag{17}$$

with:

$$C_n^p(t) = \frac{n!}{p!(n-p)!}. \tag{18}$$

The vector (s_q, t_q, u_q) reflects the position of q in local coordinates, and the c_{ijk} are the weighting coefficients corresponding to the displacements of the control points defining the lattice; in a shape optimization context, the $_{cijk}$ will play the role of the design variables. It is interesting to note that this method acts on modifications of an initial reference geometry;

with parameters c_{ijk} set to 0, the exact initial design is retrieved.

In this work, a 2-step procedure is applied to define new geometries:

- first, the NACA (hub) section is modified by means of the FFD parameterization. 2 variables x_1 and x_2 control respectively the camber of the airfoil and its chord (see Fig. 4):
  - x_1 is the relative variation of the control points at mid-chord (y-direction);
  - x_2 is the relative variation of the chord length (x-direction).
- secondly, the 2D airfoil is extruded.

The reference wing and the variable bounds are defined as follows:

- reference wing: $X_0 = [0.1; 0.1]$;
- lower bounds: $X_{lb} = [-0.2; 0.0]$;
- upper bounds: $X_{ub} = [0.4; 0.2]$.

4.3 Discipline 1: the fluid model

The first discipline to investigate is the aerodynamic behaviour of the wing. Since any (turbulent) flow variable a can be decomposed as the sum of a mean value $\bar{a}$ and a fluctuating part a', the Navier-Stokes equations governing the air flow around the wing can be written as follows (Anderson 1995):

$$\rho \frac{\partial \bar{u}_i}{\partial t} + \rho \frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} = \rho \bar{f}_i + \frac{\partial}{\partial x_j}\left[-\bar{p}\delta_{ij} + \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \rho\overline{u'_i u'_j}\right], \quad (19)$$

where:

- ρ is the density;
- u_i is the i^{th} component of the velocity vector;
- f_i is the i^{th} component of the volumic force vector;
- μ is the dynamic viscosity;
- p is the pressure;
- $\rho\overline{u'_i u'_j}$ defines the *Reynolds stress tensor*.

In this work, the finite volume method is applied: it consists in solving the conservative form of the Navier-Stokes equations for each cell of the discretized region of the fluid space (Anderson 1995).

In practice, the commercial computational fluid dynamics (CFD) software FLUENT is used for this purpose. The full fluid computation procedure is organized as follows:

- to build the fluid volume, the wing geometry is generated by extruding the parameterized 2D airfoil along the span; the rest of the geometry consists in a 3D cubic box representing the domain of interest for the flow computation;
- an unstructured grid is built by GAMBIT mesh generator. For the reference wing ($x = [0.; 0.]$), the mesh is composed of 368, 509 cells (78, 459 nodes);
- the operating and boundary conditions are set as follows:
  - Mach number $M = 0.8$;
  - incidence angle $\gamma_{inc} = 3°$;
  - steady simulation;
  - the ideal-gas law is used for the air, and pressure far-field conditions are used to model the free-stream condition at the boundaries of the flow domain;
- after 200 CFD iterations, the mesh is automatically adapted according to the pressure gradient distribution, in order to refine the regions with high pressure gradients. For the reference wing, the adapted mesh is characterized by 501, 621 cells (123, 837 nodes);
- the computations are performed with the updated mesh until the stopping criterion is reached (the

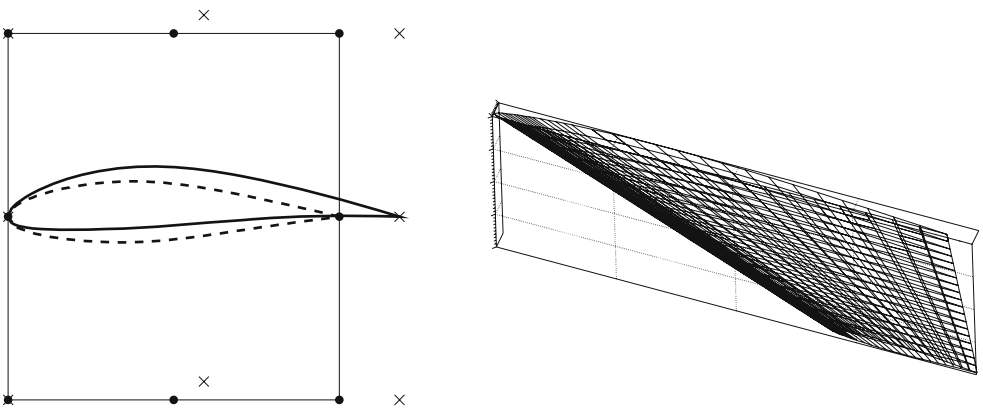

Fig. 4 Free Form Deformation: parameterization of the wing design: 2D modification (*left*) and 3D view (*right*)

convergence criterion is set to 10^{-6} for the continuity, velocity and energy residuals calculated in each finite volume cell, with a maximum of 500 iterations).

The lift L results from the pressure and viscous vertical forces, and is used to calculate the lift coefficient C_L, required as constraint in the MDO formulation. Practically, C_L is a post-processed response directly furnished by FLUENT.

4.4 Discipline 2: the structural model

In addition to the CFD run, a structural computation is performed in order to check the mechanical resistance of the wing. The structure is composed of shell elements on the wing external surface, rigidified by 5 vertical stiffeners (from hub to tip).

The governing equations include:

- the *static equilibrium* of the structure: the weak formulation of the static equilibrium leads to the equality of virtual works $\delta\pi_{int}$ and $\delta\pi_{ext}$ (calculated respectively for the internal and external forces):

$$\delta\pi_{int} = \delta\pi_{ext}; \tag{20}$$

- the Hooke's law (for an elastic material):

$$\tilde{S} = D\tilde{E}, \tag{21}$$

where $\tilde{S}$ is the second Piola-Kirchhoff stress tensor, $\tilde{E}$ is the Green-Lagrange strain tensor and D is Hooke's matrix for planar stress.

After developments (Zienkiewicz and Taylor 2005), the numerical discretization (by the finite element method) of the variational formulation derived from the virtual work principle leads to the following system (to be solved iteratively):

$$\left[K_t^{(k)}\right]\left\{\Delta q^{(k)}\right\} = \left\{\lambda^{(k)} F^{(k)}\right\} - \left\{r^{(k)}\right\}, \tag{22}$$

where k is the number of the current iteration, $[K_t^{(k)}]$ is the tangent stiffness matrix, $\{\Delta q^{(k)}\}$ is the increment of displacement between two iterations, $\lambda^{(k)}$ is the loading factor, $\{F^{(k)}\}$ is the external force vector and $r^{(k)}$ is the internal force vector. An iterative Newton-Raphson method is used to converge to the solution.

