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RESEARCH PAPER

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Designing the elastic properties of laminates as an optimisation problem: a unified approach based on polar tensor invariants

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Abstract The problem of designing the elastic properties of a laminate is considered. It is shown that a unique formulation for all the design problems with respect to elastic symmetries can be found using polar invariants of the stiffness tensors. In this way, the design of laminates having some general elastic properties is reduced to a classical optimisation problem: the search for the absolute minimum, whose value is 0, of a positive semi-definite form in the space of the polar invariants. A minimum characterisation of some important elastic properties is also given. Some numerical examples and a discussion of the results are also included in the paper.

Keywords Laminates · Polar method · Elastic symmetries · Quasi-homogeneity · Uncoupling

1 Introduction

The use of composite material laminates has considerably broadened the possibilities of designers in finding an appropriate material for a specific use. In fact, they can manage the variables governing the laminate mechanical behaviour to obtain the desired effects or to optimise a given parameter, such as the weight, the strength or the stiffness. These variables are the material of the layers, the ply number, the stacking sequence and the layer orientations. In a sense, it is perhaps more appropriate to speak of material design rather than of structural design and also if the homogenization laws used for laminates are rather "structural type" laws.

Of course, several authors have considered different laminate design problems; an organic state of the art in the field of laminate design is very difficult to be done. This is due to some reasons, the very high number of contributions being one of these, but more important is the fact that many different aspects characterise the researches in the domain: the choice of the variables or of the objective function, the basic mechanical hypothesis and the mathematical approach are the main points that distinguish the works of the scientists in the domain of laminate design and optimisation. A rather deep, but not complete, analysis of the state of the art can be found in Abrate (1994) or also in Vannucci (2002a).

Nevertheless, some general considerations can be done about laminate design. The most part of authors deal with the maximisation or minimisation of some mechanical properties: maximising the stiffness or the strength, as well as the buckling load or the fundamental frequency, or minimising the weight under some mechanical constraints are the most treated problems. Almost all the authors that deal with such problems make some basic hypothesis about the laminate; generally, they consider symmetric stacking sequences, which automatically ensures uncoupling of the in- and out-of-plane behaviours, that is $\mathbf{B}=\mathbf{O}$ [see below for a recall of the classical laminated plates theory (CLPT)]. Again, designers often impose a balanced sequence, i.e. a sequence where for each ply at the orientation α, there is another ply at $-\alpha$: this hypothesis ensures the orthotropy of $\mathbf{A}$ (see for instance Jones 1975). Another rule sometimes used by designers is the use of antisymmetric balanced sequences: this gives the orthotropy of $\mathbf{A}$ and $\mathbf{D}$, but not uncoupling, in general. Some authors have also used symmetric balanced sequences considering $\mathbf{D}$ orthotropic, too, and also if this is not correct (see Fukunaga and Vanderplaats 1991 or, again, Abrate 1994). In many cases, only some orientations are considered for the plies; typical is the case of balanced symmetric quasi-isotropic (orientations at 0°, ±45°, 90°), cross-ply (0°, 90°) or angle-ply (α, $-\alpha$) laminates to have uncoupling and orthotropy of $\mathbf{A}$.

What seems to be a general rule is that laminate designers try to optimise some parameters of the plate considering that it has already some other, generally required, properties such as orthotropy or uncoupling. To do this, they do not look for general solutions but only for solutions belonging to a special class, for instance, the one of balanced symmetric laminates, to automatically ensure the desired property. This way of doing is in a sense a short cut to solutions, especially for the computing aspects, but can lead to solutions which are opti-

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mal only in the chosen class of laminates but not globally. For instance, seeking for minimum weight orthotropic laminates in the class of balanced symmetric plates excludes automatically odd ply number laminates, which are candidates to be the optimal solution just as even ply number laminates. In addition, Vannucci and Verchery (2001a,b) have shown that the number of uncoupled symmetric laminates is very small compared with that of non-symmetric ones. So, looking for uncoupled laminates only in the class of symmetric plates can be very limiting.

Indeed, the design of general elastic properties of a laminate, such as isotropy, orthotropy or uncoupling, has not received by the scientists in the same interest that they have reserved to the above-mentioned problems. Nevertheless, some interesting works in this field can be found in the literature; see for instance the fundamental contribution of Werren and Norris (1953) on isotropy, the works of Fukunaga (1990), Paradies (1996), Grédiac (1999), Vannucci and Verchery (2002) still on isotropy, those of Caprino and Crivelli-Visconti (1982) and of Grédiac (2000) on orthotropy, and the already cited Grédiac (1999) and Vannucci and Verchery (2001a,b) on uncoupling and quasi-homogeneity.

This paper reconsiders the problem of designing general properties of a laminate, namely, its elastic symmetries, uncoupling, quasi-homogeneity and so on. An original way to unify all these problems is proposed, which preserves the absolute generality of the approach: no kind of simplifying hypothesis is made, so general solutions can be found for a given problem. In the paper, it is shown that the most part of elastic properties considered can be viewed as symmetry problems for the laminate and that they can be characterised by minimum conditions. Such a quite general approach seems to be indispensable to get absolute optimal solutions also and perhaps especially in those cases where the properties here considered are not the main goal of the optimisation process but where they must be compulsory achieved together with other quantities to be optimised.

