Struct Multidisc Optim (2006) 32: 59–64
DOI 10.1007/s00158-005-0583-4

RESEARCH PAPER

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On an optimization problem for elastic rods

Received: 22 November 2005 / Published online: 6 January 2006

Abstract Optimal shape of an elastic rod loaded by extensional force is determined. It is assumed that the rod is described by a classical Bernoulli–Euler rod theory. The optimality conditions are obtained by using Pontriyagin's maximum principle. It is shown that the optimal shape (cross-sectional area as a function of an arc length) is determined from the solution of a nonlinear second-order differential equation. The solution of this equation is given in the closed form. It is shown that for the same buckling force, the savings of the material are of the order of 30%. An interesting feature of the problem is that for certain values of parameters, there is no optimal solution.

Keywords Pontyagin's principle · buckling

1 Introduction

Consider an elastic BC column, fixed at one end and free at the other end. Suppose that at the free end (point C in Fig. 1), a straight rigid rod CD is fixed. The column is loaded by a force **F** acting at the point D having constant magnitude F and the action line always parallel to the $\overline{x}$ axis of a rectangular Cartesian coordinate system $\overline{x} - B - \overline{y}$. Let L be the length of the rod and let $\overline{\text{CD}} = b$. When the value of the force reaches a certain value, called critical value, the column can assume a buckled configuration, i.e., its axis will not coincide with the $\overline{x}$ axis of the coordinate system $\overline{x} - B - \overline{y}$.

The problem of determining the critical value of the force $bf\, F$ was treated in Bizeno and Grammel (1953). A generalization of the basic problem was treated in Atanackovic et al. (1989), where shape and load imperfections are introduced, and in Atanackovic et al. (1992), where extensibility of the rod axis was introduced. However, in all these works, it was assumed that the column cross section is constant.

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The critical load parameter F_{cr} for which the buckled configuration is possible reads

$$\frac{L}{b} = \sqrt{\frac{F_{\text{cr}} L^2}{E I_0}} \tanh \sqrt{\frac{F_{cr} L^2}{E I_0}}, \tag{1}$$

(see Bizeno and Grammel 1953) where E is the modulus of elasticity and $I_0 = \text{const.}$ is the moment of inertia of the cross-sectional area A, assumed to be constant.

Our goal is to formulate and solve the following optimization problem: Given F, L, and b, find the cross-sectional area $\widetilde{A}\,(S)$ as a function of S (S is the arc length of the column axis measured from the origin of the coordinate system B) such that F is the critical buckling force of the rod loaded, as shown in Fig. 1, and that at the same time, the volume of the rod

$$W = \int_0^L \widetilde{A}\,(S)\,\mathrm{d}S, \tag{2}$$

is smaller than the volume of any other rod having the same buckling load F. Thus, the rod with the shape $\widetilde{A}\,(S)$ may be termed *lightest rod* having prescribed buckling force F.

We first formulate the relevant equations. From Atanackovic (1997), we obtain the equilibrium equations for the rod in the form

$$\frac{\mathrm{d}H}{\mathrm{d}S} = 0, \quad \frac{\mathrm{d}V}{\mathrm{d}S} = 0, \quad \frac{\mathrm{d}M}{\mathrm{d}S} = -V \cos\vartheta + H \sin\vartheta, \tag{3}$$

where H and V are components of the contact force (i.e., the resultant force in an arbitrary cross section) along $\overline{x}$ and $\overline{y}$ axes, respectively, M is the bending moment, and ϑ is the angle between the tangent to the column axis and $\overline{x}$ axis. We adjoin to (3) the geometrical

$$\frac{\mathrm{d}x}{\mathrm{d}S} = \cos\vartheta, \quad \frac{\mathrm{d}y}{\mathrm{d}S} = \sin\vartheta, \tag{4}$$

