Brief note

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Planar articulated mechanism design by graph theoretical enumeration

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Abstract This paper deals with design of articulated mechanisms using a truss-based ground-structure representation. By applying a graph theoretical enumeration approach we can perform an exhaustive analysis of all possible topologies for a test example for which we seek a symmetric mechanism. This guarantees that one can identify the global optimum solution. The result underlines the importance of mechanism topology and gives insight into the issues specific to articulated mechanism designs compared to compliant mechanism designs.

Key words mechanism design, topology optimization, graph theory, enumeration

1 Introduction

Research on the design of compliant mechanisms (e.g., Sigmund (1997), Pedersen (2001), Bruns (2001) and Ananthasuresh (1994)) using topology optimization techniques in continuum structures suggests that it should also be possible to obtain results for articulated mechanisms by applying such techniques (Sigmund 1996; Frecker 1997).

In this paper we examine exhaustively all possible topologies of one test example by a graph theoretical enumeration approach. This allows for identifying the global optimum solution when we assume that the resultant mechanism is symmetric. We view the present work as a first step in devising optimization techniques for design of an articulated mechanism, with the achieved global optimum solution providing a rigorous comparison design with which solutions obtained by methods currently under development can be matched.

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2 Definition of test example

The test example we examine in detail is shown in Fig. 1. The figure shows the truss ground structure of possible nodes and connections, as well as all necessary boundary conditions, load cases and other related parameters. One relevant mechanism design problem is the maximization of the output displacement D_{out} for the given input forces F_{in} by picking M^* truss elements out of the M possible connections, so that the resultant mechanism has $f = 1$ degree of freedom (DOF) and is symmetric with respect to the horizontal line between nodes 2 and 11. This problem can be formulated as the following binary integer optimization problem for the truss connectivity vector $\chi = (\chi_1, \ldots, \chi_M)$:

$$\max_{\chi \in \{0,1\}^M} D_{\text{out}} \text{ s.t. } \sum_{e=1}^{M} \chi_e = M^* ; \quad f = 1 ; \quad \text{symmetric.} \tag{1}$$

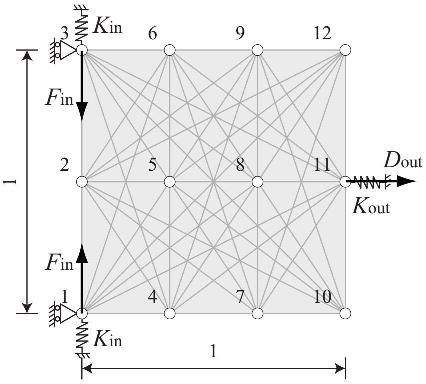

Fig. 1 Truss ground structure

Here D_{out} is given via the displacement vector U which is a solution to an equation $R(U) = 0$ that expresses the residual version of the equilibrium equations. Geometrical non-linearity should be taken into account, so we adopt a force incremental method with Newton–Raphson equilibrium iterations for evaluation of the displacement D_{out}.

3
Graph theoretical enumeration approach

There are 66 potential elements in the ground structure of Fig. 1. When the number of bars in the mechanism is set at $M^* = 8$, the complexity of the problem is ${}_{66}C_8 (\sim 6 \times 10^{10})$, so evaluating the totality of possible mechanisms is not a viable approach.[1]

An alternative way to address the problem is to first generate a complete set of topological graphs and then embed them (as shown in Fig. 2) into the ground structure in all possible ways. Assuming that the mechanism has *no redundant elements* (for brevity, we shall not discuss this here; see, e.g., Calladine (1978), Pellegrino (1986) and Fowler (2000) for details), the degrees of freedom of a mechanism that is supported in a statically determinate manner can be calculated by the following DOF equation based on Maxwell's rule in 2D:

$$f = 2n - m - 3 \tag{2}$$

where n is the number of nodes and m is the number of bars. In this case where $m = 8$ and $f = 1$, a possible mechanism has $n = 6$ nodes (note that this is only a necessary condition, see the above references). If we choose to focus on one topological configuration of eight edges and six vertices (we shall use *edge* and *vertex* for topological graphs), as shown in Fig. 2, we can reduce the design problem to a subproblem that entails picking six ordered nodes from the 12 potential nodes of the ground structure. This results in the assignment of six vertices of the topological graph to six out of 12 nodes in the ground structure. The subproblem can be illustrated by a bipartite matching problem shown in Fig. 3. The complexity of the subproblem is only ${}_{12}P_6 (\sim 6 \times 10^5)^2$. If we have p different topological configurations, the calculation time is simply multiplied by p. In the present case, $p = 24$ and the complete set of the 24 non-isomorphic graphs, which have six vertices and eight edges, can be found in Harary (1969).