When the design of elastic symmetries for a laminate is considered, a basic problem is the choice of the elastic tensors representation. In fact, the Cartesian representation is not well suited to such a purpose, as symmetries appear clearly only for some particular choices of the reference frame. In addition, the method proposed by Pedersen (1990) to detect elastic symmetries, and based upon the so-called invariants of Tsai and Pagano (1968), is not very effective, as it cannot make distinction among different types of elastic symmetries. It is, on the contrary, suitable to represent tensors by their invariants, choosing among them those whose physical meaning is directly linked to a particular symmetry of the material or of a laminate behaviour. This is the most effective way to handle problems concerning elastic symmetries. For this reason, the polar method for the representation of elastic tensors in plane elasticity is used in this paper. This technique, introduced by Verchery (1979), is the most effective one for the treatment of elastic symmetries in plane elasticity, as in this method, an elastic tensor is known by five invariants and a parameter fixing the reference frame; in addition, each one of the five invariants is linked to a particular elastic symmetry of the tensor. Verchery obtained his results by the application of a complex variable method, technique introduced by Michell (1902) and successively improved by Kolosov (1909), Muskhelishvili (1933) and Green and Zerna (1954). From a general point of view, it can be noticed that, in his approach, Verchery proposes a complex variable transformation more effectively than that used by Green and Zerna (1954); this is not the place to recall the theory of Verchery in the details, and the reader is addressed to this purpose to the original paper of Verchery (1979), which contains also the first invariant characterisation known in the literature of material symmetries in plane elasticity. An extensive presentation of the polar method can be found in Vannucci (2002a).

By the polar method, a certain number of problems concerning laminates have been treated and solved; it is not the case here to recall all the results obtained, and the reader is addressed to the literature in the subject (Valot et al. 2001, 2002, 2003, 2005; Vannucci 2001, 2002a,b; Vannucci and Verchery 2001a,b, 2002; Verchery 1979, 2000; Vincenti et al. 2001, 2002, 2003a–c).

The polar method is effectively used in this paper to obtain a unified formulation of all the problems concerning elastic symmetry properties. Under a mathematical point of view, the obtained formulation is rather classical in structural optimisation: apart from a special case of orthotropy considered in the paper, it corresponds to the search for the absolute minimum, whose value is 0, of a symmetrical positive semi-definite form in the space of the polar invariants. The objective function is non-convex, as in the most part of approaches to optimisation problems of laminates having the layer orientations among the variables, but the problem is unconstrained.

The paper is composed of eight sections besides this introduction: Sect. 2 is a quick introduction of polar invariants and of their physical meaning, Sect. 3 proposes the general formulation of the problems considered in this paper, Sect. 4 is devoted to some basic problems, Sect. 5 to some composed problems, Sect. 6 to quasi-homogeneity, Sect. 7 to orthotropy, Sect. 8 shows some numerical examples and Sect. 9 is about final remarks and conclusions.

The theoretical frame of the paper is the CLPT (see for instance Jones 1975), where the elastic behaviour of a laminate is described by three tensors: **A**, accounting for the extension behaviour; **B**, describing coupling between bending and extension; and **D**, the tensor of the bending behaviour. A laminate having **B**=**O** is said to be uncoupled: bending does not affect stretching and conversely. For the purposes of the paper, the homogeneity tensor **C** is also introduced ($1/h$ and $12/h^3$ are the factors used to homogenize **A** and **D**, respectively; see Jones 1975):

$$\mathbf{C} = \frac{\mathbf{A}}{h} - 12\frac{\mathbf{D}}{h^3}, \tag{1}$$

h being the total thickness of the laminate. **C** measures the difference between **A** and **D**, i. e. of the in- and out-of-plane behaviours; only when **C** is the null tensor, the two behaviours of the laminate are identical in each direction. If a laminate has **B**=**C**=**O**, it behaves just like a homogeneous plate; that is why such laminates are called quasi-homogeneous (Kandil and Verchery 1988; Vannucci and Verchery 2001a,b); their use can be very interesting for some applications. In the paper, no supplementary hypothesis is done: a laminate composed of n plies is considered each time; the layers can be different or identical and can be made of any kind of material, also completely anisotropic, even though orthotropic plies are usually employed.

2 The polar invariants

Let us consider a plane stiffness fourth order tensor **L**: it can be represented by the aid of six polar constants, T_0, T_1, R_0, R_1, Φ_0 and Φ_1, linked to Cartesian components by (i is the imaginary unit)

$$\begin{aligned} 8T_0 &= L_{1111} - 2L_{1122} + 4L_{1212} + L_{2222},\\ 8T_1 &= L_{1111} + 2L_{1122} + L_{2222},\\ 8R_0e^{4i\Phi_0} &= L_{1111} - 2L_{1122} - 4L_{1212} + L_{2222}\\ &\quad + 4i(L_{1112} - L_{2212})\\ 8R_1e^{2i\Phi_1} &= L_{1111} - L_{2222}\\ &\quad + 2i(L_{1112} + L_{2212}) \end{aligned} \tag{2}$$

Constants T_0, T_1, R_0, R_1, as well as the angular difference $\Phi_0 - \Phi_1$, are tensor invariants, which characterise the elastic symmetries of the material; the choice of the reference frame fixes the value of the polar angles Φ_0 and Φ_1, but not their difference, and conversely each choice of one of the two polar angles corresponds to fixing a reference frame; the usual choice for orthotropic laminae corresponds to $\Phi_1 = 0$. For what concerns symmetries, it is in particular:

(a) **L** is orthotropic if and only if

$$\Phi_0 - \Phi_1 = K\frac{\pi}{4}, K = 0, 1; \tag{3}$$

(b) **L** is square symmetric (that is, its components are invariant under rotations of $\pi/4$) if and only if

$$R_1 = 0; \tag{4}$$

(c) **L** is isotropic if and only if

$$R_0 = R_1 = 0. \tag{5}$$

The above conditions explain why T_0 and T_1 are called isotropy, while R_0 and R_1 anisotropy, invariants.
In addition to the cases above, another special type of orthotropy must be cited (Vannucci 2002b), the so-called R_0-orthotropy:
(d) **L** is R_0-orthotropic if and only if

$$R_0 = 0. \tag{6}$$

R_0-orthotropic materials are interesting under many aspects; for instance, it is apparent that in this case, the number of elastic independent constants characterising the material is only three (T_0, T_1 and R_1), like in the case of square symmetry (T_0, T_1 and R_0), and not four as for orthotropy (T_0, T_1, R_0 and R_1). Nevertheless, while square symmetry corresponds to a higher symmetry condition (four symmetry axes turned by $\pi/4$), R_0 orthotropy has the same symmetries of general orthotropy, i.e., two orthogonal axes of plane symmetry. In addition, while the square symmetry of a stiffness tensor implies the same symmetry of the corresponding compliance tensor, this is not the case for R_0-orthotropy.

