Research paper

Struct Multidisc Optim 27, 314–325 (2004)
DOI 10.1007/s00158-004-0390-3

Reliability-based design optimization using probabilistic sufficiency factor*

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Abstract A probabilistic sufficiency factor approach is proposed that combines safety factor and probability of failure. The probabilistic sufficiency factor approach represents a factor of safety relative to a target probability of failure. It provides a measure of safety that can be used more readily than the probability of failure or the safety index by designers to estimate the required weight increase to reach a target safety level. The probabilistic sufficiency factor can be calculated from the results of Monte Carlo simulation with little extra computation. The paper presents the use of probabilistic sufficiency factor with a design response surface approximation, which fits it as a function of design variables. It is shown that the design response surface approximation for the probabilistic sufficiency factor is more accurate than that for the probability of failure or for the safety index. Unlike the probability of failure or the safety index, the probabilistic sufficiency factor does not suffer from accuracy problems in regions of low probability of failure when calculated by Monte Carlo simulation. The use of the probabilistic sufficiency factor accelerates the convergence of reliability-based design optimization.

Key words Monte Carlo simulation, probabilistic sufficiency factor, reliability-based design optimization, response surface approximation

Manuscript received: 8 December 2002
Revised manuscript received: 15 August 2003
Published online: 8 June 2004

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* Preliminary version of the paper presented at the Third ISSMO/AIAA Internet Conference on Approximations in Optimization October 14–25, 2002

1 Introduction

Recently, there has been interest in using alternative measures of safety in reliability-based design optimization (RBDO). These measures are based on margin of safety or safety factors that are commonly used as measures of safety in deterministic design. Safety factor is generally expressed as the quotient of allowable over response, such as the commonly used central safety factor that is defined as the ratio of the mean value of allowable over the mean value of the response. The selection of safety factor for a given problem involves both objective knowledge such as data on the scatter of material properties and subjective knowledge such as expert opinion. Given a safety factor, the reliability of the design is generally unknown, which may lead to unsafe or inefficient design. Therefore, the use of the safety factor in reliability-based design optimization seems to be counterproductive.

Freudenthal (1962) showed that reliability can be expressed in terms of the probability distribution function of the safety factor. Elishakoff (2001) surveyed the relationship between the safety factor and reliability and showed that in some cases the safety factor can be expressed explicitly in terms of reliability. The standard safety factor is defined with respect to the response obtained with the mean values of the random variables. Thus a safety factor of 1.5 implies that with the mean values of the random variables we have a 50% margin between the response (e.g. stress) and the capacity (e.g. failure stress). However, the value of the safety factor does not tell us what the reliability is. Therefore, Birger (1970), as reported by Elishakoff (2001), introduced a factor, which we call here the probabilistic sufficiency factor that is more closely related to the target reliability. A probabilistic sufficiency factor of 1.0 implies that the reliability is equal to the target reliability, a probabilistic sufficiency factor larger than one means that the reliability exceeds the target reliability, and a probabilistic sufficiency factor less than one means that the system is not as safe as we wish. Specifically, a probabilistic sufficiency factor value of 0.9 means that we need to multiply the response by 0.9 or increase the capacity by 1/0.9 to achieve the target reliability.

Tu *et al.* (2000) used the probabilistic performance measure, which is closely related to Birger's safety factor, for RBDO using most probable point (MPP) methods (e.g. the first-order reliability method). They showed that the search for the optimum design converged faster by driving the safety margin to zero than by driving the probability of failure to its target value. Wu *et al.* (1998, 2001) used partial safety factors, which are similar to probabilistic sufficiency factors in order to replace the RBDO with a series of deterministic optimizations by converting reliability constraints to equivalent deterministic constraints.

The use of the probabilistic sufficiency factor gives a designer more quantitative measure of the resources needed to satisfy the safety requirements. For example, if the requirement is that the probability of failure is below 10^{-6} and the designer finds that the actual probability is 10^{-4}, he or she cannot tell how much change is required to satisfy the requirement. If instead the designer finds that a probability of 10^{-6} is achieved with a probabilistic sufficiency factor of 0.9, it is easier to estimate the required resources. For a stress-dominated linear problem, raising the probabilistic sufficiency factor from 0.9 to 1 typically requires a weight increase of about 10 percent of the weight of the over-stressed components.