and the constitutive equation

$$M = EI \frac{\mathrm{d}\vartheta}{\mathrm{d}S}. \tag{5}$$

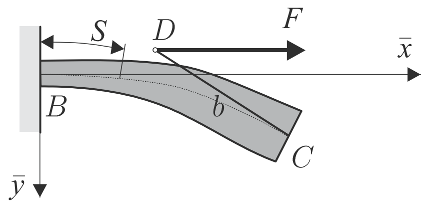

Fig. 1 Coordinate system and load configuration

In (4) and (5), we used x and y to denote coordinates of an arbitrary point on the rod axis, E is modulus of elasticity, and I is the moment of inertia of the cross section. Equations (3) and (5) correspond to the classical Bernoulli–Euler rod theory. The boundary conditions for the column shown in Fig. 1 are

$$y(0) = 0, \quad \vartheta(0) = 0, \quad M(L) = -Fa \sin\vartheta(L), \quad H(L) = 0. \tag{6}$$

The trivial solution of the systems (3)–(6), in which the rod axis remains straight, valid for any value of F, reads

$$H^0 = F, \quad V^0 = 0, \quad M^0 = 0, \quad \vartheta^0 = 0, \quad x^0 = S, \quad y^0 = 0. \tag{7}$$

Let $\Delta H, \ldots, \Delta M$ be the perturbations of the variables $H, \ldots, M$, so that $H = H^0 + \Delta H, \ldots, M = M^0 + \Delta M$. Introducing this expression into (3) and (5), omitting Δ in front of the variables, we obtain the following system

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}S} = \frac{M}{EI}, \quad \frac{\mathrm{d}M}{\mathrm{d}S} = F\sin\vartheta, \tag{8}$$

subject to

$$\vartheta(0) = 0, \quad M(L) = Fb\sin\vartheta(L). \tag{9}$$

We now specify that there is a connection between A and I as[1]

$$I = \alpha A^2, \tag{10}$$

where α is a constant. For a circular cross section, $\alpha = 1/4\pi$. By introducing the dimensionless quantities

$$m = \frac{M}{\alpha EL^3}, \quad t = \frac{S}{L}, \quad a = \frac{A}{L^2}, \quad \lambda = \frac{F}{\alpha EL^2}, \quad \xi = \frac{b}{L}, \quad w = \frac{W}{L^3}, u = \frac{y}{L}, \quad v = \frac{x}{L} \tag{11}$$

the systems (8) and (9) become

$$\dot{\vartheta} = \frac{m}{a^2}, \quad \dot{m} = \lambda\sin\vartheta, \quad \dot{u} = \sin\vartheta, \quad \dot{v} = \cos\vartheta, \tag{12}$$

subject to

$$\vartheta(0) = 0, \quad m(1) = \lambda\xi\sin\vartheta(1), \tag{13}$$

where $(\dot{\cdot}) = \frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$. With (11), the dimensionless volume is

$$w = \int_0^1 a(\xi)\mathrm{d}\xi. \tag{14}$$

The systems (12) and (13) have a trivial solution

$$\vartheta_0 = m_0 = 0. \tag{15}$$

valid for all values of λ. To determine λ for which there is a nontrivial solution to (12) and (13), we linearize it to obtain

$$\dot{\vartheta} = \frac{m}{a^2}, \quad \dot{m} = \lambda\vartheta, \tag{16}$$

subject to

$$\vartheta(0) = 0, \quad m(1) = \lambda\xi\vartheta(1). \tag{17}$$

A necessary condition that (12) and (13) have a nontrivial solution (i.e., loss of stability of the column by buckling) is that (16) and (17) have a nontrivial solution. The sufficient conditions for (8) and (9) to have a nontrivial solution may be differently formulated (see Chow and Hale 1982; Antman 1995), and we shall not analyze those conditions.