It is also apparent from (3) that for each set of values T_0, T_1, R_0 and R_1, two different kinds of orthotropic materials can exist: one with K=0 and the other with K=1. Vannucci (2002a) has shown that the first case corresponds to the so-called low-shear modulus, while the second corresponds to the high-shear modulus materials introduced by Pedersen (1993), giving in this way an interpretation of this classification in terms of tensor invariants.

It must be also emphasized that polar invariants, as well as any other material coefficients, must fulfil some conditions to have a positive definite tensor **L**, see Vannucci (2002a,b).

The polar representation can be used, of course, also for tensors **A**, **B** and **D**. In the following, the polar constants of **A** will be denoted by $\overline{T}_0$, $\overline{T}_1$ and so on, those of **B** by $\hat{T}_0$, $\hat{T}_1$ and so on and those of **D** by $\widetilde{T}_0$, $\widetilde{T}_1$ and so on. Of course, by (2), the composition laws proper to the CLPT giving the Cartesian components of **A**, **B** and **D** apply as well to polar constants

$$\begin{aligned} &\overline{T}_0, \hat{T}_0, \widetilde{T}_0 = \frac{1}{m}\sum_{k=1}^{n} T_{0_k}(z_k^m - z_{k-1}^m),\\ &\overline{T}_1, \hat{T}_1, \widetilde{T}_1 = \frac{1}{m}\sum_{k=1}^{n} T_{1_k}(z_k^m - z_{k-1}^m),\\ &\overline{R}_0e^{4i\overline{\Phi}_0}, \hat{R}_0e^{4i\hat{\Phi}_0}, \widetilde{R}_0e^{4i\widetilde{\Phi}_0} = \frac{1}{m}\sum_{k=1}^{n} R_{0_k}e^{4i(\Phi_{0_k}+\delta_k)}\\ &\qquad \times (z_k^m - z_{k-1}^m),\\ &\overline{R}_1e^{4i\overline{\Phi}_1}, \hat{R}_1e^{4i\hat{\Phi}_1}, \widetilde{R}_1e^{4i\widetilde{\Phi}_1} = \frac{1}{m}\sum_{k=1}^{n} R_{1_k}e^{4i(\Phi_{1_k}+\delta_k)}\\ &\qquad \times (z_k^m - z_{k-1}^m). \end{aligned} \tag{7}$$

In (7), m=1 for **A**, m=2 for **B** and m=3 for **D**; the subscript k indicates a quantity proper to the kth layer; so, δ_k is the orientation angle; while z_{k-1}, z_k, the distances from the middle plane z=0 of the lower and upper face of the layer k. An advantage of the polar representation on the Cartesian one is that in (7), the dependence upon the angles δ_k is explicit. In the most general case, the elastic behaviour of a laminate in a given reference frame is known when 18 parameters are known.

It is apparent from (7) that

$$R_{1_k}=0 \quad \forall k \quad \Rightarrow \quad \overline{R}_1=\hat{R}_1=\widetilde{R}_1=0; \tag{8}$$

that is, a laminate composed of square-symmetric layers will be completely square symmetric (Vincenti et al. 2001). Again,

$$R_{0_k}=0 \quad \forall k \quad \Rightarrow \quad \overline{R}_0=\hat{R}_0=\widetilde{R}_0=0, \tag{9}$$

which shows that the use of R_0-orthotropic layers ensures automatically the complete orthotropy of type R_0 of the laminate for any possible stacking sequence and orientation of the layers. In other words, unlike invariant (3) which denotes ordinary orthotropy, invariants R_0 and R_1 are strong invariants in the sense that they belong not only to the plies but to the laminate, too.

By the aid of (1) and (7), also the polar components of **C**, denoted by $\breve{T}_0$, $\breve{T}_1$, can be found;

$$\breve{T}_0=\frac{1}{h^3}\sum_{k=1}^{n}T_{0_k}(z_k-z_{k-1})[h^2-4(z_k^2+z_kz_{k-1}+z_{k-1}^2)],$$

$$\breve{T}_1=\frac{1}{h^3}\sum_{k=1}^{n}T_{1_k}(z_k-z_{k-1})[h^2-4(z_k^2+z_kz_{k-1}+z_{k-1}^2)],$$

$$\breve{R}_0e^{4i\breve{\Phi}_0}=\frac{1}{h^3}\sum_{k=1}^{n}R_{0_k}e^{4i(\Phi_{0_k}+\delta_k)}(z_k-z_{k-1}) \times[h^2-4(z_k^2+z_kz_{k-1}+z_{k-1}^2)],$$

$$\breve{R}_1e^{2i\breve{\Phi}_1}=\frac{1}{h^3}\sum_{k=1}^{n}R_{1_k}e^{2i(\Phi_{1_k}+\delta_k)}(z_k-z_{k-1}) \times[h^2-4(z_k^2+z_kz_{k-1}+z_{k-1}^2)]. \tag{10}$$

A particular but extremely important case is that of laminates composed of identical plies; for them, (7) and (10) give immediately that

$$\frac{\overline{T}_0}{h}=12\frac{\widetilde{T}_0}{h^3}=T_0,$$

$$\frac{\overline{T}_1}{h}=12\frac{\widetilde{T}_1}{h^3}=T_1,$$

$$\hat{T}_0=\breve{T}_0=\hat{T}_1=\breve{T}_1=0; \tag{11}$$

that is, the isotropic parts of the homogenized tensors **A** and **D** are the same and identical to those of the basic layer, while **B** and **C** are made only of anisotropic parts. So, for laminates made of identical layers, uncoupling and quasi-homogeneity correspond to the isotropy, and nullity, of tensors **B** and **C**. That is why in this paper, what have been called a “general elastic property” of the laminate is considered as a symmetry property. In fact, the problems of finding orthotropic, square-symmetric, R_0-orthotropic, uncoupled and quasi-homogeneous laminates are addressed herein, as well as some possible combinations of these cases, and, for the case of laminates composed by identical plies, also uncoupling and quasi-homogeneity can be regarded as symmetry properties, the isotropy of **B** and **C**. This is also generally true, but if layers are not identical, the isotropic parts of the two tensors are not automatically null.

