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RESEARCH PAPER

Harish Agarwal · Chandan K. Mozumder · John E. Renaud · Layne T. Watson

An inverse-measure-based unilevel architecture for reliability-based design optimization

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Abstract Reliability-based design optimization (RBDO) is a methodology for finding optimized designs that are characterized with a low probability of failure. Primarily, RBDO consists of optimizing a merit function while satisfying reliability constraints. The reliability constraints are constraints on the probability of failure corresponding to each of the failure modes of the system or a single constraint on the system probability of failure. The probability of failure is usually estimated by performing a *reliability analysis*. During the last few years, a variety of different formulations have been developed for RBDO. Traditionally, these have been formulated as a *double-loop (nested) optimization* problem. The upper level optimization loop generally involves optimizing a merit function subject to reliability constraints, and the lower level optimization loop(s) compute(s) the probabilities of failure corresponding to the failure mode(s) that govern(s) the system failure. This formulation is, by nature, computationally intensive. Researchers have provided *sequential* strategies to address this issue, where the deterministic optimization and reliability analysis are *decoupled*, and the process is performed iteratively until convergence is achieved. These methods, though attractive in terms of obtaining a workable reliable design at considerably reduced computational costs, often lead to premature convergence and therefore yield spurious optimal designs. In this paper, a novel *unilevel* formulation for RBDO is developed. In the proposed formulation, the lower level optimization (evaluation of reliability constraints in the double-loop formulation) is replaced by its corresponding first-order Karush–Kuhn–Tucker (KKT) necessary optimality conditions at the upper level optimization. Such a replacement is computationally equivalent to solving the original nested optimization if the lower level optimization problem is solved by numerically satisfying the KKT conditions (which is typically the case). It is shown through the use of test problems that the proposed formulation is *numerically robust (stable) and computationally efficient* compared to the existing approaches for RBDO.

Keywords Reliability-based design optimization · Unilevel methods · Deterministic optimization · Variables · Parameters

H. Agarwal (✉) · C.K. Mozumder · J.E. Renaud
Department of Aerospace and Mechanical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA
e-mail: hagarwal@nd.edu
e-mail: cmozumde@nd.edu
e-mail: jrenaud@nd.edu

L.T. Watson
Department of Computer Science, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0106, USA
e-mail: ltw@cs.vt.edu

L.T. Watson
Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0106, USA

Nomenclature

$\mathbf{d}$	design variables
$\mathbf{p}$	parameters in deterministic optimization
$\mathbf{X}$	random variables
$\mathbf{U}$	standard normal random variables
$\boldsymbol{\theta}$	distribution parameters
$\boldsymbol{\eta}$	limit-state parameters
$\mathbf{g}^R$	failure-driven or probabilistic hard constraints
$\mathbf{g}^D$	deterministic constraints
$\mathbf{y}$	deterministic state variables
$\mathbf{Y}$	random state variables
β_i	reliability index of i^{th} failure mode
$\mathbf{x}^*$	most probable point in $\mathbf{x}$-space
$\mathbf{u}^*$	most probable point in $\mathbf{u}$-space
f	merit function
$\mathbf{g}^{rc}$	reliability constraints
P_i	probability of failure of i^{th} failure mode
P_{allow_i}	allowable probability of failure for i^{th} failure mode
P_{sys}	system probability of failure
$\mathbf{d}^l$	lower bounds on design variables
$\mathbf{d}^u$	upper bounds on design variables
N_{hard}	number of hard constraints
N_{soft}	number of soft constraints
β_{reqd_i}	desired reliability index of i^{th} failure mode

1 Introduction

In a deterministic design optimization, designs are often driven to the limit of the design constraints, leaving little or no latitude for uncertainties. The resulting deterministic optimum is usually associated with a high probability of failure of the artifact being designed, due to the influence of uncertainties inherently present during the modeling and manufacturing phases of the artifact and due to the uncertainties in the external operating conditions of the artifact. The uncertainties include variations in certain parameters, which are either controllable (e.g., dimensions) or uncontrollable (e.g., material properties), and model uncertainties and errors associated with the simulation tools used for simulation-based design (Oberkampf et al. 2000, 2001). In this research, variational uncertainties are modeled as continuous random variables. Other forms of uncertainty, such as model uncertainties and errors associated with simulation tools, are assumed to be minimal and that the analysis tools can reasonably predict the actual performance behavior.

Optimized deterministic designs determined without considering uncertainties can be unreliable and might lead to catastrophic failure of the artifact being designed. Uncertainties in simulation-based design are inherently present and need to be accounted for in the design optimization process. Reliability-based design optimization (RBDO) is a methodology that addresses this problem. In designing artifacts with multiple failure modes, it is important that an artifact be designed such that it is sufficiently reliable with respect to each of the critical failure modes or to the overall system failure. The reliability index, or the probability of failure corresponding to either a failure mode or the system, can be computed by performing a *reliability analysis*. In an RBDO formulation, the critical failure modes in deterministic optimization are replaced with constraints on probabilities of failure corresponding to each of the failure-driven modes or with a single constraint on the system probability of failure (Enevoldsen and Sorensen 1994).

Traditionally, researchers have formulated RBDO as a nested optimization problem [also known as a double-loop method (DLM)]. Such a formulation is, by nature, computationally expensive because of the inherent computational expense required for the reliability analysis, which itself involves the solution to an optimization problem. Solving such nested optimization problems are cost prohibitive, especially for large-scale multidisciplinary systems, which are themselves computationally intensive. Moreover, the computational cost associated with RBDO grows exponentially as the number of random variables and the number of critical failure modes increase. To alleviate the high computational cost, researchers have developed sequential RBDO methods. In these methods, a deterministic optimization and a reliability analysis are decoupled, and the procedure is repeated until desired convergence is achieved. However, such techniques usually lead to premature convergence and hence yield spurious optimal designs.

In this research, a new unilevel formulation for performing RBDO is developed. The proposed formulation provides improved robustness and provable convergence as compared to a unilevel variant given by Kuschel and Rackwitz (2000). The formulation given by Kuschel and Rackwitz (2000) replaces the direct first-order reliability method (FORM) problems [lower level optimization in the reliability index approach (RIA)] by their first-order necessary KKT optimality conditions. The FORM problem in RIA is numerically ill-conditioned (see Tu et al. (1999) for a review); the same is true for the formulation given by Kuschel and Rackwitz (2000). In this paper, the basic idea is to replace the *inverse* FORM problem [lower level optimization in the performance measure approach (PMA)] by its first-order KKT necessary optimality conditions at the upper level optimization. It was shown in Tu et al. (1999) that PMA is robust in terms of probabilistic constraint evaluation. The proposed method is shown to be computationally equivalent to the original nested optimization problem if the lower level optimization problem is solved by satisfying the KKT necessary condition (which is what most numerical optimization algorithms actually do). The unilevel method developed in this paper is observed to be more robust and has a provably convergent structure as compared to the one given in Kuschel and Rackwitz (2000). Both unilevel methods are computationally efficient compared to the traditional nested optimization formulations for RBDO.