In practice, the Code_Aster code developed by EDF is applied to compute the displacements and stresses, by considering a linear elastic material (steel), and linear shell elements (Zienkiewicz and Taylor 2005). The mesh is characterized by 978 triangular elements (440 nodes; see Fig. 4—right), each triangular element being characterized by 6 nodes on the edges (with 3 translation and 3 rotation degrees of freedom), and 1 central node (with 3 rotation d.o.f.), leading to a total of 39 d.o.f. by element.

The pressure loading on the wing surface derives from the fluid computation. With a (geometrically) linear model, the vertical displacements obtained for the reference wing are shown in Fig. 5.

However, due to the flexibility of the wing, the deformed geometry cannot be assimilated to the initial one. Therefore, two features need to be considered:

- first, (geometrically) nonlinear structural computations must be executed to account for the deformation of the wing structure. The nonlinear shell elements available in Code_Aster are based on a Hencky-Mindlin-Naghdi kinematic formulation (Massin and Al Mikdad 2005), allowing small

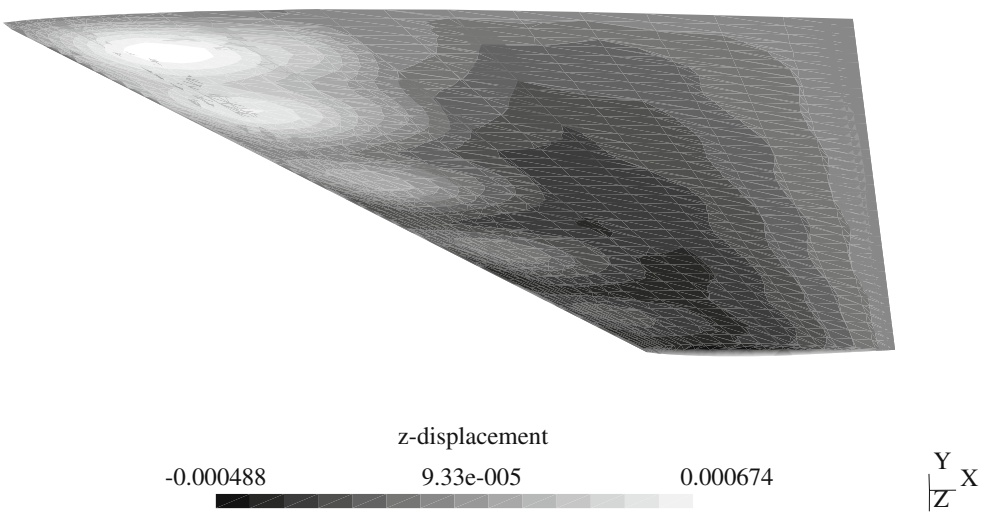

Fig. 5 Vertical displacement v obtained with the finite element (linear) structural model (Code_Aster) for the reference wing

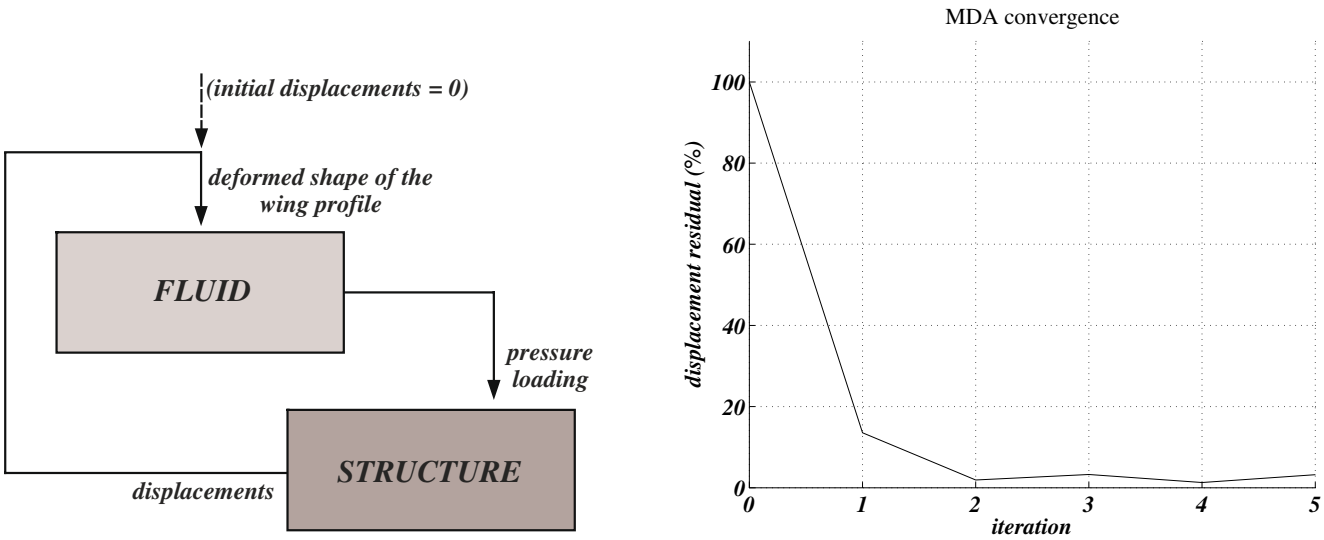

Fig. 6 Multidisciplinary analysis (MDA): procedure for the 3D wing [*left*], and convergence of the total displacement residual for the reference wing [*right*], with the fluid (FLUENT) and structure (CODE_ASTER) models

strains, large displacements and large rotations. The corresponding results show that the average deviation between the displacements obtained for the both calculations (linear and nonlinear) amounts to 6.11%, demonstrating the need to perform a nonlinear structural computation to ensure inaccurate estimation of the displacements. Therefore, in the remainder of this study, nonlinear computations are systematically applied;

– secondly, as the air flow depends on the actual wing shape, which itself relies on the pressure distribution, the fluid and the structural computations should be handled in a coupled approach to deal with these interactions. This will be discussed in the next section.

5 Bi-level reduced models for the shape optimization of the 3D wing

5.1 MDA by fixed-point procedure for the 3D wing

When two (or more) physics are involved, multidisciplinary analysis (MDA) aims at finding coupling variables coherent within all disciplines. For 2 disciplines, the most straightforward technique is the fixed-point algorithm (Alexandrov and Lewis 2000).

