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Computational procedures for plastic shakedown design of structures

F. Giambanco, L. Palizzolo, A. Caffarelli

Abstract The minimum volume design problem of elastic perfectly plastic finite element structures subjected to a combination of fixed and perfect cyclic loads is studied. The design problem is formulated in such a way that incremental collapse is certainly prevented. The search for the structural design with the required limit behaviour is effected following two different formulations, both developed on the grounds of a statical approach: the first one operates below the elastic shakedown limit and is able to provide a suboptimal design; the second one operates above the elastic shakedown limit and is able to provide the/an optimal design. The Kuhn–Tucker conditions of the two problems provide useful information about the different behaviour of the obtained structures.
An application concludes the paper; the comparison among the designs is effected, pointing out the different behaviour of the obtained structures as well as the required computational effort related to the numerical solutions.

Key words Bree diagrams, elastic plastic behaviour, elastic shakedown limit, optimization, shakedown design

1 Introduction

Often structures are subjected to a combination of fixed and cyclic loads, which are usually described as reference fixed (mechanical) and cyclic (mechanical and/or kinematical) loads, amplified by some corresponding fixed and cyclic load multipliers, ξ_0 and ξ_c, respectively. Due to such a load combination, after a transient phase which depends on the initial conditions and on the special load path, structures eventually exhibit a steady-state response which is characterized by the same periodicity features as the loads and is independent of the above referred conditions (see, e.g. Polizzotto *et al.* (1990), Polizzotto (1994a,b), Zarka and Casier (1979), Zarka *et al.* (1990)).

During the steady-state phase, a structure consisting of elastic perfectly plastic material can exhibit different behaviours depending on the values of the load multipliers. The plastic strain process which can eventually characterize the steady-state response is referred to as the plastic accumulation mechanism (PAM) (see Polizzotto *et al.* (1990)), which is a plastic strain rate cycle resulting in a compatible plastic strain field. Besides the trivial case of purely elastic behaviour, according to the type of PAM, three different types of steady-state responses are usually distinguished, and in particular:

(i) elastic shakedown (usually called simply shakedown). The PAM is a trivial one, i.e. the plastic strain rates vanish identically and the structural response is eventually elastic;
(ii) plastic shakedown (or alternating plasticity, or oligocyclic fatigue). The PAM is a non-trivial one, the plastic strains are periodic in the cycle, but the plastic strain field resulting in the cycle is nought;
(iii) ratchetting (or incremental collapse). The PAM is a nontrivial one, and the ratchet strain is non-vanishing, at least somewhere in the structure volume, causing the plastic strain to increase progressively.

On the plane of the load multipliers (ξ_0, ξ_c), it is possible to distinguish five different regions corresponding to as many different steady-state behaviours of the structure. For all the couples of load multipliers represented by points constituting regions E, S, F or R, the structure exhibits a purely elastic behaviour, an elastic shakedown behaviour, a plastic shakedown behaviour or it is exposed to incremental plastic collapse, respectively. For all the couples of load multipliers represented by points not belonging to the above regions (and constituting region I) the structure is exposed to an instantaneous plastic collapse. The graphical representation of these zones constitutes

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F. Giambanco, L. Palizzolo✉, A. Caffarelli

Dipartimento di Ingegneria Strutturale e Geotecnica, Università di Palermo, Viale delle Scienze, 90128 Palermo, Italy
e-mail: fgiamban@stru.diseg.unipa.it, palizzol@stru.diseg.unipa.it, alecaffa@msn.com

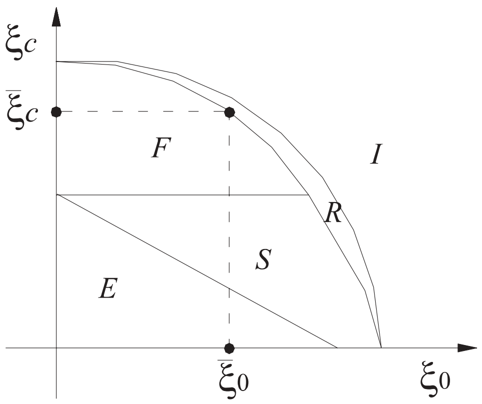

Fig. 1 Typical Bree-like diagram (mechanical cyclic load)

the Bree-like diagram (Fig. 1) relative to the assigned structure/load system; its knowledge is of crucial importance in order to establish if this system safely operates when subjected to potentially different load conditions.

In order that the structure exhibits a good behaviour for increasing values of ξ_0 and ξ_c, it is fundamental that above the elastic shakedown limit the instantaneous collapse does not occur immediately, since in this case the structure would reach collapse without a previous production of plastic dissipation, namely without utilizing its ductility properties. Furthermore, it is preferable that above the elastic shakedown limit the plastic shakedown occurs, instead of the ratchetting, since in the first case the structure can suffer a much greater number of load cycles than in the second case, and moreover a great amount of plastic dissipation can be produced with negligible second-order geometrical effects related to the plastic deformations.

In recent decades many researchers addressed their scientific efforts to structural optimization problems providing many refined and original formulations of the optimal design (see, e.g. Banichuk (1990), Brousse (1988), Gallagher and Zienkiewicz (1973), Giambanco *et al*. (1994a,b), Giambanco and Palizzolo (1995), Haftka *et al*. (1990), König (1975), Majid (1974), Rao (1978), Rozvany (1976, 1989), Save and Prager (1985)), as well as several interesting contributions related to the computational procedures (see, e.g. Cinquini *et al*. (1980), Corradi and Zavelani Rossi (1974), Giambanco *et al*. (1998) and Maier *et al*. (1972)).

The aim of the present paper is to formulate the minimum volume design problem of elastic perfectly plastic discrete structures, subjected to a load condition characterized by the multipliers $\bar{\xi}_0$ and $\bar{\xi}_c$ (Fig. 1), such that for load multipliers not greater than the assigned ones the structure is not exposed to incremental or instantaneous collapse. This goal is pursued by means of two different formulations, both developed on the grounds of a statical approach. Utilizing the first one (below the elastic shakedown limit) just a sub-optimal structure can be obtained: it is deduced as an optimal elastic shakedown design but related to a suitably reduced cyclic load multiplier and such that for the full load multiplier values it is found at the limit state of alternating plasticity. This last condition is ensured through the determination of the Bree diagram of the structure and by a parametric control on the extension of the upper plateaux of the diagram itself. On the contrary, the second formulation (above the elastic shakedown limit) provides the/an optimal design: it explicitly takes into account the elastic plastic behaviour of the structure in condition of alternating plasticity and the optimal design is obtained without recourse to any parametric solution.

2
Structural model and limit plastic shakedown multiplier

Often elastic plastic structures subjected to a combination of fixed and cyclic loads exceed the elastic shakedown limit and may be exposed to plastic shakedown or to incremental collapse (see, e.g. Caffarelli *et al.* (2001), Cohn and Parimi (1973). Giambanco (2000), Polizzotto *et al.* (2001), Ponter and Haofeng (2001)).

When a structure of volume V is subjected to a purely cyclic load and exhibits a plastic shakedown behaviour (see points like a in Fig. 2), it can be thought of as subdivided into two parts, V^F and V^E, separated by the so-called separating surface S and such that $V = V^F \cup V^E$ (see Polizzotto (1994a)). V^F is the portion of the structure where alternating plastic strains occur, while V^E is the remaining and complementary portion of the structure which exhibits an elastic behaviour. If, starting from point a, just the fixed load increases (see points like b), portion V^F remains unaltered together with the stresses and the plastic strains in it (due to the arising in the structure of suitable self-stresses which undo the fixed load effects in V^F), while in V^E a stress increment in equilibrium with the load increment arises. When the fixed load increases until it reaches point c in Fig. 2, portions V^F and V^E still remain unaltered, but the struc-

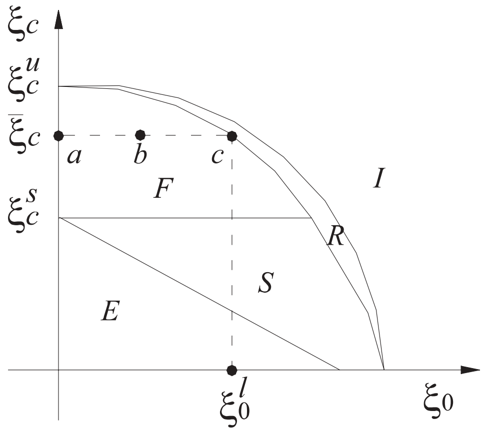

Fig. 2 Typical range of insensitivity of volume V^F to fixed load increments (path a–c) and elastic/plastic shakedown borderline determination

ture is at a limit state of plastic shakedown and, besides the *actual* plastic deformations in V^F, other *impending* plastic strains occur and the structure is exposed to an *impending* incremental collapse mechanism.

Let us consider now a structure consisting of n finite elements with elastic perfectly plastic constitutive behaviour. The ν-*th* element geometry is fully described by the s components of the vector $\mathbf{d}_\nu(\nu = 1, 2, \dots, n)$, so that $\mathbf{d} = [\tilde{\mathbf{d}}_1, \tilde{\mathbf{d}}_2, \dots, \tilde{\mathbf{d}}_\nu, \dots, \tilde{\mathbf{d}}_n]^T$ represents the $n \times s$ super-vector collecting all the design variables.

Let us denote with $\mathbf{F}_0$ and $\mathbf{F}_c$ the reference fixed mechanical load and the reference cyclic mechanical and/or kinematical load, respectively. Let us assume that the cyclic load varies in time quasi-statically and is identified with a convex polygonal-shaped loading path with vertices corresponding to a set of an even number b of mutually independent load vectors, say $\mathbf{F}_{ci}$, $i \in I(b) \equiv \{1, 2, \dots, b\}$; furthermore, let us assume the hypothesis that the cyclic load is a perfect one, namely for each basic load condition an opposite one exists in the load space. Finally, according to the previously defined symbols, $\xi_0 \mathbf{F}_0$ and $\xi_c \mathbf{F}_{ci}$, $\forall i \in I(b)$, represent the amplified fixed and cyclic loads.

For an assigned loading history the elastic plastic response of the structure can be obtained by a step-by-step analysis effected for a suitable number of cycles.

