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RESEARCH PAPER

Multi-objective optimization by genetic algorithm of structural systems subject to random vibrations

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Abstract This paper deals with a multi-objective optimization criterion for linear viscous-elastic device utilised for decreasing vibrations induced in mechanical and structural systems by random loads. The proposed criterion for the optimum design is the minimization of a vector objective function. The multi-objective optimization is carried out by means of a stochastic approach. The design variables are the device frequency and the damping ratio. As cases of study, two different problems are analysed: the base isolation of a rigid mass and the tuned mass damper positioned on a multi degree of freedom structural system subject to a base acceleration. The non-dominated sorting genetic algorithm in its second version (NSGA-II) is adopted to obtain the Pareto sets and the corresponding optima for different characterizations of the system and input.

Keywords Random vibration · Multi-objective stochastic optimization · Base isolator · Tuned mass damper · Genetic algorithm

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1 Introduction

A wide class of engineering problems deal with structural systems subject to dynamic actions, which can be suitably modelled only by using random processes (i.e., earthquakes, wind pressure, sea waves and rotating machinery induced vibrations). As a direct consequence, responses of systems are also random processes. In these environments, random dynamic analysis seems to be the most suitable method to get practical information concerning a system's response and reliability (see Lutes and Sarkani 2001 for example). It follows that structural optimization methods seem to be practically approached by means of random vibration theory. Concerning this problem, some recent works have been based on a *standard optimization problem* (*SOP*). The approach finds the optimum solution that coincides with the minimum or the maximum value of a scalar *objective function* (OF). The first problem definition of structural optimization was proposed by Nigam (1972), in which constraints were defined by using probabilistic indices of the structural response and the OF was defined by the structural weight. This led to a standard nonlinear constrained problem.

In the field of seismic engineering, the use of a stochastic defined OF was proposed for the optimum design of the damping value of a vibration control device placed on the first story of a building (Constantinou and Tadjbakhsh 1983), and was defined by the maximum displacement under a white noise excitation. A specific and more complete stochastic approach was also proposed by Takewaki (2000), aimed to stiffness-damping simultaneous optimization of structural systems. In this mentioned work the sum of system response mean squares due to a stationary random excitation was minimized under constraints on total stiffness capacity and total damping capacity.

More recently, an interesting stochastic approach for optimum design of damping devices in seismic protection was proposed by Park et al. (2004) that aimed to minimize the total building life-cycle cost. It was based on a stochastic dynamic approach for failure probability evaluation, and the OF was defined in a deterministic way. The optimization problem was formulated by adopting the location and the number of the viscous elastic dampers as design variables. The constraint was the failure probability associated to the crossing of the maximum inter-storey drift over a given allowable value. Reliability analysis was developed using the first crossing theory in stationary condition.

Other interesting works in the field of stochastic structural optimization regard the unconstrained optimization of single (Rundinger 2006) and multiple (Hoang and Warnitchai 2005) tuned mass dampers. The structural displacement covariance of the protected system is used as OF and the input is modelled by means of a stationary white noise process.

However, the *SOP* does not usually hold correctly many real structural problems, where often different and conflicting objectives may exist. In these situations, the *SOP* is utilized by selecting a single objective and then incorporating the other objectives as constraints. The main disadvantage of this approach is that it limits the choices available to the designer. This makes the optimization process a difficult task.

Instead of unique *SOP* solution with a single given constraint, a set of alternative solutions can be usually achieved, known as *Pareto optimum solutions*. These represent the *best solutions* in a wide sense that means they are superior to other solutions in the *search space*, when all objectives are considered. If any other information about the choice or preference is given, no one of the corresponding *trade-offs* can be said to be better than the others.

Many works in last decade were done by different authors in the field of multi-objective structural optimization for systems subject to static or dynamic loads (Papadrakakis et al. 2002).

This paper deals with a multi-objective optimization of linear visco-elastic devices, which are introduced in structural and mechanical systems to reduce vibration level induced by random actions applied at the support. Two different problems are considered: first, the vibration base isolation of a rigid mass subject to an acceleration of the support; second, the *tuned mass damper* (*TMD*) positioned on a multi-degree of freedom (MDoF) structural system subject to a base acceleration.

The first case concerns the problem of a vibration absorber for a rigid element isolated from a vibrating support subject to a random acceleration process. This is a typical application in many real problems in mechanical, civil and aeronautics engineering. The main system is a rigid mass linked with the support by means of a linear visco-elastic element (Fig. 1). In the multi-objective optimization, the OF is a vector that contains two elements: the first one is an index of device performance focused on reducing the vibration level, and is expressed by an acceleration reduction factor. This is assumed to be, in stochastic meaning, the ratio between the mass and the support acceleration variances.

The second objective function is the displacement of the protected mass. In probabilistic terms it is the maximum displacement which is not exceeded with a given probability in a given time interval. The design variables are the isolator damping ratio ξ_s and its pulsation ω_s, which are collected in the *design vector.* The acceleration is modelled as a filtered stationary stochastic process.

The second application concerns a *TMD* positioned in a structural system with *n* degrees of freedom. The *TMD* is composed by a mass, a spring, and a damper attached to the main structure (see Fig. 3) such that it oscillates as the same frequency as the structure. Its mechanism of attenuating vibrations within a structure is to transfer the vibration energy to the *TMD* where it is dissipated. For the *TMD* problem, the multi-objective optimization concerns the minimization of two antithetic OFs: the first one is the factor of reduction of the acceleration of the protected structure; the second one is the displacement of *TMD* relative to

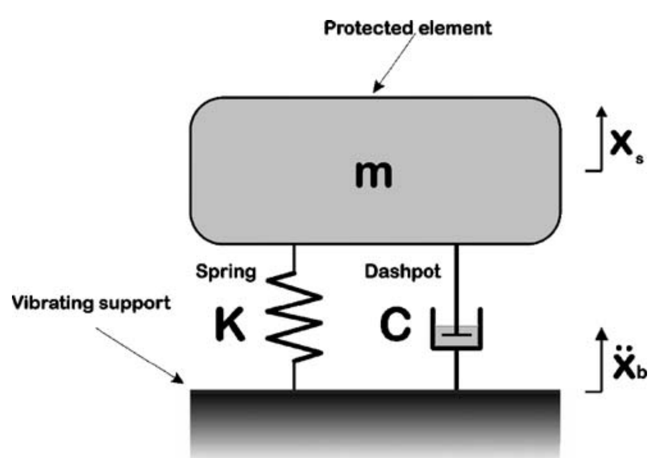

Fig. 1 Schematic model of a rigid mass isolated from a vibrating support by means of an isolation device

the top floor. The acceleration acting at the support is modelled as a filtered stationary stochastic process.

To obtain the Pareto set in the two dimensional space of the OFs and the optimum solution in the space of design variables, a specific genetic algorithm approach—the NSGA-II—is adopted. A sensitivity analysis on the optimum solution is then performed under different environmental conditions.

The novelty of the proposed method is in using a multi-dimensional criterion for the design. Nowadays, this is a very important issue in modern Technical Codes (SEAOC 1995), where several performance requirements are fixed and often conflicting. In these situations, the designer must select the design variables which make available all objectives.

The validation of the proposed method is demonstrated through two applications, in which several parameters are changed. Therefore, results attained by the proposed method can be utilised to support designers in the definition of structural vibration control strategies.

2 Multi-objective stochastic optimization of random vibrating systems

The proposed stochastic multi-objective optimization criterion is adopted in this study to define the optimum mechanical parameters in classical problems of vibration control. Two applications are investigated which consider the limitation of vibration effects in mechanical and structural systems subject to base acceleration.

The optimization problem is formulated as the search of design parameters, collected in the design vector $\mathbf{b}$ defined in the admissible domain $\mathbf{\Omega_b}$, which are able to minimize a given OF. This problem can be formulated in a standard deterministic way or in a stochastic one (using response spectral moments). This approach has limits. In fact, when designer looks for the optimum solution a choice has to be made concerning the most suitable criterion for measuring performance. It is evident that many different quantities, which have a direct influence on the performance, can be considered as efficient criteria. At the same time those quantities that must satisfy some imposed requirements and cannot be assumed as criteria, are used as constraints. Therefore, it is common in optimization problems to use a single OF subjected to some probabilistic constraints, as in the first stochastic optimization problem (Nigam 1972). Usually, inequality constraints on system failure probability are utilised.

In the multi-objective formulation, the conflict which may or may not exist between the different criteria is an essential point. Only those quantities which are competing should be considered as independent criteria. The others can be combined into a single criterion, which represents the whole group.