As illustrated in Fig. 6 (left), the fixed-point procedure is an iterative process. First, a fluid calculation is performed with the initial configuration. Then, the pressure distribution on the wing surface is collected, interpolated on the structural nodes, and a finite ele-

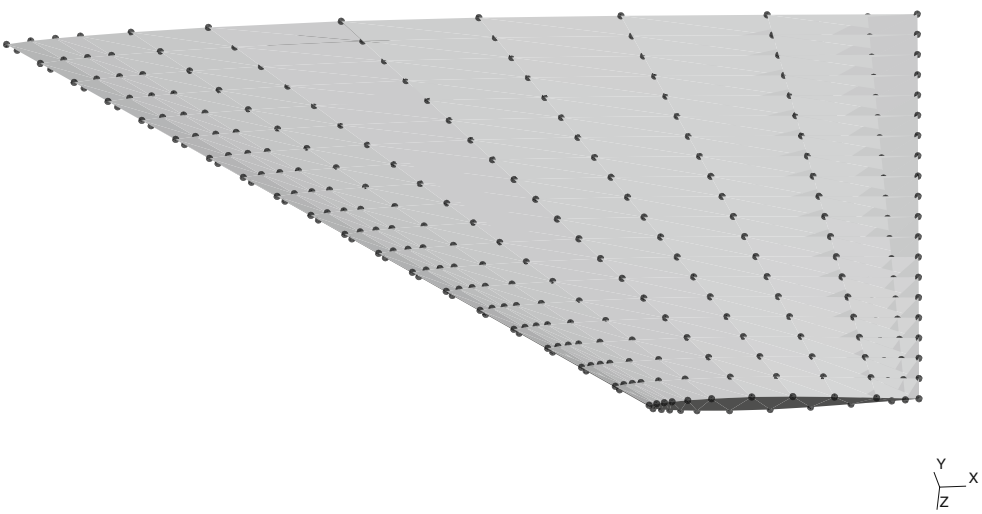

Fig. 7 Interface grid on which the pressures/displacements are interpolated: 20 × 62 nodes

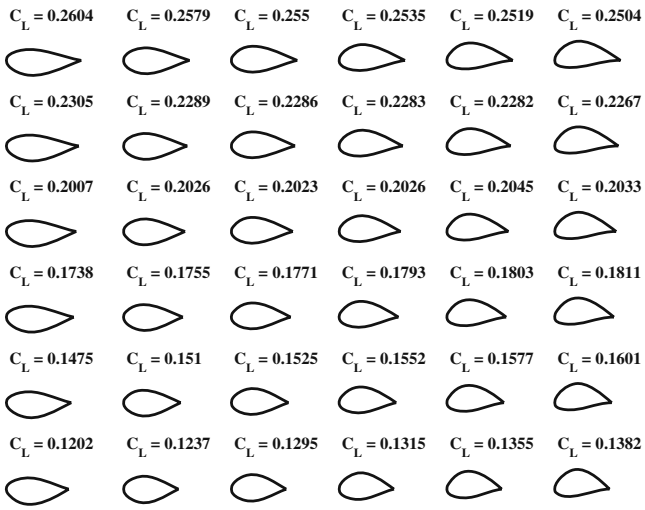

Fig. 8 Design of experiments of 36 wing geometries (with lift coefficient values)

ment calculation is done. The corresponding displacements are sent back to the fluid discipline, where a CFD run is executed with the updated geometry. The process goes on until a stopping criterion is reached, for instance the convergence of the discrepancy between the wing geometries for two successive iterations.

In this work, three particular aspects need to be noticed:

- the fluid is re-meshed at each updating of the geometry;
- when transferred from one discipline to the other, the pressures or displacements are linearly interpolated on an interface grid $S_{interface}$, defined as a 20×62 set of curvilinear coordinate nodes (see Fig. 7). To get an accurate representation of the fluid behaviour, the nodes are concentrated at the leading edge, where the higher pressure gradients are observed;
- at each iteration, due to the geometrically nonlinear shell model, the vector forces applied on the structure nodes are re-computed by taking into account the deformation of the wing (i.e. the orientation of the normals to the shell elements on which the external pressure loading is applied is updated at each fixed-point iteration).

Figure 6 (right) represents the convergence of the multidisciplinary analysis applied to the reference wing, where the residual *res* is defined as follows:

$$res = \frac{\sum_{i=1}^{nb\_nodes} \| u_i^{(k)} - u_i^{(k-1)} \|}{\sum_{i=1}^{nb\_nodes} \| u_i^{(0)} \|} \tag{23}$$