2 Deterministic design optimization

In a deterministic design optimization, the designer seeks the optimum values of design variables for which the merit function is the minimum and the deterministic constraints are satisfied. A typical deterministic design optimization problem can be formulated as

$$\min \quad f(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \tag{1}$$

$$\text{subject to} \quad g_i^R(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \geq 0, \quad i = 1, .., N_{hard}, \tag{2}$$

$$g_j^D(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \geq 0, \quad j = 1, .., N_{soft}, \tag{3}$$

$$\mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^u, \tag{4}$$

where $\mathbf{d}$ are the design variables and $\mathbf{p}$ are the fixed parameters of the optimization problem. g_i^R is the i^{th} hard constraint that models the i^{th} critical failure mechanism of the system (e.g., stress, deflection, loads, etc.). g_j^D is the j^{th} soft constraint that models the j^{th} deterministic constraint due to other design considerations (e.g., cost, marketing, etc.). The design space is bounded by $\mathbf{d}^l$ and $\mathbf{d}^u$. The merit function and the constraints are explicit functions of $\mathbf{d}$, $\mathbf{p}$, and $\mathbf{y}(\mathbf{d}, \mathbf{p})$. $\mathbf{y}(\mathbf{d}, \mathbf{p})$ are the outputs of analysis tools that are used to predict performance characteristics of an artifact.

The class of problems known as multidisciplinary problems are characterized by two or more disciplinary analyses. These problems involve complex coupled engineering

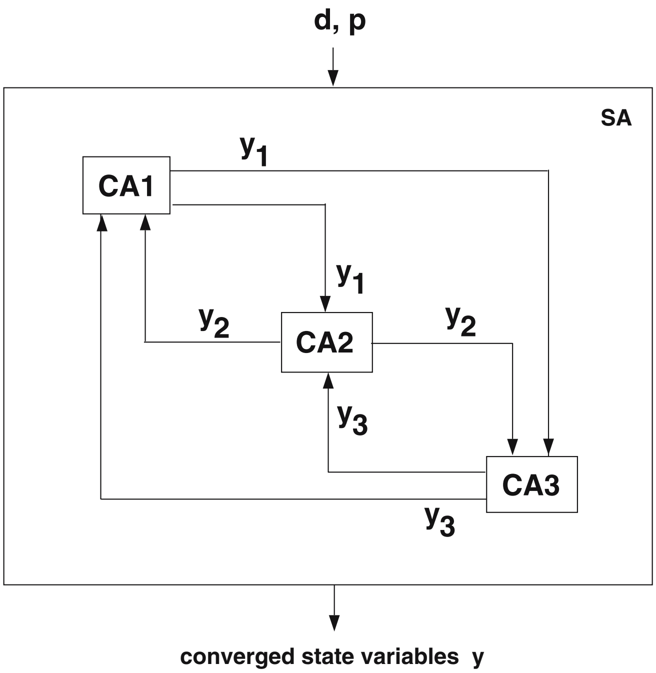

Fig. 1 Example of a multidisciplinary system analysis

simulation models. The solution of these coupled simulation models is referred to as a *system analysis*. A multidisciplinary system analysis can be illustrated by Fig. 1. Here, the system analysis consists of three disciplines or contributing analyses (CAs). Each contributing analysis (CA) makes use of a disciplinary simulation code. State information is exchanged between the three CAs, and iteration is performed until convergence is achieved. Coupling between the various simulation codes exists in the form of shared design variables and both input and output state performances. For a given set of independent variables that uniquely define the artifact, referred to as *design variables*, **d**, the system analysis is intended to determine the corresponding set of attributes which characterize the system performance, referred to as *state variables*, **y**. An analysis of such systems requires users to iterate between individual disciplines until the states are converged and consistent (Gu et al. 2000).

A deterministic optimization formulation does not account for the uncertainties in the design variables and parameters. Optimized designs based on a deterministic formulation are usually associated with a high probability of failure because of the likely violation of certain hard constraints in service. This is particularly true if the hard constraints are active at the deterministic optimum solution. In today's competitive marketplace, it is very important that the resulting designs are optimal as well as reliable. This is usually achieved by replacing a deterministic optimization formulation with a *reliability-based design optimization* formulation, where the critical hard constraints are replaced with reliability constraints.

3 Reliability-based design optimization (RBDO)

In the last two decades, researchers have proposed a variety of frameworks for efficiently performing RBDO. A careful survey of the literature reveals that the various RBDO methods can be divided into three broad categories, viz., *double-loop methods*, *sequential methods*, and *unilevel methods*.

3.1 Double-loop methods for RBDO

Traditionally, the reliability-based optimization problem has been formulated as a double-loop optimization problem. In a typical RBDO formulation, the critical hard constraints from the deterministic formulation are replaced by reliability constraints, as in

$$\min \quad f(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \tag{5}$$

$$\text{subject to} \quad \mathbf{g}^{rc}(\mathbf{X}, \boldsymbol{\eta}) \geq 0, \tag{6}$$

$$g_j^D(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \geq 0, \qquad j = 1, .., N_{soft}, \tag{7}$$

$$\mathbf{d}^l \leq \mathbf{d} \leq \mathbf{d}^u, \tag{8}$$

where $\mathbf{g}^{rc}$ are the reliability constraints. They are either constraints on probabilities of failure corresponding to each hard constraint or are a single constraint on the overall system probability of failure. In this paper, only component failure modes are considered. It should be noted that the reliability constraints depend on the random variables $\mathbf{X}$ and the limit-state parameters $\boldsymbol{\eta}$. The distribution parameters of the random variables are obtained from the design variables $\mathbf{d}$ and the fixed parameters $\mathbf{p}$ (see the section on "Reliability analysis" below). $\mathbf{g}^{rc}$ can be formulated as

$$g_i^{rc} = P_{allow_i} - P_i, \qquad i = 1, .., N_{hard}, \tag{9}$$

where P_i is the failure probability of the hard constraint g_i^R at a given design, and P_{allow_i} is the allowable probability of failure for this failure mode. The probability of failure is usually estimated by employing standard reliability techniques. A brief description of standard reliability methods is given in the next section. It has to be noted that the RBDO formulation given above [(5, 6, 7, and 8)] assumes that the violation of soft constraints due to variational uncertainties is permissible and can be traded off for more reliable designs. For practical problems, design robustness represented by the merit function and the soft constraints could be a significant issue, one that would require the solution to a hybrid robustness and RBDO formulation.

3.1.1 Reliability analysis

Reliability analysis is a tool to compute the reliability index or the probability of failure corresponding to a given failure mode or for the entire system (Haldar and Mahadevan 2000). The uncertainties are modeled as continuous random variables, $\mathbf{X} = (X_1, X_2, ..., X_n)^T$, with known (or assumed) joint cumulative distribution function (CDF), $F_{\mathbf{X}}(\mathbf{x})$. The design variables, $\mathbf{d}$, consist of either distribution parameters $\boldsymbol{\theta}$ of the random variables $\mathbf{X}$, such as means, modes, standard deviations, and coefficients of variation, or deterministic parameters, also called limit-state parameters, denoted by $\boldsymbol{\eta}$. The design parameters $\mathbf{p}$ consist of either the means, the modes, or any first-order distribution quantities of certain random variables. Mathematically, this can be represented by the statements

$$\left[\mathbf{p}, \mathbf{d}\right] = [\boldsymbol{\theta}, \boldsymbol{\eta}], \tag{10}$$

$$\mathbf{p} \text{ is a subvector of } \theta. \tag{11}$$

Random variables can be consistently denoted as $\mathbf{X}(\boldsymbol{\theta})$, and the i^{th} failure mode can be denoted as $g_i^R(\mathbf{X}, \boldsymbol{\eta})$. In the following, $\mathbf{x}$ denotes a realization of the random variables $\mathbf{X}$, and the subscript i is dropped without loss of clarity. Letting $g^R(\mathbf{x}, \boldsymbol{\eta}) \leq 0$ represent the failure domain, and $g^R(\mathbf{x}, \boldsymbol{\eta}) = 0$ be the so-called limit-state function, the time-invariant probability of failure for the hard constraint is given by