3 A unified approach to the design of laminates with given general elastic properties

Let us consider the following general problem: find the layer orientations δ_k of an n-ply laminate to obtain a given general elastic property, such as isotropy, orthotropy, uncoupling and so on. The material is known, and it can be different from one layer to another.

To state the above problem, consider the following quadratic form of the matrix **H**, defined on the 18-dimensional space of the variables P_i:

$$I(P_k)=\mathbf{P}\cdot\mathbf{HP}=H_{ij}P_iP_j, \quad i,j=1,...,18 \tag{12}$$

with

$$P_1=\frac{\overline{T}_0}{hM},\ P_2=\frac{\overline{T}_1}{hM},\ P_3=\frac{\overline{R}_0}{hM},\ P_4=\frac{\overline{R}_1}{hM},\ P_5=\overline{\Phi}_0,$$
$$P_6=\overline{\Phi}_1,$$
$$P_7=\frac{2\hat{T}_0}{h^2M},\ P_8=\frac{2\hat{T}_1}{h^2M},\ P_9=\frac{2\hat{R}_0}{h^2M},\ P_{10}=\frac{2\hat{R}_1}{h^2M},$$
$$P_{11}=\hat{\Phi}_0,\ P_{12}=\hat{\Phi}_1,$$
$$P_{13}=\frac{12\widetilde{T}_0}{h^3M},\ P_{14}=\frac{12\widetilde{T}_1}{h^3M},\ P_{15}=\frac{12\widetilde{R}_0}{h^3M},\ P_{16}=\frac{12\widetilde{R}_1}{h^3M},$$
$$P_{17}=\widetilde{\Phi}_0,\ P_{18}=\widetilde{\Phi}_1. \tag{13}$$

M is a factor used to obtain non-dimensional variables. Several definitions are possible for M, provided that it is never 0; proper choices are means of layer characteristic quantities. To this purpose, it can be used, for instance, the tensor norm proposed by Kandil and Verchery (1988) and to pose

$$M=\frac{1}{n}\sum\nolimits_{k=1}^{n}\sqrt{T_{0_k}^2+2T_{1_k}^2+R_{0_k}^2+4R_{1_k}^2}. \tag{14}$$

Moreover, if the layer orientations appear as the basic variables, the quadratic form $I(P_k)$ depends upon all the mechanical and geometrical parameters of the laminate, i.e. the orientations, the stacking sequence, the material and the thickness of the layers. In fact, by (7) and (13), the parameters P_k depend upon all these quantities.

H is a symmetric matrix of non-dimensional real numbers: the choice of its components H_{ij} determines the kind of problem to be treated; several choices are possible, and so, several different problems can be stated in the same way, simply changing the H_{ij}. **H** is positive semi-definite, and, apart from the case of orthotropy with K=1, each problem can be stated as follows: find an absolute minimum of (12) for a given **H**. As **H** is positive semi-definite, these minimums are 0-valued. Clearly, to solve such a problem means to find a vector **P** of P_k, i.e. the 18 polar components, which satisfy the above statement.

A certain number of different and technically interesting problems are detailed in Vannucci (2002b). In the following sections, only some of these problems, particularly important, are considered, and the components H_{ij} are shown for each case. It must be said, anyway, that a larger number of problems, not always technically interesting, can be treated in the same way.

Before going on, it is worth noting what was already said in Sect. 1—a unique mathematical form has been given to several different problems of laminate design: the search for the minimum of a positive semi-definite form, which is a classical problem of structural optimisation. The objective function (12) is not convex, since parameters P_k depend upon circular functions of the orientations (7), but the problem is unconstrained. The use of the parameters P_k allows a classical formulation, stated in terms of the minimisation of a quadratic positive semi-definite form in the 18-dimensional space of the P_k parameters. Nevertheless, the real design variables are the n orientations δ_k of the layers, and in this space, the problem is non-convex and higly non-linear.

It is worth noting that parameters P_k generalise the concept of lamination parameters introduced by Miki (1982) and successively widely used by several authors, but unlike these, they have some advantages: they are based upon tensor invariants linked to elastic symmetries, so their use allows a very simple statement of problems concerning these symmetries, and in addition, they make the orientation of the reference or material frame directly appear.

4 Basic problems

Let us consider first some basic problems, those concerning only one polar parameter among the 18 of the laminate. For instance, consider the search for a laminate having an R_0-orthotropic tensor **A**: the condition is

$$\overline{R}_0 = 0, \tag{15}$$

or equally

$$I(P_k) = P_3^2 = 0, \tag{16}$$

which implies that for this case in **H**, it is H_{33}=1, while the remaining H_{ij} are 0. In the same way, the following basic problems can also be stated (in the following, the components of **H** not explicitly indicated are understood to be 0; in addition, being **H** symmetric, only the components above the diagonal are shown):

- find a laminate with a square-symmetric tensor **A**,

$$\overline{R}_1 = 0; \quad I(P_k) = P_4^2 = 0 \quad \Rightarrow \quad H_{44} = 1; \tag{17}$$

- find a laminate with an R_0-orthotropic tensor **D**,

$$\widetilde{R}_0 = 0; \quad I(P_k) = P_{15}^2 = 0 \quad \Rightarrow \quad H_{1515} = 1; \tag{18}$$

- find a laminate with a square-symmetric tensor **D**,

$$\widetilde{R}_1 = 0; \quad I(P_k) = P_{16}^2 = 0 \quad \Rightarrow \quad H_{1616} = 1. \tag{19}$$

It is worth to recall that all these properties are automatically obtained in the case of a laminate made by layers having the same property. Other basic problems can be considered, but they are not so mechanically interesting as those shown hereon; their importance is eventually in the fact that they are a part of composed problems, i.e. of problems depending upon more than one polar parameter. In the next section, some of these composed problems are considered.