We assume that the cross-sectional area $a(t)$ belongs to the set $\mathcal{U}$, called the set of *admissible* cross-sectional area functions. The procedure that we will use allows for various restrictions of the type $0 \le a_{\min} \le a(t) \le a_{\max}$ to be treated, but we shall not be concerned with such restrictions here. Instead, we assume that $\mathcal{U}$ is a set of continuous nonnegative functions defined on the interval 0, 1 and having continuous first derivative, i.e., $\mathcal{U} = \{u : u \in C^1([0,1], \mathbb{R}); u(t) \ge 0\}$.[2] The optimization problem that we shall analyze may be stated as: Given λ^*, find $\widetilde{a}(t) \in \mathcal{U}$ such that when $\widetilde{a}(t)$ is used in (16) and (17), λ^* is the lowest eigenvalue of the spectral problems (16) and (17), and at the same time, the integral (14) is in minimum among all those $a(t) \in \mathcal{U}$ such that when $a(t)$ is used in (16) and (17), the value λ^* is an eigenvalue of (16) and (17).

The rod having $\widetilde{a}(t)$ will be called the *optimal* rod. We proceed now and solve the optimization problem just stated.

2 The condition defining the optimal rod

We use the Pontryagin's maximum principle (see Alekseev et al. 1979; Sage and White 1977) in solving the optimization problem. We shall treat $a(t)$ as the control variable. Thus, we set $\vartheta = x_1$, $m = x_2$, so that (16) and (17) become

$$\dot{x}_1 = \frac{x_2}{a^2}, \quad \dot{x}_2 = \lambda x_1, \tag{18}$$

[1] Some other possibilities that have practical importance are $I = b + \alpha A$, $I = \alpha A^\beta$, $\alpha = \text{const.}$, $\beta = \text{const.}$ (see Greenev and Filippov 1979).

[2] The optimality condition that follow could be obtained by setting $\mathcal{U}$ to be the set of piecewise continuous functions. However, in our analysis, we will treat the more restrictive case only.

and

$$x_1(0) = 0,\ \ x_2(1) - \lambda\xi x_1(1) = 0. \tag{19}$$

Following the standard procedure, we form the Pontryagin's function $\mathcal{H}$ as

$$\mathcal{H} = a + p_1 \frac{x_2}{a^2} - p_2 \lambda x_2, \tag{20}$$

where the co-state variables p_i, $i = 1, 2, 3$ satisfy

$$\dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = -\lambda x_2,\ \ \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = -\frac{p_1}{a^2}, \tag{21}$$

and

$$p_2(0) = 0,\ p_1(1) + \lambda\xi p_2(1) = 0. \tag{22}$$

The optimality condition, $\underset{\min a \in \mathcal{U}}{\mathcal{H}}$, leads to

$$\frac{\partial \mathcal{H}}{\partial a} = 1 - 2p_1 \frac{x_2}{a^3} = 0. \tag{23}$$

From (23), we obtain

$$a = (2p_1 x_2)^{1/3}. \tag{24}$$

To obtain a in a more suitable form, note that the systems (18) and $(19)_{1,2}$, and (21) and (22) become identical if we set

$$p_1 = x_2,\ p_2 = -x_1. \tag{25}$$

Thus, (24) becomes

$$a(t) = \left(2x_2^2(t)\right)^{1/3}. \tag{26}$$

By differentiating (23) with respect to a and by using (25), we obtain

$$\frac{\partial^2 \mathcal{H}}{\partial a^2} = 6\frac{x_2^2}{a^4} \geq 0. \tag{27}$$

