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RESEARCH PAPER

Optimality conditions and a solution scheme for fractional optimal control problems

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Abstract We formulate necessary conditions for optimality in Optimal control problems with dynamics described by differential equations of fractional order (derivatives of arbitrary real order). Then by using an expansion formula for fractional derivative, optimality conditions and a new solution scheme is proposed. We assumed that the highest derivative in the differential equation of the process is of integer order. Two examples are treated in detail.

Keywords Optimal control · Fractional order systems · Expansion formula for fractional derivative

1 Introduction

Many processes in physics and engineering are described by systems of equations in which derivatives of arbitrary (not necessarily integer) order appear. We mention problems of describing behavior of viscoelastic bodies or diffusion-wave problems. As a matter of fact, if one wants to include memory effects, i.e., the influence of the past on the behavior of the system at present time one *may* use fractional derivative to describe such an effect. In principle, there are two different approaches to "fractionalization" of the dynamics of a system (see Atanackovic and Stankovic 2007a, b). In the first procedure, integer order derivatives are simply replaced by derivatives of real order. In the second approach, considered to be more fundamental from the physical point of view, fractionalization is made on the level of Hamilton's principle (see Rekhviashvili 2004). This approach, however, leads to differential equations of the process involving both left and right fractional derivatives, thus making the effective solution procedure more difficult. For more results concerning fractional calculus and variational principles with fractional derivatives, see Special Issue of Journal of Nonlinear Dynamics (vol. 29, 2002), special issue of Nonlinear Dynamics (vol. 38, 2004) and the following papers: Baleanu and Akvar (2004), Rabei et al. (2007) and Atanackovic et al. (2008).

In a recent series of papers, Agrawal (2002, 2004, 2006) formulated necessary conditions for optimality, that is Euler-Lagrangian equations, for optimization problems involving fractional derivatives. Analysis presented in these works, are addresses to the following problem: find a minimum of a functional

$$J(u) = \int_0^1 F(x(t), u(t), t)\, dt \tag{1}$$

subject to constraints

$$\left({}_0D_t^{\alpha} x\right)(t) = G(x(t), u(t), t) \tag{2}$$

where[1] $x(t)$ is the state variable, $u(t)$ is the control variable, t is time and $F(x, u, t)$ and $G(x, u, t)$ are functions

[1]For simplicity of exposition, we treat $x(t)$ as a single variable. The case when $x(t) = [x_1(t), ..., x_n(t)]$ is a vector with n components could be treated similarly.

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possessing certain regularity properties. Also, to (2) the initial condition (see Agrawal 2004, p. 326) is prescribed

$$x(0) = x_0 \tag{3}$$

In (2) we used $\left({}_0D_t^\alpha x\right)(t)$ to denote the left Riemann–Liouville derivative of the order α defined as

$$\left({}_0D_t^\alpha x\right)(t) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1. \tag{4}$$

Minimization problem with optimality criteria given by (1), subject to the constraint (2), (3) was treated in (Agrawal 2004).

Condition (2) is a generalization of the "integer order dynamics" described as

$$x^{(1)}(t) = G_{int.}(x(t), u(t), t). \tag{5}$$

where, as usual, $x^{(1)}(t) = \frac{d}{dt}x(t)$ is the first derivative of the state function. Problems including (1), (3), (5) are the standard problems of the optimal control theory, see for example Vujanovic and Atanackovic (2004). Our intention in this work is to consider the following generalization of the optimization problems (1), (2) and (1), (5) that we state as:

Problem: Find a minimum of a functional

$$J(u) = \int_0^1 F(x(t), u(t), t)\, dt \tag{6}$$

subject to the differential constraint given by

$$x^{(1)}(t) + k\left({}_0D_t^\alpha x\right)(t) = G(x, u, t), \quad x(0) = x_0 \tag{7}$$

where $k = const$. Thus, *we assume that the highest derivative in the dynamics of the problem is of integer order*. This assumption makes the problem of satisfying the initial condition (3) by $x(t)$ satisfying (7) much easier. In order to make our point clear, we note that in Agrawal (2004), p. 331 the problem formulated by equations (1), (2) was treated with $F(x, u, t) = \left(x^2 + u^2\right)/2$, $G(x, u, t) = -x + u$, $x_0 = x(0) = 1$. Thus, the dynamics of the system is described by ${}_0D_t^\alpha x = -x + u$, subject to $x(0) = 1$. It is known (see for example Kilbas et al. 2006 p. 223) that the differential equation ${}_0D_t^\alpha x = -x + u$ has either a trivial solution $x(t) = 0$ or a solution having $\lim_{t\to 0} x(t) = \infty$. Therefore, the boundary condition $x(0) = 1$ cannot be satisfied! Our choice of constraint in the form (7) is motivated by application and has the highest derivative of integer order (equal to one). Thus, we do not have the problem with satisfying the initial condition (7). As a matter of fact the boundary conditions in our case are the same as in the classical case (without fractional derivative in (7).

2 Optimality condition

To obtain the optimality conditions for the problem formulated by (6), (7), we follow the method of Agrawal (2004). Thus, we form modified performance criteria

$$\overline{J}(u) = \int_0^1 \left[F(x, u, t) + \lambda(t)\left(G(x, u, t) - x^{(1)}(t) - k\left({}_0D_t^\alpha x\right)(t)\right)\right]dt \tag{8}$$

with $\lambda(t)$ being the Lagrange multiplier. Setting the first variation of (8) to zero, we obtain

$$\delta\overline{J}(u) = \int_0^1 \left[\frac{\partial F}{\partial u}\delta u(t) + \delta\lambda(t) \times\left[G(x, u, t) - x^{(1)}(t) - k\left({}_0D_t^\alpha x\right)(t)\right] + \lambda(t) \times\left[\frac{\partial G}{\partial u}\delta u - \delta x^{(1)} - k\left({}_0D_t^\alpha \delta x\right)(t)\right]\right]dt = 0 \tag{9}$$

where $\delta u(t)$, $\delta\lambda(t)$ and $\delta x(t)$ are Lagrange variations of corresponding dependent variables. From (7) we conclude that $\delta x(0) = 0$. By using the integration by parts formula (see Samko et al. 1993 p. 46 and Kilbas et al. 2006 p. 76) we have

$$\int_0^1 \lambda(t)\left({}_0D_t^\alpha \delta x\right)(t)\, dt = \int_0^1 \delta x(t)\left({}_tD_1^\alpha \lambda\right)(t)\, dt \tag{10}$$

where $\left({}_tD_1^\alpha \lambda\right)(t)$ denotes the right Riemann–Liouville fractional derivative defined as

$$\left({}_tD_1^\alpha \lambda\right)(t) = -\frac{1}{\Gamma(1-\alpha)}\int_t^1 \frac{\lambda(\tau)}{(\tau - t)^\alpha}d\tau.$$