5 Composed problems

First of all, let us consider the search for uncoupled laminates; this problem can be stated as follows:

$$\mathbf{B} = \mathbf{O}; \quad I(P_k) = \sum_{i=7}^{10} P_i^2 = 0 \quad \Rightarrow \quad H_{ii} = 1, \quad i = 7, ..., 10. \tag{20}$$

If the laminate is composed by identical plies, it is automatically

$$P_7 = P_8 = 0, \tag{21}$$

so H_{77} and H_{88} are meaningful; that is, they can be equally posed 0 or 1.

Another case of interest is that of uncoupled laminates with **A** or **D** isotropic, which can be respectively stated as

$$I(P_k) = P_3^2 + P_4^2 + \sum_{i=7}^{10} P_i^2 = 0 \quad \Rightarrow \quad H_{ii} = 1, \quad i = 3, 4, 7, 8, 9, 10; \tag{22}$$

$$I(P_k) = \sum_{i=7}^{10} P_i^2 + P_{15}^2 + P_{16}^2 = 0 \quad \Rightarrow \quad H_{ii} = 1, \quad i = 7, 8, 9, 10, 15, 16. \tag{23}$$

In the same way, the very treated problem of full isotropy, i.e. of the search for an uncoupled laminate with isotropic tensors **A** and **D**, can be stated as follows:

$$I(P_k) = P_3^2 + P_4^2 + \sum_{i=7}^{10} P_i^2 + P_{15}^2 + P_{16}^2 = 0 \quad \Rightarrow \quad H_{ii} = 1, i = 3, 4, 7, 8, 9, 10, 15, 16. \tag{24}$$

Another interesting problem is that of an uncoupled laminate with square symmetric **A** and **D**, having, in addition, the same axes of symmetry. Now, the coincidence of the axes must be imposed; in the polar method, this can be easily done by setting $\overline{\Phi}_0 = \widetilde{\Phi}_0$:

$$I(P_k) = P_4^2 + \sum_{i=7}^{10} P_i^2 + P_{16}^2 + (P_5 - P_{17})^2 = 0 \quad \Rightarrow$$
$$H_{ii} = 1, i = 4, 5, 7, 8, 9, 10, 16, 17; \quad H_{517} = -1. \tag{25}$$

6 Statement of quasi-homogeneity

The condition imposing **C=O** can be stated as follows:

$$\breve{T}_0^2 + \breve{T}_1^2 + \breve{R}_0^2 + \breve{R}_1^2 = 0; \tag{26}$$

taking into account for (1), (7) and (13), we get the equivalent condition

$$(P_1 - P_{13})^2 + (P_2 - P_{14})^2 + \left(P_3 e^{4iP_5} - P_{15} e^{4iP_{17}}\right)$$
$$\left(P_3 e^{-4iP_5} - P_{15} e^{-4iP_{17}}\right) + \left(P_4 e^{2iP_6} - P_{16} e^{2iP_{18}}\right)$$
$$\left(P_4 e^{-2iP_6} - P_{16} e^{-2iP_{18}}\right) = 0. \tag{27}$$

A simpler condition can be obtained if we consider that the identity of the elastic in- and out-of-plane behaviour at each direction implies the identity of the polar angles of **A** and **D** that can be stated by the condition

$$(P_5 - P_{17})^2 + (P_6 - P_{18})^2 = 0. \tag{28}$$

So, considering (27) and (28), together with (20), to take into account for uncoupling, a general condition for quasi-homogeneity (**B**=**C**=**O**) is obtained:

$$I(P_k) = (P_1 - P_{13})^2 + (P_2 - P_{14})^2 + (P_3 - P_{15})^2 +$$
$$+ (P_4 - P_{16})^2 + (P_5 - P_{17})^2 + (P_6 - P_{18})^2 +$$
$$+ P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = 0 \Rightarrow$$
$$H_{ii} = 1, i = 1, ..., 10, 13, ..., 18;$$
$$H_{113} = H_{214} = H_{315} = H_{416} = H_{517} = H_{618} = -1. \tag{29}$$

It is apparent that this approach let to the designer the possibility to add new properties to other ones. For instance, if now we look for a quasi-homogeneous square-symmetric laminate, it is sufficient to add to (29) a condition fixing square symmetry, like (17) or equally (19), the two conditions being equivalent for quasi-homogeneity:

$$I(P_k) = (P_1 - P_{13})^2 + (P_2 - P_{14})^2 + (P_3 - P_{15})^2 +$$
$$+ (P_4 - P_{16})^2 + (P_5 - P_{17})^2 + (P_6 - P_{18})^2 +$$
$$+ P_4^2 + P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = 0 \Rightarrow$$
$$H_{ii} = 1, i = 1, ..., 3, 5, ..., 10, 13, ..., 18; \quad H_{44} = 2;$$
$$H_{113} = H_{214} = H_{315} = H_{416} = H_{517} = H_{618} = -1. \tag{30}$$

7 Statement of orthotropy

Apart from R_0 orthotropy, the general condition for orthotropy is (3); this means that orthotropy of **A** and **D** can be respectively stated by

$$I(P_k) = (P_5 - P_6)^2 = \overline{K}\frac{\pi^2}{16}, \quad \overline{K} = 0, 1 \quad \Rightarrow$$
$$H_{55} = H_{66} = 1; \quad H_{56} = -1; \tag{31}$$

$$I(P_k) = (P_{17} - P_{18})^2 = \widetilde{K}\frac{\pi^2}{16}, \quad \widetilde{K} = 0, 1 \quad \Rightarrow$$
$$H_{1717} = H_{1818} = 1; \quad H_{1718} = -1. \tag{32}$$