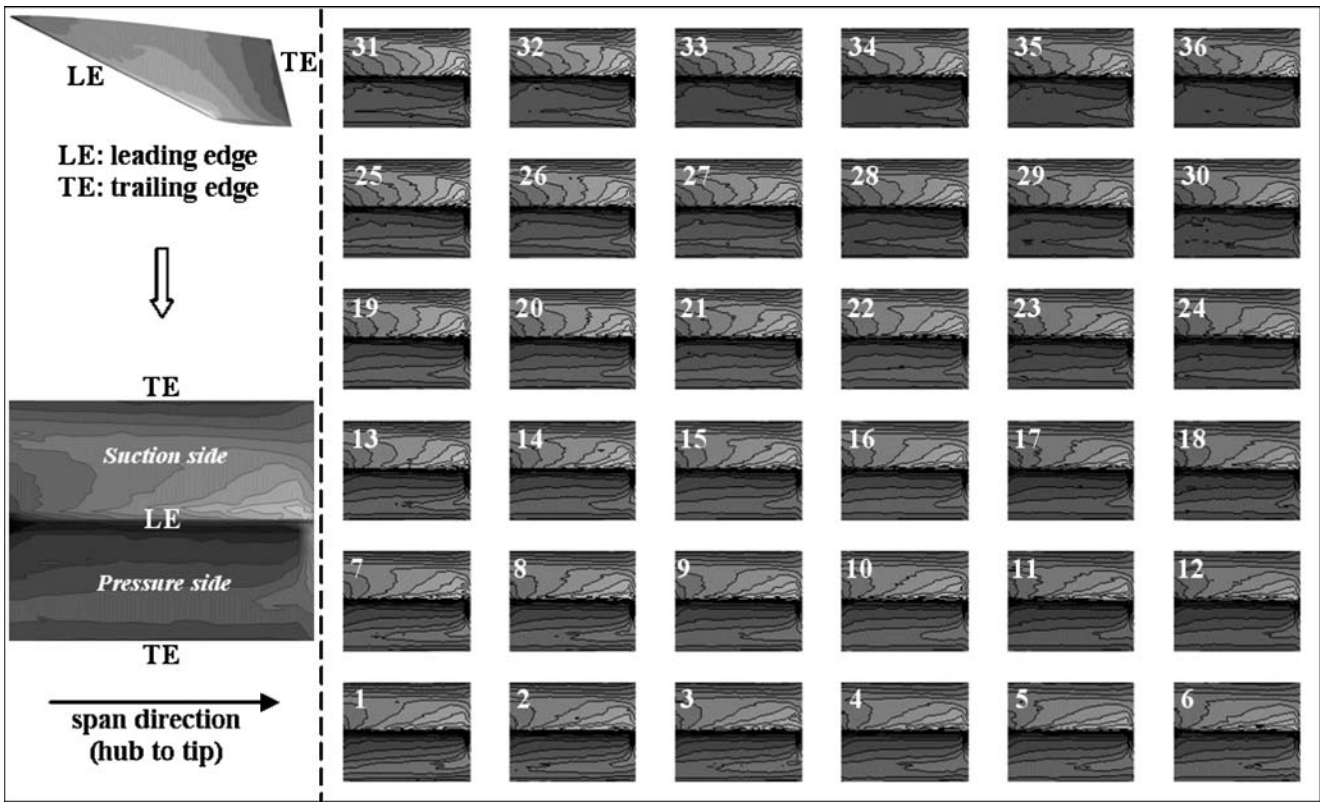

Fig. 9 Pressure snapshots of the 36 wing geometries: 2D representation

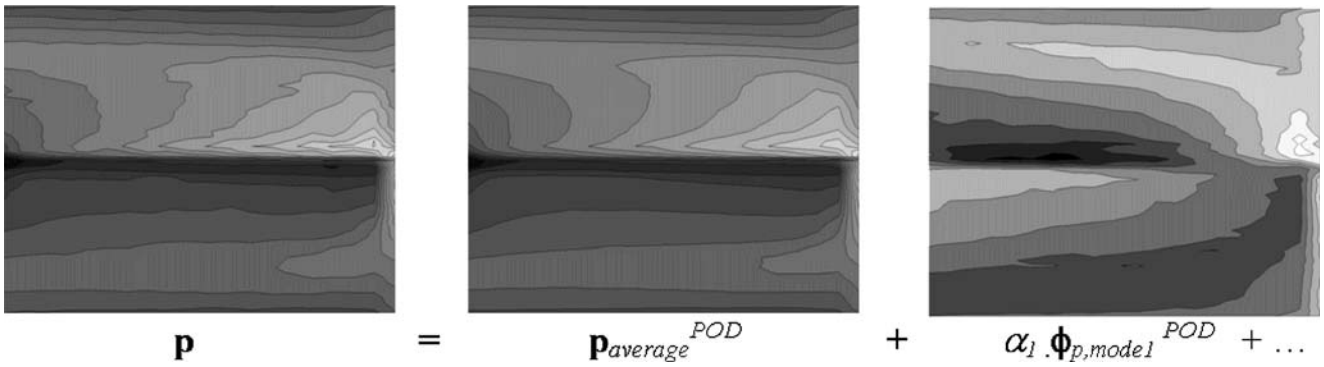

Fig. 10 POD reconstruction of a pressure snapshot

where $u_i(k)$ is the 3D displacement field of the ith FEM structural node at iteration k (the 0 index for the denominator refers to the initial iteration). A fast decrease of the residual is observed at the first 3 iterations; then, the residual oscillates around 3%. Therefore, in the remainder of the paper, the MDA convergence criterion is set to 3% of the initial displacement residual.

The 3D wing is a good example to list the difficulties that may appear while performing a multidisciplinary analysis or optimization:

- the disciplinary variables that need to be exchanged might be expressed on different meshes and even geometries. Well-suited interpolation techniques have to be specifically developed to account for the incompatibility of the data between disciplinary modules;
- to evaluate a single design, several iterations might be required, leading to a high CPU time and memory cost;
- the coupling of numerical codes might cause lots of practical issues (e.g. commercial software available on different machines or servers).

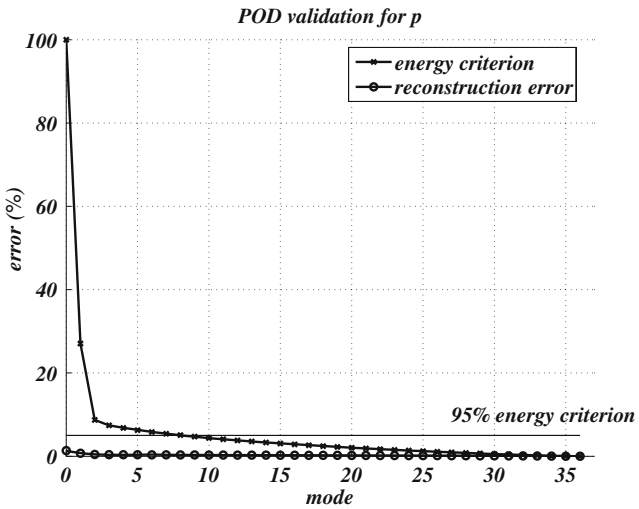

Fig. 11 POD reconstruction error averaged over the 36 pressure snapshots of the database

These reasons explain the need for model reduction techniques in the context of multidisciplinary analysis and optimization, particularly to diminish the interconnections between models. Therefore, the next sections investigate the use of the bi-level model reduction strategy for the 3D wing example.

5.2 Generation of the design of experiments

As described in Section 3.1, the first step consists in launching a set of numerical simulations, on representative wing geometries; the corresponding results will be used afterwards to build the reduced models.

Several methods are available to choose the set of design variables: this is the aim of the design of experiment techniques. When no specific knowledge about the problem is introduced, and for a small number of design variables (here: $n_{var} = 2$), the most straightforward option consists in a full factorial sampling, where the design variables are selected uniformly throughout the domain. Then, in order to provide consistent values for the pressures and displacements, a full multidisciplinary analysis is performed for each geometry.

The design of experiments used in this study is a 6-point full factorial set; the wing hub sections, along with the corresponding values of the lift coefficient obtained after a MDA, are shown in Fig. 8.