Reliability analysis of systems with multiple failure modes often employs Monte Carlo simulation (MCS), which generates numerical noise due to limited sample size. Noise in the probability of failure or safety index may cause reliability-based design optimization (RBDO) to converge to a spurious optimum. The accuracy of MCS with a given number of samples deteriorates with decreasing probability of failure. For RBDO problems with small target probability of failure, the accuracy of MCS around the optimum is not as good as in regions with high probability of failure. Furthermore, the probability of failure in some regions may be so low that it is calculated to be zero by MCS. This flat zero probability of failure does not provide gradient information to guide the optimization procedure.

The probabilistic sufficiency factor is readily available from the results of MCS with little extra computational cost. The noise problems of MCS motivate the use of response surface approximation (RSA, e.g. Khuri and Cornell 1996). Response surface approximations typically employ low-order polynomials to approximate the probability of failure or safety index in terms of design variables in order to filter out noise and facilitate design optimization. These response surfaces are called design response surface (DRS) and are widely used in the RBDO (e.g. Sues *et al.* 1996).

The probability of failure often changes by several orders of magnitude over narrow bands in design space, especially when the random variables have small coefficients of variation. The steep variation of probability of failure requires DRS to use high-order polynomials for the approximation, increasing the required number of probability calculations (Qu *et al.* 2003). An additional problem arises when Monte Carlo simulations are used for calculating probabilities. For a given number of simulations, the accuracy of the probability estimates deteriorates as the probability of failure decreases.

The numerical problems associated with steep variation of probability of failure led to the consideration of alternative measures of safety. The most common one is to use the safety index, which replaces the probability by the distance, which is measured as the number of standard deviations from the mean of a normal distribution that gives the same probability. The safety index does not suffer from steep changes in magnitude, but it has the same problems of accuracy as the probability of failure when based on Monte Carlo simulations. However, the accuracy of probabilistic sufficiency factor is maintained in the region of low probability. The probabilistic sufficiency factor also exhibits less variation than the probability of failure or safety index. Thus the probabilistic sufficiency factor can be used to improve design response surface approximations for RBDO.

The next section introduces the probabilistic sufficiency factor, followed by the computation of the probabilistic sufficiency factor by Monte Carlo simulation. The methodology is demonstrated by the reliability-based beam design problem.

2 Probabilistic sufficiency factor

The deterministic equivalent of reliability constraint in RBDO can be formulated as

$$g_r(\hat{\mathbf{x}}, \mathbf{d}) \le g_c(\hat{\mathbf{x}}, \mathbf{d}) \tag{1}$$

where g_r denotes a response quantity, g_c represent a capacity (e.g. strength allowable), $\hat{\mathbf{x}}$ is usually the mean value vector of random variables, $\mathbf{d}$ is the design vector. The traditional safety factor is defined as

$$s(\mathbf{x}, \mathbf{d}) = \frac{g_c(\mathbf{x}, \mathbf{d})}{g_r(\mathbf{x}, \mathbf{d})} \tag{2}$$

and the deterministic design problem requires

$$s(\hat{\mathbf{x}}, \mathbf{d}) \ge s_r \tag{3}$$

where s_r is the required safety factor, which is usually 1.4 or 1.5 in aerospace applications. The reliability constraint can be formulated as a requirement on the safety factor

$$\mathrm{Prob}(s \le 1) \le P_r \tag{4}$$

where P_r is the required probability of failure. Birger's probabilistic sufficiency factor P_{sf} is the solution to

$$\mathrm{Prob}(s \le P_{sf}) = P_r \tag{5}$$

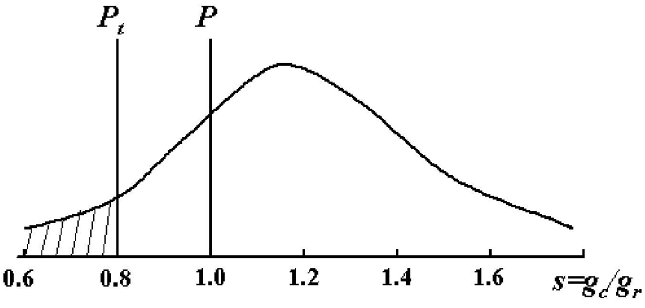

Fig. 1 Probability density of the safety factor. The area under the curve left of $s = 1$ measures the actual probability of failure, while the shaded area is equal to the required probability of failure indicating that the probabilistic sufficiency factor = 0.8

It is the safety factor that is violated with the required probability P_r.