This approach gives the designer to chose the type of orthotropy, but if K=1, the solution does not correspond with the absolute minimum of the objective function $I(P_k)$. It is easy to combine (31) and (32) with other properties to obtain for instance orthotropic and uncoupled laminates. Among these cases, the following two final problems are considered: the first one concerns a quasi-homogeneous orthotropic laminate. Taking into account for (29) and (31), or equally of (32), we get the statement of this problem:

$$\breve{I}(P_k) = (P_1 - P_{13})^2 + (P_2 - P_{14})^2 + (P_3 - P_{15})^2 +$$
$$+ (P_4 - P_{16})^2 + (P_5 - P_{17})^2 + (P_6 - P_{18})^2 +$$
$$+ P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = 0;$$
$$\overline{I}(P_k) = (P_5 - P_6)^2 - \overline{K}\frac{\pi^2}{16} = 0, \quad \overline{K} = 0, 1 \quad \Rightarrow$$
$$H_{ii} = 1, i = 1, ..., 4, 7, ..., 10, 13, ..., 18; \quad H_{55} = H_{66} = 2;$$
$$H_{56} = H_{113} = H_{214} = H_{315} = H_{416} = H_{517} = H_{618} = -1. \tag{33}$$

If $\overline{K} = 0$, the two functions $\breve{I}(P_k)$ and $\overline{I}(P_k)$ can be added to obtain again a unique quadratic form of the type $I(P_k)$; if $\overline{K} = 1$, we can still get a unique non-negative objective function $I(P_k)$, for instance, posing

$$I(P_k) = \left|\overline{I}(P_k)\right| + \breve{I}(P_k). \tag{34}$$

The second one is the case of an uncoupled laminate, orthotropic in bending and in extension and with coincident symmetry axes:

$$\overline{I}(P_k) = (P_5 - P_6)^2 - \overline{K}\frac{\pi^2}{16} = 0, \quad \overline{K} = 0, 1;$$
$$\widetilde{I}(P_k) = (P_{17} - P_{18})^2 - \widetilde{K}\frac{\pi^2}{16} = 0, \quad \widetilde{K} = 0, 1;$$
$$\hat{I}(P_k) = \sum\nolimits_{i=7}^{10} P_i^2 + (P_6 - P_{18})^2 = 0 \quad \Rightarrow$$
$$H_{ii} = 1, i = 5, 7, ..., 10, 17; \quad H_{66} = H_{1818} = 2;$$
$$H_{56} = H_{618} = H_{1718} = -1. \tag{35}$$

Like in the preceding case, if $\overline{K} = \widetilde{K} = 0$, then the three functions $\overline{I}(P_k)$, $\widetilde{I}(P_k)$ and $\hat{I}(P_k)$ can be added to obtain again a unique quadratic form of the type $I(P_k)$; if this is not the case, a unique non-negative objective function $I(P_k)$ can still be obtained, for instance, posing

$$I(P_k) = \left|\overline{I}(P_k)\right| + \left|\widetilde{I}(P_k)\right| + \hat{I}(P_k). \tag{36}$$

8 Numerical examples and discussion of the results

Different types of numerical methods can be used to solve the above-mentioned problems; in this section, some examples of laminates composed by identical layers are shown. In this hypothesis, the only variables are the ply orientations, and a descent method has been used to solve numerically the problems, namely, the steepest descent method (see for instance Arora 1989) or alternatively the conjugate gradient method of Fletcher and Reeves (1964). The initial point for the descent algorithm was established randomly, but considering that, to fix a global frame for the plate, the orientation of the first layer was always 0; so, the number of unknowns is $n-1$ and not n. We give in Table 1 some details about 10 among the several examples that we have treated. The values of the objective function shown in Table 1 are the residuals of $I(P_k)$: an obvious consequence of the use of a numerical approach is that the solution of an equation is found to a certain degree of approximation, an exact solution being in practice impossible to be obtained. To notice that $I(P_k)$ is a non-dimensional function, so the residuals shown in Table 1 are non-dimensional, too.

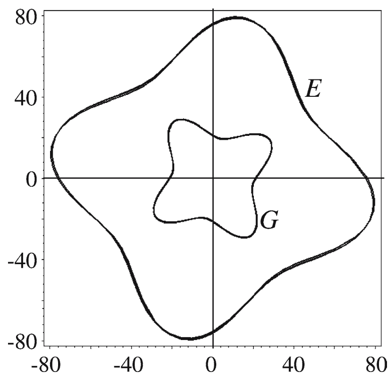

Fig. 1 The polar diagrams of E and G for example 9 in Table 1 (exact and approximate solutions; in GPa)

It is worth noting that, being in all the example the layers identical, the elastic properties of the single basic ply are meaningful, the results being valid for all the possible layers; for this reason, the mechanical properties of the ply is not necessarily specified: in fact, the true computations have been made independently of them and are valid for any material. Nevertheless, the diagrams in Figs. 1, 2 and 3 have been traced for laminates composed of T300-5208, carbon–epoxy layers, whose mechanical constants are (see Tsai and Hahn 1980) E_1=181 GPa, E_2=10.30 GPa, G_{12}=7.17 Gpa and ν_{12}=0.28.