3 First case of study: protection of a rigid mass from a vibrating support

This is the case of the isolation of a rigid mass positioned on a vibrating support. In engineering applications the mass can represent a subsystem located on a vibrating mechanical support, as motor device, airplane structure, seismic isolated building, or similar. In all these situations, the main goal is to limit the induced acceleration and to control the displacement of the rigid mass with respect to the support. The first objective is related to excessive inertial forces transmitted, for example, to electronic or mechanical devices sensitive to this effect. The second objective is related to an excessive displacement of the protected mass, which can become unacceptable. For example, this happens if the system is located close to other elements, or if the vibration isolator has a limited acceptable lateral deformation over which it collapses.

The protected element is modelled as a rigid body with a mass m. The isolator device is modelled as a simple viscous-elastic element, which connects the vibrating base with the supported mass (Fig. 1).

The stiffness k and the damping c of the isolator device are optimized to minimize the vibration of the rigid mass m.

The base acceleration is a stochastic coloured process $\ddot{X}_b(t)$, modelled by means of a second order linear filter (Tajimi 1960):

$$\ddot{X}_b(t) = \ddot{X}_\mathrm{f}(t) + w(t) = -\left(2\xi_\mathrm{f}\omega_\mathrm{f}\dot{X}_\mathrm{f} + \omega_\mathrm{f}^2 X_\mathrm{f}\right) \tag{1}$$

where $w(t)$ is a stationary Gaussian zero mean white noise process, ω_f is the filter pulsation and ξ_f is the filter damping ratio. The equations of the motion of this combined system are:

$$\ddot{X}_\mathrm{s}(t) + 2\xi_\mathrm{s}\omega_\mathrm{s}\dot{X}_\mathrm{s} + \omega_\mathrm{s}^2 X_\mathrm{s} = -\ddot{X}_b \tag{2}$$

$$\ddot{X}_\mathrm{f}(t) + 2\xi_\mathrm{f}\omega_\mathrm{f}\dot{X}_\mathrm{f} + \omega_\mathrm{f}^2 X_\mathrm{f} = -w(t) \tag{3}$$

$$\ddot{X}_b(t) = \ddot{X}_\mathrm{f} + w(t). \tag{4}$$

In the space (2)–(4) can be written as

$$\dot{\mathbf{Z}} = \mathbf{AZ} + \mathbf{F} \tag{5}$$

where the space state vector is

$$\mathbf{Z} = \left(X_\mathrm{s}\ X_\mathrm{f}\ \dot{X}_\mathrm{s}\ \dot{X}_\mathrm{f}\right)^T. \tag{6}$$

The system matrix is then

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_s^2 & \omega_f^2 & -2\xi_s\omega_s & 2\xi_f\omega_f \\ 0 & -\omega_f^2 & 0 & -2\xi_f\omega_f \end{pmatrix}, \tag{7}$$

where

$$\omega_s = \sqrt{\frac{k}{m}}; \; \xi_s = \frac{c}{2\sqrt{km}} \tag{8}$$

Finally, the input vector is

$$\mathbf{F} = -\left(0 \quad 0 \quad 0 \quad w(t)\right)^T. \tag{9}$$

The space state covariance matrix $\mathbf{R_{ZZ}} = \langle \mathbf{ZZ}^T \rangle$ is obtained by solving the *Lyapunov* equation:

$$\mathbf{AR_{ZZ}} + \mathbf{R_{ZZ}A}^T + \mathbf{B} = 0. \tag{10}$$

The variance $\sigma^2_{\ddot{y}_S}$ of the absolute mass acceleration $\ddot{y}_S = \ddot{x}_S + \ddot{x}_b$ is

$$\sigma^2_{\ddot{y}_S} = \mathbf{D}^T\mathbf{R_{ZZ}D} \tag{11}$$

where

$$\mathbf{D} = \left(-\omega_s^2 \; 0 \; -2\xi_s\omega_s \; 0\right)^T. \tag{12}$$

4 Formulation of multi-objective optimization of mechanical characteristics of the device

The multi-objective stochastic optimization problem concerns the evaluation of the design vector $\mathbf{b} = (\omega_s, \xi_s)$ able to satisfy the reduction of the transmitted inertial acceleration of the rigid mass and to limit its displacement with respect to the support. These two criteria conflict each other because as the support rigidity grows, at that time the acceleration reduction factor (i.e., the performance of the device) and the lateral displacement decrease. This situation corresponds, for example, to the design of a well known vibration control device utilized in the field of seismic engineering: the base isolator. The decoupling between the vibrating support and the protected element increases monotonically with the reduction of device stiffness. At the same time, the device displacement increases. Therefore the level of reduction of transmitted acceleration in the protected element is related to the allowable maximum value of displacement. These two conflicting criteria must be considered in the design.

The multi-objective optimization problem is

$$\min\left\{\mathrm{OF}_1, \mathrm{OF}_2\right\}. \tag{13}$$

The first objective function is

$$\mathrm{OF}_1(\mathbf{b}) = \left(\frac{\sigma_{\ddot{y}_S}(\mathbf{b})}{\sigma_{\ddot{x}_b}}\right), \tag{14}$$

where the base vibrating acceleration variance is (see ref. Crandal and Mark 1963)

$$\sigma^2_{\ddot{x}_b} = \frac{\pi}{2}\frac{S_0\omega_f}{\xi_f}\left(1 + 4\xi_f^2\right), \tag{15}$$

and S_0 is the power spectral density function of the white noise process.

This OF is a direct protection efficiency index: it tends to a null value for a totally system-base decoupling and tends to unity for a system rigidly connected with the vibrating base.

To make explicit the OF_2 the maximum displacement value $X_s^{\max}$ that is not exceeded with a given probability $\tilde{P}_f$ in an assigned time interval is adopted. Therefore

$$\begin{aligned} \mathrm{OF}_2(\mathbf{b}) &= X_S^{\max}(\mathbf{b}) : P_f\left(X_S^{\max}, \mathbf{b}\right) \\ &= P\left\{|X_S| \geq X_S^{\max}(\mathbf{b}) \,|t \in [0, T]\right\} \leq \tilde{P}_f. \end{aligned} \tag{16}$$

where T is the duration of vibration.

In case of rare failure events the Poisson hypothesis could be reasonably utilised and so (Lutes and Sarkani 2001)

$$P_f\left(X_S^{\max}, \mathbf{b}\right) = 1 - e^{-\nu\left(X_S^{\max}, \mathbf{b}\right)T}, \tag{17}$$

where the unconditioned mean crossing rate is

$$\nu = \left(X_S^{\max}, \mathbf{b}\right) = \frac{1}{\pi}\frac{\sigma_{\dot{X}_S}}{\sigma_{X_S}} e^{-\left\{\frac{1}{2}\left(\frac{X_S^{\max}}{\sigma_{X_S}}\right)^2\right\}}. \tag{18}$$

Finally one obtains

$$\mathrm{OF}_2(\mathbf{b}) = X_S^{\max} = \sqrt{-2\sigma^2_{X_S}\ln\left(-\frac{\pi}{T}\frac{\sigma_{X_S}}{\sigma_{\dot{X}_S}}\ln\left(1 - \tilde{P}_f\right)\right)}. \tag{19}$$

The two objective functions are plotted in Fig. 2 in the space ω_s/ω_f and ξ_s. The data assumed are: $\omega_f = 21\ \mathrm{rad/s}$, $\xi_f = 0.4$, $S_0 = 100\ \mathrm{cm^2/s^3}$, and $\tilde{P}_f = 10^{-2}$.

Figure 2 illustrates that the two objectives conflict each other. In fact, as the frequency ω_s increases the efficiency of the vibration control in reducing the transmitted acceleration (OF_1) reduces too. At the same time, one observes a reduction of the displacement (OF_2).

Concerning the variability of the two objectives with respect the damping ratio, from Fig. 2 it is possible to notice that OF_2 always diminishes. On the contrary, the

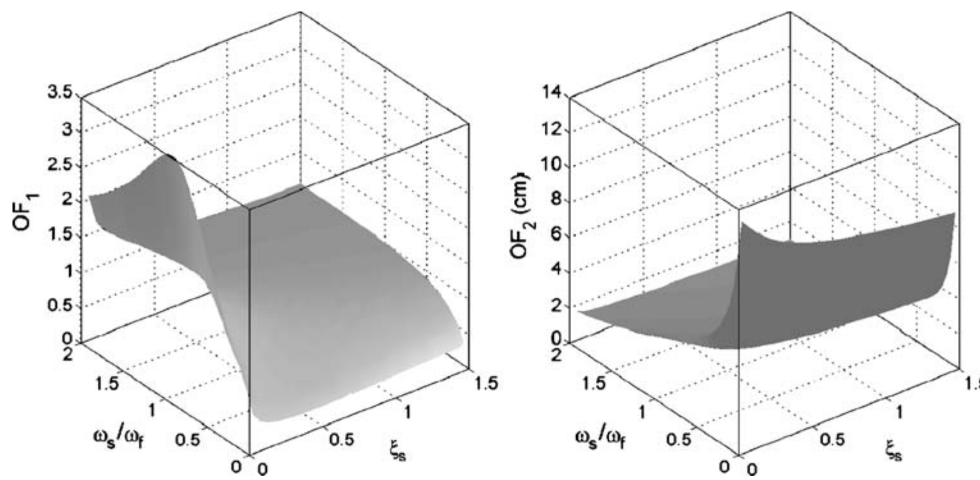

Fig. 2 Conflicting aspect of the two proposed objective functions

variability of OF_1 is different and depends on the value assumed by the system frequency.