Table 1 3D wing: errors on the POD approximation of the pressures and displacements

	1 mode	2 modes	3 modes
Error on the pressure	8.55%	5.95%	1.62%
Error on u (x-displacement)	14.00%	8.05%	3.55%
Error on v (y-displacement)	3.26%	2.15%	1.10%
Error on w (z-displacement)	18.58%	7.04%	4.59%

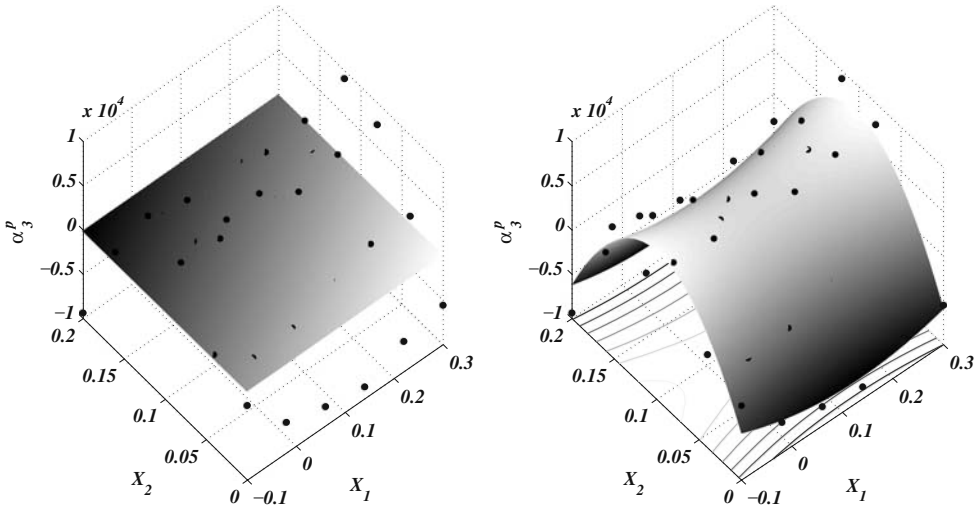

Fig. 12 Surrogate model (by *polynomial response surface*) of the POD coefficient α_3^p (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

The small number of design variables is used for a double purpose: limiting the size of the design of experiments required for building the surrogate models, and also enabling eventually a visualization of these models (cf. Section 5.4).

5.3 Level 1: proper orthogonal decomposition to reduce the response variables

To get a better insight of the POD reduction, the pressure distribution on the wing surface is illustrated for the 36 snapshots of the database. First, the pressures interpolated on the interface grid are projected on a 2D rectangle (see Fig. 9). The reconstruction of a single pressure snapshot is synthesized in Fig. 10.

The average energy criterion error ε_{energy} and the reconstruction error ε_{rec} obtained for the 36 elements of the database is depicted in Fig. 11 for the pressure field, showing a rapid decrease of the error for a low number of modes. The difference between the 2 curves is due to the contribution of the average pressure, which is by definition not taken into account in the covariance matrix, and explains the larger values obtained for the energy criterion.

The errors of the pressures and displacements (i.e. the values obtained by the MDA process compared to their POD approximation, averaged over the 36 points of the DOE) are collected in Table 1, showing that 3 modes are sufficient to get a satisfactory representation of the fields (the larger errors observed for the

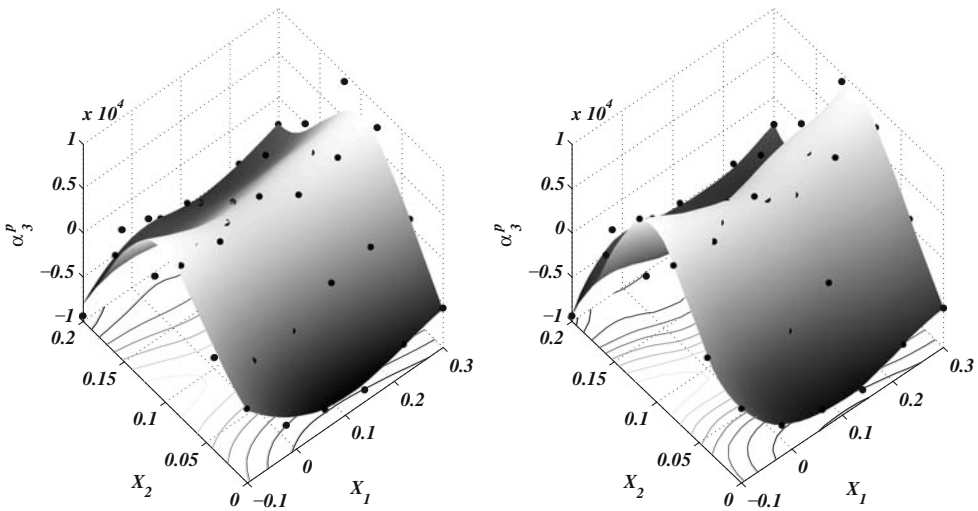

Fig. 13 Surrogate model (by *MLS approximation*) of the POD coefficient α_3^p (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

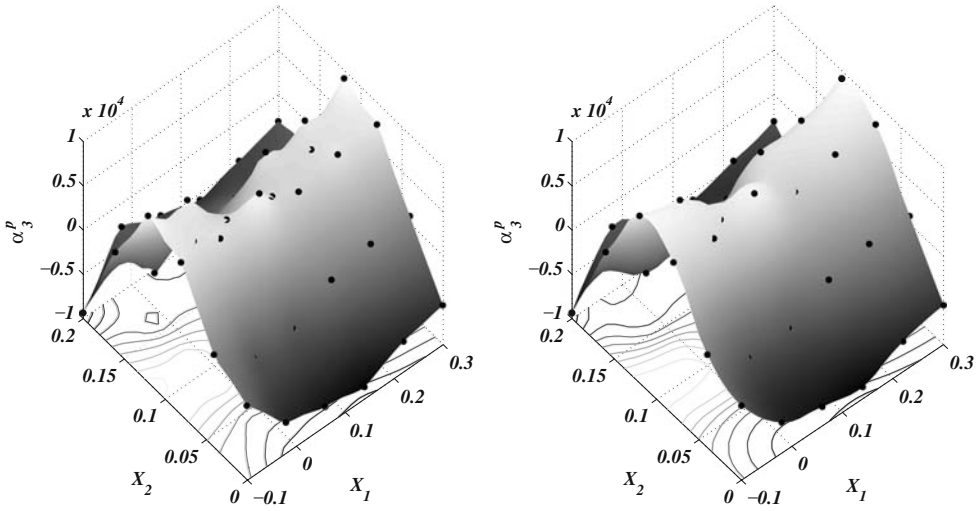

Fig. 14 Surrogate model (by *MLS interpolation*) of the POD coefficient α_3^p (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