By using (10) in (9) and by invoking the fundamental lemma of Variational calculus, we obtain

$$\begin{aligned} x^{(1)} &= -k\left({}_0D_t^\alpha x\right)(t) + G(x, u, t), \\ \frac{\partial F}{\partial u} + \lambda(t)\frac{\partial G}{\partial u} &= 0, \\ \lambda^{(1)} &= k\left({}_tD_1^\alpha \lambda\right)(t) - \lambda(t)\frac{\partial G}{\partial x}, \end{aligned} \tag{11}$$

subject to

$$x(0) = x_0, \qquad \lambda(1) = 0. \tag{12}$$

In general, the system of (11), (12) solves the optimization problem. However, the presence of both left and right fractional derivatives in (11) makes the process of analytical solutions difficult (see, for example, Atanackovic and Stankovic 2007a, b; Agrawal 2008). Therefore, we use a numerical scheme, based on approximation of fractional derivatives proposed in (Atanackovic and Stankovic 2004) for the solution of minimization problem defined by (6), (7).

3 Expansion formulas for the left fractional derivative

We derive the expansion formula of Atanackovic and Stankovic (2004), without reference to the distribution theory. Let $V_n\left(f^{(p)}\right), n\in\mathbb{N}$, denote the $n-th$ moment of the function $f^{(p)}$,where $f^{(p)}$, $p\in\mathbb{N}$ is the $p-th$ derivative of f, i.e.,

$$V_n\left(x^{(p)}\right)(t)=\int_0^t x^{(p)}(\tau)\,\tau^n d\tau,\quad n\in\mathbb{N},\quad t\geq 0. \tag{13}$$

In the following procedure, it is assumed that $x\in AC^2\left([0,b]\right)$ and $0<\alpha<1$. Recall that $x\in AC^2([0,b])$ if x and $x^{(1)}$ are continuous on $[0,b]$ and $x^{(2)}\in L^1(0,b)$. By partial integration in (4) we obtain

$$\left({}_0D_t^{\alpha}x\right)(t)=\frac{1}{\Gamma(1-\alpha)}x(0)\,t^{-\alpha}+\frac{1}{\Gamma(2-\alpha)}x^{(1)}(0)\,t^{1-\alpha}+\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-\tau)^{1-\alpha}x^{(2)}(\tau)\,d\tau,\ 0<t\leq b. \tag{14}$$

We now make use of the binomial formula

$$(1+z)^{\gamma}=\sum_{p=0}^{\infty}\binom{\gamma}{p}z^p=\sum_{p=0}^{\infty}\frac{(-1)^p\,\Gamma(p-\gamma)}{\Gamma(-\gamma)\,p!}z^p,\quad |z|<1. \tag{15}$$

Expression (15), holds also for $z=1$ if and only if $\gamma>-1$ and $z=-1$, $\gamma\neq 0$ if and only if $\gamma>0$. Also it is well known that

$$\left|\binom{\gamma}{p}\right|\leq C\frac{1}{p^{1+\gamma}},\quad \gamma\neq -1,-2,\ldots \text{ and } p\to\infty. \tag{16}$$

With (5), expression for $\left({}_0D_t^{\alpha}f\right)(t)$ becomes $(\gamma=1-\alpha)$

$$\left({}_0D_t^{\alpha}x\right)(t)=\frac{1}{\Gamma(1-\alpha)}x(0)\,t^{-\alpha}+\frac{1}{\Gamma(2-\alpha)}x^{(1)}(0)\,t^{1-\alpha}+t^{1-\alpha}\int_0^t x^{(2)}(\tau)\left(\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\,p!}\left(\frac{\tau}{t}\right)^p\right)d\tau,\ t>0. \tag{17}$$

Because of (16) we can integrate series in (17) term by term. Also, by using the relation

$$\int_0^t x^{(2)}(\tau)\,\tau^p d\tau=t^p x^{(1)}(t)-p\int_0^t x^{(1)}(\tau)\,\tau^{p-1}d\tau,\quad p\geq 1$$

in (17), we obtain

$$\begin{aligned}\left({}_0D_t^{\alpha}x\right)(t)&=\frac{1}{\Gamma(1-\alpha)}x(0)\,t^{-\alpha}+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\,p!}\\&\quad\times\left(x^{(1)}(t)-\frac{p}{t^p}V_{p-1}\left(x^{(1)}\right)(t)\right)\\&\quad+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}x^{(1)}(t),\quad t>0.\end{aligned} \tag{18}$$

Integrating by parts in (18) and rearranging the result, we obtain

$$\begin{aligned}\left({}_0D_t^{\alpha}x\right)(t)&=\frac{x(t)\,t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(2-\alpha)}\\&\quad\times\left\{x^{(1)}(t)\left[1+\sum_{p=1}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\,p!}\right]t^{1-\alpha}\right.\\&\quad\left.-\sum_{p=2}^{\infty}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\left(\frac{x(t)}{t^{\alpha}}+\frac{\widetilde{V}_p}{t^{p-1+\alpha}}\right)\right\},\end{aligned} \tag{19}$$

where

$$\widetilde{V}_p=-(p-1)\int_0^t\tau^{p-2}f(\tau)\,d\tau,\qquad p=2,3,\ldots. \tag{20}$$