5 The second case of study: the tuned mass damper problem

TMD is modelled as a mass-dashpot-spring system (the secondary system) attached at the top of a linear MDoF system (Fig. 3). As in the previous case of study, the excitation at the base is represented by a stationary filtered stochastic process.

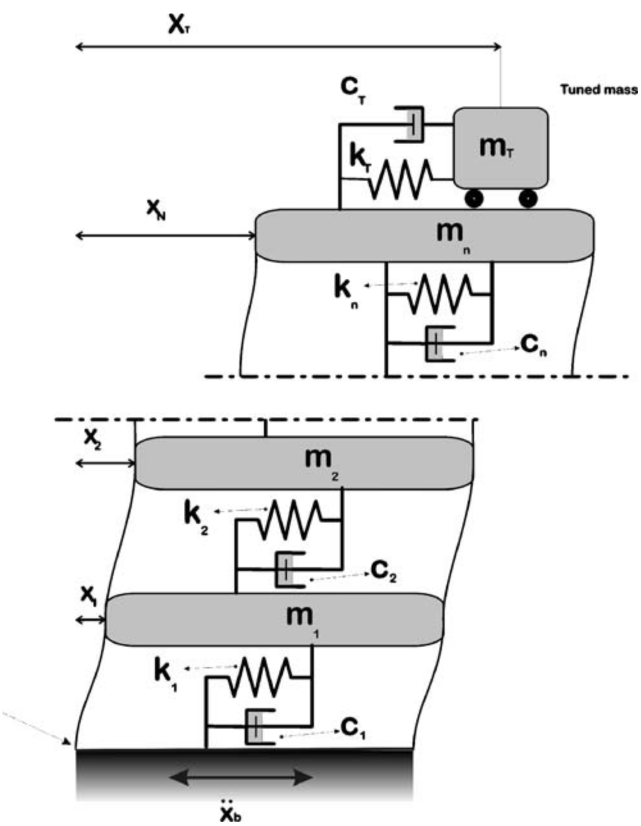

Fig. 3 Schematic model of a MDoF structural system equipped with a TMD

For the system in Fig. 3 the equations of the motion is

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = -\mathbf{Mr}\,\ddot{X}_b, \tag{20}$$

where **M**, **C** and **K** are the deterministic mass, damping and stiffness matrices. The vectors $\mathbf{X} = (x_1\ x_2 \ldots.. x_n\ x_T)^T$, $\dot{\mathbf{X}} = (\dot{x}_1\ \dot{x}_2 \ldots.. \dot{x}_n\ \dot{x}_T)^T$ and $\ddot{\mathbf{X}} = (\ddot{x}_1\ \ddot{x}_2 \ldots.. \ddot{x}_n\ \ddot{x}_T)^T$ collect the displacements, velocities and accelerations of n floors and of the *TMD* relative to the ground. Finally $r = (1 \ldots 1)^T$.

The mechanical characteristics of the *TMD* are described by parameters m_T, k_T, and c_T.These are, respectively, the mass, stiffness and damping of the *TMD*.

By adding the equation of the motion of the filter in (20) one obtains

$$\ddot{\mathbf{X}}(t) = -\left[\mathbf{M}^{-1}\mathbf{C}\right]\dot{\mathbf{X}} - \left[\mathbf{M}^{-1}\mathrm{K}\right]\mathbf{X} + \mathbf{r}\left(2\xi_f\omega_f X_f + \omega_f^2 X_f\right). \tag{21}$$

Introducing the space state vector

$$\mathbf{Z} = \left(\mathbf{X}\ \ X_f\ \dot{\mathbf{X}}\ \ \dot{X}_f\right)^T, \tag{22}$$

in the state space, (21) becomes

$$\dot{\mathbf{Z}} = \mathbf{AZ} + \mathbf{F}, \tag{23}$$

where the system matrix **A** is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}^{(n+2)(n+2)} & \mathbf{I}^{(n+2)(n+2)} \\ -\mathbf{H}_K & -\mathbf{H}_C \end{bmatrix}. \tag{24}$$

The two sub-matrices $\mathbf{H}_K$ and $\mathbf{H}_C$ are

$$\mathbf{H}_K = \left[\begin{array}{ccc:c} & & & \omega_f^2 \\ & (\mathbf{M}^{-1}\mathbf{K})^{(n+1)(n+1)} & & \omega_f^2 \\ & & & \cdots \\ & & & \omega_f^2 \\ \hdashline 0 & \ldots & 0 & -\omega_f^2 \end{array}\right], \tag{25}$$

$$\mathbf{H}_C = \left[\begin{array}{ccc:c} & & & 2\xi_f\omega_f \\ & (\mathbf{M}^{-1}\mathbf{C})^{(n+1)(n+1)} & & 2\xi_f\omega_f \\ & & & \cdots \\ & & & 2\xi_f\omega_f \\ \hdashline 0 & \ldots & 0 & -2\xi_f\omega_f \end{array}\right], \tag{26}$$

respectively.

Sub-matrices in (25) and (26) are

$$\mathbf{M}^{-1}\mathbf{K} = \begin{pmatrix} \frac{k_1}{m_1}+\frac{k_2}{m_1} & -\frac{k_2}{m_1} & 0 & & & \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2}+\frac{k_3}{m_2} & -\frac{k_3}{m_2} & & & \\ 0 & -\frac{k_3}{m_3} & \ddots & \ddots & & \\ & & \ddots & \ddots & & \\ & & & -\frac{k_n}{m_n} & \frac{k_n}{m_n}+\frac{k_T}{m_n} & -\frac{k_T}{m_n} \\ & & & & -\frac{k_T}{m_T} & \frac{k_T}{m_T} \end{pmatrix}; \tag{27}$$

and

$$\mathbf{M}^{-1}\mathbf{C} = \begin{pmatrix} \frac{c_1}{m_1}+\frac{c_2}{m_1} & -\frac{c_2}{m_1} & 0 & & & \\ -\frac{c_2}{m_2} & \frac{c_2}{m_2}+\frac{c_3}{m_2} & -\frac{c_3}{m_2} & & & \\ 0 & -\frac{c_3}{m_3} & \ddots & \ddots & & \\ & & \ddots & \ddots & & \\ & & & -\frac{c_n}{m_n} & \frac{c_n}{m_n}+\frac{c_T}{m_n} & -\frac{c_T}{m_n} \\ & & & & -\frac{c_T}{m_T} & \frac{c_T}{m_T} \end{pmatrix}. \tag{28}$$

The *Liapunov* equation, whose solution supplies the response covariance of the system, has the same form of (10). The covariance matrix $\mathbf{R}_{\ddot{\mathbf{Y}}\ddot{\mathbf{Y}}}$ of inertial accelerations is obtained by means of following relation

$$\mathbf{R}_{\ddot{\mathbf{Y}}\ddot{\mathbf{Y}}} = \mathbf{D}\mathbf{R}_{\mathbf{ZZ}}\mathbf{D}^T, \tag{29}$$

where $\mathbf{D} = \left[\mathbf{H_K}\ \mathbf{H_C}\right]$.

6 Formulation of multi-objective optimization of TMD

The multi-objective optimization for the *TMD* problem is formulated as the evaluation of design vector $\mathbf{b} = (\omega_T, \xi_T)$, where $\omega_T = \sqrt{k_T/m_T}$ and $\xi_T = c_T/2\omega_T m_T$. These variables must be able to satisfy both the reduction of the transmitted inertial acceleration in the main structure and the relative displacement of the *TMD* with respect to the *n*th floor. The *TMD* mass ratio (i.e., the ratio of the added mass to the mass of the top floor) is assigned. With reference to this assumption, recent works (Lin et al. 2001) showed that when this parameter is considered as a design variable, the optimum values attained are very high if compared with the range of usual structural damping. Therefore, they are incompatible with real applications, due to economic consideration.

In contrast to the previous example, in the *TMD* problem the acceleration reduction factor of the main structure reaches a minimum value, but to this one a maximum value of *TMD* displacement respect to the top floor corresponds. This maximum displacement can become unacceptable in real engineering applications.