$$P(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_{g^R(\mathbf{x}, \boldsymbol{\eta}) \leq 0} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}, \tag{12}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of $\mathbf{X}$. It is usually impossible to find an analytical expression for the above integral. In standard reliability techniques, a probability distribution transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is usually employed. An arbitrary n-dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ is mapped into an independent standard normal vector $\mathbf{U} = (U_1, U_2, ..., U_n)^T$. This transformation is known as the *Rosenblatt transformation* (Rosenblatt 1952). The standard normal random variables are characterized by a zero mean and unit variance. The limit-state function in $\mathbf{U}$-space can be obtained as $g^R(\mathbf{x}, \boldsymbol{\eta}) = g^R(T^{-1}(\mathbf{u}), \boldsymbol{\eta}) = G^R(\mathbf{u}, \boldsymbol{\eta}) = 0$. The failure domain in $\mathbf{U}$-space is $G^R(\mathbf{u}, \boldsymbol{\eta}) \leq 0$. Equation (12) thus transforms to

$$P_i(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_{G^R(\mathbf{u}, \boldsymbol{\eta}) \leq 0} \phi_{\mathbf{U}}(\mathbf{u})\, d\mathbf{u}, \tag{13}$$

where $\phi_{\mathbf{U}}(\mathbf{u})$ is the standard normal density. If the limit-state function in $\mathbf{U}$-space is affine, i.e., if $G^R(\mathbf{u}, \boldsymbol{\eta}) = \boldsymbol{\alpha}^T\mathbf{u} + \beta$, then an exact result for the probability of failure is $P_f = \Phi\left(-\frac{\beta}{\|\boldsymbol{\alpha}\|}\right)$, where $\Phi(\cdot)$ is the cumulative Gaussian distribution function. If the limit-state function is close to being affine, i.e., if $G^R(\mathbf{u}, \boldsymbol{\eta}) \approx \boldsymbol{\alpha}^T\mathbf{u} + \beta$ with $\beta = -\boldsymbol{\alpha}^T\mathbf{u}^*$, where $\mathbf{u}^*$ is the solution of the following optimization problem,

$$\min \ ||\mathbf{u}|| \tag{14}$$

$$\text{subject to } \ G^R(\mathbf{u}, \boldsymbol{\eta}) = 0, \tag{15}$$

then the first-order estimate of the probability of failure is $P_f = \Phi\left(-\frac{\beta}{\|\boldsymbol{\alpha}\|}\right)$, where $\boldsymbol{\alpha}$ represents a normal to the manifold (15) at the solution point. The solution $\mathbf{u}^*$ of the above optimization problem, the so-called design point, β-point, or the *most probable point* (MPP) of failure, defines the reliability index $\beta_p = -\frac{\boldsymbol{\alpha}^T\mathbf{u}^*}{\|\boldsymbol{\alpha}\|}$. This method of estimating the probability of failure is known as the FORM (Haldar and Mahadevan 2000).

In the second-order reliability method, the limit-state function is approximated as a quadratic surface. A simple closed form solution for the probability computation using a second-order approximation was given by Brietung (1984) using the theory of asymptotic approximations as

$$P_f(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_{G^R(\mathbf{u}, \boldsymbol{\eta}) \leq 0} \phi_{\mathbf{U}}(\mathbf{u})\, d\mathbf{u}$$
$$\approx \Phi(-\beta_p) \prod_{l=1}^{n-1} (1 - \kappa_l)^{-1/2}, \tag{16}$$

where the κ_l is related to the principal curvatures of the limit-state function at the minimum distance point $\mathbf{u}^*$, and β_p is the reliability index using FORM. Brietung (1984) showed that the second-order probability estimate asymptotically approaches the first-order estimate as β_p approaches infinity if $\beta_p \kappa_l$ remains constant.

The first-order approximation, $P_f \approx \Phi(-\beta_p)$, is sufficiently accurate for most practical cases. Thus, only first-order approximations of the probability of failure are used in practice. Using the FORM estimate, the reliability constraints in (9) can be written in terms of reliability indices as

$$g_i^{rc} = \beta_i - \beta_{reqd_i}, \tag{17}$$

where β_i is the first-order reliability index, and $\beta_{reqd_i} = -\Phi^{-1}(P_{allow_i})$ is the desired reliability index for the i^{th} hard constraint. When the reliability constraints are formulated as given in (17), the approach is referred to as the RIA.

It should be noted that the reliability analysis involves a probability distribution transformation, the search for the MPP, and the evaluation of the cumulative Gaussian distribution function. To solve the FORM problem (14, 15, and 16), various algorithms have been reported in the literature (see Liu and Kiureghian (1991) for a review). The solution typically requires many system analysis evaluations. Moreover, there might be cases where the optimizer may fail to provide a solution to the FORM problem, especially when the limit-state surface is far from the origin in $\mathbf{U}$-space or when the case $G^R(\mathbf{u}, \boldsymbol{\eta}) = 0$ never occurs at a particular design variable setting.

In design automation, it cannot be known a priori what design points the upper level optimizer will visit; therefore, it is not known if the optimizer for the FORM problem will provide a consistent result. This problem was addressed recently by Padmanabhan and Batill (2002) by using a trust region algorithm for equality-constrained problems. For cases when $G^R(\mathbf{u}, \boldsymbol{\eta}) = 0$ does not occur, the algorithm provided the best possible solution for the problem through

$$\min \ \|\mathbf{u}\| \tag{18}$$

$$\text{subject to } \ G^R(\mathbf{u}, \boldsymbol{\eta}) = c. \tag{19}$$

The reliability constraints formulated by the RIA are therefore not robust. RIA is usually more effective if the probabilistic constraint is violated, but it yields a singularity if the design has zero failure probability (Tu et al. 1999). To overcome this difficulty, Tu et al. (1999) provided an improved formulation to solve the RBDO problem. In this method, known as the PMA, the reliability constraints are stated by an inverse formulation as

$$g_i^{rc} = G_i^R(\mathbf{u_i}^*_{\beta=\rho}, \boldsymbol{\eta}) \quad i = 1, .., N_{hard}. \tag{20}$$

$\mathbf{u}_i^*$ is the solution to the inverse reliability analysis (IRA) optimization problem

$$\min\ G_i^R(\mathbf{u}, \boldsymbol{\eta}) \tag{21}$$

$$\text{subject to} \quad \|\mathbf{u}\| = \rho = \beta_{reqd_i}, \tag{22}$$

where the optimum solution $\mathbf{u_i}^*_{\beta=\rho}$ corresponds to MPP in IRA of the i^{th} hard constraint. Solving RBDO by the PMA formulation is usually more efficient and robust than the RIA formulation where the reliability is evaluated directly. The efficiency lies in the fact that the search for the MPP of an inverse reliability problem is easier to solve than the search for the MPP corresponding to an actual reliability. The RIA and the PMA approaches for RBDO are essentially inverse of one another and would yield the same solution if the constraints are active at the optimum (see Tu et al. (1999) for a review). If the constraint on the reliability index (as in the RIA formulation) or the constraint on the optimum value of the limit-state function (as in the PMA formulation) is not active at the solution, the reliable solution obtained from the two approaches might differ. Similar RBDO formulations were independently developed by other researchers (Polak et al. 2000; Royset et al. 2001; Kirjner-Neto et al. 1998). In these RBDO formulations, constraint (22) is considered as an inequality constraint ($\|\mathbf{u}\| \leq \beta_{reqd_i}$), which is a more robust way of handling the constraint on the reliability index. The major difference lies in the fact that in these papers, semi-infinite optimization algorithms were employed to solve the RBDO problem. Semi-infinite optimization algorithms solve the inner optimization problem approximately. However, the overall RBDO is still a nested double-loop optimization procedure. As mentioned earlier, such formulations are computationally intensive for problems where the function evaluations are expensive. Moreover, the formulation becomes impractical when the number of hard constraints increase, which is often the case in real-life design problems. To alleviate the computational cost associated with the nested formulation, sequential RBDO methods have been developed.