As already pointed out, for the described load conditions the steady-state response of the structure possesses, in terms of stresses and strains, the same periodicity features as the cyclic loads and it is independent of the initial conditions and of the chosen loading path. Moreover, for each cycle of the loading history, the steady-state response just depends on the sequence of the b amplified basic load conditions $\mathbf{F}_i = \xi_0 \mathbf{F}_0 + \xi_c \mathbf{F}_{ci}$, $\forall i \in I(b)$, obtained as a combination of the amplified fixed and cyclic loads. As a consequence, the elastic plastic steady-state response of the structure in the cycle can be obtained by an analysis effected just for the b basic load conditions.

For the purposes of the present paper it can be very useful to consider the steady-state elastic plastic response of the structure subjected just to the amplified perfect cyclic loads $\bar{\xi}_c \mathbf{F}_{ci}$, $\forall i \in I(b)$, where $\bar{\xi}_c$ is a selected cyclic load multiplier such that $0 \le \bar{\xi}_c < \xi_c^u$ results (Fig. 2), being ξ_c^u the ultimate purely cyclic load multiplier.

For an assigned design $\mathbf{d}$ and for an assigned purely cyclic loading history as above described, in the hypothesis of small displacements and assuming that the elastic domain of the finite elements is a convex and temperature-independent hyperpolyhedric function, the elastic plastic steady-state behaviour of the structure is described by the following equations:

$$\mathbf{P}_{ci} = \tilde{\mathbf{B}} \mathbf{u}_{ci} + \mathbf{P}_{ci}^*, \quad \mathbf{K} \mathbf{u}_{ci} - \mathbf{F}_{ci} = \mathbf{0} \quad \forall i \in I(b) \tag{1}$$

$$\mathbf{Z}_i = \mathbf{R} - \bar{\xi}_c \tilde{\mathbf{N}} \mathbf{P}_{ci} + \mathbf{S} \mathbf{Y}_i \quad \forall i \in I(b) \tag{2}$$

$$\mathbf{Z}_i \ge \mathbf{0}, \quad \mathbf{Y}_i \ge \mathbf{0}, \quad \tilde{\mathbf{Y}}_i \mathbf{Z}_i = 0 \quad \forall i \in I(b) \tag{3}$$

In (1) $\mathbf{u}_{ci}$ and $\mathbf{P}_{ci}$ are the purely elastic response to the reference cyclic loads in terms of structure node displacement and generalized stress vectors evaluated at the strain points, respectively; $\mathbf{P}_{ci}^*$ is the generalized stress vector evaluated at the strain points due to the loads directly acting upon the elements in the absence of nodal displacements; $\mathbf{F}_{ci} = \mathbf{F}_{ci}^n + \mathbf{F}_{ci}^*$, $\forall i \in I(b)$, is the structure equivalent nodal load vector (which collects nodal loads that by themselves provide structure node displacements coincident with those occurring because of the real acting loads) obtained as the sum of $\mathbf{F}_{ci}^n$, vector of the loads directly acting upon the structure nodes, and $\mathbf{F}_{ci}^*$, nodal load vector equivalent to the actions applied upon the elements; $\mathbf{B} = \tilde{\mathbf{C}}\, \mathbf{D}_e\, \mathbf{G}_p$ is the so-called pseudo-force matrix which transforms plastic strains evaluated at the strain points in equivalent structure node loads, $\mathbf{C}$ is the compatibility matrix, $\mathbf{D}_e$ is the block diagonal finite element internal elastic stiffness matrix and $\mathbf{G}_p$ a compatibility matrix which applied to plastic strains provides finite element nodal displacements; $\mathbf{K} = \tilde{\mathbf{C}}\, \mathbf{D}_e\, \mathbf{C}$ is the structure external elastic stiffness matrix.

The steady-state elastic plastic response of the structure to the amplified cyclic loads is described and governed by (2)–(3), where $\mathbf{Y}_i$, $\forall i \in I\,(b)$, is the plastic activation intensity vectors (being the typical entry $\mathbf{Y}_{ij}$ the coefficient which amplifies the unit plastic strain related to the j-*th* hyperplane of the yield surface for the i-*th* basic load condition), $\mathbf{Z}_i$, $\forall i \in I(b)$, is the opposite of the plastic potential vectors (collecting the spread with respect to the yield surface in the space of the generalized stresses related to the relevant amplified load conditions), $\mathbf{R}$ is the plastic resistance vector (collecting the distances from the origin to the yield surface in the space of the generalized stresses), $\mathbf{N}$ is the block diagonal matrix of the unit external normal to the yield surface and $\mathbf{S} = -\tilde{\mathbf{N}}(\tilde{\mathbf{B}}\, \mathbf{K}^{-1} \mathbf{B} - \mathbf{D})\mathbf{N}$ is a symmetric structural matrix which transforms plastic activation intensities into the opposite of the plastic potentials, with $\mathbf{D} = \tilde{\mathbf{G}}_p \mathbf{D}_e \mathbf{G}_p$ finite element internal stiffness matrix related to the strain points. In the present case $\mathbf{S}$ is positive semi-definite (see Maier (1968)) and, as a consequence, the uniqueness of the solution $\mathbf{Y}_i$ is not guaranteed (see Cottle *et al*. (1992)).

If $0 \le \bar{\xi}_c \le \xi_c^s$ is assumed, being ξ_c^s (Fig. 2) the elastic shakedown limit load multiplier, (2)–(3) admit the vanishing solution $\mathbf{Y}_i = \mathbf{0}$, $\forall i \in I(b)$, and in the steady-state phase the whole structure behaves elastically.

If $\xi_c^s < \bar{\xi}_c < \xi_c^u$ is assumed (see point a in Fig. 2), (2)–(3) admit a non-vanishing solution $\mathbf{Y}_i$ and the structure exhibits an elastic plastic behaviour. If the solution $\mathbf{Y}_i$ is not unique, then each couple of solutions to the same problem can differ just by a stressless (i.e. compatible, corresponding to a mechanism) set of plastic deformations (see Maier (1968)), and so the solution is unique in terms of $\mathbf{Z}_i$.

Taking into account (1)–(3), for a selected value of the cyclic load multiplier $\bar{\xi}_c$ (Fig. 2), the fixed load elastic/plastic shakedown multiplier at the incremental/instantaneous collapse limit, ξ_0^ℓ, can be determined solving the following problem (see Polizzotto *et al.* (2001)):

$$\mathbf{P}_0 = \tilde{\mathbf{B}}\mathbf{u}_0 + \mathbf{P}_0^*, \quad \mathbf{K}\mathbf{u}_0 - \mathbf{F}_0 = \mathbf{0} \tag{4}$$

$$\xi_0^\ell\left(\bar{\xi}_c\right) = \max_{(\xi_0,\boldsymbol{\rho})} \xi_0 \quad \text{subject to} \tag{5a}$$

$$\mathbf{Z}_i - \xi_0\tilde{\mathbf{N}}\mathbf{P}_0 - \tilde{\mathbf{N}}\boldsymbol{\rho} \geq \mathbf{0} \quad \forall i \in I(b) \tag{5b}$$

$$\tilde{\mathbf{A}}\,\boldsymbol{\rho} = \mathbf{0} \tag{5c}$$

In (4), (5b) $\mathbf{F}_0 = \mathbf{F}_0^n + \mathbf{F}_0^*$ is the reference equivalent fixed nodal force vector, with $\mathbf{F}_0^n$ representing the loads directly acting upon the structure nodes and $\mathbf{F}_0^*$ the nodal loads equivalent to the actions applied upon the elements, $\mathbf{u}_0$ and $\mathbf{P}_0$ are the purely elastic response of the structure to the reference fixed load in terms of structure node displacements and generalized stresses at the strain points, respectively, with $\mathbf{P}_0^*$ denoting the generalized stress vector due to the loads directly acting upon the elements. In (5b,c) $\boldsymbol{\rho}$ is a self-stress vector and $\mathbf{A} = \mathbf{C}_p\mathbf{C}$ is a compatibility matrix, being $\mathbf{C}_p$ a matrix which applied to element node displacements provides plastic strains evaluated at the strain points ($\mathbf{C}_p\mathbf{G}_p = \mathbf{I}$).

If (2)–(3) provide the vanishing solution $\mathbf{Y}_i = \mathbf{0}$, $\forall i \in I(b)$, (5) become a classic elastic shakedown limit load multiplier problem, otherwise the elastic plastic response to the purely cyclic load is involved in problem (5), which becomes a plastic shakedown limit load multiplier problem.

Assuming as a parameter the cyclic load multiplier $\bar{\xi}_c$, choosing a discrete, suitable set of values not exceeding the allowed range $\left(0 \leq \bar{\xi}_c < \xi_c^u\right)$ and solving problems (1)–(3) and (4)–(5) for each one of them, the borderline $\xi_0^\ell\left(\bar{\xi}_c\right)$ between elastic/plastic shakedown (zones $S+F$ in the Bree diagram) and incremental/instantaneous collapse (zones $R+I$ in the Bree diagram) can be determined.

3 Plastic shakedown design

This section is devoted to the formulation of the optimal plastic shakedown design problem in such a way that the structure is not exposed to incremental or instantaneous collapse as far as the basic load amplifiers do not exceed some prescribed safety factors $(\bar{\xi}_0, \bar{\xi}_c)$.

This goal can be reached by means of two different formulations, both developed on the grounds of a statical approach, but one working below and the other working above the elastic shakedown limit.

Following the first formulation, it is necessary to choose the separating surface S and the portions V^F and V^E, in such a way that portion V^E is redundant or statically determinate and possesses the same degrees of freedom as the whole structure. Furthermore, the choice of an assigned level $\bar{\xi}_c^s < \bar{\xi}_c$ for the upper plateaux of the elastic shakedown (Fig. 3(a)) is necessary, in correspondence to which only plastic strains are impending in the portion V^F. Furthermore, the knee of the upper plateaux of the elastic shakedown is imposed to have the coordinates $(\alpha\bar{\xi}_0, \bar{\xi}_c^s)$, where $\alpha > 1$ is a scalar. A structure with such a behaviour can be obtained by formulating an elastic shakedown design problem, modified in such a way that some suitably chosen entries of the plastic potential vector, related to portion V^F subjected just to the purely perfect cyclic load amplified by $\bar{\xi}_c^s$, are satisfied as equality, while the self-stresses in V^F undo the stresses related to the fixed load. Parametrically acting on the scalar α the upper plateaux may be suitably enlarged or reduced until the borderline between the alternating plasticity and the ratchetting/instantaneous collapse zones contains the selected point $(\bar{\xi}_0, \bar{\xi}_c)$. On changing S, it is possible to obtain other designs and, finally, to choose the one which possesses the minimum volume. In practice, for well-known structural typologies, it is quite easy to identify a good surface S and just one attempt can suffice in order to obtain a good design.