The multi-objective optimization problem is

$$\min\{\mathrm{OF}_1, \mathrm{OF}_2\}. \tag{30}$$

The first objective is to minimize the ratio between the maximum value of standard deviation $\sigma_{\ddot{y}_i}$ of inertial acceleration of the protected structure and the same quantity $\sigma^0_{\ddot{y}_i}$ of the unprotected structure, therefore:

$$\mathrm{OF}_1(\mathbf{b}) = \max\left(\frac{\sigma_{\ddot{y}_i}(\mathbf{b})}{\sigma^0_{\ddot{y}_i}}\right). \tag{31}$$

This parameter is a direct measure of the performance of the *TMD* in reducing structural vibrations.

The second objective is to minimize the failure probability of the *TMD,* which is associated to the crossing of the tuned-top floor displacement $\Delta_T = x_T - x_n$ over a value $\Delta_T^{\max}$ during the duration T of the vibration. Therefore

$$\begin{aligned} \mathrm{OF}_2(\mathbf{b}) &= \Delta_T^{\max}(\mathbf{b}) : P_f\left(\Delta_T^{\max}, \mathbf{b}\right) \\ &= P\left\{|\Delta_T| \geq \Delta_T^{\max}(\mathbf{b})\,|t \in [0, T]\right\} \leq \tilde{P}_f. \end{aligned} \tag{32}$$

7 An overview on methods for multi-objective optimization using genetic algorithm

Many real engineering problems often involve several OFs, each other in conflict. In these situations it is not possible to define a universally approved criterion of “optimum” as in single objective optimization. Instead of aiming to find a single solution, one can try to produce a set of good compromises. In a typical minimization-based *MOOP* (multi-objectives optimization), given two candidate solutions $\{\mathbf{b}_j, \mathbf{b}_k\}$, if

$$\forall i \in \{1, ..., M\}, \mathrm{OF}_i\left(\mathbf{b}_j\right) \leq \mathrm{OF}_i\left(\mathbf{b}_k\right) \wedge \exists i \in \{1, ..., M\} : \mathrm{OF}_i\left(\mathbf{b}_j\right) < \mathrm{OF}_i\left(\mathbf{b}_k\right), \tag{33}$$

and defined the two objective vectors

$$\mathbf{v}\left(\mathbf{b}_j\right) = \left\{\mathrm{OF}_1\left(\mathbf{b}_j\right), ..., \mathrm{OF}_M\left(\mathbf{b}_j\right)\right\}, \tag{34}$$

$$\mathbf{v}\left(\mathbf{b}_k\right) = \left\{\mathrm{OF}_1\left(\mathbf{b}_k\right), ..., \mathrm{OF}_M\left(\mathbf{b}_k\right)\right\}, \tag{35}$$

the vector $\mathbf{v}(\mathbf{b}_j)$ is said to dominate vector $\mathbf{v}(\mathbf{b}_k)$ (denoted by $\mathbf{v}\left(\mathbf{b}_j\right) \prec \mathbf{v}\left(\mathbf{b}_k\right)$).

Moreover, if no feasible solution $\mathbf{v}(\mathbf{b}_k)$ exists that dominates solution $\mathbf{v}(\mathbf{b}_j)$, then $\mathbf{v}(\mathbf{b}_j)$ is classified as a *non-dominated* or *Pareto optimal solution*. The collection of all Pareto optimal solutions are known as the *Pareto optimal set* or *Pareto efficient set*, instead, the corresponding objective vectors are described as the *Pareto front* or *Trade-off surface*. Unfortunately, the Pareto optimum concept typically does not give a single solution, but a set of possible solutions that cannot be used directly to find the final design solution by an analytic way. On the contrary, usually the decision about the "best solution" to be adopted is formulated by so-called (human) *decision maker* (*DM*). On the other hand, several *preference-based methods* exist in literature. A more general classification of the *preference-based method* is considered when the preference information is used to influence the search (Coello Coello 2000). Thus in *a priori methods, DM's* preferences are incorporated before the search begins. Therefore, based on the *DM's* preferences, it is possible to avoid producing the whole *Pareto optimal set*. In *progressive methods* the *DM's* preferences are incorporated during the search: this scheme offers the advantage of driving the search process, but the *DM* may be unsure of his/her preferences at the beginning of the procedure and may be informed and influenced by information that becomes available during the search. A last class of methods is *a posteriori*: in this case the optimiser carries out the *Pareto optimal set* and the *DM* chooses a solution ("searches first and decides later"). Many researchers view this last category as standard, so that in the greater part of the circumstances a *MOOP* is considered resolved once that all *Pareto optimal solutions* are recognized.

In the category of *a posteriori approaches*, several methods are available in literature to treat multi-objective optimization problems using conventional single objective algorithms. The so-called ε-constraint method, due to its simplicity, is one of the most used techniques. This method is based on minimizing a single objective function and considering the other objectives as constraints bound by some admissible levels ε. Another way to treat multi-objectives optimization by a standard SOP is by weighting the different OFs by normalized coefficients.

Moreover, it has been stated that this algorithm may find weakly non-dominated solutions, so that the more common way to approach *MOOP* is by using different Evolutionary Algorithms (*EA*). In Luh and Chuen (2004) an algorithm for finding constrained Pareto-optimal solutions based on the characteristics of a biological immune system (Constrained Multi-Objective Immune Algorithm, *CMOIA*) is proposed. Other adopted algorithms are the Multiple Objective Genetic Algorithm (*MOGA*) (Fonseca and Fleming 1993) and the Non-Dominated Sorting in Genetic Algorithm (*NSGA*) (Srinivas and Deb 1994). In this work the *NSGA-II* (Deb et al. 2000) is adopted in order to obtain the Pareto sets and the corresponding optimum *DV* values for different systems and input configurations. Particularly, the *Real Coded GA* (Raghuwanshi and Kakde 2004), *Binary Tournament Selection* (Blickle and Thiele 1995), *Simulated Binary Crossover* (SBX) (Deb and Agrawal 1995) and *polynomial mutation* (Raghuwanshi and Kakde 2004) are used (see Appendix: Genetic operators adopted in NSGA-II).

8 Results analysis

8.1 Multi-objective optimization of isolator mechanical characteristics

In this section the results of this first optimization problem are analysed. The *NSGA-II* algorithm illustrated in Table 1 has been adopted.

It is assumed that the admissible domain for $\mathbf{b}$ is the following

$$\mathbf{\Omega_b} = \{\xi_s, \omega_s : 0.01 \leq \xi_s \leq 2.5 \vee 1\ \mathrm{rad/s} \leq \omega_s \leq 30\ \mathrm{rad/s}\}. \tag{36}$$

Input parameters are listed in Table 2.

Concerning *NSGA-II* setup, parameters reported in Table 3 have been adopted for the analysis. These have been obtained trial and error, where the choice derives from considerations about the equilibrium of computing cost and solution stability. The population size is 500, which is sufficient to obtain a continuum Pareto front. The maximum iteration number is 100

Table 1 NSGA-II algorithm for multi-objective optimization problem

Load data
For GA
Population size and maximum generation
Crossover probability
Mutation probability
For the main structure, the TMD and the filter
Fixed parameters
Initialize population
Generate random population in the specified admissible domain
Calculate OFs values
Sort the initialized population
Sort the population using non-domination-sort. For each individual, rank and crowding distance are assigned
Loop for each generation
Select the parents which are fit for reproduction
Binary tournament selection based on the rank and crowding distance
Genetic Operators on selected parents
Simulated Binary Crossover
Polynomial mutation
The offspring population is combined with parents (size of intermediate population is double)
Selection is performed to set the individuals of the next generation
Once the intermediate population is sorted, only the best individuals are selected based on its rank and crowding distance
Create a new generation
Constant population size
Close loop if stop criteria for max number of generation is verified, otherwise return on the top of loop
Report on results

and was determined after several continuous numerical experiments to obtain stable solutions.

Figure 4a, b show, respectively, the Pareto front and the space of elements of the design vector in first case of study. The optimum frequency ω_s^{opt} and the optimum damping ratio ξ_s^{opt} are shown in Fig. 4b on the x- and y-axes. respectively. The vertical line corresponds to the filter frequency ω_f. Table 4 shows some numerical data that correspond to some points on the Pareto set. The symbols in Table 4 correspond to points recognized on Pareto set in Fig. 4a.