3.2 Sequential methods for RBDO

Sequential RBDO methods include a variety of different approaches proposed by different researchers. Chen and Du (2002) developed a decoupled sequential probabilistic design methodology. In this framework, the deterministic optimization and the reliability assessment are decoupled from one another. During each cycle, a deterministic optimization problem is solved, followed by reliability assessment and a convergence check. Chen et al. (1997) also developed a sequential RBDO methodology that was recently generalized for nonnormal distributions by Wang and Kodiyalam (2002) and extended for multidisciplinary systems by Agarwal et al. (2003). In this methodology, the lower level optimization is eliminated, and the MPP of failure corresponding to the probabilistic constraints is estimated implicitly by using a nonlinear transformation based on the direction cosines of the hard constraints at the mean values of the random variables. This methodology is shown to be extremely efficient. However, for highly nonlinear limit-state functions, the estimate of the MPP of failure given by the nonlinear transformation might be very different from the actual MPP of failure, and the framework may fail to converge to the true solution. The drawback of sequential RBDO methodologies is that a local optimum cannot be guaranteed. Such methodologies can lead to spurious optimal designs.

It has been noted that the traditional reliability-based optimization problem is a nested optimization problem. Solving such nested optimization problems for a large number of failure-driven constraints and/or nondeterministic variables is extremely expensive. Researchers have developed sequential approaches to speed up the optimization process and to obtain a consistent reliability-based design. To address the issue of obtaining spurious optimal designs, a new sequential optimization strategy for reliability-based design is developed in Agarwal and Renaud (2004).

3.3 Unilevel methods for RBDO

RBDO is typically formulated as a nested optimization problem requiring a large number of system analyses. The major concern in evaluating reliability constraints is the fact that reliability analysis methods are formulated as optimization problems (Rackwitz 2001). To overcome this difficulty, a unilevel formulation was developed by Kuschel and Rackwitz (2000). In their method, the direct FORM problem [lower level optimization—(14), (15), and (16)] is replaced by the corresponding first-order Karush–Kuhn–Tucker (KKT) optimality conditions of the first-order reliability problem. As mentioned earlier, the direct FORM problem can be ill-conditioned, and the same may be true for the unilevel formulation given by Kuschel and Rackwitz (2000). The reason is that the probabilistic hard constraints might have a zero failure probability at a particular design setting, and hence the optimizer might not converge due to the hard constraints (which are posed as equality constraints) being not satisfied. Moreover, the conditions under which such a replacement is equivalent to the original bilevel formulation were not detailed in Kuschel and Rackwitz (2000).

4 Proposed unilevel method for RBDO

In this research, a new unilevel formulation is developed. The conditions under which the proposed formulation is computationally equivalent to the traditional bilevel formulation are KKT constraint qualifications and convexity assumptions. Initial studies indicate that the proposed formulation is inherently robust and computationally efficient compared to other RBDO formulations.

The main focus of this research has been to develop a robust and efficient unilevel formulation for performing RBDO. As mentioned earlier, the probabilistic constraint specification using the PMA is robust compared to the RIA. However, the methodology is still nested and is hence expensive. In this research, the IRA optimization problem is replaced by the corresponding first-order necessary KKT optimality conditions. The KKT conditions for the reliability constraints similar to PMA (21 and 22) are used. The treatment of (22) is a bit subtle. No simple modification of (22) will result in an equality constraint that is both quasiconvex and quasiconcave, which would be required for the sufficiency of the KKT conditions. For necessity of the KKT conditions, observe that $\|\mathbf{u}\| - \rho$ is convex and $\|\mathbf{u}\| - \rho \le 0$ trivially satisfies Slater's constraint qualification (feasible set has a strictly interior point) (Mangasarian 1994). Assume that $G^R(\mathbf{u}, \boldsymbol{\eta})$ is pseudoconvex with respect to $\mathbf{u}$ for each fixed $\boldsymbol{\eta}$. Now $G^R(\mathbf{u}, \boldsymbol{\eta})$ pseudoconvex and $\|\mathbf{u}\| - \rho$ convex mean that the KKT conditions are also sufficient, hence the original and KKT formulation will be equivalent. Therefore, to facilitate development of the current method, the inverse FORM can be restated as

$$\min\ G_i^R(\mathbf{u}, \boldsymbol{\eta}) \tag{23}$$

$$\text{subject to}\ \ \|\mathbf{u}\| \le \rho. \tag{24}$$

The Lagrangian corresponding to the optimiation problem is

$$L = G^R(\mathbf{u}, \boldsymbol{\eta}) + \lambda(\|\mathbf{u}\| - \rho), \tag{25}$$

where λ is the scalar Lagrange multiplier. The first-order necessary conditions for the problem are

$$\nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta}) + \lambda \nabla_{\mathbf{u}}(\|\mathbf{u}^*\| - \rho) = 0, \tag{26}$$

$$\|\mathbf{u}^*\| - \rho \le 0, \tag{27}$$

$$\lambda \ge 0, \tag{28}$$

$$\lambda(\|\mathbf{u}^*\| - \rho) = 0 \tag{29}$$

where $\mathbf{u}^*$ is the solution point $\mathbf{u}^*_{\beta=\rho}$ of the inverse reliability optimization problem when $\|\mathbf{u}^*\| = \rho$. $\mathbf{u}^* = 0$ is a special degenerate case, so assume henceforth that $\mathbf{u}^* \neq 0$. From (26), we have (assuming $\lambda \neq 0$)

$$\mathbf{u}^* = -\frac{1}{\lambda}\ \|\mathbf{u}^*\|\ \nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta}). \tag{30}$$

Observe that (30) implies

$$\lambda = \|\nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta})\| \ge 0, \tag{31}$$

which is consistent with (28) and is valid even if $\lambda = 0$. Substituting for λ in (30) and rearranging,

$$\frac{\nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta})}{\|\nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta})\|} = -\frac{\mathbf{u}^*}{\|\mathbf{u}^*\|}. \tag{32}$$

Equation (32) says that $\mathbf{u}^*$ and $\nabla_{\mathbf{u}} G^R(\mathbf{u}^*, \boldsymbol{\eta})$ point in opposite directions, which is consistent with $\mathbf{u}^*$ being the closest point in the manifold $G^R(\mathbf{u}, \boldsymbol{\eta}) =$ constant to the origin.