Therefore, the necessary condition for $\underset{\min a \in U}{\mathcal{H}}$ is satisfied with the choice (25).[3] With (26), the system (18) may be written as $(x_2 = m)$

$$\ddot{m} = \frac{\lambda}{2^{2/3}} m^{-1/3}. \tag{28}$$

and

$$\dot{m}(0) = 0, \qquad m(1) - \xi\dot{m}(1) = 0. \tag{29}$$

(28) possesses a first integral that after the use $\dot{m}(0) = 0$ becomes

$$\dot{m}^2 = 3\Lambda\left[m^{2/3} - m_0^{2/3}\right]. \tag{30}$$

where $m_0 = m(0)$, $\Lambda = \frac{\lambda}{2^{2/3}}$. By separating variables and integrating, we obtain from (28)

$$\frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\left\{\left(\frac{m}{m_0}\right)^{1/3}\sqrt{\left(\frac{m}{m_0}\right)^{2/3} - 1} + \ln\left[2\left(\frac{m}{m_0}\right)^{1/3} + 2\sqrt{\left(\frac{m}{m_0}\right)^{2/3} - 1}\right]\right\} = t + C, \tag{31}$$

where C is another constant. Setting $t = 0$ in (31), it follows that $C = \frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\ln 2$. Finally, we note that from (19), it follows that $m(1) = \xi\dot{m}(1)$. By using (30), this condition leads to

$$m_0 = \left[\frac{3\xi^2\Lambda(m_1)^{2/3} - (m_1-)^2}{3\xi^2\Lambda}\right]^{3/2}, \tag{32}$$

where $m_1 = m(1)$. By substituting $t = 1$ into (31), we obtain another condition that m_0 and m_1 must satisfy

$$\frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\left\{\left(\frac{m_1}{m_0}\right)^{1/3}\sqrt{\left(\frac{m_1}{m_0}\right)^{2/3} - 1} + \ln\left[2\left(\frac{m_1}{m_0}\right)^{1/3} + 2\sqrt{\left(\frac{m_1}{m_0}\right)^{2/3} - 1}\right]\right\} = 1 + \frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\ln 2. \tag{33}$$

Thus, for given λ and ξ, we determine $\Lambda = \frac{3\lambda}{2^{2/3}}$ and solve the systems (32) and (33) for m_1 and m_0. The moment of the buckled optimal rod $m(t)$ is determined from (31) as

$$\frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\left\{\left(\frac{m(t)}{m_0}\right)^{1/3}\sqrt{\left(\frac{m(t)}{m_0}\right)^{2/3} - 1} + \ln\left[2\left(\frac{m(t)}{m_0}\right)^{1/3} + 2\sqrt{\left(\frac{m_t}{m_0}\right)^{2/3} - 1}\right]\right\} = t + \frac{\sqrt{3}m_0^{2/3}}{2\Lambda^{1/2}}\ln 2. \tag{34}$$

Finally, from (32) and (33), we obtain a single equation for determining m_1 as

$$\frac{1}{2}\sqrt{\frac{3}{\Lambda}}\left[(m_1)^{2/3} - \frac{(m_1)^2}{3\Lambda\xi^2}\right]\left\{\frac{(m_1)^{1/3}}{\left[(m_1)^{2/3} - \frac{(m_1)^2}{3\Lambda\xi^2}\right]^{1/2}}\sqrt{\frac{(m_1)^{2/3}}{\left[(m_1)^{2/3} - \frac{(m_1)^2}{3\Lambda\xi^2}\right]} - 1} + \ln\left[2\frac{(m_1)^{1/3}}{\left[(m_1)^{2/3} - \frac{(m_1)^2}{3\Lambda\xi^2}\right]^{1/2}} + 2\sqrt{\frac{(m_1)^{2/3}}{\left[(m_1)^{2/3} - \frac{(m_1)^2}{3\Lambda\xi^2}\right]} - 1}\right] - \ln 2\right\} = 1. \tag{35}$$

If (35) has a real solution for m_1 from (32), we determine m_0 and the optimal shape follows from (34).

Finally, the optimal cross section $a(t)$ follows from (26) as

$$a(t) = 2^{1/3} m^{2/3}(t). \tag{36}$$

[3] In general, we may set $p_1 = cx_2$, $p_2 = -cx_1$, with c being a constant different from zero. However, if $c < 0$ is chosen, then the condition (27) is not satisfied.