Table 2 System parameters

Filter damping ratio ξ_f	0.6
Filter pulsation ω_f	20.94 (rad/s)
Power spectral density S_0	11000 (cm^2/s^3)
T	10^3 (s)
Max probability of failure $\widetilde{P}_f$	10^{-2}

Table 3 NSGA-II setup

Maximum generation	500
Population size	100
Crossover probability	0.9
Mutation probability	0.1

Figure 4a points out that a larger level of protection is related to an increase of allowable displacement. The figure also shows an asymptotic limit of performance, which means that the maximum reduction of transmitted acceleration is about 0.2. Moreover, some interesting observations are carried out by examining the slope of Pareto front, which is a convex curve. It is possible to distinguish three different portions of the Pareto front that correspond to different criteria in vibration control strategy. Specifically, on the left section of the Pareto front, which corresponds to a low efficiency, by means of a small increase of maximum allowable displacement one can obtain a large increase of performance (i.e. the slope is high). In the second portion of the Pareto set the slope reduces. Finally, the right part shows that an increase in performance is obtained only by means of a large increase of maximum admissible displacement. In this last situation, only small variations of optimum design variables take place (Fig. 4b). On the contrary, the reduction of maximum displacement is reached by increasing both frequency and damping. The variation is fast as the displacement reduces. Moreover, if the imposed displacement is very low, the control strategy acts by increasing the system frequency and damping ratio (the quick increase of damping is associated to energy dissipation).

Figures 5, 7 and 9 show different Pareto fronts obtained for different values of power spectral density, filter damping ratio, and filter pulsation. Figures 6, 8, and 10 show the corresponding optimum design variables. All the other parameters adopted are the same of Fig. 4a.

From Fig. 5 it is possible to notice that a variation of power spectral density induces a variation of optimum Pareto front, due to non-linearity of OF_2. It is evident that higher performance is associated with low values of S_0, but the maximum level of vibration reduction (expressed by the asymptotic value of OF_1) is about the same in all cases. However, in this situation, to higher values of S_0, larger displacements correspond. This conclusion is quite clear, because the requirement on the maximum displacement is associated to S_0 by means of a non-linear formulation, meanwhile the vibration reduction is a linear function of this parameter. Anyway, one can conclude that the strategy adopted for the optimal solution in terms of design variables is

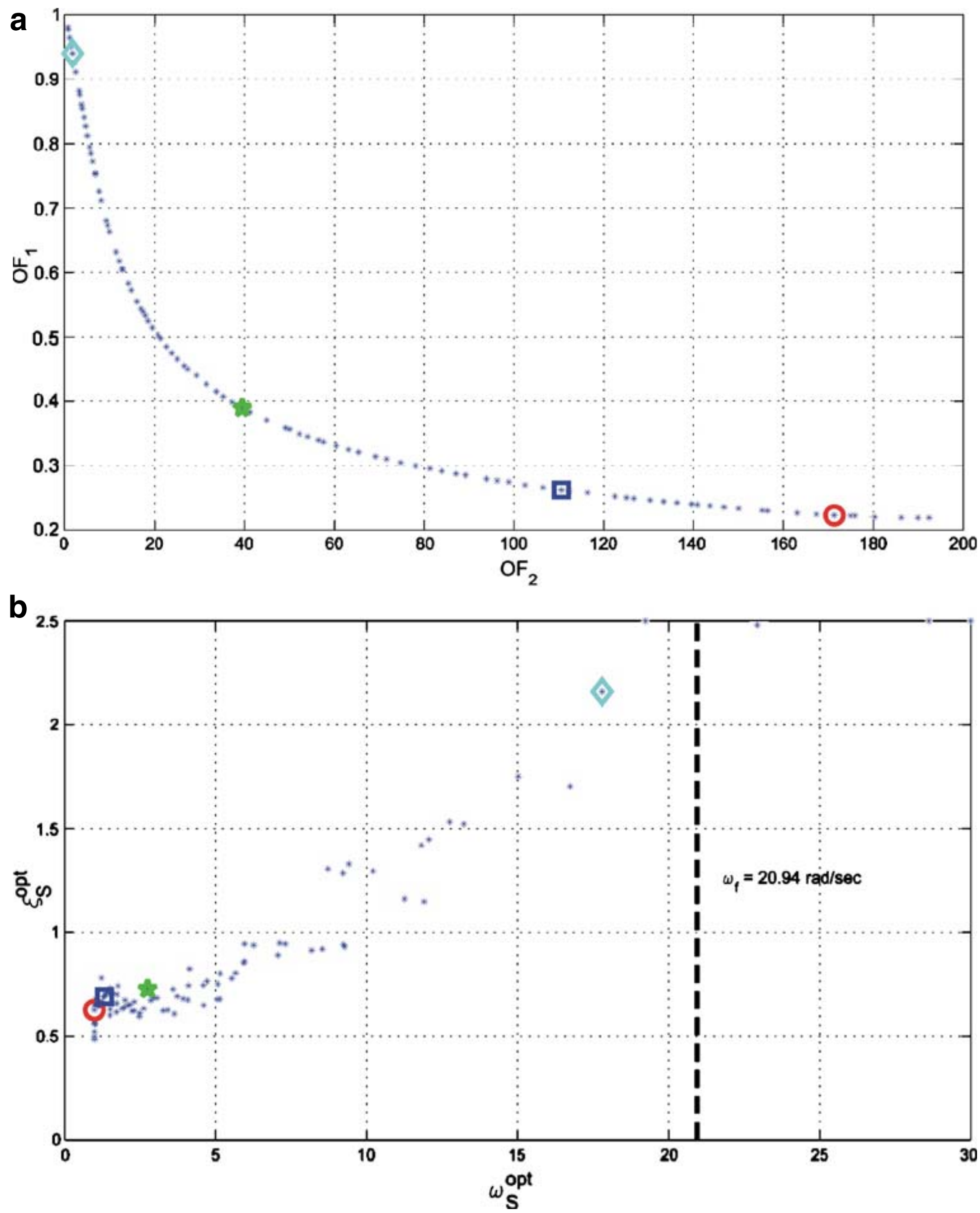

Fig. 4 **a** Pareto front for SDoF system **b** Space of design vector elements of multi-objective problem

Table 4 Some numerical data from Fig. 4 (symbols correspond to points in Fig. 4)

Symbols	OF_2 (cm)	OF_1	ω_S^{opt}	ξ_S^{opt}
○	171.3159	0.2227	1	0.6256
✱	39.5099	0.3896	2.7629	0.7276
□	110.5646	0.2624	1.3313	0.6910
◇	1.7741	0.9402	17.8002	2.1599

about the same for all values of S_0. This is shown in Fig. 5, where one can observe the same variability of the Pareto set for all values of S_0.

Concerning the variability of Pareto set versus input characterization, one should observe that both the parameters modify the Pareto set, but major sensitivities take place as ω_f varies (Figs. 9 and 10) than with respect to ξ_f (Figs. 7 and 8).

Figures 8 and 9 show that, the filter damping ratio influences the optimum solution. First, the initial slope of the Pareto set increases as ξ_f increases, but the asymptotic value of OF_1 shows only little variation as this parameter varies.

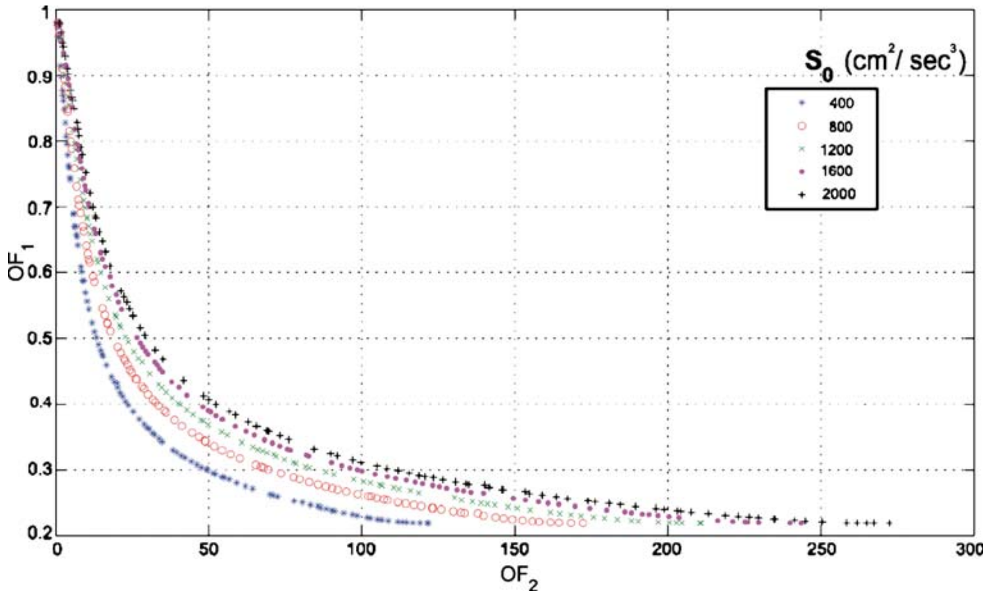

Fig. 5 Sensitivity of Pareto front for different values of power spectral density

Figure 9 shows the maximum performance of the device in reducing the acceleration level of the mass through ω_f. Moreover, the initial slopes (which corresponds to very small admissible displacements) are quite different. In detail, the variation of OF_1 is greater for higher ω_f and tends to decrease as this parameter increases. Also the optimization strategy in terms of optimum design variables varies (see Fig. 10). On the left portion of design vector space only small variations of the optimum solution are observed, which correspond to the points located at the bottom-right of Pareto front in Fig. 9. These values correspond to the asymptotic value of OF_2, where the minimum is attained for each displacement. These points tend to be located in a small region of the *DV* space, quite closer to this unconditional optimum solution point.