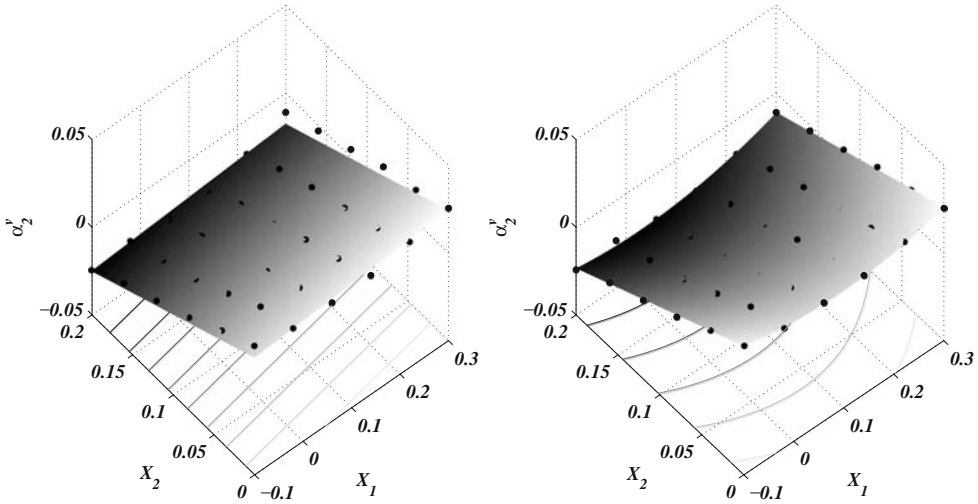

Fig. 15 Surrogate model (by *polynomial response surface*) of the POD coefficient α_2^v (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

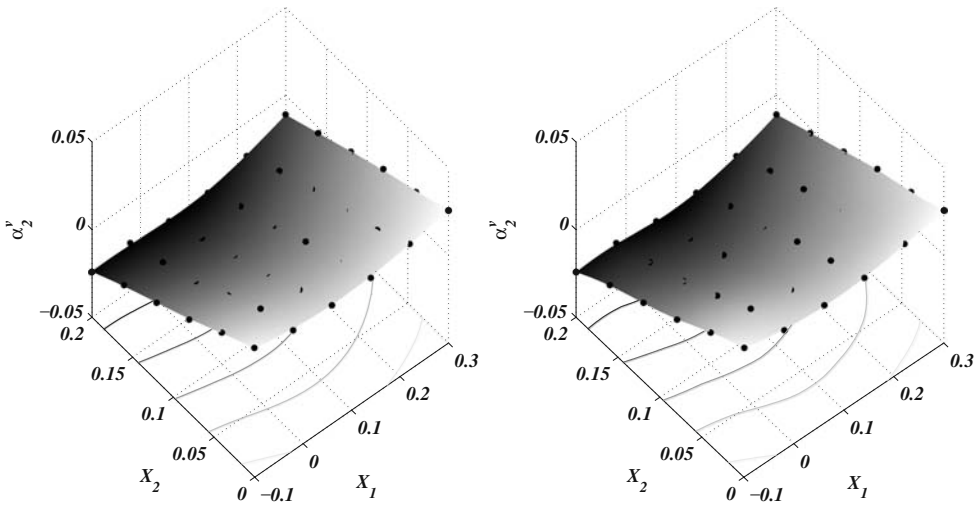

Fig. 16 Surrogate model (by *MLS approximation*) of the POD coefficient α_2^v (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

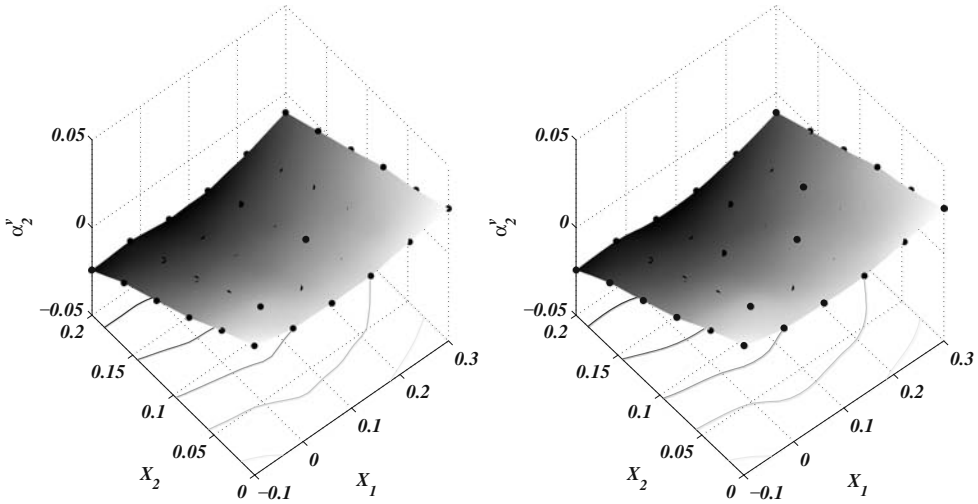

Fig. 17 Surrogate model (by *MLS interpolation*) of the POD coefficient α_2^v (*left*: 1st-order polynomial; *right*: 2nd-order polynomial)

displacements u and w are due to their smaller magnitude with respect to the vertical displacement v).

5.4 Level 2: building and validating the surrogate models

Once the behaviour of a coupling variable has been reduced by the POD technique, the m scalar values contain enough information to build an approximation of the whole field. Therefore, the idea proposed here is to build inexpensive surrogate surfaces of the scalar POD coefficients of the flow and the structure with respect to the design variables.

Before using the reduced model for the shape optimization of the wing design, its ability to predict accurate pressure and displacements fields should be validated. To accomplish this task, a leave-1-out procedure is applied: for each point of the design of experiments (DOE), a surrogate surface is built with the $(M-1)$ other points of the DOE (M being the number of points in the DOE = 36), and the pressure field p_{approx} obtained by the approximate model is compared with the results contained in the DOE database.