Figure 1 shows the probability density of the safety factor for a given design. The area under the curve left of $s = 1$ represents the probability that $s < 1$, hence it is equal to the actual probability of failure. The shaded area in the figure represents the required probability of failure, P_r. For this example, since it is the area left of the line $s = 0.8$, $P_{sf} = 0.8$. The value of 0.8 indicates that the target probability will be achieved if we reduced the response by 20% or increased the capacity by 25% $(1/0.8 - 1)$. For many problems this provides sufficient information for a designer to estimate the additional structural weight. For example, raising the safety factor from 0.8 to 1 of a stress-dominated linear problem typically requires a weight increase of about 20% of the weight of the over-stressed components.

2.1 The use of the probabilistic sufficiency factor to estimate additional structural weight to satisfy the reliability constraint

The following cantilever beam example (Fig. 2) is taken from Wu *et al.* (2001) to demonstrate the use of the probabilistic sufficiency factor.

There are two failure modes in the beam design problem. One failure mode is yielding, which is most critical at the corner of the rectangular cross-section at the fixed end of the beam

$$g_S(R, X, Y, w, t) = R - \sigma = R - \left(\frac{600}{wt^2}Y + \frac{600}{w^2t}X\right) \qquad (6)$$

where R is the yield strength, and X and Y are the independent horizontal and vertical loads. Another failure

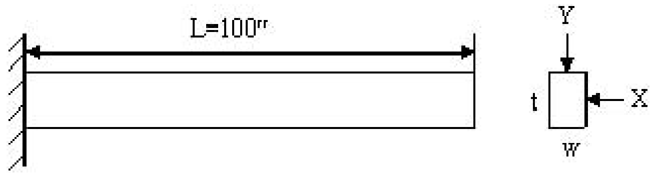

Fig. 2 Cantilever beam subject to vertical and lateral bending

mode is the tip deflection exceeding the allowable displacement, D_0

$$g_D(E, X, Y, w, t) = D_0 - D =$$

$$D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \qquad (7)$$

where E is the elastic modulus. The random variables are defined in Table 1.

The cross-sectional area is minimized subjected to a reliability constraint, which requires the safety index to be larger than three (probability of failure less than 0.00135). The reliability-based design optimization problem, with the width w and thickness t of the beam as design variables that are deterministic, can be formulated as

minimize $A = wt$

such that

$$p - 0.00135 \leq 0 \qquad (8)$$

based on probability of failure, or

minimize $A = wt$

such that

$$3 - \beta \leq 0 \qquad (9)$$

based on the safety index, where β is the safety index, or

minimize $A = wt$

such that

$$1 - P_{sf} \leq 0 \qquad (10)$$

based on the probabilistic sufficiency factor. The reliability constraints are formulated in the above three forms, which are equivalent in terms of safety. The details of the beam design are given later in the paper.

In order to demonstrate the utility of the P_{sf} for estimating the required weight for correcting a safety defi-

Table 1 Random variables in the beam design problem

Random variables	X	Y	R	E
Distribution	Normal (500, 100) lb	Normal (1000, 100) lb	Normal (40 000, 2000) psi	Normal $(29 \times 10^6, 1.45 \times 10^6)$ psi

ciency, it is useful to see how the stresses and the displacements depend on the weight (or cross-sectional area) for this problem. If we have a given design with dimensions w_0 and t_0 and a P_{sf} of P_{sf0}, which is smaller than one, we can make the structure safer by scaling both w and t uniformly by a constant c

$$w = cw_0 \,, \quad t = ct_0 \tag{11}$$

It is easy to check from (6) that the stress will then change by a factor of c^3, and the area by a factor of c^2. Since the P_{sf} is inversely proportional to the most critical stress or displacement, it is easy to obtain the relationship

$$P_{sf} = P_{sf0} \left(\frac{A}{A_0} \right)^{1.5} \tag{12}$$

where $A_0 = w_0 t_0$. This indicates that a one percent increase in area (corresponding to 0.5 percent increase in w and t) will improve the P_{sf} by about 1.5 percent. Since nonuniform increases in the width and thickness may be more efficient than uniform scaling, we may be able to do better than 1.5 percent. Thus, if we have $P_{sf} = 0.97$, we can expect that we can make the structure safe with a weight increase under two percent. For displacement, the factor at the exponent of (7) is 2. For general structures, engineering judgment is requirement to estimate the factor.

The probabilistic sufficiency factor gives a designer a measure of safety that can be used more readily than the probability of failure or the safety index to estimate the required weight increase to reach a target safety level. The P_{sf} of a beam design, presented in Sect. 4 in detail, is 0.9733 for a target probability of failure of 0.00135, (12) indicate that the deficiency in the P_{sf} can be corrected by scaling up the area by a factor of 1.0182. Since the area A is equal to $c^2 wt$, the dimensions should be scaled by a factor c of 1.0091 ($= 1.0182^{0.5}$) to $w = 2.7123$ and $t = 3.5315$. Thus the objective function of the scaled design is 9.5785. The probability of failure of the scaled design is 0.001302 (safety index of 3.0110 and probabilistic sufficiency factor of 1.0011) evaluated by MCS with 1 000 000 samples. Such estimation is not readily available using the probability of failure (0.00314) and the safety index (2.7328) of the design.