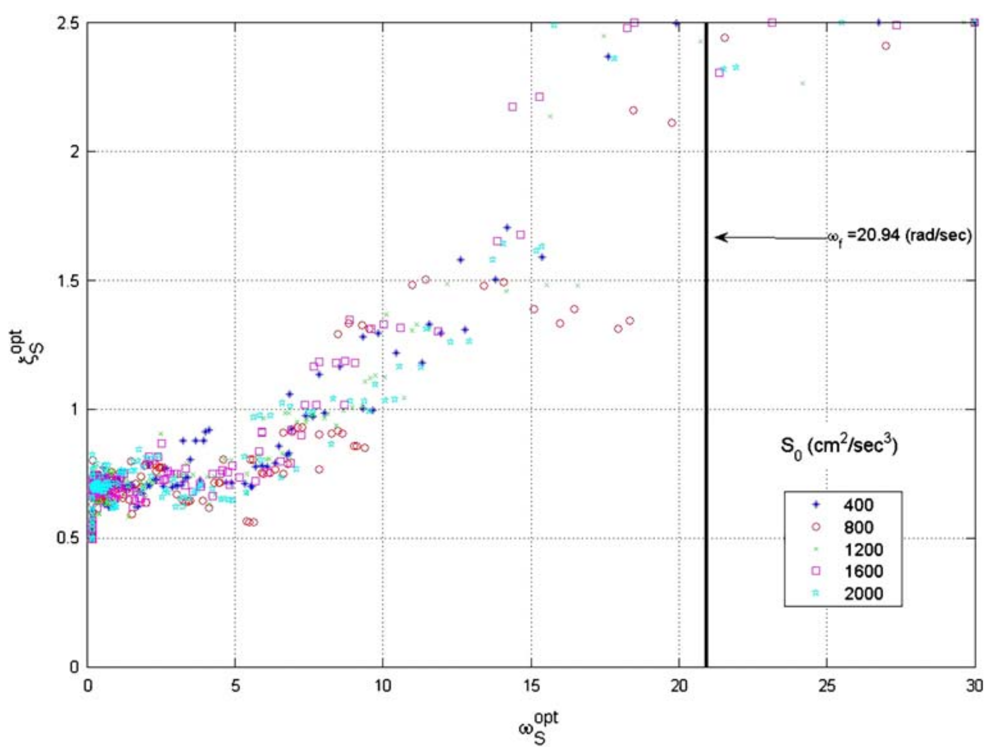

Fig. 6 Space of design vector elements of multi-objective problem for different values of power spectral density

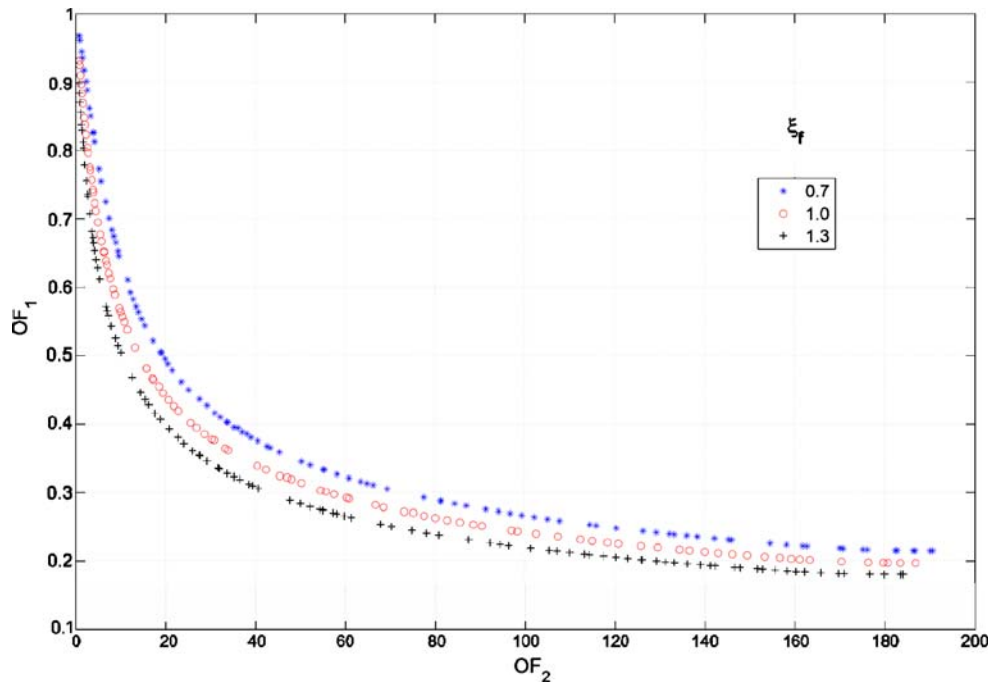

Fig. 7 Sensitivity of Pareto front for different values of filter damping ratio

8.2 Multi-objective optimization of TMD

This section discusses the results of the multi-objective optimization carried out on a building having n floors. The structural model has a constant mass at each floor equal to $m_i = 3.50 \times 10^5$(kg), a linear variation of k_i between $4.5 \times 10^8 (N/m)$ at the first floor and 1.5×10^8 (N/m) at the top floor (Fig. 11). Structural storey damping is $c_i = 2\xi_i\sqrt{k_i m_i}$, where $\xi_i = 0.05$ for all storeys. The maximum admissible displacement of

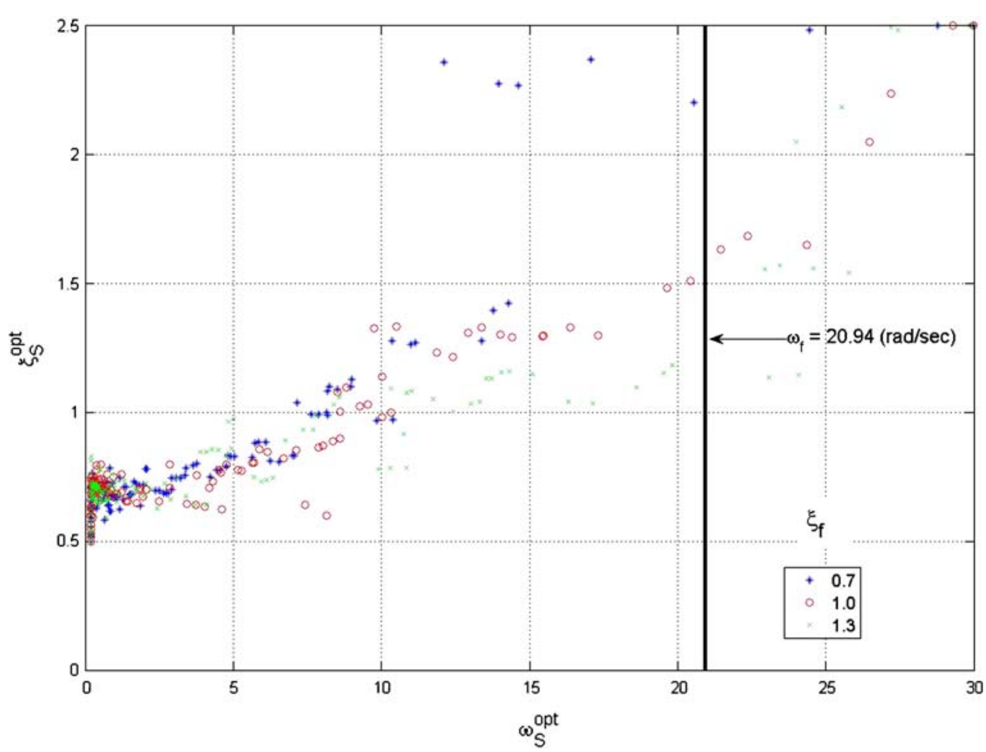

Fig. 8 Space of design vector elements of multi-objective problem for different values of filter damping ratio