The first case has been treated to test the effectiveness of the method; it is a five-layer laminate composed of plies reinforced by balanced fabrics (that is, having the same amount of fibres in warp and weft, what ensures the square symmetry, R_1=0) and designed to be uncoupled. The complete solution

Table 1 Some numerical examples

	Type of laminate	Layers number	Stacking sequence (°)	Objective function
1	Uncoupled (ply R_1=0)	5	[0/4.57$_2$/−3.04/1.54]	6.7 10^{-4}
2	Uncoupled, **A** isotropic	7	[0/−60.10/59.27/61.19/−59.37/0.45]	1.6 10^{-7}
3	Fully isotropic (ply R_0=0)	7	[0/87.59/ 82.32/−39.69/20.81/5.14/−83.42]	1.4 10^{-4}
4	Fully isotropic (ply R_0=0)	8	[0/−78.94/63.34/−62.83/31.28/−4.42/−29.36/74.74]	1.1 10^{-8}
5	Fully isotropic	12	[0/−55.89/49.92/65.58/−65.74/79.91/ −2.85/−36.58/11.36/26.75/−47.75/76.46]	2.8 10^{-7}
6	Uncoupled, **A** square symmetric	7	[0/−81.03/35.79/−83.14/−22.93/−86.73/12.66]	1.5 10^{-7}
7	Uncoupled, **D** square symmetric	7	[0/88.36/−85.89/56.24/4.19/−1.89/−89.84]	4.5 10^{-6}
8	Quasi-homogeneous	12	[0/14.75/2.34/10.59/−2.75/17.55/ 13.80/3.99/2.57/8.08/−3.98/13.38]	1.2 10^{-7}
9	Quasi-homogenous square symmetric	12	[0/61.76/−52.12/82.67/−18.21/−78.31/ 64.64/1.10/−2.52/44.63/−29.90/−89.65]	1.1 10^{-13}
10	Uncoupled orthotropy	12	[0/44.67/15.70/−39.98/−25.46/−37.21/ 59.04/54.28/36.92/−38.16/18.58/−5.23]	4.7 10^{-5}

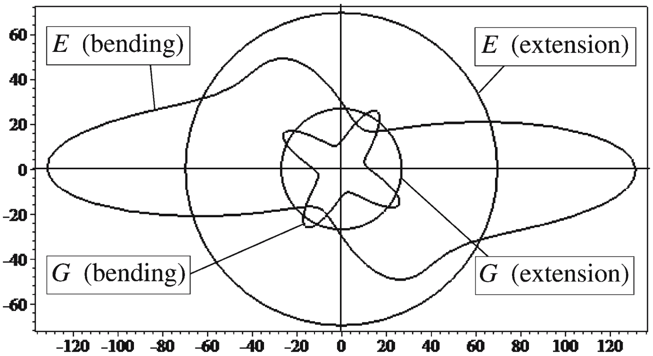

Fig. 2 Polar diagrams of E and G for example 2 in Table 1 (in GPa)

to this case has been analytically found by Vincenti et al. (2001), so the comparison of the results of Table 1 for case 1 and the theoretical solution are possible. The exact solution is, in degrees and up to the fourth decimal digit,

$$[0°/4.5735°_2/-3.0374°/1.5361°]; \tag{37}$$

it can be noticed that the approximate numerical solution is very close to the exact one. A remark about the precision: if the exact solution is given with only the first two decimal digits, the residual of the objective function passes immediately from 0 to the value of the numerical solution in Table 1. In effects, a rather great sensitivity of the objective function to the precision has always been observed, but this sensitivity concerns much more the value of the residual than the quality of the solution. Indeed, it has been observed that residuals in the range $10^{-6} \div 10^{-4}$ are very well suited for technically acceptable solutions. Another example confirming this is the solution of case 9, which is actually an exact solution, being the residual extremely small. When, to have a stacking sequence well suited for applications, the solution is approximated to the nearest integer angle,

$$[0°/62°/-52°/83°/-18°/-78°/65°/1°/-2°/45°/$$
$$-30°/90°], \tag{38}$$

the residual becomes 8.56 10^{-5}, i.e. 7.78 10^8 times greater. Nevertheless, the solution remains very close to the exact one, as it can be seen on the diagrams of moduli E and G for extension and bending of the exact and approximated solution shown in Fig. 1: the four curves of E and the four of G are almost perfectly superposed (there are four curves because the laminate was also required to be quasi-homogeneous).

Example 2 is another interesting case, since for this problem, an exact solution exists, the Werren and Norris (1953) symmetric solution is

$$[0°/-60°/60°/60°/-60°/0°]. \tag{39}$$

It can be noticed that the solution found numerically is very near to the exact one; in Fig. 2, the polar diagrams of moduli E and G, for bending and extension, are shown; it can be noticed that the extension behaviour is in practice isotropic (the ratio $E_{\max}/E_{\min}$ is equal to 1.0015, i.e. a difference of 0.15%), while the bending one is completely anisotropic.

Examples 3 and 4 concern the search for fully isotropic laminates composed by R_0-orthotropic layers. The use of layers having one of the two anisotropic invariants null simplifies the search and increases considerably the number of solutions. In fact, it has been possible to find hundreds of fully isotropic laminates composed of R_0-orthotropic layers (those shown in Table 1 are the best ones in the sense that their residual is the lowest), while it is well known that it is rather difficult to find fully isotropic laminates composed of generally orthotropic layers (see Grédiac 1999; Vannucci and Verchery 2002). In addition, the solutions with seven R_0-orthotropic layers, to which example 3 belongs, are, up to now, the known fully isotropic solutions with the lowest number of layers.

Finally, example 10 is the case of an uncoupled in- and out-of-plane orthotropic laminate with identical symmetry axes in extension and in bending; to test the effectiveness of the approach, two different types of orthotropy for **A** and **D** have been imposed, simply choosing $\overline{K}=1$ and $\widetilde{K}=0$. In other words, the laminate in example 3 is low-shear modulus in bending and high-shear modulus in extension. The diagrams of E and G are shown in Fig. 3. The conditions imposed to the solution have been satisfied with a good approximation: in fact, $\overline{\Phi}_0-\overline{\Phi}_1=-45.03$, $\widetilde{\Phi}_0-\widetilde{\Phi}_1=0.20$ and the angular gap between the extension and bending symmetry axes is only $\left|\overline{\Phi}_1-\widetilde{\Phi}_1\right|=0.02$. The two curves of E in Fig. 3 show clearly the different type of orthotropy in bending and in extension: contrarily to what happens for bending, the Young's modulus E in extension is not the highest on the orthotropy axes.