3 Reliability analysis using Monte Carlo simulation

Let $g(\mathbf{x})$ denote the limit state function of a performance criterion (such as strength allowable larger than stress), so that the failure event is defined as $g(\mathbf{x}) < 0$, where $\mathbf{x}$ is a random variable vector. The probability of failure of a system can be calculated as

$$P_f = \int\limits_{g(\mathbf{x})<0} f_X(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{13}$$

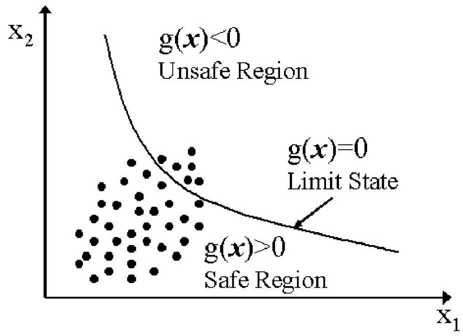

Fig. 3 Monte Carlo simulation of problem with two random variables

where $f_X(\mathbf{x})$ is the joint probability distribution function (JPDF). This integral is hard to evaluate, because the integration domain defined by $g(\mathbf{x}) < 0$ is usually unknown, and integration in high dimensions is difficult. Commonly used probabilistic analysis methods are either moment-based methods such as the first-order-reliability method (FORM) and the second-order-reliability method (SORM), or simulation techniques such as Monte Carlo simulation (MCS) (e.g. Melchers 1999). Monte Carlo simulation is a good method to use for system reliability analysis with multiple failure modes. The present paper focuses on the use of MCS with response surface approximation in RBDO.

Monte Carlo simulation utilizes randomly generated samples according to the statistical distribution of the random variables, and the probability of failure is obtained by calculating the statistics of the sample simulation. Figure 3 illustrated the Monte Carlo simulation of a problem with two random variables. The probability of failure of the problem is calculated as the ratio of the number of samples in the unsafe region over the total number of samples.

A small probability requires a large number of samples for MCS to achieve low relative error. Therefore, for a fixed number of simulations, the accuracy of MCS deteriorates with the decrease of probability of failure. For example, with 10^6 simulations, a probability estimate of 10^{-3} has a relative error of a few percent, while a probability estimate of 10^{-6} has a relative error of the order of 100 percent. In RBDO, the required probability of failure is often very low, thus the probability (or safety index) calculated by MCS is inaccurate near the optimum. Furthermore, the probabilities of failure in some design regions may be so low that they are calculated as zero by MCS. This flat zero probability of failure or infinite safety index cannot provide useful gradient information for the optimization.

3.1 Calculation of probabilistic sufficiency factor by Monte Carlo simulation

Here we propose the use of the probabilistic sufficiency factor to solve the problems associated with probability calculation by MCS. P_{sf} can be estimated by MCS as follows. Define the n^{th} safety factor of the MCS as

$$s_{(n)} = n^{th} \min_{i=1}^{M} (s(\mathbf{x_i})) \quad (14)$$

where M is the sample size of the MCS, and the n^{th} min means the n^{th} smallest safety factor among M safety factors from the MCS. Thus $s_{(n)}$ is the n^{th}-order statistics of M safety factors from the MCS, which corresponds to a probability of n/M of $s(\mathbf{x}) \leq s_{(n)}$. That is, we seek to find the safety factor that is violated with the required probability P_r. The probabilistic sufficiency factor is then given as

$$P_{sf} = s_{(n)} \quad \text{for} \quad n = P_r M \quad (15)$$

For example, if the required probability P_r is 10^{-4} and the sample size of Monte Carlo simulation M is 10^6, P_{sf} is equal to the highest safety factor among the 100 samples ($n = P_r M$) with the lowest safety factors. The calculation of P_{sf} only requires sorting the lowest safety factors in the Monte Carlo samples. While the probability of failure changes by several orders of magnitude the probabilistic sufficiency factor usually varies by less than one order of magnitude in a given design space.