Equation (32) is true for all $\boldsymbol{\eta}$, if $\mathbf{u}^* = \mathbf{u}^*_{\beta=\rho}$ is the solution to the inverse reliability optimization problem, because $\rho - \|\mathbf{u}\| \le 0$ satisfies the reverse convex constraint qualification [the equality constraint (22) is equivalent to the convex constraint (24) and $\rho - \|\mathbf{u}\| \le 0$, hence constraint qualifications are satisfied and the KKT condition (32) is necessary]. In general, without the pseudoconvexity assumption on G^R, solving (32) does not necessarily imply that $\mathbf{u}^*$ is the optimal solution to the optimization problem.

It should be noted that the KKT conditions for the direct and inverse FORM problems differ only in terms of what constraints are being presented as equality constraints to the upper level optimizer. When using the KKT conditions of the direct FORM problem in the upper level optimization, the limit-state function is presented as an equality constraint, and the constraint on the reliability index is an inequality constraint. As mentioned earlier, it is possible to have cases where the limit-state function never becomes zero. In other words, it is associated with zero (or one) failure probability. When such a case occurs, the formulation given by Kuschel and Rackwitz (2000) might fail to yield a solution. In other words, it is numerically unstable.

In this research, the first-order conditions of the *inverse FORM* problem are used. The corresponding KKT conditions for the inverse reliability problem (23 and 24) are

$$h1_i \equiv \nabla_{\mathbf{u}} G_i^R(\mathbf{u}_i, \boldsymbol{\eta}) + \lambda_i \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|} = 0, \tag{33}$$

$$g1_i \equiv ||\mathbf{u}_i|| - \beta_{reqd_i} \le 0, \tag{34}$$

$$h2_i \equiv \lambda_i(\|\mathbf{u}_i\| - \beta_{reqd_i}) = 0, \tag{35}$$

$$g2_i \equiv \lambda_i \ge 0. \tag{36}$$

Using these first-order optimality conditions, the unilevel RBDO architecture can be stated as follows

$$\min_{\mathbf{d}_{aug}}\quad f(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \tag{37}$$

$$\mathbf{d}_{aug} = [\mathbf{d}, \mathbf{u}_1, .., \mathbf{u}_{N_{hard}}, \lambda_1, .., \lambda_{N_{hard}}]$$

$$\text{sub. to}\quad \mathbf{G}_i^R(\mathbf{u}, \boldsymbol{\eta}) \ge 0 \quad i = 1, .., N_{hard}, \tag{38}$$

$$h1_i = 0 \quad i = 1, .., N_{hard}, \tag{39}$$

$$h2_i = 0 \quad i = 1, .., N_{hard}, \tag{40}$$

$$g1_i \le 0 \quad i = 1, .., N_{hard}, \tag{41}$$

$$g2_i \ge 0 \quad i = 1, .., N_{hard}, \tag{42}$$

$$g_j^D(\mathbf{d}, \mathbf{p}, \mathbf{y}(\mathbf{d}, \mathbf{p})) \ge 0 \quad j = 1, .., N_{soft}, \tag{43}$$

$$\mathbf{d}^l \le \mathbf{d} \le \mathbf{d}^u. \tag{44}$$

If $\mathbf{d}^*$ is a solution of (5), (6), (7), and (8), then there exist $\mathbf{u}_i^*$ and λ_i^* such that $[\mathbf{d}^*, \mathbf{u}_1^*, .., \mathbf{u}_{N^*_{hard}}, \lambda_1^*, .., \lambda^*_{N_{hard}}]$ is a solution of (37), (38), (39), (40), (41), (42), (43), and (44).

The converse is true under the mild assumption that all the functions $G_i^R(\mathbf{u}, \boldsymbol{\eta})$ are pseudoconvex in $\mathbf{u}$ for each fixed $\boldsymbol{\eta}$.

It should be noted that the dimensionality of the problem has increased as in the unilevel method given in Kuschel and Rackwitz (2000). The optimization is performed with respect to the design variables $\mathbf{d}$, the MPPs of failure, and the Lagrange multipliers, simultaneously. At the beginning of the optimization, $\mathbf{u}_i$ does not correspond to the true MPP at the design $\mathbf{d}$. The exact MPPs of failure $\mathbf{u}_i^*$ and the optimum design $\mathbf{d}^*$ are found at convergence.

5 Test problems

The proposed method is implemented for a simple analytical problem to illustrate the method. A higher dimensional multidisciplinary structures control problem is used to illustrate the efficacy of the method for real engineering problems. The double-loop methods (DLM) for RBDO, which use the RIA or the PMA for reliability constraint evaluation, are compared to the unilevel methods on the analytical problem. The unilevel method developed by Kuschel and Rackwitz (2000) is referred as *unilevel-RIA*, and the method developed in this paper is referred as *unilevel-PMA*. The proposed methodology (*unilevel-PMA*) is also compared with the double-loop PMA approach on the structures control problem.

5.1 Analytical problem

The method is illustrated with a small analytical multidisciplinary test problem. This problem is chosen to illustrate the robustness of the proposed formulation compared to other methods. Even though the problem is just two-dimensional, it is sufficiently nonlinear and has the attributes of a general multidisciplinary problem. This problem has two design variables, d_1 and d_2, and two parameters, p_1 and p_2. There are two random variables, X_1 and X_2. The design parameters, p_1 and p_2, are the means of the random variables, X_1 and X_2, respectively. This problem involves a coupled system analysis and has two CAs. The problem has two hard constraints, g_1^R and g_2^R. The RBDO problem in standard form is as follows.

$$\begin{aligned}
&\text{Minimize} : d_1^2 + 10d_2^2 + y_1\\
&\text{subject to} : g_1^R = Y_1(\mathbf{X}, \eta)/8 - 1 \geq 0\\
&\qquad g_2^R = 1 - Y_2(\mathbf{X}, \eta)/5 \geq 0\\
&\qquad -10 \leq d_1 \leq 10\\
&\qquad 0 \leq d_2 \leq 10\\
&\text{where} \quad d_1 = \eta_1, d_2 = \eta_2\\
&\text{and} \quad p_1 = \mu x_1 = 0,\ p_2 = \mu x_2 = 0\\
&CA_1 : \quad Y_1(\mathbf{X}, \eta) = \eta_1^2 + \eta_2 - 0.2Y_2(\mathbf{X}, \eta) + X_1;\\
&\qquad y_1(\mathbf{d}, \mathbf{p}) = d_1^2 + d_2 - 0.2y_2(\mathbf{d}, \mathbf{p}) + p_1;\\
&CA_2 : \quad Y_2(\mathbf{X}, \eta) = \eta_1 - \eta_2^2 + \sqrt{Y_1(\mathbf{X}, \eta)} + X_2;\\
&\qquad y_2(\mathbf{d}, \mathbf{p}) = d_1 - d_2^2 + \sqrt{y_1(\mathbf{d}, \mathbf{p})} + p_2.
\end{aligned}$$

It is assumed that the random variables X_1 and X_2 have a uniform distribution over the intervals $[-1, 1]$ and $[-0.75, 0.75]$, respectively. The desired value of the reliability index β_{reqd_i} (for $i = 1,2$) is chosen as 3 for both the hard constraints.

Figure 2 shows the contours of the merit function and the constraints. The zero contours of the hard constraints are plotted at the design parameters, p_1 and p_2 (mean of the random variables, X_1 and X_2). It should be noted that in deterministic optimization, two local optima exist for this problem. At the global solution, only the first hard constraint is active, whereas at the local solution, both the hard constraints are active. They are shown by star symbols. Both of these solutions can be located easily by choosing different starting points in the design space.