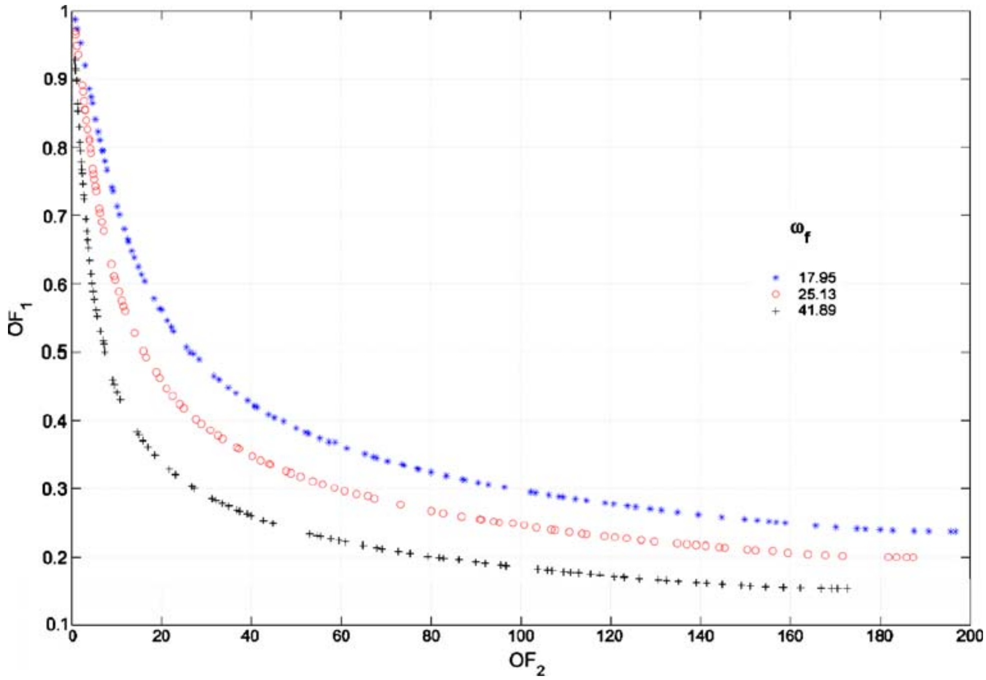

Fig. 9 Pareto front for different values of filter pulsation expressed in rad/s

TMD with respect to the top floor is $\Delta_T^{\max} = 40\,(\mathrm{cm})$. The power spectral density is $S_0 = 150\left(\mathrm{cm}^2/\mathrm{s}^3\right)$ whereas $\omega_\mathrm{f} = 7.39\left(\mathrm{rad}/\mathrm{s}\right)$ and $\xi_\mathrm{f} = 0.45$.

Figure 12 shows the Pareto sets obtained for different values of the parameter $\psi_\omega = \frac{\omega_1}{\omega_\mathrm{f}}$, where ω_1 is the natural frequency of the structure. From the shape of the Pareto sets one can deduce that when the performances are maximum (OF_1 is minimum), the failure probability reaches the maximum value. In addition, results show that the shape of the Pareto set (i.e., the relative importance of two antithetic OFs) depends on the parameter ψ_ω. More precisely, the Pareto sets that

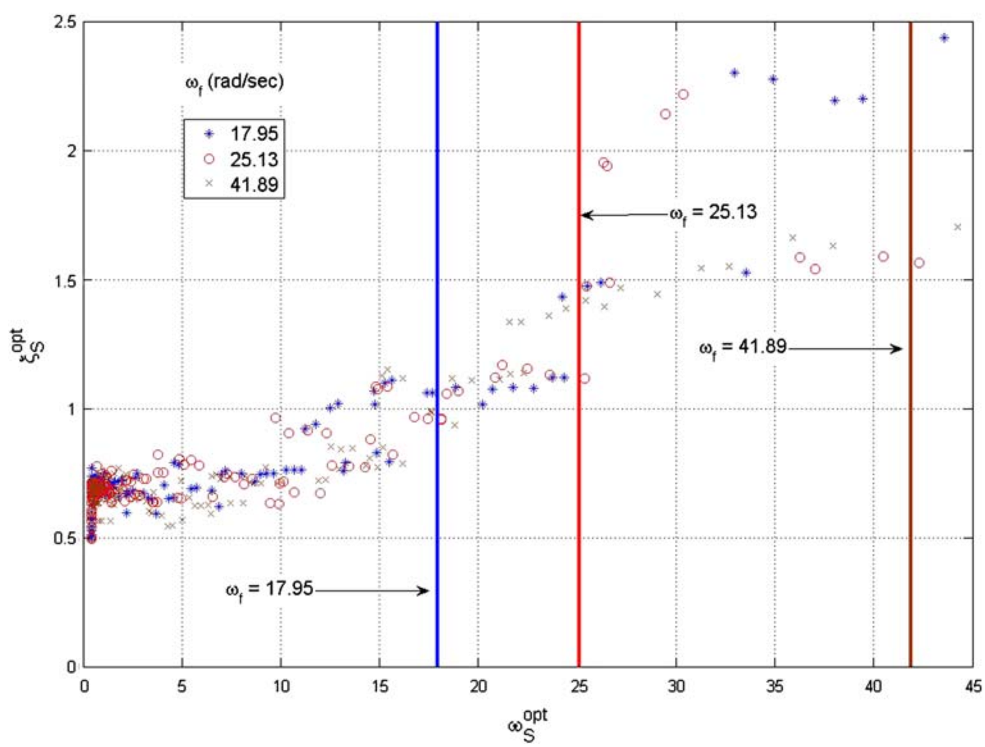

Fig. 10 Space of design vector elements of multi-objective problem for different values of filter pulsation

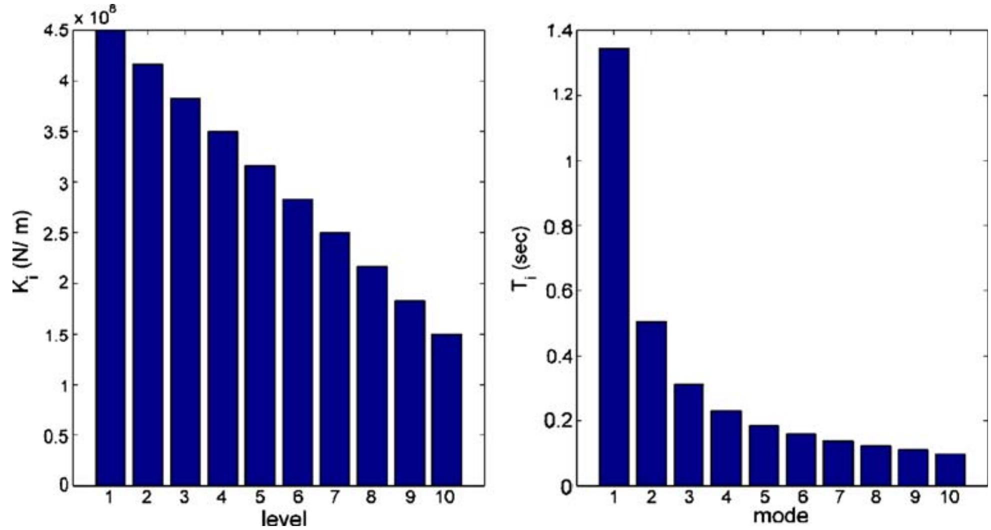

Fig. 11 Storey building stiffness configuration and relative modal periods

correspond to *near−resonance* condition ($\psi_\omega = 0.75$) and *resonance* condition ($\psi_\omega = 1$), respectively, are dissimilar from Pareto sets that correspond to conditions far from these ones. It is evident from Fig. 12 that in *near-resonance* and *resonance* conditions an increase of performance can be attained by growing the failure probability (related to crossing of the limit displacement of the *TMD*). Moreover, a more large failure probability does not produce a sensible increase of performance. At limit (for $\psi_\omega = 5$) any variation of OF_1 with respect to OF_2 takes place.

This outcome is important because the designer must select the strategy for the optimum design, as the two antithetic criteria are strictly related in *near-resonance* and resonance conditions.

Figure 13 shows the optimal solution in terms of design variables. More precisely, the ratio $\rho_\omega^{\mathrm{opt}} = \frac{\omega_T^{\mathrm{opt}}}{\omega_1}$ is reported on the y-axis, and the optimum TMD damping ratio ξ_T^{opt} is on the x-axis.