Note that the moments, $\widetilde{V}_p$, $p=1,2,\ldots$ are solutions to the following system of differential equations

$$\begin{aligned}&\widetilde{V}_p^{(1)}(t)=-(p-1)\,t^{p-2}x(t),\\&\widetilde{V}_p(0)=0,\qquad p=2,3,\ldots\end{aligned} \tag{21}$$

In application we shall use (19) with finite number of terms that is

$$\begin{aligned}\left({}_0D_t^{\alpha}f\right)(t)&=\frac{x(t)\,t^{-\alpha}}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(2-\alpha)}\\&\quad\times\left\{x^{(1)}(t)\left[1+\sum_{p=1}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\,p!}\right]t^{1-\alpha}\right.\\&\quad\left.-\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)(p-1)!}\left(\frac{x(t)}{t^{\alpha}}+\frac{\widetilde{V}_p}{t^{p-1+\alpha}}\right)\right\},\end{aligned} \tag{22}$$

with N suitably chosen.

Another approximation of the fractional derivative may be obtained as follows. First, by substituting $z = -1$ in (15) we obtain

$$1 + \sum_{p=1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)\, p!} = 0, \tag{23}$$

so that (19) becomes

$$\left({}_0D_t^{\alpha} x\right)(t) = \frac{x(t)}{t^{\alpha}} \left[\frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \times \sum_{p=2}^{\infty} \frac{\Gamma(p-1+\alpha)}{(p-1)!} \right] - \frac{1}{\Gamma(\alpha-1)\Gamma(2-\alpha)} \sum_{p=2}^{\infty} \frac{\Gamma(p-1+\alpha)}{(p-1)!} \frac{\widetilde{V}_p}{t^{p-1+\alpha}}. \tag{24}$$

Again, by using a finite number of terms in (24) we have

$$\left({}_0D_t^{\alpha} f\right)(t) \approx \frac{x(t)}{t^{\alpha}} \left[\frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \times \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} \right] - \frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} \frac{\widetilde{V}}{t^{p-1+\alpha}}. \tag{25}$$

Note that the first sum on the right hand side can be written as

$$S(\alpha, N) = \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} = \frac{\Gamma(N+\alpha)}{(N-1)!} - \frac{\Gamma(1+\alpha)}{\alpha} \tag{26}$$

Then, (25) becomes

$$\left({}_0D_t^{\alpha} f\right)(t) \approx \frac{x(t)}{t^{\alpha}} \left[\frac{1}{\Gamma(1-\alpha)} - \frac{\frac{\Gamma(N+\alpha)}{(N-1)!} - \frac{\Gamma(1+\alpha)}{\alpha}}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \right] - \frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \times \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} \frac{\widetilde{V}_p}{t^{p-1+\alpha}}, \tag{27}$$

or

$$\left({}_0D_t^{\alpha} f\right)(t) \approx \left[\frac{x(t)}{t^{\alpha}} \mathcal{A}(\alpha) + \sum_{i=2}^{N} \mathcal{B}(\alpha, i) \frac{\widetilde{V}_i}{t^{i-1+\alpha}} \right], \tag{28}$$

where we used $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha, p)$ to denote

$$\mathcal{A}(\alpha) = \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \times \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!},$$
$$\mathcal{B}(\alpha, i) = -\frac{1}{\Gamma(\alpha-1)\,\Gamma(2-\alpha)} \frac{\Gamma(i-1+\alpha)}{(i-1)!}, \tag{29}$$

with $\widetilde{V}_p$ satisfying (21). Equations (21), (22), (27), represent the basic relations that we use in numerical solutions to fractional differential equations. In numerical applications we shall take $N \le 7$.

Expansion formula (24) may be interpreted as an expression for the fractional derivative $\left({}_0D_t^{\alpha} x\right)(t)$ in terms of $x(t)$ and the "internal variables" $\widetilde{V}_p$, $p = 2, \ldots, N$.

4 Numerical scheme for the solution of (6), (7)

The main idea of the method that we propose is to reduce the system (6), (7) to an equivalent system with integer order derivatives only. For that purpose, we use expansion formula (27) for fractional derivatives in (7) to obtain

$$x^{(1)} = G(x, u, t) - k \left[\frac{x(t)}{t^{\alpha}} \mathcal{A}(\alpha) + \sum_{i=2}^{N} \mathcal{B}(\alpha, i) \frac{\widetilde{V}_i}{t^{i-1+\alpha}} \right],$$
$$\widetilde{V}_i^{(1)}(t) = -(i-1)\, t^{i-2} x(t), \quad i = 2, \ldots, N \tag{30}$$

with the initial conditions

$$x(0) = x_0, \quad \widetilde{V}_i(0) = 0, \quad i = 2, 3, \ldots, N. \tag{31}$$

Optimization problem described with (6), (30), (31) is a standard problem with integer order derivatives.

We use Pontryagin's principle to solve it. Thus, we form Hamiltonian $\mathcal{H}$ as follows

$$\mathcal{H}=F(x,u,t)+p_1\left\{G(x,u,t)-k\left[\frac{x(t)}{t^\alpha}\mathcal{A}(\alpha)+\sum_{i=2}^{N}\mathcal{B}(\alpha,i)\frac{\widetilde{V}_i}{t^{i-1+\alpha}}\right]\right\}+\sum_{i=2}^{N}p_i\left\{-(p-1)t^{i-2}x(t)\right\}, \tag{32}$$

where the co–state variables p_j, $j=1,\dots,N$ are determined from

$$p_1^{(1)}=-\frac{\partial\mathcal{H}}{\partial x}=-\frac{\partial F}{\partial x}-p_1\left[\frac{\partial G}{\partial x}-k\frac{\mathcal{A}(\alpha)}{t^\alpha}\right]+\sum_{i=2}^{N}p_i\left\{(p-1)t^{i-2}\right\},\quad p_1(1)=0$$
$$p_i^{(1)}=-\frac{\partial\mathcal{H}}{\partial\widetilde{V}_i}=-p_1\frac{k\mathcal{B}(\alpha,i)}{t^{i-1+\alpha}},\quad p_i(1)=0,\quad i=2,\dots,N. \tag{33}$$

Optimality condition $\lim_{u\in\mathcal{U}}\mathcal{H}$ leads to

$$\frac{\partial F}{\partial u}+p_1\frac{\partial G}{\partial u}=0. \tag{34}$$

Note that optimality condition $(11)_2$ and (34) are the same (with $p_1=\lambda$). System of (30), (31), (32)–(34) may be solved by standard procedures for two point boundary-value problems.