For problems with k reliability constraints, the most critical safety factor is calculated first for each Monte Carlo sample,

$$s(\mathbf{x_i}) = \min_{i=1}^{k} \left(\frac{g_c^i}{g_r^i} \right) \quad (16)$$

Then the sorting of the n^{th} minimum safety factor can proceed as in (14). When n is small, it may be more accurate to calculate P_{sf} as the average between the n^{th} and $(n+1)^{th}$ lowest safety factor in the Monte Carlo samples.

The probabilistic sufficiency factor provides more information than the probability of failure or the safety index. Even in the regions where the probability of failure is so small that it cannot be estimated accurately by the MCS with given sample size M, the accuracy of P_{sf} is maintained. Using the probabilistic sufficiency factor also gives designers useful insights on how to change the design to satisfy safety requirements as shown in Sect. 2.1. The estimate is not readily available from the probability of failure or the safety index. The probabilistic sufficiency factor is based on the ratio of allowable to response, which exhibits much less variation than the probability of failure or safety index. Therefore, approximating the probabilistic sufficiency factor in design optimization is easier than approximating the probability of failure or the safety index as discussed in the next section.

3.2 Monte Carlo simulation using response surface approximation

Monte Carlo simulation is easy to implement, robust, and accurate with sufficiently large samples, but it requires a large number of analyses to obtain a good estimate of small failure probabilities. Monte Carlo simulation also produces a noisy response and hence is difficult to use in optimization. Response surface approximations solve the two problems, namely simulation cost and noise from random sampling.

Response surface approximations fit a closed-form approximation to the limit state function to facilitate reliability analysis. Therefore, response surface approximation is particularly attractive for computationally expensive problems such as those requiring complex finite element analyses. Response surface approximations usually fit low-order polynomials to the structural response in terms of random variables

$$\hat{g}(\mathbf{x}) = Z(\mathbf{x})^T \mathbf{b} \quad (17)$$

where $\hat{g}(\mathbf{x})$ denotes the approximation to the limit state function $g(\mathbf{x})$, $Z(\mathbf{x})$ is the basis function vector that usually consists of monomials, and $\mathbf{b}$ is the coefficient vector estimated by least-square regression. The probability of failure can then be calculated inexpensively by Monte Carlo simulation or moment-based methods using the fitted polynomials.

Response surface approximations (RSA) can be used in different ways. One approach is to construct a local RSA around the most probable point (MPP) that contributes most to the probability of failure of the structure. The statistical design of experiment (DOE) of this approach is iteratively performed to approach the MPP on the failure boundary. For example, Bucher and Bourgund (1990), and Sues (1996, 2000) constructed a progressively refined local RSA around the MPP by an iterative method. This local RSA approach can produce satisfactory results given enough iterations. Another approach is to construct a global RSA over the entire range of random variables, i.e. design of experiment around the mean values of the random variables. Fox (1993, 1994, 1996) used Box-Behnken design to construct global response surfaces and summarized 12 criteria to evaluate the accuracy of the RSA. Romero and Bankston (1998) employed progressive lattice sampling as the design of experiments to construct global RSA. With this approach, the accuracy of response surface approximation around the MPP is unknown, and caution must be taken to avoid extrapolation near the MPP. Both approaches can be used to perform reliability analysis for computationally expensive problems.

However, reliability analysis needs to be performed and hence the RSA needs to be constructed at every design point visited by the optimizer, which requires a fairly large number of response surface constructions and thus limit state evaluations. The local RSA approach is even more computationally expensive than the global approach in the design environment. Qu *et al.* (2003) developed a global analysis response surface (ARS) approach in unified space of design and random variables to reduce the number of RSA substantially and achieve higher efficiency than the previous approach. This analysis response surface can be written as

$$\hat{g}(\mathbf{x}, \mathbf{d}) = Z(\mathbf{x}, \mathbf{d})^T \mathbf{b} \quad (18)$$

x and **d** are the random variable and design variable vectors, respectively. They recommended Latin hypercube sampling as the statistical design of experiments. The number of response surface approximations constructed in the optimization process is reduced substantially by introducing design variables into the response surface approximation formulation.

The selection of the RSA approach depends on the limit state function of the problem and target probability of failure. The global RSA approach is more efficient than local RSA, but it is limited to problems with relatively high probability or limit state function that can be well approximated by regression analysis based on simple basis functions. To avoid the extrapolation problems, RSA generally needs to be constructed around the important region or MPP to avoid large errors in the results of MCS induced by fitting errors in RS. Therefore, an iterative RSA is desirable for general reliability analysis problem.