The second formulation does not require either any choice of the separating surface S, or any selected level

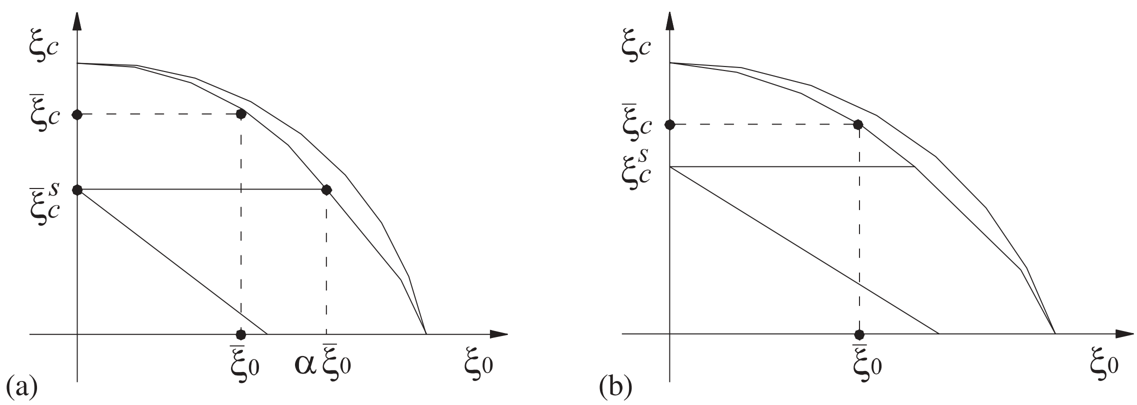

Fig. 3 Typical Bree-like diagram of the designs obtained: (a) working below the elastic shakedown limit; (b) working above the elastic shakedown limit

for the upper plateaux of the elastic shakedown, which are provided by the solution. The design is obtained simply imposing that the actual plastic strains due to the action of the purely cyclic load amplified by $\bar{\xi}_c$ (see point (0, $\bar{\xi}_c$) in Fig. 3(b)) coincide with those that arise in the whole structure subjected to the same cyclic load, to the full fixed load and to an arbitrary set of self-stresses (see point ($\bar{\xi}_0$, $\bar{\xi}_c$) in Fig. 3(b)). Therefore, the formulation of the search problem involves among its constraints the steady-state elastic plastic response of the structure to the cyclic loads and, in the authors' opinion, it represents the most correct version of the optimal plastic shakedown design.

3.1 Working below the elastic shakedown limit

Let us make reference to the discretized structure as previously defined, subjected to the described combination of fixed and perfect cyclic loads. Let us choose a surface S which separates the structure into the two parts V^F and V^E, characterized by the already known features.

The following partitions are effected, requiring some appropriate row reordering:

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}^F \\ \mathbf{d}^E \end{bmatrix}, \quad \boldsymbol{\varphi}_i = \begin{bmatrix} \boldsymbol{\varphi}_i^F \\ \boldsymbol{\varphi}_i^E \end{bmatrix}, \quad \tilde{\mathbf{N}} = \begin{bmatrix} \tilde{\mathbf{N}}^F & 0 \\ 0 & \tilde{\mathbf{N}}^E \end{bmatrix}, \tag{6a–c}$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}^F \\ \mathbf{R}^E \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}^F \\ \mathbf{B}^E \end{bmatrix} \tag{6d, e}$$

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0^F \\ \mathbf{P}_0^E \end{bmatrix}, \quad \mathbf{P}_{ci} = \begin{bmatrix} \mathbf{P}_{ci}^F \\ \mathbf{P}_{ci}^E \end{bmatrix}, \quad \boldsymbol{\rho} = \begin{bmatrix} \boldsymbol{\rho}^F \\ \boldsymbol{\rho}^E \end{bmatrix}, \tag{6f–h}$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \tilde{\mathbf{A}}^F & \tilde{\mathbf{A}}^E \end{bmatrix} \tag{6i}$$

where symbols like $(\cdot)^F$ and $(\cdot)^E$ represent quantities related to parts V^F and V^E, respectively. Furthermore, due to the discrete nature of the yield function, a further sub-partition must be effected, i.e.:

$$\boldsymbol{\varphi}_i^F = \begin{bmatrix} \boldsymbol{\varphi}_i^{Feq} \\ \boldsymbol{\varphi}_i^{Fle} \end{bmatrix}, \quad \tilde{\mathbf{N}}^F = \begin{bmatrix} \tilde{\mathbf{N}}^{Feq} \\ \tilde{\mathbf{N}}^{Fle} \end{bmatrix}, \quad \mathbf{R}^F = \begin{bmatrix} \mathbf{R}^{Feq} \\ \mathbf{R}^{Fle} \end{bmatrix} \tag{7a–c}$$

where apex *eq* means *equal to zero* and symbols like $(\cdot)^{Feq}$ are related to the imposed active set of limit yield functions within region V^F, while apex *le* means *less than or equal to zero* and symbols like $(\cdot)^{Fle}$ refer to the remaining set within the same region. These last partitions imply a further row reordering in (6a–e)–(7a–c). The set $\boldsymbol{\varphi}_i^{Feq}$ can be chosen by utilizing an appropriate computational strategy.

With the above described results, for a selected separating surface S and an assigned scalar α, assuming $\boldsymbol{\rho}^F = -\xi_0 \mathbf{P}_0^F$, the minimum volume plastic shakedown design problem, on the grounds of a statical approach, has the following form:

$$V_S(S) = \min_{(\mathbf{d}, \xi_0, \xi_c, \mathbf{u}_0, \mathbf{u}_{ci}, \boldsymbol{\rho}^E)} V \tag{8a}$$

subject to:

$$\mathbf{d} - \bar{\mathbf{d}} \geq \mathbf{0} \tag{8b}$$

$$\mathbf{T}\mathbf{d} - \bar{\mathbf{t}} \geq \mathbf{0} \tag{8c}$$

$$\alpha \bar{\xi}_0 - \xi_0 \leq 0 \tag{8d}$$

$$\bar{\xi}_c^s - \xi_c \leq 0 \tag{8e}$$

$$\mathbf{P}_0 = \tilde{\mathbf{B}}\mathbf{u}_0 + \mathbf{P}_0^*, \quad \mathbf{K}\mathbf{u}_0 - \mathbf{F}_0 = \mathbf{0} \tag{8f}$$

$$\mathbf{P}_{ci} = \tilde{\mathbf{B}}\mathbf{u}_{ci} + \mathbf{P}_{ci}^*, \quad \mathbf{K}\mathbf{u}_{ci} - \mathbf{F}_{ci} = \mathbf{0} \quad \forall i \in I(b) \tag{8g}$$

$$\boldsymbol{\varphi}_i^{Feq} \equiv \tilde{\mathbf{N}}^{Feq} \xi_c \mathbf{P}_{ci}^F - \mathbf{R}^{Feq} = \mathbf{0} \quad \forall i \in I(b) \tag{8h}$$

$$\boldsymbol{\varphi}_i^{Fle} \equiv \tilde{\mathbf{N}}^{Fle} \xi_c \mathbf{P}_{ci}^F - \mathbf{R}^{Fle} \leq \mathbf{0} \quad \forall i \in I(b) \tag{8i}$$

$$\boldsymbol{\varphi}_i^E \equiv \tilde{\mathbf{N}}^E \left(\xi_c \mathbf{P}_{ci}^E + \xi_0 \mathbf{P}_0^E + \boldsymbol{\rho}^E \right) - \mathbf{R}^E \leq \mathbf{0} \quad \forall i \in I(b) \tag{8j}$$

$$\tilde{\mathbf{A}}^E \boldsymbol{\rho}^E - \tilde{\mathbf{A}}^F \mathbf{P}_0^F \xi_0 = \mathbf{0} \tag{8k}$$

In (8b) the minimal values of the design variables, $\bar{\mathbf{d}}$, are considered; (8c) represents the technological constraints.

In order to investigate the special features of the solution of the relevant search problem (8) the related Kuhn–Tucker conditions must be deduced. With this aim the following enlarged functional is considered:

$$\begin{aligned} \Psi_b = {} & gV + \tilde{\gamma} \left(\mathbf{d} - \bar{\mathbf{d}} \right) + \tilde{\boldsymbol{\theta}} \left(\mathbf{T}\mathbf{d} - \bar{\mathbf{t}} \right) + e_0 \left(\alpha \bar{\xi}_0 - \xi_0 \right) + \\ & e_c \left(\bar{\xi}_c^s - \xi_c \right) - \xi_0 \tilde{\boldsymbol{\nu}}_0 \left(\mathbf{K}\mathbf{u}_0 - \mathbf{F}_0 \right) - \\ & \sum_{i=1}^{b} \xi_c \tilde{\boldsymbol{\nu}}_{ci} \left(\mathbf{K}\mathbf{u}_{ci} - \mathbf{F}_{ci} \right) + \sum_{i=1}^{b} \tilde{\boldsymbol{\lambda}}_i^{Feq} \boldsymbol{\varphi}_i^{Feq} + \\ & \sum_{i=1}^{b} \tilde{\boldsymbol{\lambda}}_i^{Fle} \boldsymbol{\varphi}_i^{Fle} + \sum_{i=1}^{b} \tilde{\boldsymbol{\lambda}}_i^E \boldsymbol{\varphi}_i^E - \tilde{\boldsymbol{\nu}} \left(\tilde{\mathbf{A}}^E \boldsymbol{\rho}^E - \tilde{\mathbf{A}}^F \mathbf{P}_0^F \xi_0 \right) \end{aligned} \tag{9}$$