5 Examples

5.1 Example 1

Firstly, we treat as a test example a modification of the problem presented in Agrawal (2004), where the following optimality criteria was used

$$J=\frac{1}{2}\int_0^1\left[x^2(t)+u^2(t)\right]dt, \tag{35}$$

subject to

$$x^{(1)}+k\left({}_0D_t^\alpha x\right)(t)=-x+u,\qquad x(0)=1. \tag{36}$$

Note that in this case $F(x,u,t)=\frac{1}{2}\left[x^2(t)+u^2(t)\right]$, $G(x,u,t)=-x+u$ so that (34) leads to

$$u=-p_1 \tag{37}$$

Therefore, the system of (30), (33) becomes

$$\begin{aligned}
&x^{(1)}=-x-p_1-k\left[\frac{x(t)}{t^\alpha}\mathcal{A}(\alpha)+\sum_{i=2}^{N}\mathcal{B}(\alpha,i)\frac{\widetilde{V}_i}{t^{i-1+\alpha}}\right],\\
&x(0)=1\quad p_1(0)=1\\
&\widetilde{V}_i^{(1)}(t)=-(i-1)t^{p-2}x(t),\quad \widetilde{V}_i(0)=0,\quad i=2,\dots,N,\\
&p_1^{(1)}=-x-p_1\left[-1-k\frac{\mathcal{A}(\alpha)}{t^\alpha}\right]+\sum_{i=2}^{N}p_i\left\{(p-1)t^{i-2}\right\},\quad p_1(1)=0\\
&p_i^{(1)}=-p_1\frac{k\mathcal{B}(\alpha,i)}{t^{i-1+\alpha}},\quad p_i(1)=0,\quad i=2,\dots,N.
\end{aligned} \tag{38}$$

The system (38) represents the standard two point boundary-value problem that is solved numerically.

First we take $\alpha=0.5$, $k=1$ and $N=4,5,7$ and examine the influence of the number of terms in (30) on the solution. Results for $x(t)$ are shown in Fig. 1.

Similar results for different values of N, obtained for $\alpha=0.9$, $k=1$, and $N=4,5,7$ in time interval $0.1\le t\le 1$, are shown in Fig. 2.

As can be seen in the above figures, the number of terms may be taken to be $N=4$, since the convergence is sufficiently good. Solution displayed in Figs. 1 and 2 has the same character as the one presented in Agrawal (2004), where the same optimality criterion J was used and where, instead of (36), system dynamics was described by ${}_0D_t^\alpha x=-x+u$, $x(0)=1$.

For smaller values of α, differences between solutions become even smaller, as shown in Fig. 3 for $\alpha=0.1$, $k=1$, and $N=4,5,7$ in time interval $0.1\le t\le 1$. In Fig. 4 we show the influence of α on the solution.

As there are no significant differences between the solutions for $N=7,5$ and 4 we have prescribed $N=4$, and decreased the lower bound of the time interval so that $0.0001\le t\le 1$. We use the initial time close to zero, and still avoid singularity for $t=0$ in (38). In Fig. 4, we show numerical solutions for $\alpha=0.1$, $k=0.1$, $N=4$ and $\alpha=1$, $k=0.1$ in time interval $0.0001\le t\le 1$. The optimal control law, corresponding to the solution shown in Fig. 4, is given in Fig. 5.

Finally, we examine cases where $\alpha=0.2$, $k=0.5$, $N=4$ and $\alpha=1$, $k=0.5$ in time interval $0.0001\le t\le 1$. Results show significant difference between integer and real order derivative solution. In Fig. 6 we present the optimal state trajectory for this case.

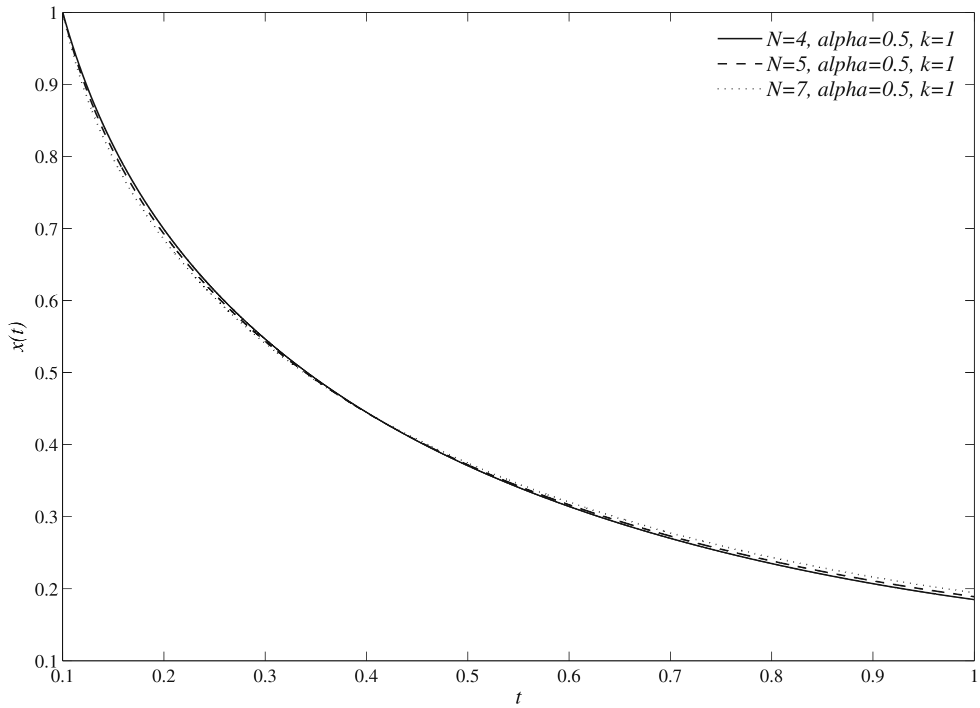


Fig. 1 Solution of (35), (36) for $\alpha = 0.5$, $k = 1$ and $N = 4, 5, 7$

Finally, in Fig. 7 we show the corresponding control law.