The examples given above do not consider but elastic symmetries. The main goal of this approach was to consider the design of a laminate, including the design of the elastic symmetries in the procedure, to not use some short cuts usually accepted by designers, like, for instance, the use of symmetric stacking sequences. The method proposed hereon can treat all the problems concerning elastic symmetries in the same way and can also take into account any combination of these problems.

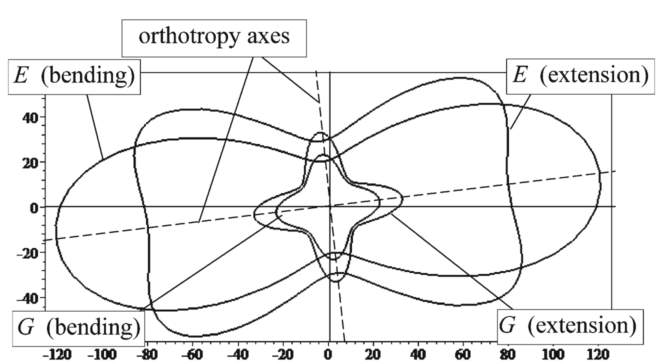

Fig. 3 Diagrams of E and G for example 10 in Table 1 (in GPa)

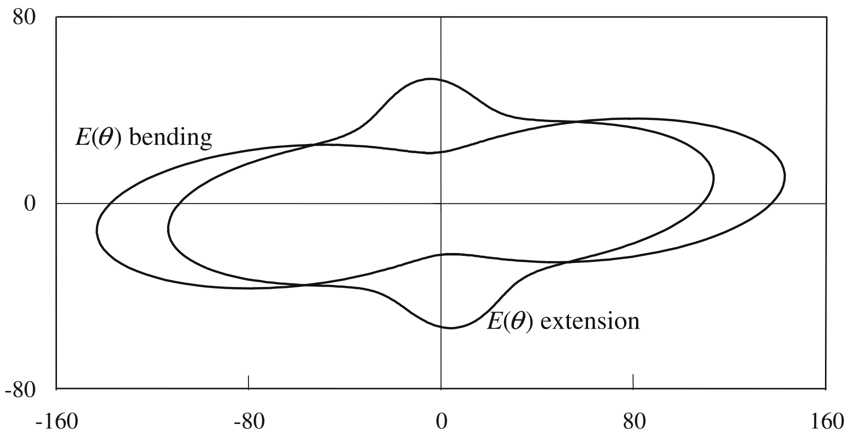

Fig. 4 Polar diagrams of E for laminate (Eq. 41) (in GPa)

Nevertheless, the main interest of designers is to obtain a laminate which, in addition to some specified general properties, i.e. some elastic symmetries in the sense specified in Sect. 2, maximises some other property, for instance, stiffness or strength and so on, or respects certain given conditions imposed to the design.

In addition, the formalisation of the problem, which conserves the orientation angles as main variables, makes use of a very non-linear objective function. No general rules are known to establish a good starting point for a numerical search of the solution; in addition, Vincenti et al. (2001) have shown that the set of solutions of several problems concerning laminates is not discrete; that is, solutions vary continuously with respect to design variables (see also Valot and Vannucci 2005). So, descent methods, like those used for finding the solutions in Table 1, are not the best suited for this kind of problems.

Nevertheless, the method proposed in this paper is effective also in treating more complicated problems, and of course, it can be adapted to different numerical methods for the search of the solution. Let us consider, for instance, the following example: find a laminate, composed by 12 identical plies, designed to be uncoupled and orthotropic in extension and with the orientation angles belonging to a discrete set (namely, 0°, ±15°, ±30°, ±45°, ±60°, ±75° and 90°). The plies are made with the same material of the previous examples, i.e. carbon–epoxy T300-5208. Two constraints are imposed to the solution: the Young's modulus in extension must fulfil the following requirements:

$$E_{\max} \geq 100 \text{ GPa};$$
$$E_{\min} \geq 40 \text{ GPa}. \tag{40}$$

To solve this problem, which is a discrete optimisation problem, a genetic algorithm has been used (see Vincenti et al. 2003b,c). The best solution found is

$$[0°/30°/-15°/15°/90°/-75°/0°/45°/-75°/0°/-15°/15°], \tag{41}$$

to whom the residual 1.3×10^{-3} corresponds; the values of the extension Young's modulus for this laminate are $E_{\min}$=45.8 GPa at 61.22° and $E_{\max}$=114 GPa at 6.05°, respecting the imposed minimal values. The polar diagrams of the Young's modulus in bending and in extension are shown in Fig. 4. More details on the numerical algorithm and about the handling of constraints can be found in Vincenti et al. (2003b,c) or in Vannucci (2002a).

9 Final remarks and conclusion

A new method for designing laminates with given stiffness properties has been proposed. Some basic problems concerning the design of laminates, often discarded by designers, are considered in the most general way. This opens the way to find optimal solutions to some particular problems in the most general case, without the use of simplifying but rather limiting hypothesis, such as the use of symmetric balanced stacking sequences; an example in this sense has been given.

Under a mathematical point of view, this method makes use of the polar invariants, introduced by Verchery et al. (2000), and several distinct problems concerning the properties of a laminate, namely, its elastic symmetries, are condensed in a unique formulation, which is a classical problem of structural optimisation: the search for the minimum of a positive semi-definite form in the space of polar invariants. The objective function is non-convex, but the problem is unconstrained. This new formulation takes a great advantage by the use of polar tensor invariants, as they are the most effective way to represent material symmetries.

The examples given in the paper show the effectiveness of the method, and a discussion of the order of approximation of the solutions has also been given.

In perspective, some points must still be solved: the use of the proposed method into a general approach to the optimisation of laminates and the inclusion of the number of plies among the design variables. Researches are going on in these directions.

A final remark, the paper shows that some basic properties not only of a laminate but also of a layer can be characterised as being minimum properties, such as isotropy. This is a quite new formulation of these properties, as the form to be minimised is a function of parameters having a physical meaning, the polar invariants.

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