where $\boldsymbol{\gamma} \leq \mathbf{0}$, $\boldsymbol{\theta} \leq \mathbf{0}$, $e_0 \geq 0$, $e_c \geq 0$, $\xi_0 \boldsymbol{\nu}_0$ and $\xi_c \boldsymbol{\nu}_{ci}$ (with ξ_0 and ξ_c not subjected to variations), $\boldsymbol{\lambda}_i^{Feq} \geq \mathbf{0}$, $\boldsymbol{\lambda}_i^{Fle} \geq \mathbf{0}$, $\boldsymbol{\lambda}_i^E \geq \mathbf{0}$, $\boldsymbol{\nu}$ are the Lagrange multipliers and $g = 1$ is a constant introduced for dimensionality sake. Taking the first variation of (9) with respect to all the variables, since Ψ_b must have a minimum with respect to the variables of problem (8) and a maximum with respect to the Lagrange multipliers, assuming $\boldsymbol{\lambda}_i = \left[\tilde{\boldsymbol{\lambda}}_i^{Feq} \quad \tilde{\boldsymbol{\lambda}}_i^{Fle} \quad \tilde{\boldsymbol{\lambda}}_i^{E}\right]^T$, the following Kuhn–Tucker conditions are deduced:

$$\mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0}\,, \quad \boldsymbol{\gamma} \leq \mathbf{0}\,, \quad \tilde{\boldsymbol{\gamma}}\left(\mathbf{d} - \overline{\mathbf{d}}\right) = 0 \tag{10a}$$

$$\mathbf{Td} - \overline{\mathbf{t}} \geq \mathbf{0}\,, \quad \boldsymbol{\theta} \leq \mathbf{0}\,, \quad \tilde{\boldsymbol{\theta}}\left(\mathbf{Td} - \overline{\mathbf{t}}\right) = 0 \tag{10b}$$

$$\alpha\bar{\xi}_0 - \xi_0 \leq 0\,, \quad e_0 \geq 0\,, \quad e_0\left(\alpha\bar{\xi}_0 - \xi_0\right) = 0 \tag{10c}$$

$$\bar{\xi}_c^s - \xi_c \leq 0\,, \quad e_c \geq 0\,, \quad e_c\left(\bar{\xi}_c^s - \xi_c\right) = 0 \tag{10d}$$

$$\mathbf{P}_0 = \tilde{\mathbf{B}}\mathbf{u}_0 + \mathbf{P}_0^*\,, \quad \mathbf{K}\mathbf{u}_0 - \mathbf{F}_0 = \mathbf{0} \tag{10e}$$

$$\mathbf{P}_{ci} = \tilde{\mathbf{B}}\mathbf{u}_{ci} + \mathbf{P}_{ci}^*\,, \quad \mathbf{K}\mathbf{u}_{ci} - \mathbf{F}_{ci} = \mathbf{0} \quad \forall i \in I(b) \tag{10f}$$

$$\boldsymbol{\varphi}_i^{Feq} \equiv \tilde{\mathbf{N}}^{Feq}\xi_c\mathbf{P}_{ci}^F - \mathbf{R}^{Feq} = \mathbf{0}\,, \quad \boldsymbol{\lambda}_i^{Feq} > \mathbf{0} \quad \forall i \in I(b) \tag{10g}$$

$$\boldsymbol{\varphi}_i^{Fle} \equiv \tilde{\mathbf{N}}^{Fle}\xi_c\mathbf{P}_{ci}^F - \mathbf{R}^{Fle} \leq \mathbf{0}\,, \quad \boldsymbol{\lambda}_i^{Fle} \geq \mathbf{0}\,,$$
$$\tilde{\boldsymbol{\varphi}}_i^{Fle}\boldsymbol{\lambda}_i^{Fle} = 0 \quad \forall i \in I(b) \tag{10h}$$

$$\boldsymbol{\varphi}_i^{E} \equiv \tilde{\mathbf{N}}^{E}\left(\xi_c\mathbf{P}_{ci}^E + \xi_0\mathbf{P}_0^E + \boldsymbol{\rho}^E\right) - \mathbf{R}^{E} \leq \mathbf{0}\,, \quad \boldsymbol{\lambda}_i^{E} \geq \mathbf{0}\,,$$
$$\tilde{\boldsymbol{\varphi}}_i^{E}\boldsymbol{\lambda}_i^{E} = 0 \quad \forall i \in I(b) \tag{10i}$$

$$\tilde{\mathbf{A}}^E\boldsymbol{\rho}^E - \tilde{\mathbf{A}}^F\mathbf{P}_0^F\xi_0 = \mathbf{0} \tag{10j}$$

$$\alpha\bar{\xi}_0 - \xi_0 \leq 0\,, \quad \left(\tilde{\mathbf{P}}_0^E\sum_{i=1}^{b}\mathbf{N}^E\boldsymbol{\lambda}_i^E + \tilde{\mathbf{P}}_0^F\mathbf{A}^F\boldsymbol{\nu} - e_0\right) \geq 0$$
$$\left(\tilde{\mathbf{P}}_0^E\sum_{i=1}^{b}\mathbf{N}^E\boldsymbol{\lambda}_i^E + \tilde{\mathbf{P}}_0^F\mathbf{A}^F\boldsymbol{\nu} - e_0\right)\left(\alpha\bar{\xi}_0 - \xi_0\right) = 0 \tag{10k}$$

$$\bar{\xi}_c^s - \xi_c \leq 0\,, \quad \left(\sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\mathbf{N}\,\boldsymbol{\lambda}_i - e_c\right) \geq 0\,,$$
$$\left(\sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\mathbf{N}\boldsymbol{\lambda}_i - e_c\right)\left(\bar{\xi}_c^s - \xi_c\right) = 0 \tag{10l}$$

$$\mathbf{K}\boldsymbol{\nu}_0 = \sum_{i=1}^{b}\mathbf{B}^E\mathbf{N}^E\boldsymbol{\lambda}_i^E + \mathbf{B}^F\mathbf{A}^F\boldsymbol{\nu} \tag{10m}$$

$$\mathbf{K}\boldsymbol{\nu}_{ci} = \mathbf{B}^F\mathbf{N}^F\boldsymbol{\lambda}_i^F + \mathbf{B}^E\mathbf{N}^E\boldsymbol{\lambda}_i^E \quad \forall i \in I(b) \tag{10n}$$

$$\sum_{i=1}^{b}\mathbf{N}^E\boldsymbol{\lambda}_i^E = \mathbf{A}^E\boldsymbol{\nu} \tag{10o}$$

$$\tilde{\boldsymbol{\Gamma}}_b \equiv \frac{\partial V}{\partial \mathbf{d}} + \tilde{\boldsymbol{\gamma}} + \tilde{\boldsymbol{\theta}}\mathbf{T} - \xi_0\tilde{\boldsymbol{\nu}}_0\frac{\partial \mathbf{K}}{\partial \mathbf{d}}\mathbf{u}_0 - \xi_c\sum_{i=1}^{b}\tilde{\boldsymbol{\nu}}_{ci}\frac{\partial \mathbf{K}}{\partial \mathbf{d}}\mathbf{u}_{ci} +$$
$$\xi_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\tilde{\mathbf{N}}\frac{\partial \mathbf{P}_{ci}}{\partial \mathbf{d}} - \sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial \mathbf{R}}{\partial \mathbf{d}} + \xi_0\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i^E\tilde{\mathbf{N}}^E\frac{\partial \mathbf{P}_0^E}{\partial \mathbf{d}} +$$
$$\xi_0\tilde{\boldsymbol{\nu}}\tilde{\mathbf{A}}^F\frac{\partial \mathbf{P}_0^F}{\partial \mathbf{d}}$$

$$\boldsymbol{\Gamma}_b \leq \mathbf{0}\,, \quad \mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0}\,, \quad \tilde{\boldsymbol{\Gamma}}_b\left(\mathbf{d} - \overline{\mathbf{d}}\right) = 0 \tag{10p}$$

In virtue of (10m,o) and taking into account that:

$$\mathbf{B}^F\mathbf{A}^F + \mathbf{B}^E\mathbf{A}^E = \mathbf{K} \tag{11}$$

one obtains:

$$\mathbf{K}\,\boldsymbol{\nu}_0 = \mathbf{K}\,\boldsymbol{\nu}\,, \quad \text{i.e.}\ \boldsymbol{\nu} = \boldsymbol{\nu}_0 \tag{12}$$

Furthermore, from (10k–n) the following relation holds:

$$\mathbf{K}\left(\sum_{i=1}^{b}\boldsymbol{\nu}_{ci} - \boldsymbol{\nu}_0\right) = \mathbf{B}^F\left(\sum_{i=1}^{b}\mathbf{N}^F\boldsymbol{\lambda}_i^F - \mathbf{A}^F\boldsymbol{\nu}_0\right) \tag{13}$$

and taking into account that, due to the compatibility between the resulting plastic mechanism in V^F and the displacements ν_0, the right-hand side of (13) is nought, it is possible to deduce:

$$\sum_{i=1}^{b}\boldsymbol{\nu}_{ci} = \boldsymbol{\nu}_0 \tag{14}$$

With the help of these last results, taking into account that the *j-th* condition of (10a,p) influences the optimal design only if $d_j > \bar{d}_j$, in the hypothesis that $\mathbf{d}^E - \bar{\mathbf{d}}^E \neq \mathbf{0}$ and $\mathbf{d}^F - \bar{\mathbf{d}}^F \neq \mathbf{0}$, $e_0 > 0$ and $e_c > 0$ result and, besides (10a,b) and (10e–j), the Kuhn–Tucker conditions read:

$$\xi_0 = \alpha\,\bar{\xi}_0 \tag{15a}$$

$$\xi_c = \bar{\xi}_c^s \tag{15b}$$

$$W_0 \equiv \tilde{\mathbf{P}}_0\,\mathbf{A}\,\boldsymbol{\nu}_0 > 0 \tag{15c}$$

$$W_c \equiv \sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\,\mathbf{N}\,\boldsymbol{\lambda}_i > 0 \tag{15d}$$