Design response surface approximations (DRS) are fitted to probability of failure to filter out noise in MCS and facilitate optimization. Based on past experience, high-order DRS (such as quintic polynomials) are needed in order to obtain a reasonably accurate approximation of the probability of failure. Constructing highly accurate DRS is difficult because the probability of failure changes by several orders of magnitude over small distance in design space. Fitting to the safety index $\beta = -\Phi^{-1}(p)$, where p is the probability of failure and Φ is the cumulative distribution function of normal distribution, improves the accuracy of the DRS to a limited extent. The probabilistic sufficiency factor can be used to improve the accuracy of DRS approximation.

4 Beam design example

The details of the beam design problem mentioned in Sect. 2 are presented here. Since the limit state of the problem is available in closed form, as shown by (6) and (7), the direct Monte Carlo simulation with a sufficiently large number of samples is used here (without analysis response surface) in order to better demonstrate the advantage of probabilistic sufficiency factors over the probability of failure or safety index. By using the exact limit state function, the errors in the results of Monte Carlo simulation are purely due to the convergence errors, which can be easily controlled by changing the sample size. In applications where analysis response surface approximation must be used, the errors introduced by approximation can be reduced by sequentially improving the approximation as the optimization progresses.

The reliability constraints, shown in (8) to (10), are approximated by design response surface approximates that fit to the probability of failure, safety index, and probabilistic sufficiency factor. The accuracy of the design response surface approximations is then compared. The design response surface approximations are in two design variables w and t. A quadratic polynomial in two variables has six coefficients to be estimated. Since face center central composite design (FCCCD, e.g. Khuri and Cornell 1996) is often used to construct quadratic response surface approximation, an FCCCD with 9 points was employed here first with poor results. Based on our previous experience, higher-order design response surface approximations are needed to fit the probability of failure or the safety index, and the number of points of a typical design of experiments should be about twice the number of coefficients. A cubic polynomial in two variables has 10 coefficients that require about 20 design points. Latin hypercube sampling can be used to construct the higher-order response surface (Qu *et al.* 2003). We found that Latin hypercube sampling might fail to sample points near some corners of the design space, leading to poor accuracy around these corners. To deal with this extrapolation problem, all four vertices of the design space were added to 16 Latin hypercube sampling points for a total of 20 points. Mixed stepwise regression (e.g. Myers and Montgomery 1995) was employed to eliminate poorly characterized terms in the response surface models.

4.1 Design with strength failure mode

The range for the design response surface, shown in Table 2, was selected based on the mean-based deterministic design, $w = 1.9574''$ and $t = 3.9149''$. The probability of failure was calculated by direct Monte Carlo simulation with 100 000 samples based on the exact stress in (6).

Cubic design response surfaces with 10 coefficients were constructed and their statistics are shown in Table 3. An R^2_{adj} close to one and an average percentage error (defined as the ratio of root-mean-square error (RMSE) predictor and mean of response) close to zero indicate good accuracy of the response surfaces. It is seen that the design response surfaces for the probabilistic sufficiency factor has the highest R^2_{adj} and the smallest average percentage error. The standard error in probability calculated by Monte Carlo simulation can be estimated as

$$\sigma_p = \sqrt{\frac{p(1-p)}{M}} \quad (19)$$

where p is the probability of failure, and M is the sample size of the Monte Carlo simulation. If a probability

Table 2 Range of design variables for the design response surface

System variables	w	t
Range	$1.5''$ to $3.0''$	$3.5''$ to $5.0''$

Table 3 Comparison of cubic design response surface approximations of probability of failure, safety index and probabilistic sufficiency factor for single strength failure mode (based on Monte Carlo simulation of 100 000 samples)

Error Statistics	16 Latin hypercube sampling points + 4 vertices		
	Probability RS	Safety index RS	Probabilistic sufficiency factor RS
R^2_{adj}	0.9228	0.9891	0.9999
RMSE predictor	0.1103	0.3027	0.002409
Mean of response	0.2844	1.9377	1.0331
APE (Average percentage error = RMSE predictor/Mean of response)	38.78%	15.62%	0.23%
APE in Pof (= RMSE predictor of Pof/Mean of Pof)	38.78%	12.04%	N/A

of failure of 0.2844 is to be calculated by Monte Carlo simulation of 100 000 samples (the mean probability of failure in Table 3), the standard error due to the limited sampling is 0.00143. The RMSE error of the probability design response surface is of 0.1103. Thus the error induced by the limited sampling (100 000) is much smaller than the error of the response surface approximation to the probability of failure.