So far, we have not utilized the symmetry constraint and we illustrate now how to use this to reduce the problem further. We begin with vertex configurations without edges. There are two possible symmetric vertex configurations with three rows (see Fig. 4). Here we introduce a symmetric embedding that assigns the vertices to the nodes maintaining the symmetry around the horizontal mid-axis, which means that the vertices on the first row go to the nodes on the first row, the second row in F1 or F2 goes to the second row in Fig. 1, and so on.

[1] If one function evaluation (i.e., one non-linear analysis) takes one second, the total calculation time amounts to about 180 years.

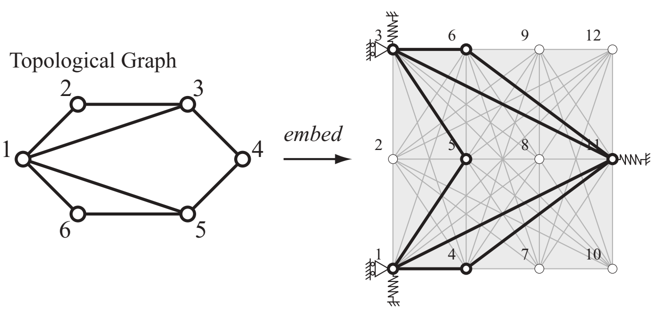

Fig. 2 Embedding of a topological graph (eight edges and six vertices) into a truss ground structure

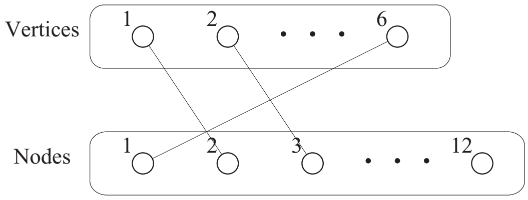

Fig. 3 Bipartite matching problem between vertices and nodes

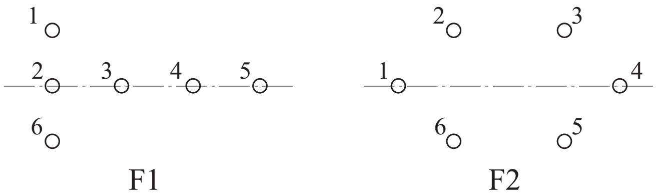

Fig. 4 Two possible vertex configurations

3.1
Case F1

Note that all the possible topological graphs generated from the vertex configuration F1 should satisfy the following condition on the number of incident edges with a vertex, namely vertex degree v_i $(i = 1, \ldots, 6)$:

$$v_1 = v_6 \tag{3}$$

due to the restriction of the symmetry. Besides, if the embeddings involve all possible swaps among vertices 2 through 5, it suffices to consider the cases where vertex degrees meet the constraints:

$$v_2 \geq v_3 \geq v_4 \geq v_5 \,. \tag{4}$$

[2] This amounts to 7.7 days of computational time for evaluating all designs.

This is because vertices 1 and 6 should be assigned to the input ports in the ground structure of Fig. 1 anyway.

We can thus classify the topological graphs generated from the configuration F1 by the vertex degree $v_1 (= v_6)$ running from 1 to 4; these correspond to the classes (i) to (iv) in Fig. 5. Note that the number of possible graphs varies in the different classes. Graph (i) is actually irrelevant as in reality it has two degrees of freedom even though (2) gives $f = 1$ DOF. This is a special case where the structure has one redundant bar in the mid axis. In this case, the structure has extra states of self-stress and mechanism mode. Thus the mechanism DOF increases; see Calladine (1978), Pellegrino (1986) and Fowler (2000) for details. Graphs (ii)-{1, ... ,5}, (iii)-{1,2} and (iv)-1 are also all irrelevant due to the symmetric loading and support conditions; all these graphs have asymmetric mechanism modes with respect to the horizontal mid-axis.

With reference to the mechanisms that have one isolated bar, since the rest of nodes and bars form a structure, these mechanisms cannot transfer energy or work from the input ports to the output port. Also, we have observed that the isolated bars cause the mechanical DOF to increase in graph (i). We are not interested in all of these irrelevant mechanisms, and will consider the topological graphs that satisfy $v_i \geq 2$ as the candidate graphs henceforth.