5.2 Example 2

In this sub-section, we consider the fractional derivative model of a light amplification in Erbium-doped fiber amplifiers (EDFA). EDFA is one of the most commonly used type of fiber amplifiers in both long-haul and metro optical networks. Fractional derivative type model provides a natural framework for generalization of a nonlinear model first derived by Bononi, in the controllable form used for describing amplified

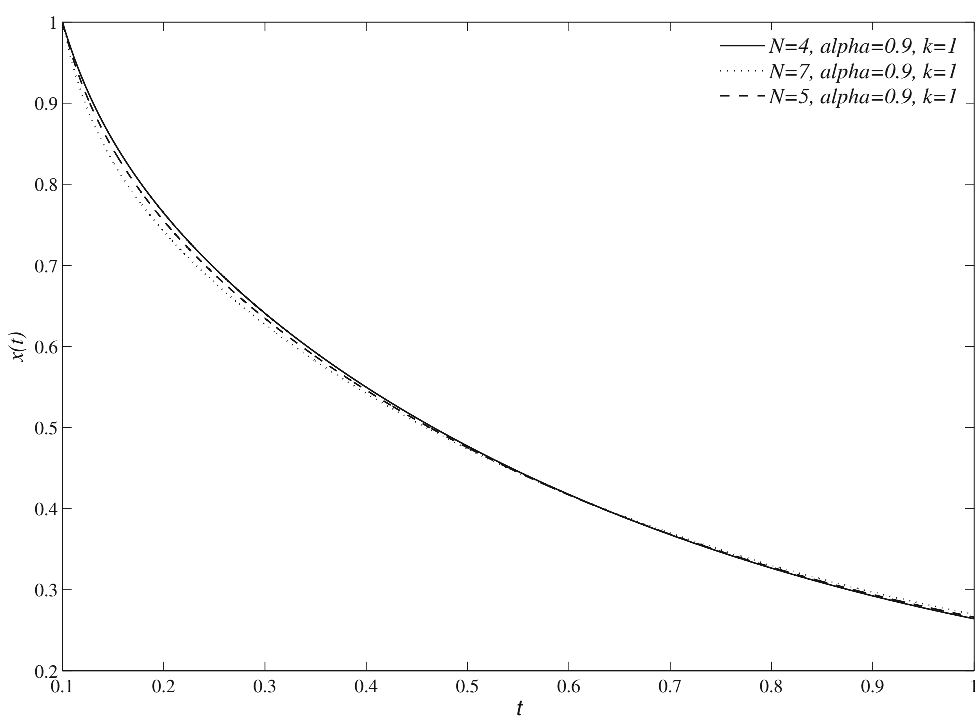


Fig. 2 Solutions of (35), (36) for $\alpha = 0.9$, $k = 1$, and $N = 4, 5, 7$

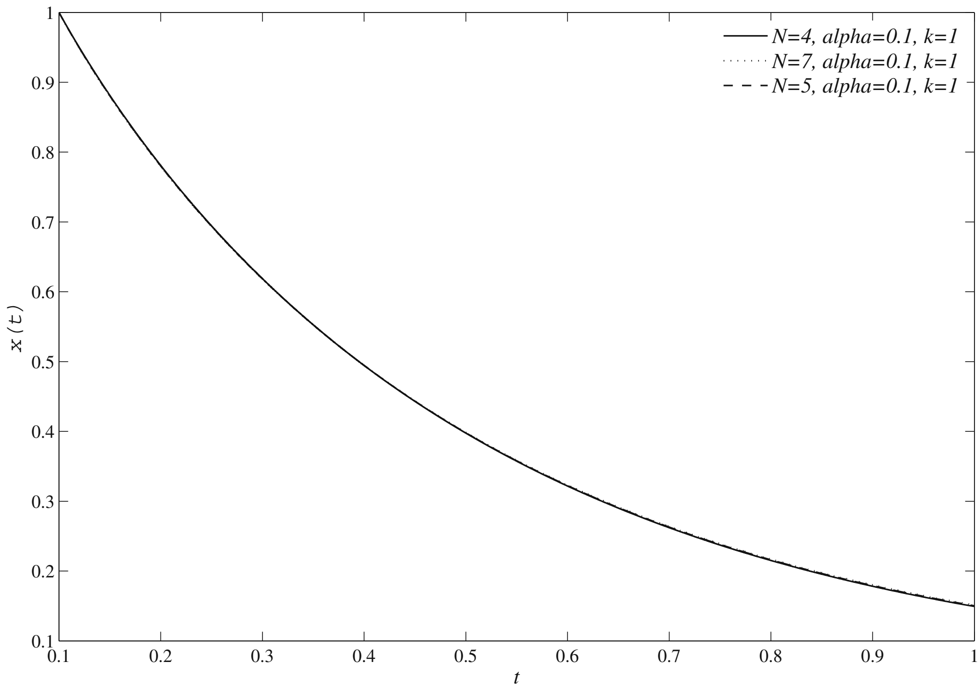


Fig. 3 Solutions of (35), (36) for $\alpha = 0.1$, $k = 1$, and $N = 4, 5, 7$

spontaneous emission (ASE) of ions, see Jelicic and Petrovacki (2007).