To determine the volume of the optimal rod, we note that from (36), $a(0) = a_0 = 2^{1/3}(m_0)^{2/3}$, $a(1) = a_1 = 2^{1/3}(m_1)^{2/3}$. By using (30), we obtain

$$\mathrm{d}t = \frac{1}{2^{4/3}}\sqrt{\frac{3}{\Lambda a_0}\frac{a}{a-a_0}}\mathrm{d}a. \tag{37}$$

Multiplying the last equation by a and integrating, it follows that

$$w = \int_0^1 a(t)\,\mathrm{d}t = \frac{1}{2^{4/3}}\sqrt{\frac{3}{\Lambda a_0}}\int_{a_0}^{a_1}\frac{a^{3/2}}{\sqrt{a-a_0}}\mathrm{d}a. \tag{38}$$

From (38), the following relation between the volume w and values of the cross-sectional area at the end points is obtained

$$w = \frac{a_0^{3/2}}{2^{1/3}}\sqrt{\frac{3}{\Lambda}}\left\{\frac{3}{8}\frac{a_1}{a_0}\sqrt{\left(\frac{a_1}{a_0}\right)^2-1} + \frac{1}{4}\left(\frac{a_1}{a_0}\right)^3\sqrt{\left(\frac{a_1}{a_0}\right)^2-1} + \frac{3}{8}\ln\left[2\frac{a_1}{a_0} + 2\sqrt{\left(\frac{a_1}{a_0}\right)^2-1}\right]\right\}. \tag{39}$$

Since from (36) we have $\frac{a_1}{a_0} = \left(\frac{m_1}{m_0}\right)^{2/3}$, (39) may be written as

$$w = \frac{m_0}{2^{1/3}}\sqrt{\frac{3}{\Lambda}}\left\{\frac{3}{8}\left(\frac{m_1}{m_0}\right)^{2/3}\sqrt{\left(\frac{m_1}{m_0}\right)^{4/3}-1} + \frac{1}{4}\left(\frac{m_1}{m_0}\right)^2 \sqrt{\left(\frac{m_1}{m_0}\right)^{4/3}-1} + \frac{3}{8}\ln\left[2\left(\frac{m_1}{m_0}\right)^{2/3} + 2\sqrt{\left(\frac{m_1}{m_0}\right)^{4/3}-1}\right]\right\}. \tag{40}$$

If w is given, then the critical load Λ for the optimally shaped column is determined by solving (32), (33), and (40) for Λ, m_0, and m_1. The shape of the rod is then determined from (34).

3 Numerical results

In this section, we present numerical results for several values of parameters. We used standard procedures to solve the boundary value problems (28) and (29) to determine m and then from (36) to determine the cross-sectional area. The accuracy of the procedure was checked by comparing the results with (34).

There are three cases that we distinguish.

3.1 The case $\xi > \frac{2}{3}$, $\lambda > 0$

In this case, (35) has a single solution. We present results of several concrete examples:

1. In the first experiment, we choose $\xi = 1$, $\lambda = \lambda_{\text{optimal}} = 5$. The resulting cross-sectional area is shown in Fig. 2. The values of the cross section at the end points are
$$a(0) = a_0 = 1.476252, \qquad a(1) = a_1 = 2.413545. \tag{41}$$
The volume of the optimal rod is $w = \int_0^1 a(t)\,\mathrm{d}t = 1.810158$. Thus, we can compare the critical force of the optimal rod with the critical force of the rod with constant cross section having the same volume.
2. Next, we consider the case of large λ. Thus, we assume $\xi = 1, \lambda = 300$. The optimal shape of the rod is shown in Fig. 3. The values of the cross section at the end points in this case are $a(0) = 11.435006$, $a(1) = 18.695241$, and the volume is $w = \int_0^1 a(t)\mathrm{d}t = 14.02143097$.
3. Finally, we analyze the optimal shapes for large ξ. Thus, we take $\xi = 100, \lambda = 10$. The shape of the optimal rod is shown in Fig. 4. As could be seen in this case, the rod has small change in the cross section since $a_0 = 31.5348053$, $a_1 = 31.6403908$. The volume is $w = 31.5700162$.