The probabilistic sufficiency factor design response surface has an average error less than one percent, while the safety index design response surface has an average error of about 15.6 percent. It must be noted, however, that the average percent errors of the three design response surface cannot be directly compared, because one percent error in probabilistic sufficiency factor does not correspond to one percent error in probability of failure or safety index. Errors in safety index design response surface were transformed to errors in terms of probability as shown in Table 3. It is seen that the safety index design response surface approximation is more accurate than the probability design response surface approximation.

Besides the average errors over the design space, it is instructive to compare errors measured in probability of failure in the important region of the design space. For optimization problems, the important region is the region containing the optimum. Here it is the curve of target reliability according to each design response surface, on which the reliability constraint is satisfied critically, and the probability of failure should be 0.00135 if the design response surface approximation does not have errors. For each design response surface approximation, 11 test points were selected along a curve of target reliability and given in the Appendix. The average percentage errors at these test points, shown in Table 4, demonstrate the ac-

Table 4 Averaged errors in cubic design response surface approximations of probabilistic sufficiency factor, safety index and probability of failure at 11 points on the curves of target reliability

Design response surface of	Probability of failure	Safety index	Probabilistic sufficiency factor
Average percentage error in probability of failure	213.86%	92.38%	10.32%

Table 5 Comparisons of optimum designs based on cubic design response surface approximations of probabilistic sufficiency factor, safety index and probability of failure

Design response surface of	Minimize objective function F while $\beta \geq 3$ or $0.00135 \geq$ pof		
	Optima	Objective function $F = wt$	Pof/Safety index/Safety factor from MCS of 100 000 samples
Probability	$w = 2.6350$, $t = 3.5000$	9.2225	0.00690/2.4624/0.9481
Safety index	$w = 2.6645$, $t = 3.5000$	9.3258	0.00408/2.6454/0.9663
Probabilistic sufficiency factor	$w = 2.4526$, $t = 3.8884$	9.5367	0.00128/3.0162/1.0021
Exact optimum (Wu et al. 2001)	$w = 2.4484$, $t = 3.8884$	9.5204	0.00135/3.00/1.00

curacy advantage of the probabilistic sufficiency factor approach. For the target reliability, the standard error due to Monte Carlo simulation of 100 000 samples is 8.6%, which is comparable to the response surface error for the P_{sf}. For the other two response surfaces, the errors are apparently dominated by the modelling errors due to the cubic polynomial approximation.

The optima found by using the design response surface approximations of Table 3 are compared in Table 5. The probabilistic sufficiency factor design response surface clearly led to a better design, which has a safety index of 3.02 according to Monte Carlo simulation. It is seen that the design from probabilistic sufficiency factor design response surface approximation is very close to the exact optimum. Note that the values of P_{sf} for the probability-based optimum and safety-index-based optimum provide a good estimate to the required weight increments. For example, with a $P_{sf} = 0.9663$ the safety-index-based design has a safety factor shortfall of 3.37 percent, indicating that it should not require more than 2.25 percent weight increment to remedy the problem. Indeed the optimum design is 2.08 percent heavier. This would have been difficult to infer from a probability of failure of 0.00408, which is three times larger than the target probability of failure.

4.2 Design with strength and displacement failure modes

For system reliability problems with strength and displacement constraints, the probability of failure is calculated by direct Monte Carlo simulation with 100 000 samples based on the exact stress and exact displacement in (6) and (7). The allowable tip displacement D_0 is chosen to be 2.25″ in order to have two competing constraints (Wu *et al.* 2001). The three cubic design response surface approximations in the range of design variables shown in Table 2 were constructed and their statistics are shown in Table 6.

It is seen that the R^2_{adj} of the probabilistic sufficiency factor response surface approximation is the highest among the three response surface approximations, which implies that the probabilistic sufficiency factor design response surface approximation is the most accurate in terms of averaged errors in the entire design space defined in Table 2. The critical errors of the three design response surfaces are also compared. For each design response surface approximation, 51 test points were selected along a curve of target reliability (probability of failure = 0.00135). The average percentage errors at these test points, shown in Table 7, demonstrate that the probabilistic sufficiency factor design response surface approximation is more accurate than the probability of failure and safety index response surface approximations.