Mathematical model of absorption/emission of ions in EDFA is obtained as follows. We treat the reservoir of ions as the first state (x1) and the laser pump power as the second state (x2). The derivative of this second state (rate of change of pump power) equals to the control law (for details, see again Jelicic and Petrovacki 2007). Then, system dynamics reads as follows

$$\dot{x}_1 + k\left({}_0D_t^{\alpha}x_1\right) = -x_1 + x_2\xi\left(1 - \exp\left(B_p x_1 - A_p\right)\right) + \eta\left(1 - \exp\left(B_s x_1 - A_s\right)\right),$$
$$\dot{x}_2 = u. \tag{39}$$

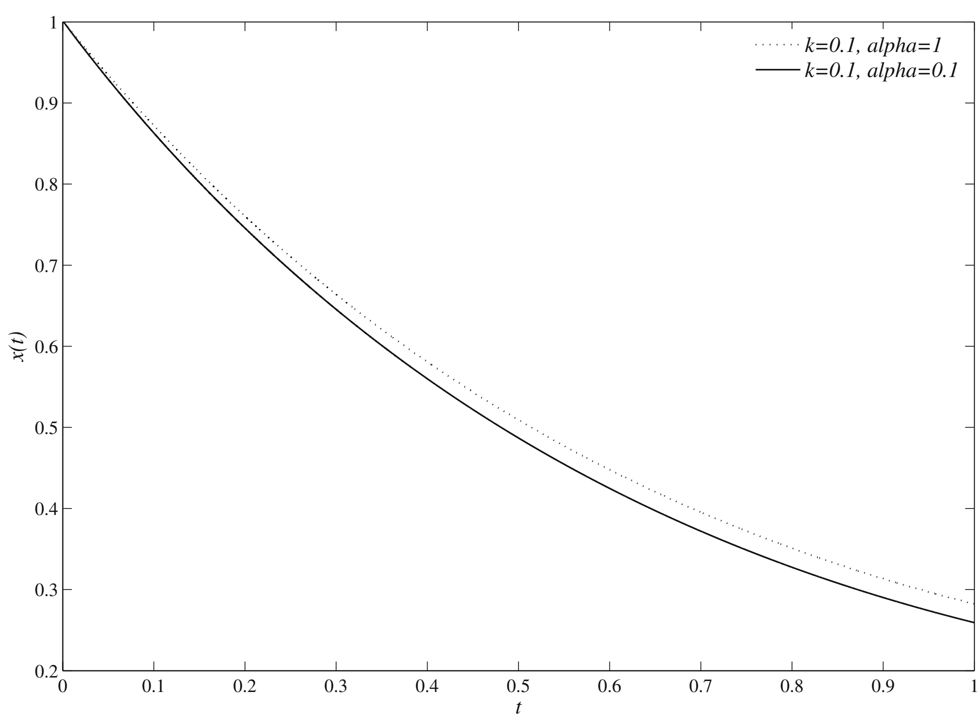


Fig. 4 Solutions of (35), (36) for $\alpha = 0.1$ and $\alpha = 1$, $k = 1$, and $N = 4$

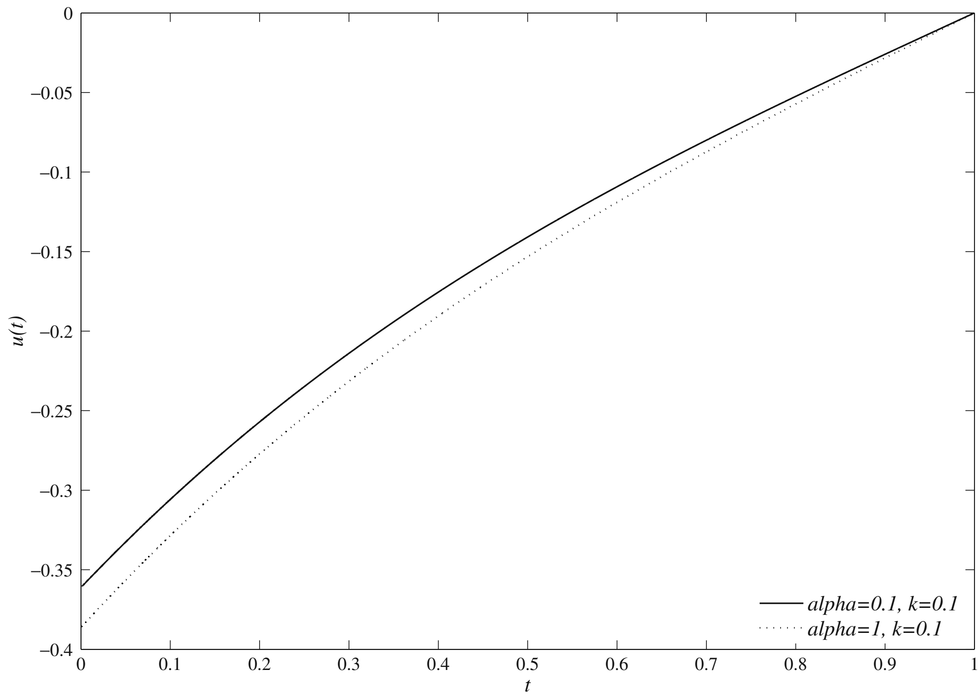


Fig. 5 Control law $u(t)$ of (35), (36) for $\alpha \in \{0.1, 1\}$, $k = 1$, and $N = 4$

where A_i and B_i are dimensionless coefficients, dependent on working frequency and the Erbium doped fiber's absorption and emission cross sections; parameters ξ, η are depending on working frequency, number of input channels, and time constant of EDFA.