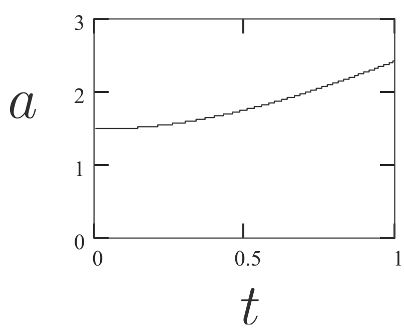

Fig. 2 Optimal cross section for $\xi = 1$, $\lambda = 5$

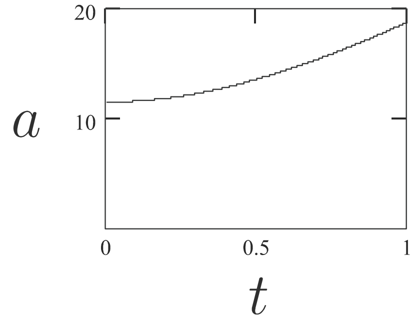

Fig. 3 Optimal cross-sectional area for $\xi = 1$, $\lambda = 300$

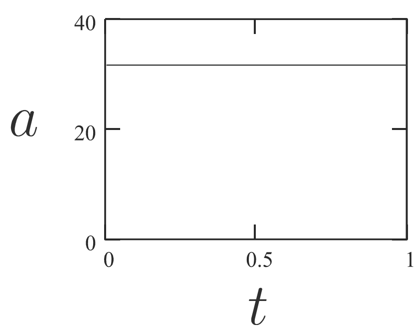

Fig. 4 Optimal cross section for $\xi = 100$, $\lambda = 10$

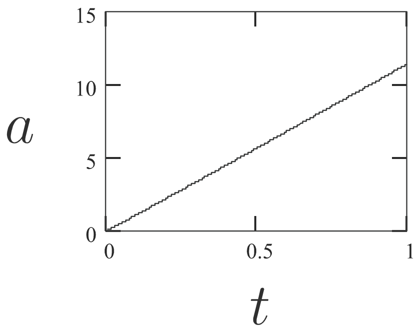

Fig. 5 Cross-sectional area for the first solution for $\lambda = 100,\ \xi = \frac{2}{3}$

3.2 The case $\xi = \frac{2}{3},\ \lambda > 0$

In this case, there are two solutions of (35).

The first one reads

$$m_0 = 0, \qquad m_1 = \left(2\sqrt{\frac{\Lambda}{3}}\right)^{3/2}, \tag{42}$$

This solution has $a\,(0) = a_0 = 0$.

The second solution, obtained numerically from (32) and (35), has $m_0 > 0$ and $m_1 < \left(2\sqrt{\frac{\Lambda}{3}}\right)^{3/2}$, $a\,(0) = a_0 > 0$. We illustrate the case $\xi = \frac{2}{3}$ by concrete examples.

4. Let $\lambda = 100,\ \xi = \frac{2}{3}$. The first solution has $m_0 = 0,\ m_1 = 27.745276335252$ and $a_0 = 0,\ a_1 = 11.5137381287606$. We show the cross-sectional area of the optimal rod in Fig. 5. It is a linearly increasing function. The volume of the rod in this case is $w = 5.74468825$.

Next, we show the second solution for $\lambda = 100,\ \xi = \frac{2}{3}$. The second solution of the systems (32) and (35) is $m_0 = 3.49874532$, $m_1 = 21.11502816$ and $a_0 = 2.9036987176$, $a_1 = 9.62522187$. We show the cross-sectional area in Fig. 6.The volume in this case is $w = 5.53013372$. Since this volume is smaller than the volume corresponding to the first solution, we conclude that the cross-sectional area shown in Fig. 6 corresponds to optimal rod.