Similarly, two local optimum designs exist for the RBDO problem as well. Both reliable designs get pushed into the feasible region, characterized with a higher merit function value and a lower probability of failure. They are shown by the shaded squares in Fig. 3.

To locate the two local optimal solutions of this problem, two different starting points, $[-5, 3]$ and $[5, 3]$, are chosen. The results corresponding to the starting point $[-5, 3]$ are listed in Table 1.

Starting at the design $\mathbf{d} = [-5, 3]$, the proposed unilevel method, which uses the KKT conditions of the PMA to prescribe the probabilistic constraints, converges to the reliable optimum point without any difficulty. The proposed unilevel method requires 24 system analysis evaluations as compared to 225 when using the traditional double-loop PMA method. Analytical gradients were used in implementing this problem for all methods. Note that the DLM that uses the RIA to prescribe the probabilistic constraints does not converge. For the designs that are visited by the upper level optimizer (say, $\mathbf{d}^k$ at the k^{th} iteration), the FORM problem does not have a solution (because of zero failure probability at these

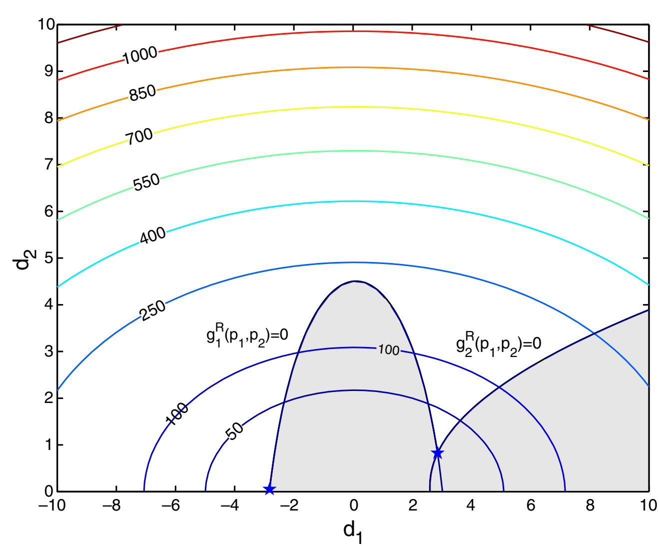

Fig. 2 Contours of objective and constraints

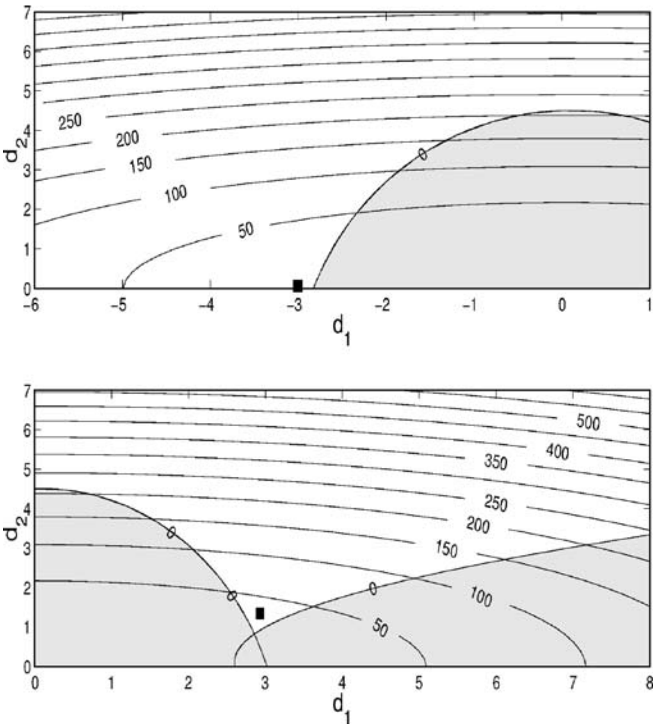

Fig. 3 Plot showing two reliable optima

designs). Similar conclusions can be drawn for the unilevel-RIA method. Starting from the design [−5, 3], the optimizer tries to find the local design [−3.006, 0.049]. However, it turns out that at this design, the second hard constraint, g_2^R, is never zero in the space of uniformly distributed random variables, **X**. Because in the unilevel-RIA method the limit-state function is enforced as an equality constraint, the optimizer does not converge.

The results corresponding to the starting point [5, 3] are listed in Table 2. Note that the DLM that uses the RIA for probabilistic constraint evaluation fails to converge for this starting point, too. Again, the reason for this is that there is zero failure probability (infinite reliability index) at the designs visited by the upper level optimizer and therefore the lower level optimizer does not provide any true solution. All the other methods converge to the same local optimum solution. The computational cost associated with the two unilevel methods is comparable. However, the unilevel-PMA method developed in this study is numerically robust compared to the unilevel-RIA method. Both unilevel methods show remarkable computational savings as compared to the DLMs.

Table 1 Starting point [−5, 3], solution [−3.006, 0.049]

Cost measure	Double-loop		Unilevel	
	RIA	PMA	RIA	PMA
SA calls	Not converged	225	Not converged	24

Table 2 Starting point [5, 3], solution [2.9277, 1.3426]

Cost measure	Double-loop		Unilevel	
	RIA	PMA	RIA	PMA
SA calls	Not converged	184	24	21

5.2 Control-augmented structures problem

Figure 4 shows the control-augmented structure as proposed by Sobieszczanski-Sobieski et al. (1990). This test problem has been used by Padmanabhan and Batill (2002) for testing RBDO methodologies. For the sake of completeness, the problem as described in Padmanabhan and Batill (2002) is given here. There are two coupled disciplines (CAs) in this problem. They are the structures subsystem and the controls subsystem. The structure is a five-element cantilever beam, numbered 1–5 from the free end to the fixed end, as shown in the Fig. 4. Each element is of equal length, but the breadth and height are design variables. Three static loads, T_1, T_2, and T_3, are applied to the first three elements. The beam is also acted on by a time varying force P, which is a ramp function. Controllers A and B are designed as an optimal linear quadratic regulator to control the lateral and rotational displacements of the free end of the beam, respectively. The analysis is coupled because the weights of the controllers, which are assumed to be proportional to the control effort, are required for the mass matrix of the structure, and one requires the eigenfrequencies and eigenvectors of the structure in the modal analysis for designing the controller as shown in Fig. 5. The damping matrix is taken to be proportional to the stiffness matrix by a factor c for the dynamic analysis of the structure. This damping parameter is also a design variable. The constraints arise due to constraints on static stresses, static and dynamic displacements, and the natural frequencies. The main objective is to minimize the total weight of the beam and the controllers.