The following parameters of the approximation are investigated:

- the nature of the surrogate surfaces:
  - polynomial RSM on the POD coefficients;
  - MLS approximation on POD coefficients;
  - MLS interpolation on POD coefficients;
- the influence of the number of POD modes (1, 2 or 3) kept for the approximation;
- the degree of the polynomial bases (first-order or second-order).

As an illustration, the surrogate models for α_3^p (third POD coefficient for the pressure) and α_2^v (second POD coefficient for the y-displacement) are depicted in Figs. 12, 13, 14, 15, 16, and 17, with respect to the design variables.

The flow calculations must furnish 2 outputs: the pressure distribution on the wing surface, which can be reconstructed by the reduced fluid model (p_{approx}), and the lift coefficient C_L. In order to estimate C_L for any wing geometry, the resulting vertical force $f_{vertical}$ due to the loading of p_{approx} on the wing is calculated as follows:

$$C_{L,approx}(X) = \frac{f_{vertical}}{\frac{1}{2}\rho v_{ref}^2 A}, \tag{24}$$

where v_{ref} is the speed of the airplane and A is the characteristic area of the wing. Over the 36 instances of the DOE, the average error committed by this approximation of C_L is equal to 0.58%.

Tables 2 and 3 collect the average errors between the values of the scalar POD coefficients and their approx-

Table 2 3D wing: errors on the approximation of the POD coefficients (for the fluid)

	Relative contribution	Approximation method					
		Polyn. basis – degree = 1			Polyn. basis—degree = 2		
		PRSM	MLS app.	MLS interp.	PRSM	MLS app.	MLS interp.
α_1^p	15.9% of p	1.54%	0.35%	0.35%	0.41%	0.33%	0.34%
α_2^p	9.0% of p	1.66%	0.71%	0.71%	0.69%	0.69%	0.68%
α_3^p	4.8% of p	2.56%	1.16%	1.16%	1.23%	1.15%	1.15%

Table 3 3D wing: errors on the approximation of the POD coefficients (for the structure)

	Relative contribution	Approximation method					
		Polyn. basis – degree = 1			Polyn. basis – degree = 2		
		PRSM	MLS app.	MLS interp.	PRSM	MLS app.	MLS interp.
β_1^u	42.9% of u	10.57%	4.18%	4.17%	3.36%	3.46%	3.47%
β_2^u	10.2% of u	13.55%	5.21%	5.19%	3.76%	3.93%	3.95%
β_3^u	4.6% of u	13.64%	5.26%	5.24%	3.83%	3.98%	4.00%
β_1^v	41.8% of v	1.75%	1.51%	1.51%	1.33%	1.84%	1.88%
β_2^v	2.8% of v	1.88%	1.56%	1.57%	1.38%	1.88%	1.94%
β_3^v	1.5% of v	2.29%	1.64%	1.65%	1.55%	1.91%	1.96%
β_1^w	55.6% of w	10.36%	4.84%	4.82%	4.64%	4.34%	4.34%
β_2^w	11.9% of w	13.28%	5.63%	5.60%	5.80%	4.70%	4.69%
β_3^w	8.8% of w	14.43%	6.38%	6.35%	7.04%	5.42%	5.41%

imation by the surrogate models (using the leave-1-out validation technique on the 36 designs of the DOE).

The results lead to the following remarks:

- for all approximation methods, a second order polynomial basis gives lower error values. This is logically explained by the fact that a richer basis, containing more terms, allows for a better representation of the surface. It is interesting to notice that the MLS is a general approximation method which enables the use of bases other than polynomial (e.g. sinusoidal, spline), depending on available information about the physics involved;
- the PRSM gives a good trend of the points to approximate, but cannot capture the local phenomena (as clearly illustrated in Figs. 12 to 14 for α_3^p), demonstrating the benefit of using MLS approximation in a general case;
- as with other interpolating meta-models like the kriging, the MLS interpolation, by forcing the surrogate surface to pass through all sample points, tends to degrade the smoothness of the surface, in comparison with a MLS approximation;
- the largest errors concern displacements u and w, whose scale of magnitudes is much lower than the (vertical) displacement v;
- for all responses, the lowest error values are observed for the first mode, and slightly increase with the mode rank. This is due to the fact that higher rank modes bring a smaller contribution to the field, and are more polluted by numerical noise. Practically, the response surfaces of the corresponding POD coefficients are more irregular, hence also more difficult to approximate. However, since the relative contribution of the successive modes rapidly decreases, the larger errors obtained for those modes cause no harm to the approximation.

From these remarks, the combination of MLS approximation (with second order polynomial basis) and POD reduction is advised. The major advantage of this approach is to supply full surrogate models for

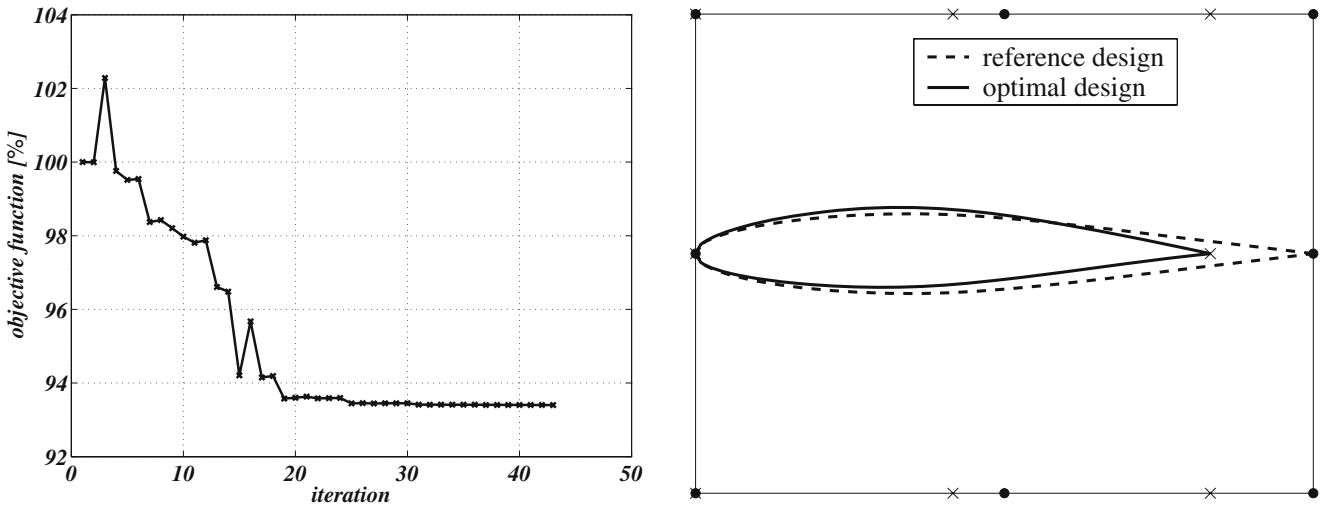

Fig. 18 Multidisciplinary optimization with reduced model of the wing design: evolution of the objective function (mass) with respect to the iterations (in % of the mass of the reference wing) [*left*] and optimal design [*right*]