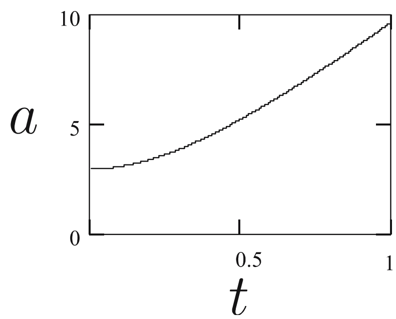

Fig. 6 Cross-sectional area for the second solution for $\lambda = 100,\ \xi = \frac{2}{3}$

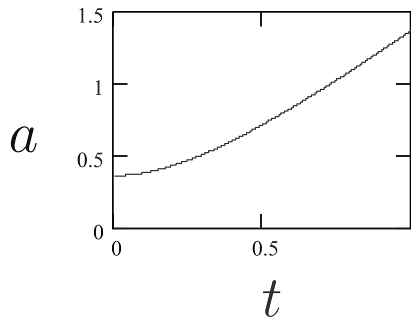

Fig. 7 Cross-sectional area for $\lambda = 2,\ \xi = 0.649$

3.3 The case $0.620831458369682 < \xi < \frac{2}{3},\ \lambda > 0$

When ξ is decreased further, we found that for $0.620831458369682 < \xi < \frac{2}{3}$, (35) has again two solutions, $\overline{m}_1$ and $\hat{m}_1$. However, these solutions lead to two values of $m_0 = m\,(0)$ that are both larger than zero. We illustrate this by a concrete example.

Let $\lambda = 2,\ \xi = 0.649$. (35) has two solutions, $\overline{m}_1 = 1.126426$ and $\hat{m}_1 = 1.400735$. This leads to $\overline{a}_0 = 0.35985063$, $\overline{a}_1 = 1.3639921$ and $\hat{a}_0 = 0.024549654$, $\hat{a}_1 = 1.57729840$. The corresponding volumes are $\overline{w} = 0.75865382$ and $\hat{w} = 0.774212481$. Therefore, the first solution $\overline{a}$ is optimal. It is shown in Fig. 7.

3.4 The case $\xi = 0.620831458369682$

In this case, (35) has a single solution for m_1 independently of the value of λ. For example, for $\lambda = 2$, we have $m_0 = 0.0505993, m_1 = 1.2039095$, and the shape of the rod is shown in Fig. 8.

3.5 The case $\xi < 0.620831458369682$

If

$$\xi < \xi_{\text{critical}} = 0.620831458369682, \tag{43}$$

there is no solution of (28) and (29) for any value of λ Numerical examination of (35) shows that it does not have real

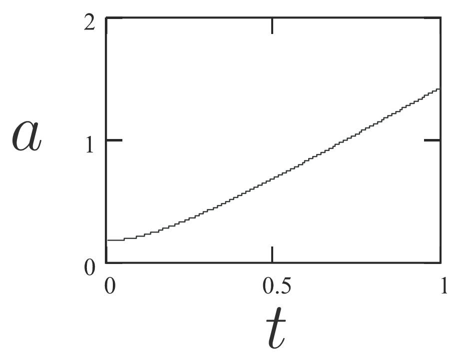

Fig. 8 Cross-sectional area for $\xi = 0.620831458369682$

solution when $\xi < 0.620831458369682$ for any λ, while for $\xi \geq 0.620831458369682$, there is always a real solution of (28) and (29).