5.3 Computation of sensitivities of the SA

Sensitivities for the control-augmented system analysis can be estimated using a finite difference scheme, or one can use analytic techniques. The sensitivities obtained from the analytical techniques are superior to those calculated from finite difference techniques especially when used for coupled systems because the use of finite difference techniques can give inaccurate and "noisy" derivatives and also because it is difficult to obtain accurate sensitivities for certain outputs

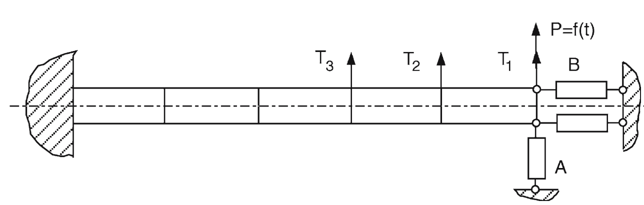

Fig. 4 Control-augmented structures problem

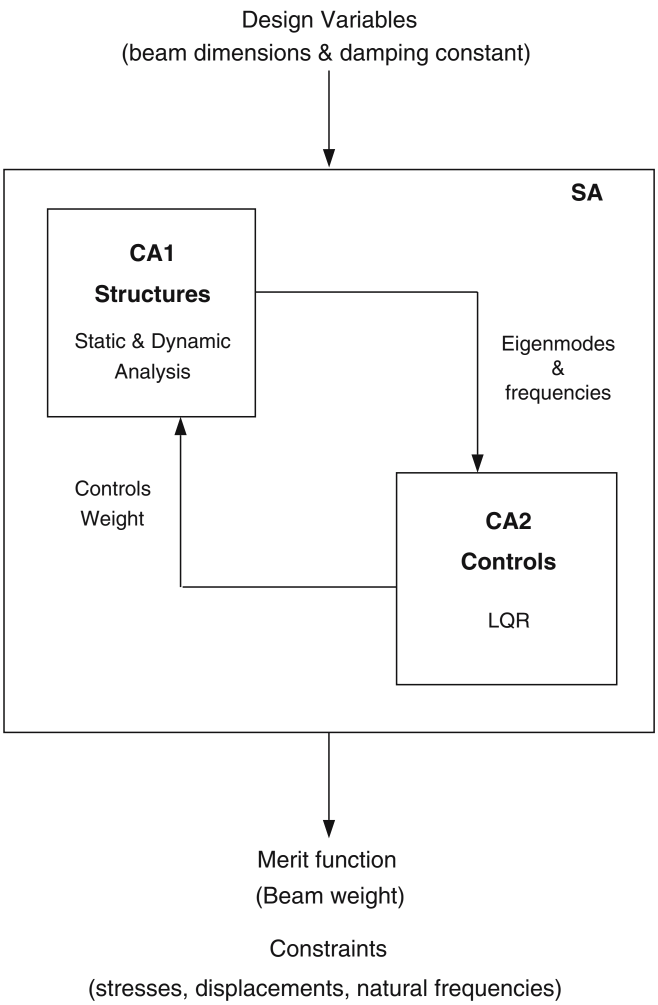

Fig. 5 Coupling in control-augmented structures problem

like the natural frequencies and the mode shapes using finite differencing.

Because the problem being considered is coupled, one needs to use global sensitivity equations, which are based on the implicit function differentiation rule. Sensitivities of the outputs of the structures' module can be found using analytic and numerical techniques (Haftka et al. 1990). The sensitivities of the static displacements and stresses are quite easy to compute, but the computation of sensitivities of natural frequencies and corresponding mode shapes is more involved and can be done using various methods like Nelson's method, the modal method, and the modified modal method. Sutter et al. (1988) compare these methods in terms of computational costs and rate of convergence. Analytic sensitivities of the controls' output require the computation of sensitivities of the solution of an algebraic Riccati equation used for obtaining a linear quadratic regulator as described by Khot (1994).

The design variables for this test problem are

$$\mathbf{d} = [b_1, b_2, b_3, b_4, b_5, h_1, h_2, h_3, h_4, h_5, c]^T, \quad (45)$$

where

b_i, h_i = breadth and height of the i^{th} element, respectively, and
c = damping matrix to stiffness matrix ratio (scalar).

The random variables in the problem are

ρ = density of the beam material,
E = modulus of elasticity of the beam material, and
σ_a = ultimate static stress.

The constraints for the problem are formulated in terms of the allowable displacements (lateral and rotational), the first and second natural frequencies, and the stresses. They are

$$g_i = 1 - \left(\frac{dl_i}{dla}\right)^2, \quad i = 1, .., 5,$$

$$g_{i+5} = 1 - \left(\frac{dr_i}{dra}\right)^2, \quad i = 1, .., 5,$$

$$g_{11} = \frac{\omega_1}{\omega_{1a}} - 1,$$

$$g_{12} = \frac{\omega_2}{\omega_{2a}} - 1,$$

$$g_{2i+11} = 1 - \frac{\sigma_i^r}{\sigma_a}, \quad i = 1, .., 5,$$

$$g_{2i+12} = 1 - \frac{\sigma_i^l}{\sigma_a}, \quad i = 1, .., 5,$$

$$g_{i+22} = 1 - \left(\frac{ddl_i}{dla}\right)^2, \quad i = 1, .., 5,$$

$$g_{i+27} = 1 - \left(\frac{ddr_i}{dra}\right)^2, \quad i = 1, .., 5,$$

where

dl_i, dr_i = static lateral and rotational displacements of i^{th} element, respectively,
dla, dra = maximum allowable static lateral and rotational displacements,
ω_1, ω_2 = first and second natural frequencies,
ω_{1a}, ω_{2a} = minimum required value for the first and second natural frequencies,
σ_i^r, σ_i^l = maximum static stresses at the right and left ends of i^{th} element,
σ_a = maximum allowable static stress,

Table 3 Statistical information for the random variables

Random variable	Mean	Standard deviation
σ_a (psi)	30,000	3,000
ρ (lb/in^3)	0.1	0.01
E (ksi)	10,500	1050

Table 4 Merit function at the initial and final designs

	Initial design	Final design
b_{1-5}	3	3
h_1	3.703	3.5805
h_2	7.040	8.5816
h_3	9.807	12.136
h_4	11.998	14.863
h_5	13.840	17.162
c	0.06	0.06
f	1493.97	1753.5

ddl_i, ddr_i = dynamic lateral and rotational displacements of i^{th} element, and
$ddla, ddra$ = maximum allowable dynamic lateral and rotational displacements.

The random variables for this problem, σ_a, ρ, and E, are assumed to be independent and normally distributed with statistical parameters given in Table 3.

For RBDO test studies, constraints g_1, g_6, g_{14}, g_{16}, g_{18}, g_{20}, g_{22}, and g_{28} are considered to be more important and therefore only these are considered as hard constraints. The rest of the constraints are considered as soft (deterministic) constraints. The system probability of failure ($P_{allow_{sys}}$) was required to be 0.001, which was equally distributed among the eight failure modes. This gives a desired reliability index of $\beta_{reqd_i} = -\Phi^{-1}\left(\frac{P_{allow_{sys}}}{8}\right) = 3.6623$ for each failure mode.

The RBDO was performed using two different methods, the DLM that uses the PMA approach to prescribe the probabilistic constraint and the unilevel-PMA method described earlier in the paper. The unilevel-PMA method for this test case results in nontrivial problem. Because there are eight failure modes and three random variables for each failure mode, there are 32 equality constraints imposed by the unilevel method. Also, because the unilevel method is solved in an augmented design space consisting of the original design variables, the MPP of failure for each failure mode, and the Lagrange multipliers for each failure mode, the dimensionality of the design vector for this test case is 43. It should be noted that the sensitivities of the first-order KKT conditions (26) require calculation of second-order information for the failure modes with respect to the augmented design variable vector. In the present implementation, a damped BFGS update is used to obtain the second-order information (Nocedal and Wright 1999). This method is defined by

$$r_k = \psi_k y_k + (1 - \psi_k) H_k s_k, \tag{46}$$

where the scalar

$$\psi_k = \left\{ \begin{array}{ll} 1 & : s_k^T y_k \geq 0.2 s_k^T H_k s_k \\ \dfrac{0.8 s_k^T H_k s_k}{s_k^T H_k s_k - s_k^T y_k} & : s_k^T y_k \leq 0.2 s_k^T H_k s_k \end{array} \right\}, \tag{47}$$

and y_k and s_k are the differences in the function and gradient values of the previous iteration from the current iteration, respectively. The Hessian update is

$$H_{k+1} = H_k - \frac{H_k s_k s_k^T H_k}{s_k^T H_k s_k} + \frac{r_k r_k^T}{s_k^T r_k}. \tag{48}$$

The values of the hard constraints at the final design are given in Table 5. It should be noted that the constraints g_6, g_{16}, g_{18}, g_{20}, and g_{22} dictate the system failure. The reliability constraints corresponding to these constraints are the only active constraints in the RBDO. The other hard constraints have a value greater than zero, which means that the reliability index corresponding to those constraints is much higher than the desired index.