All parameters where determined in a complex experiment (see Petrovacki and Jelicic 2006) as: $\alpha = 0.5$, $B_p = 1.8$, $A_p = 1.8$, $B_s = 5.85$, $A_s = 1.8$, $\xi = 15403$ and $\eta = 29.41$, $k = 5$. The optimality criterion was chosen to be

$$J = \int_0^T \left(q_1 x_1^2 + q_2 x_2^2 + r u^2\right) dt, \tag{40}$$

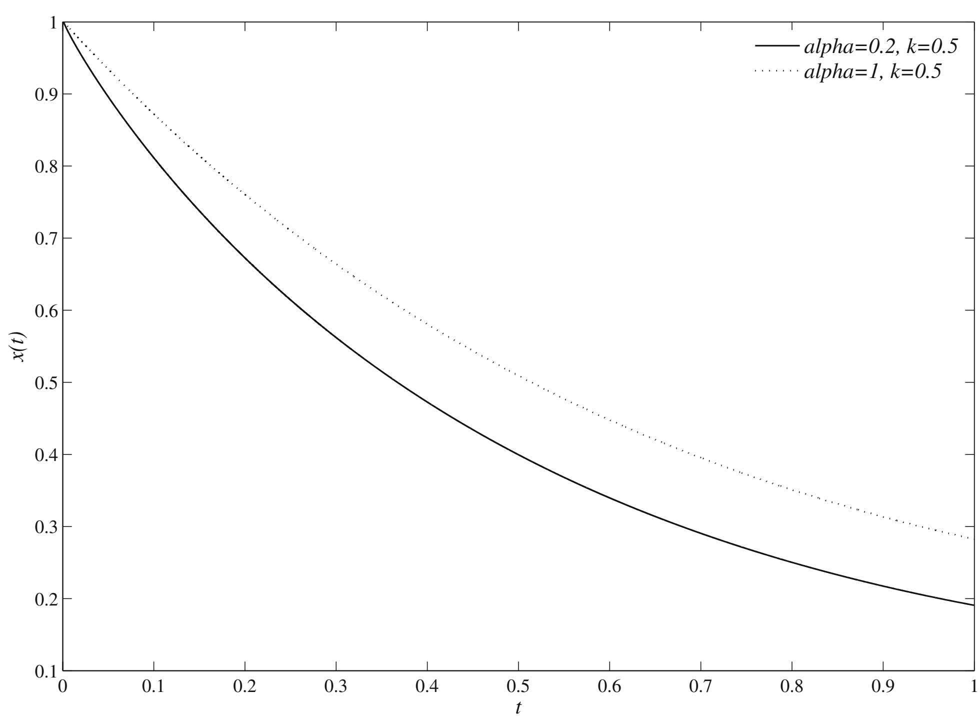
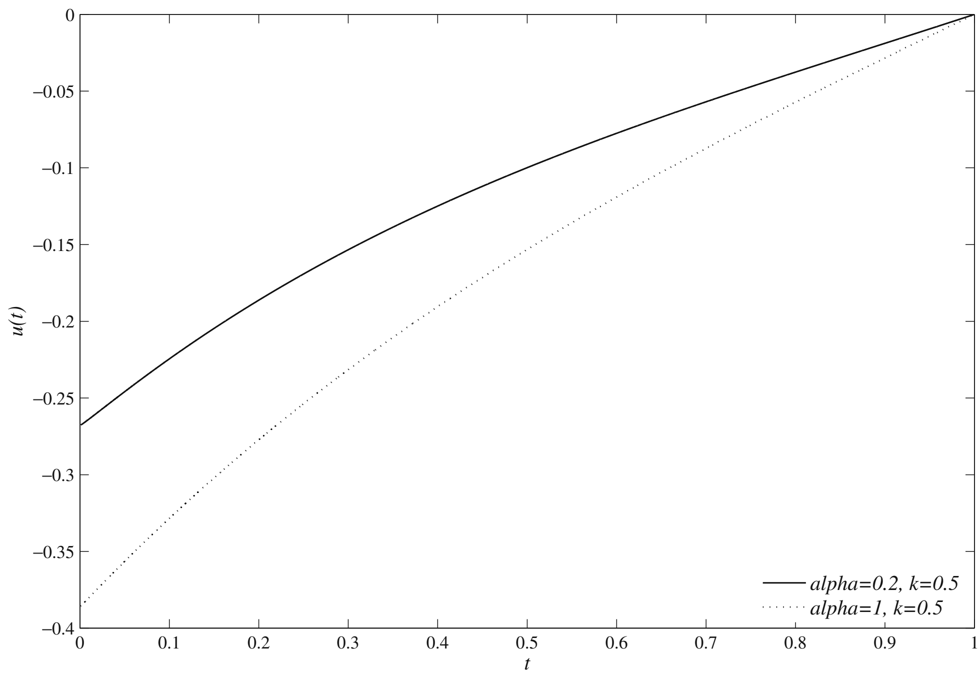


Fig. 6 Solutions of (35), (36) for $\alpha = 0.2$ and $\alpha = 1$, $k = 0.5$, and $N = 4$

Fig. 7 Control law $u(t)$ of (35), (36) for $\alpha = 0.2$ and $\alpha = 1$, $k = 1$, and $N = 4$



where $T = 1.5, r = 10; q_1 = 1; q_2 = 1$. By using (28) in (39) we obtain

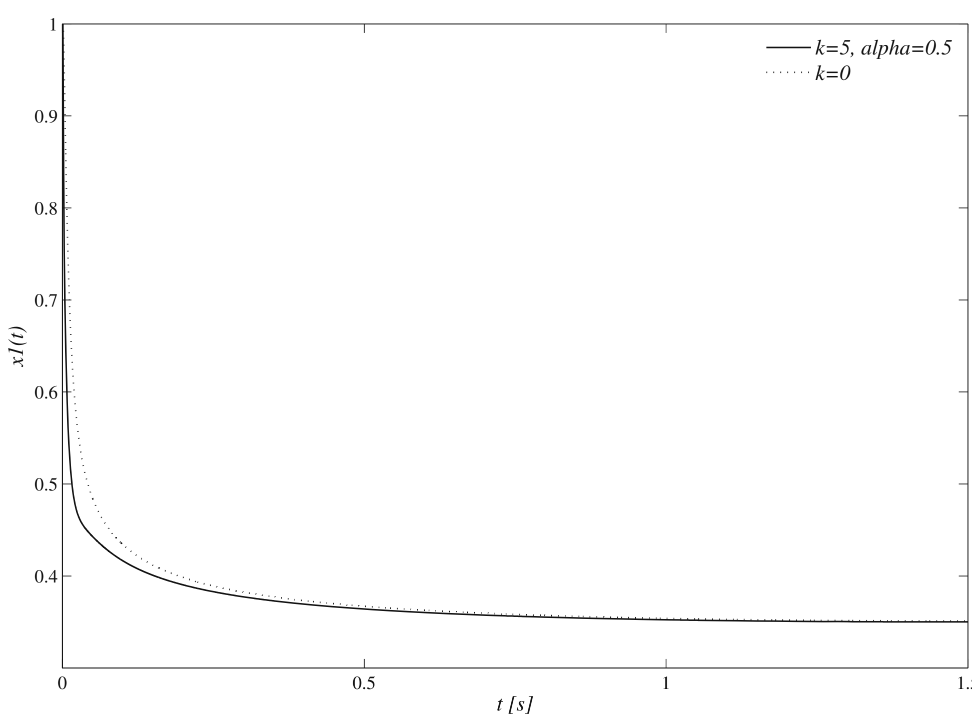
$$\dot{x}_1 = -x_1 - k\left[\frac{x_1(t)}{t^{\alpha}}\mathcal{A}(\alpha) + \sum_{i=2}^{N}\mathcal{B}(\alpha,i)\frac{\widetilde{V}_i}{t^{i-1+\alpha}}\right] + x_2\xi\left(1-\exp\left(B_p x_1 - A_p\right)\right) + \eta\left(1-\exp(B_s x_1 - A_s)\right) \quad (41)$$