The final graph (iv)-2 essentially has only one possible embedding into the ground structure because vertices 1 and 6 of the graph are assigned to the input ports in Fig. 1, and swapping any pair of vertices 2 through 5 makes no difference in terms of the embedding. This mechanism is, however, difficult to move because the edge between vertices 1 and 2 and the edge between vertices 2 and 6 are on the straight line. The mechanism relies on buckling, which would only occur due to some imperfection; thus we discard this graph, too.

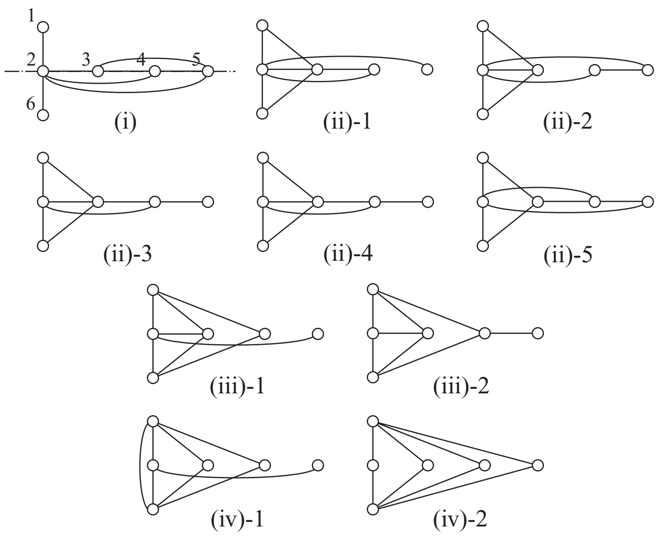

Fig. 5 Topological graphs generated from the vertex configuration F1

3.2 Case F2

There is a formula for the relation between the total vertex degrees and the total number of edges m in a graph (see, e.g., Kaveh (1991)):

$$\sum_{i=1}^{n} v_i = 2m \,. \tag{5}$$

Considering the symmetry of the vertex configuration F2, we obtain

$$v_1 + 2v_2 + 2v_3 + v_4 = 16 \,; \; v_2 = v_6 \,; \; v_3 = v_5 \,. \tag{6}$$

Embedding a graph from the vertex configuration F2 into the ground structure requires that one pair of the vertices 2 and 6 and the vertices 3 and 5 retain their symmetry. If the embedding involves any vertical reflection, it suffices to investigate the following cases.

$$v_1 \leq v_4 \,; \quad v_2 \leq v_3 \,. \tag{7}$$

Besides, because we are not interested in isolated edges and the maximum vertex degree is five (as the number of vertices is six), we can bound each variable as

$$2 \leq v_i \leq 5 \quad (i = 1, \ldots , 6) \,. \tag{8}$$

Based on (6) together with (7) and (8), the vertex configuration F2 produces six classes described in Table 1. Each class has at least one possible graph as shown in Fig. 6. In order to obtain a valid mechanism among the embeddings one has to require that a relative move between vertices 3 and 5 is possible (or, topologically equivalent, between vertices 2 and 6). For that reason, we can discard graphs (I)-1 and (I)-2 immediately because both pairs, vertex 2–6 and vertex 3–5, are linked directly. Likewise, graph (V) can be discarded because vertices 2 and 6 are linked directly and vertices 3 and 5 are on the adjacent two triangles, which form a rigid substructure in the graph. Graphs (IV)-1 and (IV)-2 are irrelevant because the motion of the mechanism is asymmetric. All in all, we can conclude that all the relevant graphs amounts to only five, namely graphs (I)-3, (II), (III)-1, (III)-2 and (VI). The original problem is thus reduced to five embedding subproblems, corresponding to each of the five

Table 1 Classification of topological graphs from F2 by vertex degrees

Class	v_1	v_2	v_3	v_4
I	2	3	3	2
II	2	2	4	2
III	2	2	3	4
IV	3	2	3	3
V	3	2	2	5
VI	4	2	2	4