The starting and the final designs for RBDO are given in Table 4. The deterministic optimum design was chosen to be the starting design for RBDO. It was noted that this significantly reduces the required number of system analysis evaluations. For the proposed method, the designer has to choose a starting point for the MPPs in the augmented design vector. Different choices for the initial MPPs may lead to different final designs. In this study, an IRA was performed at the design **d** to identify a suitable starting point for the MPP of failure for each hard constraint. At the initial design, g_1 is active. Both the DLM-PMA and the unilevel-PMA methods yield the same final design. Note that the value of the merit function (weight of the beam) has increased considerably at the final design. This is expected for a more reliable structure to account for the variation in the random variables.

The computational cost of the two methods is compared in Table 6. The proposed method is observed to be twice as fast as the nested approach. Therefore, the proposed method is not only a robust formulation for RBDO problems but is also computationally efficient.

Table 5 Hard constraints at the final design

g_i	Value at optimum
g_1	0.2232
g_6	4.7×10^{-8}
g_{14}	0.1794
g_{16}	1.1×10^{-16}
g_{18}	1.1×10^{-16}
g_{20}	1.1×10^{-16}
g_{22}	3.3×10^{-16}
g_{28}	0.3749

Table 6 Comparison of computational cost of RBDO methods

Method	SA Calls
DLM-PMA	493
Unilevel-PMA	261

6 Conclusions

In this research, a new unilevel formulation for RBDO is developed. The first-order KKT conditions corresponding to the probabilistic constraint (as in PMA) are enforced directly at the system-level optimizer, thus eliminating the lower level optimizations used to compute the probabilistic constraints. The proposed formulation is solved in an augmented design space that consists of the original decision variables, the MPP of failure corresponding to each failure-driven mode, and the Lagrange multipliers. It is computationally equivalent to the original nested optimization formulation if the inner optimization problem is solved by satisfying the KKT conditions (which is effectively what most numerical optimization algorithms do). Under mild pseudoconvexity assumptions on the hard constraints, the proposed formulation is mathematically equivalent to the original nested formulation. The method is tested using a simple analytical problem and a multidisciplinary structures control problem and is observed to be numerically robust and computationally efficient compared to the existing approaches for RBDO.

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References

Agarwal H, Renaud JE (2004) Decoupled methodology for probabilistic design optimization. ASCE joint specialty conference on probabilistic mechanics and structural reliability. Albuquerque, NM (July)

Agarwal H, Renaud JE, Mack JD (2003) A decomposition approach for reliability-based multidisciplinary design optimization. Proceedings of the 44th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference & exhibit, AIAA 2003-1778. Norfolk, Virginia, 7–10 April 2003

Brietung K (1984) Asymptotic approximations for multinormal integral. J Eng Mech 110(3):357–366

Chen W, Du X (2002) Sequential optimization and reliability assessment method for efficient probabilistic design. ASME design engineering technical conferences and computers and information in engineering conference, DETC2002/DAC-34127. Montreal, Canada

Chen XC, Hasselman TK, Neill DJ (1997) Reliability based structural design optimization for practical applications. Proceedings of the 38th AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference, AIAA-97-1403, pp 2724–2732

Enevoldsen I, Sorensen JD (1994) Reliability-based optimization in structural engineering. Struct Saf 15(3):169–196

Gu X, Renaud JE, Batill SM, Brach RM, Budhiraja AS (2000) Worst case propagated uncertainty of multidisciplinary systems in robust design optimization. Struct Multidisc Optim 20(3):190–213

Haftka RT, Gurdal Z, Kamat MP (1990) Elements of structural optimization, vol 1, 2nd edn. Kluwer, Norwell, MA

Haldar A, Mahadevan S (2000) Probability, reliability and statistical methods in engineering design. Wiley, New York

Khot NS (1994) Optimization of controlled structures. In: Adeli H (ed) Advances in design automation. Chapman & Hall, London, UK, 588 pp

Kirjner-Neto C, Polak E, der Kiureghian A (1998) An outer approximations approach to reliability-based optimal design of structures. J Optim Theory Appl 98(1):1–16

Kuschel N, Rackwitz R (2000) A new approach for structural optimization of series systems. Applications of Statistics and Probability 2(8):987–994

Liu P-L, Kiureghian AD (1991) Optimization algorithms for structural reliability. Struct Saf 9(3):161–177

Mangasarian OL (1994) Nonlinear programming classics in applied mathematics. SIAM, Philadelphia, PA

Nocedal J, Wright SJ (1999) Numerical optimization. Springer, Berlin Heidelberg New York

Oberkampf WL, DeLand SM, Rutherford BM, Diegert KV, Alvin KF (2000) Estimation of total uncertainty in modeling and simulation. Technical Report SAND2000-0824, Sandia National Laboratories, Livermore, CA

Oberkampf WL, Helton JC, Sentz K (2001) Mathematical representation of uncertainty. Proceedings of the 42nd AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference & exhibit, AIAA 2001-1645. Seattle, WA. 16–19 April 2001

Padmanabhan D, Batill SM (2002) Decomposition strategies for reliability based optimization in multidisciplinary system design. Proceedings of the 9th AIAA/USAF/NASA/ISSMO symposium on multidisciplinary analysis & optimization, AIAA 2002-5471. Atlanta, GA. 4–6 September 2002

Polak E, Wets RJ-B, der Kiureghian A (2000) On an approach to optimization problems with a probabilistic cost and or constraints. In: Pillo GD, Giannessi F (eds) Nonlinear optimization and related topics, volume 36 of applied optimization, pp 299–316. Kluwer, Norwell, MA

Rackwitz R (2001) Reliability analysis—a review and some perspectives. Struct Saf 23(4):365–395

Rosenblatt M (1952) Remarks on a multivariate transformation. Ann Math Stat 23(3):470–472

Royset JO, Kiureghian AD, Polak E (2001) Reliability based optimal structural design by the decoupling approach. Reliab Eng Syst Saf 73(3):213–221

Sobieszczanski-Sobieski J, Bloebaum CL, Hajela P (1990) Sensitivity of control-augmented structure obtained by a system decomposition method. AIAA J 29(2):264–270

Sutter TR, Camarda CJ, Walsh JL, Adelman HM (1988) Comparison of several methods for calculating vibration mode shape derivatives. AIAA J 26:1506–1511

Tu J, Choi KK, Park YH (1999) A new study on reliability-based design optimization. J Mech Des 121:557–564

Wang L, Kodiyalam S (2002) An efficient method for probabilistic and robust design with non-normal distribution. Proceedings of the 43rd AIAA/ASME/ASCE/AHS/ASC structures, structural dynamics, and materials conference, AIAA 2002-1754. Denver, Colorado. 22–25 April 2002