Problem of minimizing (40) with constraints (41) and (21) that in the present case takes the form

$$\widetilde{V}_p^{(1)}(t) = -(p-1)\,t^{p-2}x_1(t), \quad \widetilde{V}_p(0) = 0, \quad p = 2, 3, \ldots \quad (42)$$

is the standard one as described above. In Fig. 8 we present $x_1(t)$ with $N = 4$ and $k = 5, \alpha = 0.5$

Fig. 8 The optimal solution of optimization problem defined by (40), (41) for $\alpha = 0.5$, $k = 0$, and $k = 5$ with $N = 4$



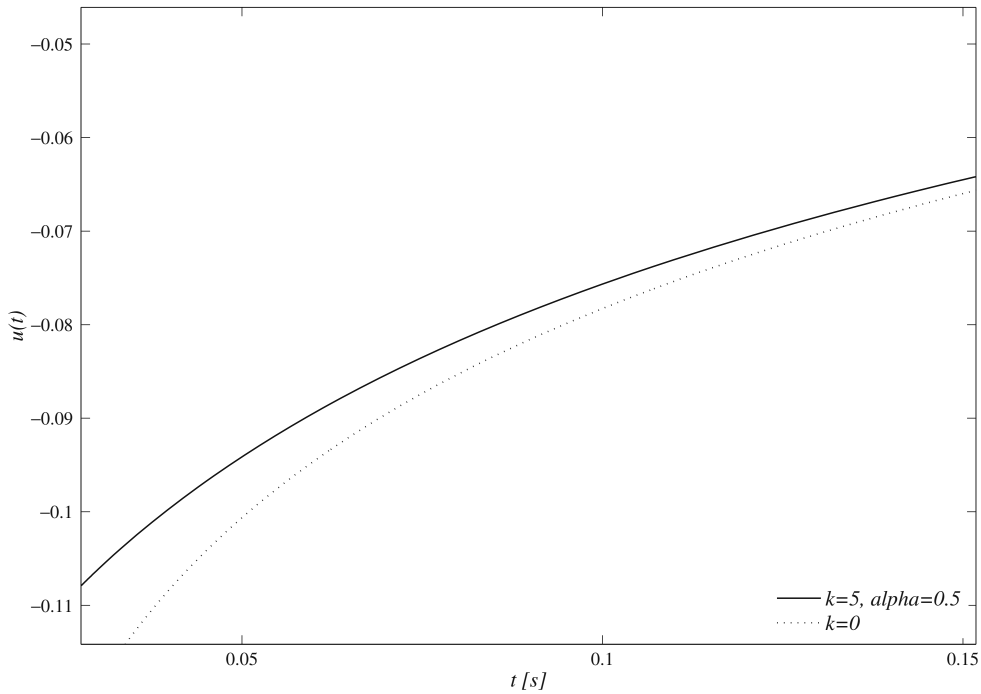


Fig. 9 Control law $u(t)$ of optimization problem defined by (40), (41) for $\alpha = 0.5$, $k = 0$, and $k = 5$ with $N = 4$

Finally in Fig. 9 we present the optimal control corresponding to $x_1(t)$ shown in Fig. 8.

6 Conclusion

In this work we analyzed the optimization problem for the case when the dynamics is described by a differential equation containing integer and fractional order derivatives. In our analysis, we assumed that the highest order derivative is of integer order. Thus we generalize the optimality conditions presented by Agrawal (2004). Our main results may be stated as:

1. For the problem of minimization of the functional (6) subject to the constraint (7) we obtained the optimality conditions in the form (11), (12). These optimality conditions involve both left and right Riemann–Liouville derivatives.
2. By using expansion formula for fractional derivatives, obtained by Atanackovic and Stankovic (2004) we transformed the optimization problem to the form involving integer derivatives only. Central result are expressions (19), (21) that gave the possibility to transform the optimization problem to a standard form to which Pontryagin's principle may be applied. Expressions (19), (21) for the fractional derivatives are of the type used earlier (not for optimization) by Yuan and Agrawal (1998, 2002) and Ruge and Trinks (2004).
3. We analyzed in detail two examples and showed the influence of the fractional derivative, i.e., memory effects on the solution.
4. We assumed that the highest order derivative in the equation describing the dynamics of the system (7) is of integer order. This assumption leads to the possibility of satisfying the initial conditions for all values of the fractional derivatives $0 < \alpha < 1$.
5. In the expansion (28) we used several values of N to examine sensitivity of the procedure on the number of terms (see Example 1). We found that $N = 4$ gives, in both examples treated here satisfactory convergence of the numerical solutions.
6. We comment of the values of α used. In the Example 1 we used several values of α to test the procedure. In Example 2 the value of α was determined from physical considerations (Petrovacki and Jelicic 2006).

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