From these figures it is evident that the optimality of the solution is reached by different strategies (in terms of design variables) as a function of the parameter ψ_ω. In fact, in *near-resonance* and *resonance* conditions, the optimization is reached always for a value $\rho_\omega^{\mathrm{opt}}$ near to unity. In this case, a low value of ξ_T^{opt} corresponds to very high failure probability (and at the same time

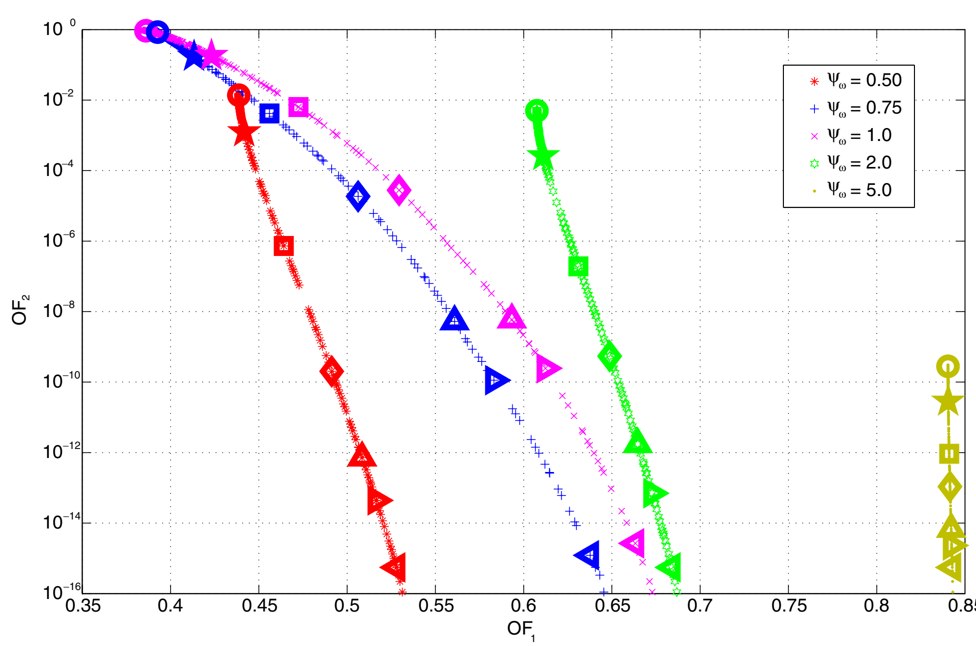

Fig. 12 Pareto fronts for different values $\psi_\omega = \frac{\omega_1}{\omega_f}$

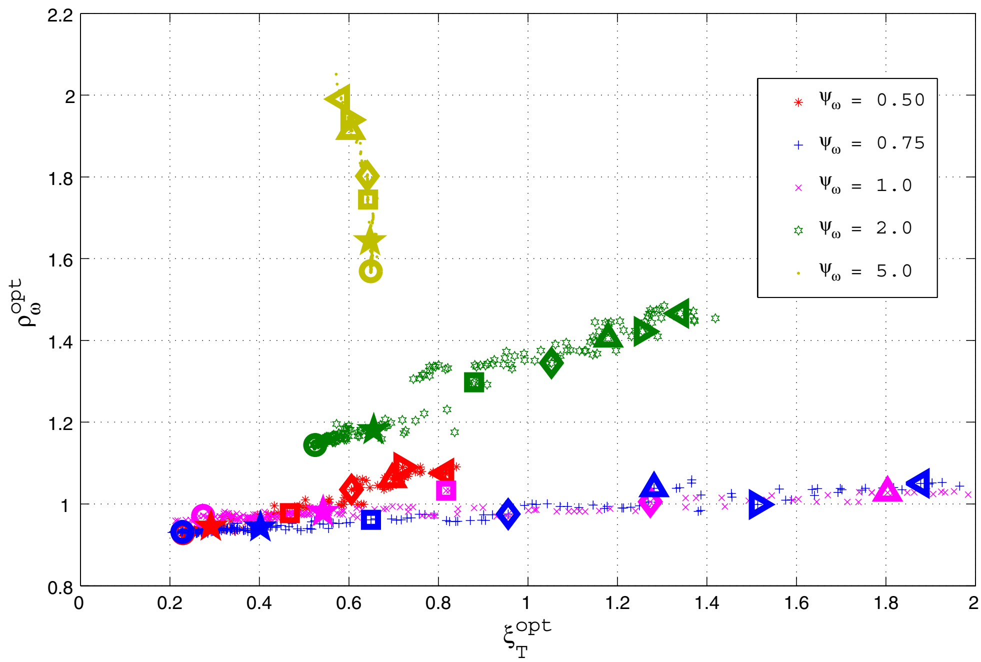

Fig. 13 Optimal design variables for different $\psi_\omega = \frac{\omega_1}{\omega_f}$

the *TMD* performance in terms of OF_1 is maximum). Lower values can be attained by growing ξ_T^{opt}, because this strategy corresponds to an increase of the energy dissipation. In the other situations the strategy is not unique, and both the optimum *TMD* frequency and damping ratio change to attain the optimum solution. As extreme situation ($\psi_\omega = 5$), ξ_T^{opt} attains any variation and $\rho_\omega^{\mathrm{opt}}$ varies to reach a lower *TMD* failure probability. The *TMD* performance in terms of OF_1 (Fig. 12) is constant.

9 Conclusions

In the present work a multi-objective optimization design criterion for linear viscous-elastic vibration control devices has been proposed. More in detail, the problem of an isolator device for a single rigid mass, and the *TMD* problem for a MDoF structural system, have been analysed.

The analyses have been carried out by adopting a stochastic approach, assuming that the excitations acting on the base of the protected systems are stationary stochastic coloured processes.

In the multi-objective optimization problems two antithetic objectives are considered: the maximization of control strategy performance (expressed in stochastic terms by the reduction of transmitted acceleration in the protected systems), and the limitation of the displacement of the vibration control device. The design variables are frequency and damping ratio of the device.

To perform the stochastic multi-objective optimization the non dominated sorting genetic algorithm in its second version (*NSGA-II*) has been adopted. This method supplies the *Pareto set* and the corresponding optimum design variables for different systems and input configurations.

The sensitivity analysis carried out has showed that the optimum solution (i.e., the maximization of control strategy—expressed in terms of reduction of the response of the main system—and the limitation of the device displacement) is reached by adopting different strategies, and this is a function of the input and system characterization. These strategies act by varying the optimum frequency and the damping ratio of the device differently, as a function of the allowable performance.

Appendix: Genetic operators adopted in NSGA-II

Simulated Binary Crossover

In Simulated Binary Crossover *SBX,* the probability distribution used to create a child solution is derived to have a similar search power as that in a single-point crossover in binary-coded genetic algorithm, and is given as follows

$$P(\beta) = \begin{cases} \frac{1}{2}(\eta_c+1)\,\beta_k^{\eta_c} & \text{if} \quad 0 \le \beta \le 1 \\ \frac{1}{2}(\eta_c+1)\,\frac{1}{\beta_k^{\eta_c+2}} & \text{otherwise} \end{cases}, \tag{37}$$

where η_c is the distribution index for crossover operator. Therefore, first a random number $u_k \in [0, 1]$ is generated and using expression (37) then β_k is calculated with this formulation

$$\beta_k = \begin{cases} (2u_k)^{\frac{1}{\eta_c+1}} & \text{if} \quad u_k \leq 0.5 \\ \frac{1}{[2(1-u_k)]^{\frac{1}{\eta_c+1}}} & \text{otherwise} \end{cases}. \tag{38}$$

After obtaining β from (38) the children solutions are calculated as follows

$$c_{1,k} = \frac{1}{2}\left[(1-\beta_k)\, p_{1,k} + (1+\beta_k)\, p_{2,k}\right], \tag{39}$$

$$c_{2,k} = \frac{1}{2}\left[(1+\beta_k)\, p_{1,k} + (1-\beta_k)\, p_{2,k}\right]. \tag{40}$$

In (39) and (40) $c_{i,k}$ is the ith child with kth component, $P_{i,k}$ is the selected parent.

Polynomial mutation

Polynomial mutation is performed on one string as follows

$$c_k = p_k + \left(p_k^u - p_k^l\right)\delta_k, \tag{41}$$

In (41) P_k is the parent, p_k^u and p_k^l are the upper bound and lower bound on the parent component and finally c_k is the child. Mutation operator is based on δ_k which is calculated from a polynomial distribution. A random number $r_k \in [0, 1]$ first is generated, and δ_k is calculated with this formulation

$$\delta_k = \begin{cases} (2r_k)^{\frac{1}{\eta_m+1}} - 1 & \text{if} \quad r_k < 0.5 \\ 1 - [2(1-r_k)]^{\frac{1}{\eta_m+1}} & \text{if} \quad r_k \geq 0.5 \end{cases}. \tag{42}$$

In (42) η_m is the mutation distribution index.

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