$$\mathbf{K}\,\boldsymbol{\nu}_{ci} = \mathbf{B}\,\mathbf{N}\,\boldsymbol{\lambda}_i \quad \forall i \in I(b) \tag{15e}$$

$$\sum_{i=1}^{b}\mathbf{N}\boldsymbol{\lambda}_i = \mathbf{A}\boldsymbol{\nu}_0 \tag{15f}$$

$$\tilde{\boldsymbol{\Gamma}}_b \equiv \frac{\partial V}{\partial \mathbf{d}} + \tilde{\boldsymbol{\gamma}} + \tilde{\boldsymbol{\theta}}\mathbf{T} + \xi_0\tilde{\boldsymbol{\nu}}_0\tilde{\mathbf{A}}\frac{\partial \mathbf{P}_0}{\partial \mathbf{d}} +$$
$$\xi_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\tilde{\mathbf{N}}\frac{\partial \mathbf{P}_{ci}}{\partial \mathbf{d}} - \sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial \mathbf{R}}{\partial \mathbf{d}} - \xi_0\tilde{\boldsymbol{\nu}}_0\frac{\partial \mathbf{K}}{\partial \mathbf{d}}\mathbf{u}_0 -$$

$$\xi_c \sum_{i=1}^{b} \tilde{\boldsymbol{\nu}}_{ci} \frac{\partial \mathbf{K}}{\partial \mathbf{d}} \mathbf{u}_{ci}$$

$$\boldsymbol{\Gamma}_b \leq \mathbf{0}\,, \quad \mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0}\,, \quad \tilde{\boldsymbol{\Gamma}}_b \left(\mathbf{d} - \overline{\mathbf{d}}\right) = 0 \tag{15g}$$

From the Kuhn–Tucker conditions (10), (15) it is possible to deduce the meaning of the introduced Lagrange multipliers. Actually, $\boldsymbol{\lambda}_i^F$, and $\boldsymbol{\lambda}_i^E$ are plastic coefficients related to the regions V^F and V^E, respectively, and $\boldsymbol{\nu}_{ci}$ and $\boldsymbol{\nu}_0$ are structure node displacements. From (15) it is possible to state that the obtained optimal structure ($\mathbf{d} = \mathbf{d}_{opt}$) is found at the limit state of elastic shakedown behaviour under the prescribed loads affected by the assigned load multipliers $\alpha\,\bar{\xi}_0$, $\bar{\xi}_c^s$, i.e. the point $(\alpha\,\bar{\xi}_0, \bar{\xi}_c^s)$ on the Bree-like diagram of the structure belongs to the boundary of the elastic shakedown domain. With reference to the design $\mathbf{d}_{opt}$, it is worth noticing that (10g–i) show that a variation $\delta\xi_c < 0$ of the cyclic load multiplier ξ_c produces an elastic return in the whole volume V of the structure, being $\boldsymbol{\varphi}_i < \mathbf{0}$, $\boldsymbol{\lambda}_i = \mathbf{0}$, $\forall i \in I(b)$, $W_0 = 0$, $W_c = 0$, while a variation $\delta\xi_0 < 0$ of the steady load multiplier ξ_0 produces an elastic return just in the volume V^E of the structure, being $\boldsymbol{\varphi}_i^E < \mathbf{0}$, $\boldsymbol{\lambda}_i^E = \mathbf{0}$, and remaining unaltered (10g,h). In this last case, the plastic accumulation mechanism resulting in the cycle characterized by $\boldsymbol{\nu}_0$ must be nought and as a consequence $W_0 = 0$ and $W_c > 0$ result, namely the impending plastic mechanism is an alternating plasticity one (see, e.g. Giambanco and Palizzolo (1994c, 1996)). As a consequence of the above properties, the point $(\alpha\,\bar{\xi}_0, \bar{\xi}_c^s)$ is found at the knee of the upper plateaux of the Bree-like diagram of the designed structure.

The solution to problem (8) provides the design variables $\mathbf{d}$ and for the obtained structure it is possible to determine the relevant Bree diagram. If the borderline between plastic shakedown and ratchetting contains the selected point $(\bar{\xi}_0, \bar{\xi}_c)$, the search is concluded; otherwise, problem (8) must be solved again modifying the parameter α until the referred boundary contains the selected point (Fig. 3(a)).

If $\mathbf{d}^E - \bar{\mathbf{d}}^E \neq \mathbf{0}$ and $\mathbf{d}^F - \bar{\mathbf{d}}^F = \mathbf{0}$ result, then $e_0 > 0$ and $e_c \geq 0$, $\xi_0 = \alpha\,\bar{\xi}_0$ and $\xi_c > \bar{\xi}_c^s$, $W_0 > 0$ and $W_c \geq 0$ also result, the upper plateaux is found at the ξ_c level and may be below or above the prescribed point $(\bar{\xi}_0, \bar{\xi}_c)$. In the first case it is still possible to obtain a plastic shakedown design acting on the value of the parameter α, while in the second case an elastic shakedown design is obtained.

Finally, the cases characterized by the solution $\mathbf{d}^E - \bar{\mathbf{d}}^E = \mathbf{0}$ may occur, but these undesirable cases are not considered here, since in practice they may be easily avoided.

Anyway, considering all the admissible separating surfaces S (as previously defined) and the related volumes $V_S\,(S)$, the absolute minimum volume $V^* = V^{F*} \cup V^{E*}$ of the structure can be reached by solving the following problem:

$$V^* = \min_{(S)} \; V_S\,(S) = \min_{\left(\mathbf{d}, \xi_0, \xi_c, \mathbf{u}_0, \mathbf{u}_{ci}, \boldsymbol{\rho}^E, S\right)} V \quad \text{subject to} \tag{16a}$$

$$\text{constraints (8b–k)} \tag{16b}$$

$$\text{rank}\left(\tilde{\mathbf{A}}^E\right) = \text{freedom degree} \tag{16c}$$

If $V^{E*} > \bar{V}^E$ $(\bar{V}^E = V^E(\bar{\mathbf{d}}^E))$ and $V^{F*} > \bar{V}^F$ $(\bar{V}^F = V^F(\bar{\mathbf{d}}^F))$, the designed optimal structure is a plastic shakedown one. If $V^{F*} = \bar{V}^F$, the structure/load system may not be susceptible to the production of a plastic shakedown design and so, in order to prevent the incremental plastic collapse, in the steady state the structure must eventually behave elastically.

3.2 Working above the elastic shakedown limit

With the results described in the previous sections, the/a plastic shakedown minimum volume design of an elastic perfectly plastic finite element structure, subjected to a combination of amplified fixed and cyclic loads, can be formulated on the grounds of a statical approach as follows:

$$\min_{(\mathbf{d}, \xi, \mathbf{u_0}, \mathbf{u}_{ci}, \mathbf{Z}_i, \mathbf{Y}_i, \boldsymbol{\rho})} V \tag{17a}$$

subject to

$$\mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0} \tag{17b}$$

$$\mathbf{T}\mathbf{d} - \overline{\mathbf{t}} \geq \mathbf{0} \tag{17c}$$

$$\bar{\xi} - \xi \leq 0 \tag{17d}$$

$$\mathbf{P}_0 = \tilde{\mathbf{B}}\mathbf{u}_0 + \mathbf{P}_0^*\,, \quad \mathbf{K}\mathbf{u}_0 - \mathbf{F}_0 = \mathbf{0} \tag{17e}$$

$$\mathbf{P}_{ci} = \tilde{\mathbf{B}}\mathbf{u}_{ci} + \mathbf{P}_{ci}^*\,, \quad \mathbf{K}\mathbf{u}_{ci} - \mathbf{F}_{ci} = \mathbf{0} \quad \forall i \in I(b) \tag{17f}$$

$$\mathbf{Z}_i = \mathbf{R} - \xi\bar{\xi}_c\tilde{\mathbf{N}}\,\mathbf{P}_{ci} + \mathbf{S}\,\mathbf{Y}_i \quad \forall i \in I(b) \tag{17g}$$

$$\mathbf{Z}_i \geq \mathbf{0}\,, \quad \mathbf{Y}_i \geq \mathbf{0}\,, \quad \tilde{\mathbf{Y}}_i \mathbf{Z}_i = 0 \quad \forall i \in I(b) \tag{17h}$$

$$\mathbf{Z}_i - \xi\bar{\xi}_0\tilde{\mathbf{N}}\,\mathbf{P}_0 - \tilde{\mathbf{N}}\,\boldsymbol{\rho} \geq \mathbf{0} \quad \forall i \in I(b) \tag{17i}$$

$$\tilde{\mathbf{A}}\,\boldsymbol{\rho} = \mathbf{0} \tag{17j}$$

In (17d) $\bar{\xi}$ is the minimal value of the load multiplier ξ, which amplifies both fixed and cyclic loads. The elastic response to the reference loads is provided by (17e,f), while (17g,h) represent the steady-state elastic plastic response to the amplified purely perfect cyclic loads and (17i,j) are the limit conditions of actual plastic shakedown (and impending incremental or instantaneous collapse) for the structure subjected to both fixed and cyclic loads and to the self-stresses.