4 Conclusion

In this work, we analyzed the problem of determining optimal shape of an elastic rod loaded, as shown in Fig. 1. This problem is known as the buckling by extension problem. Our main results are as follows:

1. The optimal shape, i.e., cross-sectional area as a function of arc length, is determined from (28), (29), and (36). The function $m(t)$ (actually, t as a function m) can be obtained from (34) after the constants m_0 and m_1 are determined from (32) and (33). The volume of the optimally shaped rod is expressed as (40). We assumed that the cross-sectional area function $a(t)$, which in our analysis is "control," belongs to the set of continuously differentiable functions. In terms of "state variables," it has representation (26).
2. We found that there are three distinct regions defined by the value of the parameter ξ representing the ratio between the length of the rigid and elastic rod.
    a. For $\xi > 2/3$, there exists a single real solution to (35) that determines $m_1 = m(1)$, and through (32), $m_0 = m(0)$.
    b. When $\xi = 2/3$, (35) has two solutions, one having linear change of a as a function of t and having $a(0) = 0$. This solution is shown in Fig. 5. The other solution has $a(0) > 0$ and smaller volume than the first one. Therefore, it is optimal. It is shown in Fig. 6.
    c. When $0.620831458369682 < \xi < \frac{2}{3}$, there are two solutions of (35). Both have $a(0) > 0$. The optimal between the two (one that has smaller volume) has larger $a(0)$.
    d. For $\xi = 0.620831458369682$, there is a single solution to (35). The optimal shape for a specific value of λ is shown in Fig. 8.
    e. Finally, when $\xi < 0.620831458369682$, there is no real solution to (35). Thus, we conclude that for this case, there is no $a \in C^1([0, 1], \mathbb{R})$ solution to the optimization problem.
3. Since our optimality conditions are derived by using the Pontriyagin's maximum principle, we could impose the restriction $0 < a_{\min} \leq a(t) \leq a_{\max}$. In this case, instead of (23), we would choose the optimal cross-sectional area $a*$ from (see Vujanovic and Atanackovic 2004)

$$\min_{0<a_{\min}\leq a(t)\leq a_{\max}} \mathcal{H}(a, x_1, x_2, p_1, p_2) = \mathcal{H}(a^* x_1, x_2, p_1, p_2) \quad (44)$$

   However, such an analysis will not be conducted here.
4. We found that for increasing values of the parameter $\xi = b/L$, the rod has cross section that becomes almost cylindrical (see Fig. 4).
5. The savings of the material depend on the values of ξ and λ. From $(11)_4$, we have $F = \lambda\alpha E L^2$, and since for rod of constant cross section $I = \alpha A_{\text{const}}^2$, we obtain

$$\lambda_{\text{const}} = \frac{FL^2}{EI} = \frac{\lambda\alpha EL^4}{E\alpha A_{\text{const}}^2} = \frac{\lambda}{a_{\text{const}}^2} = \frac{\lambda}{w_{\text{const}}^2}, \quad (45)$$

   where $a_{\text{const}} = A_{\text{const}}/L^2$ and w_{const} is the dimensionless volume of the rod with constant cross section. With (11) and (45), (1) may be written as

$$\frac{1}{\xi} = \sqrt{\lambda_{\text{const}}} \tanh \sqrt{\lambda_{\text{const}}}. \quad (46)$$

   With (46), we determine the savings in the material: for case 1, where $\xi = 1$, $\lambda = 5$, it follows that $w/w_{\text{const}} = 0.781527216$; for case 2, where $\xi = 1$, $\lambda = 300$, we have $w/w_{\text{const}} = 0.75449603$; for case 3, where $\xi = 100$, $\lambda = 10$, we have $w/w_{\text{const}} = 0.997502$. Thus, with increasing ξ, the savings decrease since the optimal shape approaches to the shape of rod with constant cross section.
6. We found that for the case when $\xi < 0.620831458369682$, there is no continuously differentiable cross-sectional area function $u(t)$ that minimizes the volume. In this case, the optimization should be performed in the space of continuous or piecewise continuous functions. This will be the subject of our further work.

Acknowledgements We are grateful to Alexander von Humboldt Foundation for financial help during this research which was conducted at Technische Universität Berlin.

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