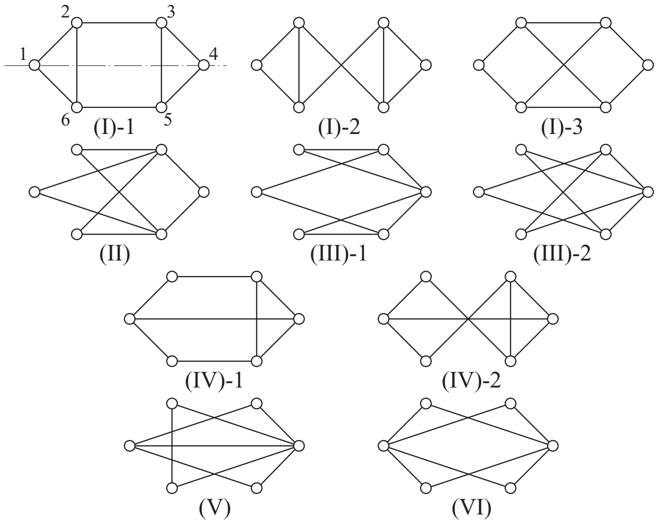

Fig. 6 Topological graphs generated from the vertex configuration F2

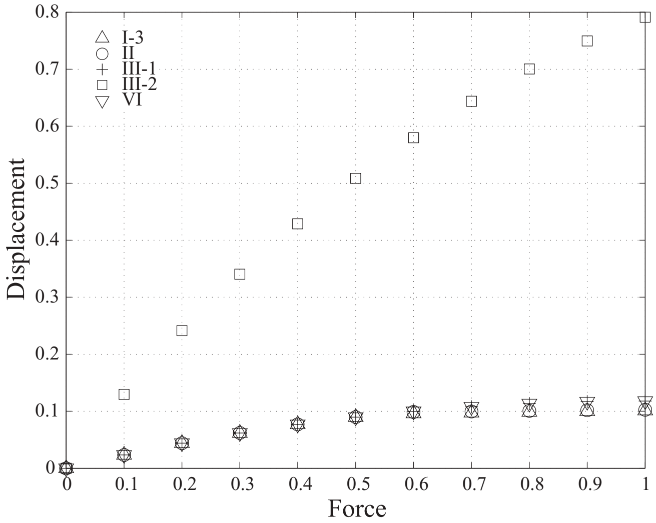

Fig. 8 Comparison of the performance

graphs. (II), (III)-1 and (III)-2 require 36 calculations for each. (I)-3 and (VI) need 18 calculations for each because of their vertical symmetry. The total enumeration corresponds to the evaluation of 144 possible mechanisms, one of which then constitutes the globally optimal (symmetric) mechanism.

Figure 7 shows the best mechanism among the embeddings of graph (III)-2 with the undeformed state depicted in grey and the deformed state in black. Also, this mechanism is the global optimum solution to the problem. Figure 8 compares the performance among the best mechanisms from the five different graphs. Graph (III)-2 is significantly better than the rest of the topologies, which all have a similar performance that is significantly lower than that of the optimal mechanism. This shows the great importance of also choosing good topologies when designing mechanisms – bad topologies cannot be expected to be improved significantly by changing the shape. Moreover, note that the optimal topology (III)-2 forms a scissors-like mechanism when embedded as the globally optimal mechanism and this is the reason for its outstanding performance. This kind of mechanism can not be obtained by the continuum-based topology optimization technique that has been very successful for compliant mechanism, suggesting that an inclusion of hinges in the "ground structure" is crucial for obtaining efficient articulated mechanisms.

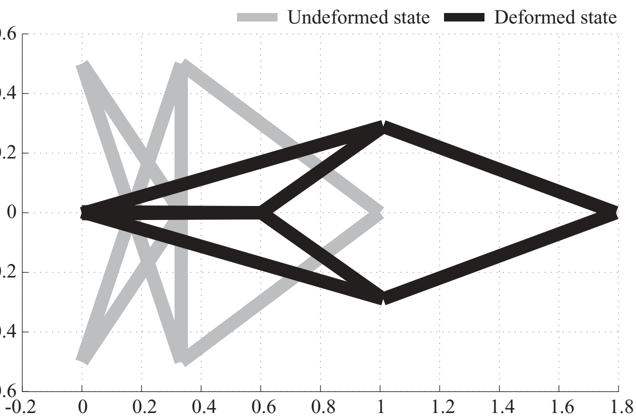

Fig. 7 Best mechanism from graph (III)-2

We close this technical note by remarking that the development above should be seen as a method to obtain a benchmark design that will be useful for further developments in the area; the key is that the *globally optimal solution* has been identified. It is not clear if the approach can be generalized to more complicated settings – it is the identification of the topological graphs that is a cornerstone of the method and here number synthesis, as described by Tischler (1995), may provide a viable path ahead.

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