In order to investigate on the special features of the relevant search problem solution, the related Kuhn–Tucker conditions must be deduced. With this aim the following enlarged functional is considered:

$$\Psi_a = gV + \tilde{\boldsymbol{\gamma}}\left(\mathbf{d}-\bar{\mathbf{d}}\right) + \tilde{\boldsymbol{\theta}}\left(\mathbf{Td}-\bar{\mathbf{t}}\right) + e\left(\bar{\xi}-\xi\right) -$$

$$\xi\bar{\xi}_0\tilde{\boldsymbol{\nu}}_0\left(\mathbf{Ku}_0-\mathbf{F}_0\right) - \sum_{i=1}^{b}\xi\bar{\xi}_c\tilde{\boldsymbol{\nu}}_i\left(\mathbf{Ku}_{ci}-\mathbf{F}_{ci}\right) -$$

$$\sum_{i=1}^{b}\tilde{\mathbf{y}}_i\left(-\mathbf{R}+\xi\bar{\xi}_c\tilde{\mathbf{N}}\,\mathbf{P}_{ci}-\mathbf{S}\,\mathbf{Y}_i\right) +$$

$$a\sum_{i=1}^{b}\tilde{\mathbf{Y}}_i\left(-\mathbf{R}+\xi\,\bar{\xi}_c\tilde{\mathbf{N}}\,\mathbf{P}_{ci}-\mathbf{S}\,\mathbf{Y}_i\right) +$$

$$\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\left(-\mathbf{R}+\xi\,\bar{\xi}_c\tilde{\mathbf{N}}\,\mathbf{P}_{ci}-\mathbf{S}\,\mathbf{Y}_i+\xi\,\bar{\xi}_0\tilde{\mathbf{N}}\,\mathbf{P}_0+\tilde{\mathbf{N}}\boldsymbol{\rho}\right)-\tilde{\boldsymbol{\nu}}\tilde{\mathbf{A}}\boldsymbol{\rho} \tag{18}$$

where $\boldsymbol{\gamma}\le\mathbf{0}$, $\boldsymbol{\theta}\le\mathbf{0}$, $e\ge 0$, $\xi\bar{\xi}_0\boldsymbol{\nu}_0$ and $\xi\bar{\xi}_c\boldsymbol{\nu}_i$, (with ξ not subjected to variations), $\mathbf{y}_i\ge\mathbf{0}$, a, $\boldsymbol{\lambda}_i\ge\mathbf{0}$, $\boldsymbol{\nu}$ are the Lagrange multipliers and $g=1$ is a constant introduced for dimensionality sake. Taking the first variation of functional (18) with respect to all the variables, since Ψ_a must have a minimum with respect to the variables of problem (17), and a maximum with respect to the Lagrange multipliers, the following Kuhn–Tucker conditions are deduced:

$$\mathbf{d}-\overline{\mathbf{d}}\ge\mathbf{0}\,,\quad \boldsymbol{\gamma}\le\mathbf{0}\,,\quad \tilde{\boldsymbol{\gamma}}\left(\mathbf{d}-\overline{\mathbf{d}}\right)=0 \tag{19a}$$

$$\mathbf{Td}-\bar{\mathbf{t}}\ge\mathbf{0}\,,\quad \boldsymbol{\theta}\le\mathbf{0}\,,\quad \tilde{\boldsymbol{\theta}}\left(\mathbf{Td}-\bar{\mathbf{t}}\right)=0 \tag{19b}$$

$$\bar{\xi}-\xi\le 0\,,\quad e\ge 0\,,\quad e\left(\bar{\xi}-\xi\right)=0 \tag{19c}$$

$$\mathbf{P}_0=\tilde{\mathbf{B}}\mathbf{u}_0+\mathbf{P}_0^*\,,\quad \mathbf{Ku}_0-\mathbf{F}_0=\mathbf{0} \tag{19d}$$

$$\mathbf{P}_{ci}=\tilde{\mathbf{B}}\mathbf{u}_{ci}+\mathbf{P}_{ci}^*\,,\quad \mathbf{Ku}_{ci}-\mathbf{F}_{ci}=\mathbf{0}\quad \forall i\in I(b) \tag{19e}$$

$$\mathbf{Z}_i=\mathbf{R}-\xi\bar{\xi}_c\tilde{\mathbf{N}}\,\mathbf{P}_{ci}+\mathbf{S}\,\mathbf{Y}_i\quad \forall i\in I(b) \tag{19f}$$

$$\mathbf{Z}_i\ge\mathbf{0}\,,\quad \mathbf{y}_i\ge\mathbf{0}\,,\quad \tilde{\mathbf{y}}_i\mathbf{Z}_i=0\quad \forall i\in I(b) \tag{19g}$$

$$\boldsymbol{\varphi}_i\equiv-\mathbf{Z}_i+\xi\,\bar{\xi}_0\tilde{\mathbf{N}}\,\mathbf{P}_0+\tilde{\mathbf{N}}\boldsymbol{\rho}\quad \forall i\in I(b) \tag{19h}$$

$$\boldsymbol{\varphi}_i\le\mathbf{0}\,,\quad \boldsymbol{\lambda}_i\ge\mathbf{0}\,,\quad \tilde{\boldsymbol{\lambda}}_i\boldsymbol{\varphi}_i=0\quad \forall i\in I(b) \tag{19i}$$

$$\tilde{\mathbf{A}}\,\boldsymbol{\rho}=\mathbf{0} \tag{19j}$$

$$\mathbf{K}\boldsymbol{\nu}_0=\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{G}_p\sum_{i=1}^{b}\mathbf{N}\,\boldsymbol{\lambda}_i \tag{19k}$$

$$-\mathbf{K}\boldsymbol{\nu}_i-\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{G}_p\,\mathbf{N}\,\mathbf{y}_i+\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{G}_p\,\mathbf{N}\,\boldsymbol{\lambda}_i+$$
$$a\,\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{G}_p\,\mathbf{N}\,\mathbf{Y}_i=\mathbf{0}\quad \forall i\in I(b) \tag{19l}$$

$$-a\,\mathbf{R}+a\,\xi\,\bar{\xi}_c\,\tilde{\mathbf{N}}\,\mathbf{P}_{ci}-2\,a\,\mathbf{S}\,\mathbf{Y}_i+\mathbf{S}\,\mathbf{y}_i-\mathbf{S}\,\boldsymbol{\lambda}_i\le\mathbf{0}\,,\quad \mathbf{Y}_i\ge\mathbf{0}\,,$$
$$\tilde{\mathbf{Y}}_i\left(-a\,\mathbf{R}+a\xi\,\bar{\xi}_c\,\tilde{\mathbf{N}}\,\mathbf{P}_{ci}-2\,a\,\mathbf{S}\,\mathbf{Y}_i+\mathbf{S}\,\mathbf{y}_i-\mathbf{S}\,\boldsymbol{\lambda}_i\right)=0$$
$$\forall i\in I(b) \tag{19m}$$

$$\sum_{i=1}^{b}\mathbf{N}\,\boldsymbol{\lambda}_i=\mathbf{A}\,\boldsymbol{\nu} \tag{19n}$$

$$-e+\bar{\xi}_c\sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\,\mathbf{N}\,\boldsymbol{\lambda}_i+\bar{\xi}_0\tilde{\mathbf{P}}_0\sum_{i=1}^{b}\mathbf{N}\,\boldsymbol{\lambda}_i\ge 0\,,\quad \left(\bar{\xi}-\xi\right)\le 0\,,$$
$$\left(\bar{\xi}-\xi\right)\left(-e+\bar{\xi}_c\sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\,\mathbf{N}\,\boldsymbol{\lambda}_i+\bar{\xi}_0\tilde{\mathbf{P}}_0\sum_{i=1}^{b}\mathbf{N}\,\boldsymbol{\lambda}_i\right)=0 \tag{19o}$$

$$\tilde{\boldsymbol{\Gamma}}_a\equiv\frac{\partial V}{\partial\mathbf{d}}+\tilde{\boldsymbol{\gamma}}+\boldsymbol{\theta}\tilde{\mathbf{T}}-\xi\bar{\xi}_0\tilde{\boldsymbol{\nu}}_0\,\tilde{\mathbf{C}}\,\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}\mathbf{C}\,\mathbf{u}_0-$$
$$\xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\nu}}_i\,\tilde{\mathbf{C}}\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}\mathbf{C}\,\mathbf{u}_{ci}+\sum_{i=1}^{b}\tilde{\mathbf{y}}_i\frac{\partial\mathbf{R}}{\partial\mathbf{d}}-$$
$$\xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\mathbf{y}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}\mathbf{C}\,\mathbf{u}_{ci}+\sum_{i=1}^{b}\tilde{\mathbf{y}}_i\frac{\partial\mathbf{S}}{\partial\mathbf{d}}\mathbf{Y}_i-$$
$$a\sum_{i=1}^{b}\tilde{\mathbf{Y}}_i\frac{\partial\mathbf{R}}{\partial\mathbf{d}}+a\xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\mathbf{Y}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}\mathbf{C}\,\mathbf{u}_{ci}-$$
$$a\sum_{i=1}^{b}\tilde{\mathbf{Y}}_i\frac{\partial\mathbf{S}}{\partial\mathbf{d}}\mathbf{Y}_i-\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial\mathbf{R}}{\partial\mathbf{d}}+\xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}\mathbf{Cu}_{ci}-$$
$$\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial\mathbf{S}}{\partial\mathbf{d}}\mathbf{Y}_i+\xi\bar{\xi}_0\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\frac{\partial\mathbf{D}_e}{\partial\mathbf{d}}a\mathbf{Cu}_0$$
$$\boldsymbol{\Gamma}_a\le\mathbf{0}\,,\quad \mathbf{d}-\overline{\mathbf{d}}\ge\mathbf{0}\,,\quad \tilde{\boldsymbol{\Gamma}}\left(\mathbf{d}-\overline{\mathbf{d}}\right)=0 \tag{19p}$$

In (19p) the derivatives of the quantities $\mathbf{P}_0^*$, $\mathbf{P}_{ci}^*$, $\mathbf{F}_0^*$, $\mathbf{F}_{ci}^*$ do not appear because the simplifying hypothesis that they are independent of the design variables $\mathbf{d}$ has been assumed.

From (19) it is possible to deduce the meaning of the introduced Lagrange multipliers.