Table 4 3D wing: comparison between MDO strategies (without and with surrogate models)

Computational time	CFD/FEM models	Surrogate models
1 fluid computation	1 h 20 min	< 1 s
1 structure computation	8 min	< 1 s
1 optimizer iteration	4 h 24 min (1 MDA)	3 s
Time for DOE	—	6 days 14 h (36 MDA)
Number of optimizer iterations	50 (stopped)	43
Total computational time	9 days 3 h	6 days 19 h (DOE + 1 final MDA)

the fluid and the structure, allowing the calculation of the pressures, displacements, weight, lift and von Mises stress at a very low cost.

Furthermore, the 36 snapshots of numerical FVM or FEM simulations (requiring a multidisciplinary analysis of the coupled models) can be run independently, which makes the DOE calculations ideally suited for parallel computing.

After this validation procedure, the final step consists in optimizing the wing design by taking benefit of the fluid model reduction procedure introduced hereby.

5.5 Shape optimization of the 3D wing with reduced models

As mentioned in Section 4.1, the shape design variables are optimized in order to minimize the weight of the structure.

It has been demonstrated on a 2D wing application that a direct approximation of the responses C_L and $\sigma_{vonMises,max}$ (i.e. not derived from the pressures and displacements) is generally not reliable enough to represent the complexity of the problem: a preliminary reduced model of the coupling variables of interest is mandatory (Filomeno Coelho et al. 2008). Therefore, in this study, the constraints (related to the lift coefficient and von Mises maximum stress) are post-processed values directly derived from the pressures and displacements fields calculated by the bi-level reduction strategy.

To optimize the 3D wing shape, the COBYLA (Constrained Optimization BY Linear Approximation) algorithm available in SciPy has been used, since it is derivative-free and well adapted to problems with a low number of variables (< 10) and the presence of nonlinear constraints (Powell 1994). The reference wing (Section 4.2) is used as the starting point; the evolution of the objective function (the weight) with respect to the optimizer iterations as well as the optimal design are depicted in Fig. 18. The surrogate models are based on a POD decomposition with 3 modes for each variable, and the response surfaces are based on MLS approximation (with a second order polynomial basis).

The solution found by this method ($X^{(opt)} = [-0.1775 - 0.0048]$) decreases the weight by 6.6%, and a verification of the lift and von Mises stress with the accurate numerical solvers confirm that the solution is feasible (i.e. all constraints are satisfied). Moreover, the fact that no Finite Volume computation has to be performed accelerates considerably the CPU time of the optimization process in comparison with a standard approach requiring the execution of the accurate numerical codes. Finally, no coupling of codes is necessary during the optimization.

The comparison of MDO strategies with accurate (CFD, FEM) models and surrogate models is summarized in Table 4. For the optimization performed with the full scale models, the process is stopped after 50 iterations (the algorithm has not converged yet and the weight reduction at the 50th iteration is 4.3%, to be compared with the 6.6% found with the surrogate models). The results were obtained on a single PC (1 CPU at 1 GHz frequency); the benefit of using surrogate models in terms of overall computational time is clearly demonstrated.

6 Conclusions

In this paper, model reduction strategies for coupled systems have been discussed. In particular, in the context of multidisciplinary analysis and optimization, the challenges of decreasing the interconnections between models and the computational time need to be addressed.

Therefore, the original contribution of this work is to propose a bi-level model reduction technique: first, the coupling variables are reduced by means of the Proper Orthogonal Decomposition (POD), allowing to express any large vector as a linear combination of a few modes. Then, surrogate models are carried out by approximating the scalar coefficients of the POD linear expansion (by the Polynomial Response Surface or the Moving Least Squares methods). Applied to the shape optimization of a 3D flexible wing, the benefit of the reduction methodology has been demonstrated to eliminate the need to couple the codes while optimiz-

ing. Moreover, because of its inherent handling of the local effects, the MLS approximation reveals itself as a reliable and flexible technique to model various natures of response surfaces.

However, it must be emphasized that the bi-level reduction method has been tested for a small number of variables (here: $n_{var} = 2$). To be efficient with more design variables, both following characteristics must be encountered:

- *construction of the design of experiments*: with more variables, a full-factorial design of experiments (DOE) is no longer permitted. Therefore, other techniques, as the Latin Hypercube Sampling for instance, must be applied to obtain a representative set of wings;
- *efficiency of the POD decomposition*: the solutions (e.g.: pressures, displacements) found for all designs of the DOE still have to be approximated accurately for a limited number of POD modes;
- *accuracy of the surrogate surfaces of the POD coefficients*: the response surfaces of the POD coefficients must have minimal properties of smoothness and regularity in order to be represented accurately by a surrogate model.

From the promising results obtained both for the 2D airfoil (Filomeno Coelho et al. 2008) and the 3D wing, the next research axes will focus on the following topics:

- *improvement of the POD*: the quality of the POD depends on the snapshots of the database (obtained by a design of experiments). To enrich the POD basis, specific strategies could be developed, for instance by adding snapshots resulting from non-converged multidisciplinary analyses (e.g. computed during the optimization process);
- *enrichment of the MLS approximation*: according to each specific problem, the MLS parameters can be adapted, by incorporating new terms in the basis (to allow for a better representation of the physical variable), or by forcing the MLS approximation to satisfy additional constraints;
- *combination with multi-level MDO strategies*: several multi-level MDO strategies have been proposed in the literature (Collaborative Optimization, Concurrent subspace optimization, BLISS, MORDACE, DIVE, ...). All multi-level strategies try to separate as most as possible the handling of the different disciplines, and local optimization tasks are also performed at the discipline level. Therefore, the reduced models proposed in this paper could be adapted and tested with the existing multi-level methods, in sequential and parallel environments.

Acknowledgements This work has been supported by the National Agency of French Research/National Network of Software Technologies (ANR/RNTL), in the framework of the *OMD* project.

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