In virtue of (17g,h) and taking into account (19f,g), it is possible to obtain the following relation:

$$\mathbf{y}_i=\mathbf{Y}_i\quad \forall i\in I(b) \tag{20}$$

With the help of this last result and assuming $a=1$, the following relations can be deduced from (19l), (19m) and (19p), respectively:

$$\mathbf{K}\,\boldsymbol{\nu}_i = \tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{G}_p\,\mathbf{N}\,\boldsymbol{\lambda}_i \quad \forall i \in I(b) \tag{21}$$

$$\mathbf{Z}_i + \mathbf{S}\,\boldsymbol{\lambda}_i \geq \mathbf{0}\,, \quad \mathbf{Y}_i \geq \mathbf{0}\,, \quad \tilde{\mathbf{Y}}_i\,(\mathbf{Z}_i + \mathbf{S}\,\boldsymbol{\lambda}_i) = 0 \quad \forall i \in I(b) \tag{22}$$

$$\tilde{\boldsymbol{\Gamma}}_a \equiv \frac{\partial V}{\partial \mathbf{d}} + \tilde{\boldsymbol{\gamma}} + \tilde{\boldsymbol{\theta}}\mathbf{T} - \xi\bar{\xi}_0\tilde{\boldsymbol{\nu}}_0\tilde{\mathbf{C}}\frac{\partial \mathbf{D}_e}{\partial \mathbf{d}}\mathbf{C}\mathbf{u}_0 + \xi\bar{\xi}_0\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\tilde{\mathbf{N}}\tilde{\mathbf{G}}_p\frac{\partial \mathbf{D}_e}{\partial \mathbf{d}}\mathbf{C}\mathbf{u}_0 - \xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\nu}}_i\tilde{\mathbf{C}}\frac{\partial \mathbf{D}_e}{\partial \mathbf{d}}\mathbf{C}\mathbf{u}_{ci} + \xi\bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\tilde{\mathbf{N}}\tilde{\mathbf{G}}_p\frac{\partial \mathbf{D}_e}{\partial \mathbf{d}}\mathbf{C}\mathbf{u}_{ci} - \sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial \mathbf{R}}{\partial \mathbf{d}} - \sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\frac{\partial \mathbf{S}}{\partial \mathbf{d}}\mathbf{Y}_i$$

$$\boldsymbol{\Gamma}_a \leq \mathbf{0}\,, \quad \mathbf{d} - \overline{\mathbf{d}} \geq \mathbf{0}\,, \quad \tilde{\boldsymbol{\Gamma}}\left(\mathbf{d} - \overline{\mathbf{d}}\right) = 0 \tag{23}$$

Remembering the third of (17h), from the third relation of (22) it is possible to state that, if the *j-th* component of vector $\mathbf{Y}_i$ is positive ($Y_{ij} > 0$), then the *j-th* component of vector $\mathbf{S}\,\boldsymbol{\lambda}_i$ must vanish, and as a consequence the plastic coefficients $\boldsymbol{\lambda}_i$ must produce a nought variation of $\boldsymbol{\varphi}$ in all those elements of the structure which find themselves already in a condition of plastic shakedown under the purely cyclic loads. In other words, the plastic coefficients $\boldsymbol{\lambda}_i$ determine an impending ratchet or instantaneous plastic mechanism which does not modify the plastic behaviour of those elements which behave in conditions of actual alternating plasticity, described by vectors $\mathbf{Y}_i$, $\forall i \in I(b)$, and with zero actual plastic strain resulting in the cycle.

In the last set of Kuhn–Tucker conditions (23), the influence of the design variables $\mathbf{d}$ on V, $\mathbf{D}_e$, $\mathbf{R}$ and $\mathbf{S}$ is taken into account. If the *j-th* component d_j of vector $\mathbf{d}$ equals the corresponding component $\bar{d}_j$ of vector $\overline{\mathbf{d}}$, then correspondingly $\Gamma_{aj} < 0$ results and, thus, no constrain of this sort is imposed on the design; otherwise, if $d_j > \bar{d}_j$, then $\Gamma_{aj} = 0$ and $\gamma_j = 0$ result, and the *j-th* condition of (23) can also be written as:

$$\frac{\partial}{\partial d_j}\left(\tilde{\mathbf{R}}\sum_{i=1}^{b}\boldsymbol{\lambda}_i\right) = \frac{\partial}{\partial d_j}\left\{V + \tilde{\boldsymbol{\theta}}\mathbf{T}_j - \xi\left(\bar{\xi}_0\tilde{\boldsymbol{\nu}}_0\,\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{C}\,\mathbf{u}_0 + \bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\nu}}_i\,\tilde{\mathbf{C}}\,\mathbf{D}_e\,\mathbf{C}\,\mathbf{u}_{ci} - \bar{\xi}_0\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\,\mathbf{D}_e\,\mathbf{C}\,\mathbf{u}_0 - \bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\tilde{\mathbf{G}}_p\,\mathbf{D}_e\,\mathbf{C}\,\mathbf{u}_{ci}\right)\right\} \tag{24}$$

Therefore, it is possible to state that at the optimum the sensitivity (with respect to the design variable d_j) of the *impending* total plastic dissipation promoted in the structure by the plastic accumulation mechanism related to the limit state of plastic shakedown equals the analogous sensitivity of the modified cost of the structure, the latter being the quantity in brackets on the right-hand side of (24). This last equality constitutes a generalization at the present context of the theorem of Drucker–Prager–Rozvany–Shield (see Rozvany (1976)).

Again, (19n) means that the impending resulting plastic mechanism, described by the plastic strains $\mathbf{N}\,\boldsymbol{\lambda}_i$ in the limit state of elastic shakedown of part V^E and plastic shakedown of part V^F, is compatible with displacements $\boldsymbol{\nu}$.

From (19k–n) and (21) the following relation can be deduced:

$$\boldsymbol{\nu} = \boldsymbol{\nu}_0 = \sum_{i=1}^{b}\boldsymbol{\nu}_i \tag{25}$$

The first of (19o) constitutes a constraint on the external work. Since the latter is surely positive, also the multiplier e must be positive, and so (19c) allows us to obtain:

$$\xi = \bar{\xi} \tag{26}$$

Furthermore, since all the kinematical variables are determined within an arbitrary factor, it is possible to assume $e=1$ and to write (19o) as the impending unit external work in the cycle:

$$\bar{\xi}_c\sum_{i=1}^{b}\tilde{\mathbf{P}}_{ci}\,\mathbf{N}\,\boldsymbol{\lambda}_i + \bar{\xi}_0\tilde{\mathbf{P}}_0\sum_{i=1}^{b}\mathbf{N}\,\boldsymbol{\lambda}_i = 1 \tag{27}$$

Taking into account the third condition of (19i), the following identity can be deduced:

$$-\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\mathbf{R} + \xi\left(\bar{\xi}_c\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\mathbf{P}_{ci} + \bar{\xi}_0\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\mathbf{P}_0\right) - \sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\mathbf{S}\,\mathbf{Y}_i = -\sum_{i=1}^{b}\tilde{\boldsymbol{\lambda}}_i\,\tilde{\mathbf{N}}\,\boldsymbol{\rho} \tag{28}$$

In virtue of the virtual work principle the quantity on the right-hand side of (28) is nought, and considering (19g), (20), (22), (26) and (27) it is possible to rewrite (28) in the following way:

$$\sum_{i=1}^{b}\tilde{\mathbf{R}}\,\boldsymbol{\lambda}_i = \bar{\xi} \tag{29}$$

This last equation shows that the total plastic dissipation in the impending ratchet or instantaneous collapse mechanism equals the limit value of the load multiplier $\bar{\xi}$.

4
Application

As an application the three-floor steel frame plotted in Fig. 4(a), constituted by rectangular box section elements (Fig. 4(b)), has been studied. The width B and the height H of the typical structural element are assigned (Table 1), while its thickness s is assumed as the design variable.

The material has the following properties: Young's modulus $E = 21\,000\,\mathrm{kN/cm^2}$, yield stress $\sigma_y = 36\,\mathrm{kN/cm^2}$.

Two rigid perfectly plastic hinges are located at the extremes of all elements which are considered to be purely elastic, while an additional plastic hinge is located in the middle point of the beams. The interaction between the bending moment M and the axial force N has been taken into account, and in Fig. 4(c) the relevant dimensionless rigid plastic domain of the typical plastic hinge is plotted in the plane (N/N_y, M/M_y), being N_y and M_y the yield generalized stress corresponding to N and M, respectively.

The structure is subjected to a fixed uniformly distributed vertical load on the beams, $q_0 = 50\,\mathrm{kN/m}$, and to perfect cyclic concentrated horizontal loads applied on the nodes as reported in Fig. 4(a). The total horizontal cyclic load has been evaluated at each floor by means of the following formulas (according to the current Italian seismic rules):

$$\gamma_i = h_i \frac{\sum_{j=1}^{n} w_j}{\sum_{j=1}^{n} h_j w_j} \quad \forall i, j \in I(n) \tag{30a}$$

$$F_{ci} = c\gamma_i w_i \quad \forall i \in I(n) \tag{30b}$$

where $n = 3$ is the number of the floors, F_{ci} the total load applied at the *i-th* floor, γ_i the partition coefficient of the *i-th* floor, h_i the height of the *i-th* floor with respect to the soil, w_j the total weight of the *j-th* floor and $c = 0.1$ the seismic coefficient. The total force F_{ci} has been equally partitioned among the nodes of the *i-th* floor.

The fixed and cyclic load multipliers $\bar{\xi}_0 = 1$ and $\bar{\xi}_c = 4$ have been assumed as limit values for the designs. A further condition related to the instantaneous collapse under the purely fixed load amplified by the multiplier $\xi_0^u = 2.5$ has been also taken into account, with ξ_0^u as the ultimate fixed load multiplier.

As technological constraints the following conditions have been taken into account: at each floor the beams have the same thickness and the columns do not have increasing thickness from a floor to the higher one.

The limit design, the plastic shakedown design working above the elastic shakedown limit and the plastic shakedown design working below the elastic shakedown limit have been carried out.

For the latter effected design the level of the upper plateaux of the elastic shakedown, $\bar{\xi}_c^s = 2.75$, has been imposed, and, furthermore, the beams of the second floor (3–4) have been chosen as pertaining to the portion V^F.

Selecting some suitable values of parameter α, related to the knee of the upper plateaux of the elastic shakedown, as many designs were obtained and the related Bree-like diagrams were constructed. Few interpolations allowed us to obtain the value $\alpha = 1.49$, such that the borderline, which separates the zones F and R, crosses very near to the prefixed point ($\bar{\xi}_0 = 1$, $\bar{\xi}_c = 4$). In Fig. 5 the Bree diagrams of the three relevant obtained structures are plotted. The comparison of such diagrams shows that just the plastic shakedown designs, working above and below the elastic shakedown limit, ensure an alternating plasticity behaviour above the elastic shakedown until the prefixed point ($\bar{\xi}_0 = 1$, $\bar{\xi}_c = 4$), while the standard limit design is exposed to an incremental collapse condition.

The optimal thicknesses are reported in Table 1, together with the optimal volume and the plastic dissipation of the obtained frames. Table 1 shows that plastic

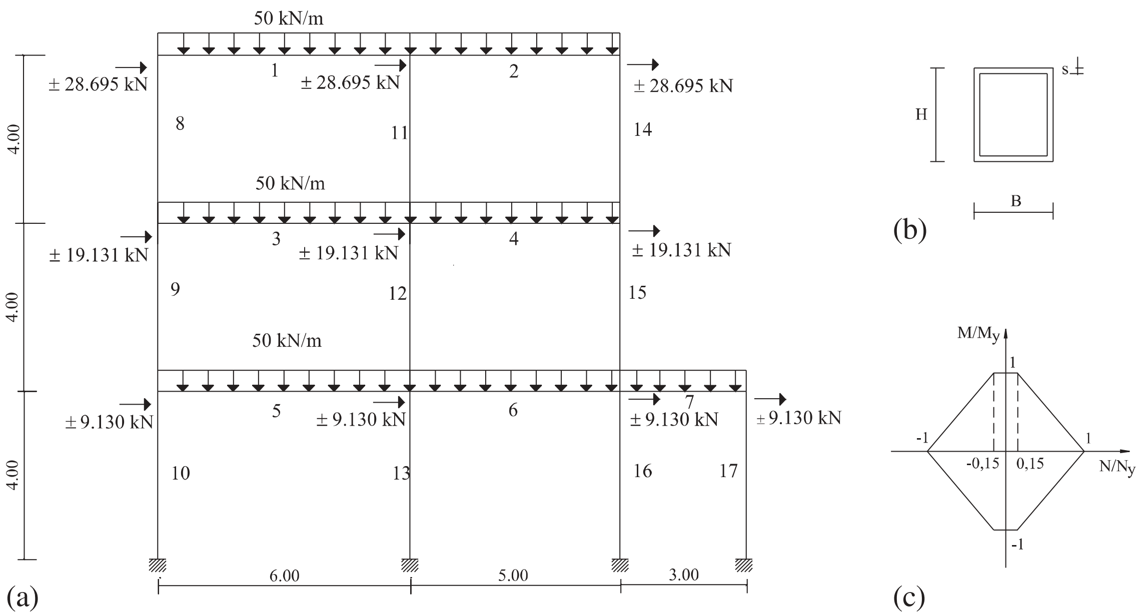

Fig. 4 Steel frame: (a) geometry and load condition; (b) typical cross-section; (c) typical dimensionless rigid plastic hinge domain

Table 1 Cross-section dimensions ($B \times H$) and thicknesses (s), structure volumes and plastic dissipations related to the limit design (LD), the plastic shakedown design working above the elastic shakedown limit (PSD above) and working below the elastic shakedown limit (PSD below)

Element	B (mm)	H (mm)	LD s (mm)	PSD above s (mm)	PSD below s (mm)
1 = 2	200	300	7.96	8.64	8.17
3 = 4	200	300	12.13	12.30	7.98
5 = 6 = 7	200	300	10.78	11.84	12.09
8	200	400	5.08	4.00	4.74
9	200	400	6.34	5.42	7.93
10	200	400	9.08	8.49	11.04
11	200	400	6.27	5.63	9.96
12	200	400	8.96	10.15	11.19
13	200	400	10.26	11.61	11.19
14	200	400	4.00	4.00	5.08
15	200	400	5.92	7.51	7.86
16	200	400	7.68	9.48	10.17
17	200	400	6.80	5.73	6.87
Volume (m^3)			0.686	0.715	0.733
Plastic dissipation (kN m)			–	22.43	51.91

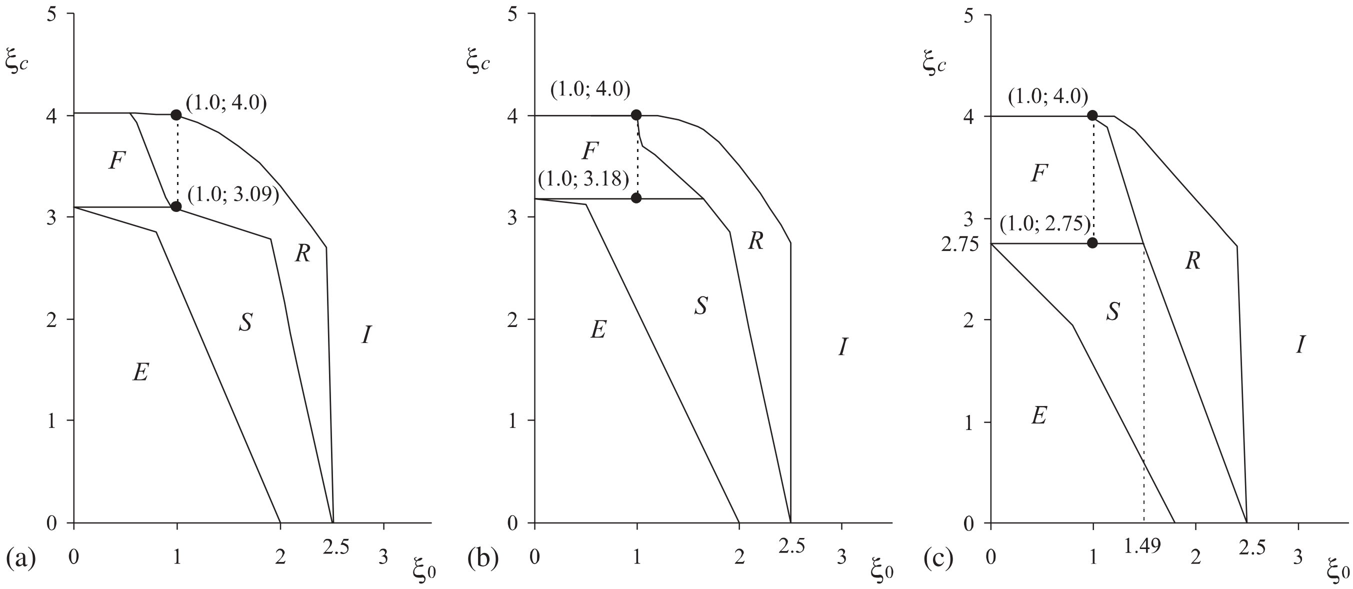

Fig. 5 Bree diagrams of the steel frame: (a) limit design; (b) plastic shakedown design working above the elastic shakedown limit; (c) plastic shakedown design working below the elastic shakedown limit

dissipation in the cycle exhibited by the plastic shakedown design working below the elastic shakedown limit is more than double the one exhibited by the plastic shakedown design working above the elastic shakedown limit. Actually, at level $\xi_c = 2.75$ just beams of the second floor of the former design plastify, but, successively, for increasing cyclic load multiplier ξ_c values ($2.75 < \xi_c \leq 4$) other elements also plastify.

In Table 1 the plastic dissipation in the cycle referring to the limit design is not reported because it includes the work of the fixed load, and it is not comparable with the one obtained in the case of the plastic shakedown designs.

Table 1 and Fig. 5 show that, with small volume increments, the obtained plastic shakedown designs are definitely preferable in terms of behaviour with respect to the limit design.

It is worth noticing that, as was expected, the plastic shakedown design shows the special trend of preserving an elastic portion of the structure able to suffer the fixed load as well as the increment of the cyclic load above the elastic shakedown limit. Such a good trend is quite general and it has been observed in all the effected applications.

From a computational point of view, it is possible to state that: the limit design requires the lowest effort involving just the solution of typical linear programming problems; the plastic shakedown optimal design working below the elastic shakedown limit also requires the solution of linear programming problems, but some additional constraints must be added and a computational strategy is needed to select the ones to be satisfied as equality; the plastic shakedown optimal design working above the elastic shakedown limit involves the elastic plastic response and requires the solution of linear complementarity and linear programming problems.

5 Conclusions

The plastic shakedown design problem of structures constituted by elastic perfectly plastic finite elements and subjected to a combination of fixed and cyclic loads has been studied. The cyclic load has been considered as a perfect one, namely for each cyclic basic load condition an opposite one exists in the load space. A suitable form of the equations related to the steady-state response of the structure has been reported and the problem of the determination of the limit (elastic and/or plastic) shakedown boundary in the Bree diagram is treated.

The optimal design problem has been formulated according to a plastic shakedown criterion, i.e. so as to prevent the incremental collapse of the structure, as long as the loads are not greater than some prescribed values. Two different formulations have been proposed for the design problem, both developed on the grounds of a statical approach and devoted to the search for the minimum volume structure design with the required limit behaviour; the first one operates below the elastic shakedown limit and represents a suitably modified elastic shakedown design problem, the second one operates above the elastic shakedown limit and involves the elastic plastic response of the desired design.

Since both formulations identify with strongly nonlinear mathematical programming problems, their solution can be reached developing suitable iterative techniques and related computational algorithm.

As is obvious, there are a lot of differences between the two formulations, also with respect to the computational stage.

The first one provides a suboptimal design, but in order to obtain such a solution it suffices just to solve linear mathematical programming problems, with constraints of equality and inequality. Assigning a suitably low level for the upper plateaux, it is possible to ensure the production of a great amount of plastic dissipation before reaching the collapse condition.

On the contrary, the second formulation provides an optimal design, but in order to obtain such a solution it is necessary to solve linear complementarity and linear programming problems. In this case it is not possible to assign the level at the limit of the elastic shakedown and consequently to control the production of plastic dissipation. Nevertheless, a modified formulation useful to increase the plastic dissipation is in progress.

The related Kuhn–Tucker equations of the two search problems are deduced through a variational approach and discussed, in order to point out the basic features of the obtained designs.

In the framework of the numerical applications, reference has been made to a steel frame. The limit standard optimal design and the plastic shakedown optimal designs following the two different formulations have been developed. The two different plastic shakedown design problems obviously provide different solutions and some interesting comparison is discussed in the application stage. Finally, the relevant Bree diagrams of the obtained structures have been determined, providing useful information on the behaviour of the optimal structures and allowing us to effect further useful comparison among them.

The results obtained allow us to confirm the theoretical expectations in terms of behavioural features of the optimal designs and, furthermore, they provide useful indication about the required computational effort related to the numerical solutions.

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