The optima found by using the design response surface approximations of Table 6 are compared in Table 8. The probabilistic sufficiency factor design response surface led to a better design than the probability or safety index design response surface in terms of reliability. The probability of failure of the P_{sf} design is 0.00314 evaluated by Monte Carlo simulation, which is higher than the target probability of failure of 0.00135. The deficiency in reliability in the P_{sf} design is induced by the errors

Table 6 Comparison of cubic design response surface approximations of the first design iteration for probability of failure, safety index and probabilistic sufficiency factor for system reliability (strength and displacement)

Error statistics	16 Latin hypercube sampling points + 4 vertices		
	Probability response surface	Safety index response surface	Probabilistic sufficiency factor response surface
R^2_{adj}	0.9231	0.9887	0.9996
RMSE predictor	0.1234	0.3519	0.01055
Mean of response	0.3839	1.3221	0.9221
APE (Average percentage error = RMSE predictor/Mean of response)	32.14%	26.62%	1.14%
APE in Pof (= RMSE predictor of Pof/Mean of Pof)	32.14%	10.51%	N/A

Table 7 Averaged errors in cubic design response surface approximations of probabilistic sufficiency factor, safety index and probability of failure at 51 points on the curves of target reliability

Design response surface of	Probability of failure	Safety index	Probabilistic sufficiency factor
Average percentage error in probability of failure	334.78%	96.49%	39.11%

Table 8 Comparison of optimum designs based on cubic design response surface approximations of the first design iteration for probabilistic sufficiency factor, safety index and probability of failure

Design response surface of	Minimize objective function F while $\beta \geq 3$ or $0.00135 \geq$ pof		
	Optima	Objective function $F = wt$	Pof/Safety index/Safety factor from MCS of 100 000 samples
Probability	$w = 2.6591$, $t = 3.5000$	9.3069	0.00522/2.5609/0.9589
Safety index	$w = 2.6473$, $t = 3.5000$	9.2654	0.00630/2.4949/0.9519
Probabilistic sufficiency factor	$w = 2.6881$, $t = 3.500$	9.4084	0.00314/2.7328/0.9733

Table 9 Range of design variables for design response surface approximations of the second design iteration

System variables	w	t
Range	$2.2''$ to $3.0''$	$3.2''$ to $4.0''$

in the probabilistic sufficiency factor design response surface approximation. The probabilistic sufficiency factor can be used to estimate the additional weight to satisfy the reliability constraint. A scaled design of $w = 2.7123$ and $t = 3.5315$ was obtained in Sect. 2.1. The objective function of the scaled design is 9.5785. The probability of failure of the scaled design is 0.001302 (safety index of 3.0110 and probabilistic sufficiency factor of 1.0011) evaluated by MCS with 1 000 000 samples.

The design can be improved by performing another design iteration, which would reduce the errors in design response surface by shrinking the design space around the current design. The reduced range of design response surface approximations is shown in Table 9 for the next design iteration. The design response surface approximations constructed are compared in Table 10. It is observed again that the probabilistic sufficiency factor response surface approximation is the most accurate.

Table 10 Comparison of cubic design response surface approximations of the second design iteration for probability of failure, safety index and probabilistic sufficiency factor for system reliability (strength and displacement)

Error statistics	16 Latin Hypercube sampling points + 4 vertices		
	Probability response surface	Safety index response surface	Probabilistic sufficiency factor response surface
R^2_{adj}	0.9569	0.9958	0.9998
RMSE predictor	0.06378	0.1329	0.003183
Mean of response	0.1752	2.2119	0.9548
APE (Average percentage error = RMSE predictor/Mean of response)	36.40%	6.01%	0.33%

Table 11 Comparisons of optimum designs based on cubic design response surfaces of the second design iteration for probabilistic sufficiency factor, safety index and probability of failure

Design response surface of	Minimize objective function F while $\beta \geq 3$ or $0.00135 \geq$ pof		
	Optima	Objective function $F = wt$	Pof/Safety index/Safety factor from MCS of 100 000 samples
Probability	$w = 2.7923$, $t = 3.3438$	9.3368	0.00511/2.5683/0.9658
Safety index	$w = 2.6878$, $t = 3.5278$	9.4821	0.00177/2.9165/0.9920
Probabilistic sufficiency factor	$w = 2.6041$, $t = 3.6746$	9.5691	0.00130/3.0115/1.0009

The optima based on design response surface approximations for the second design iteration shown in Table 10 are compared in Table 11. It is seen that the design converges in two iterations with the probabilistic sufficiency factor response design surface due to its superior accuracy over the probability of failure and safety index design response surfaces.

5
Concluding remarks

The paper presented a probabilistic sufficiency factor as a measure of the safety level relative to a target safety level, which can be obtained from the results of Monte Carlo simulation with little extra computation. It was shown that a design response surface approximation can be more accurately fitted to the probabilistic sufficiency factor than to the probability of failure or the safety index. Using the beam design example with single or system reliability constraints, it was demonstrated that the design response surface approximation based on the probabilistic sufficiency factor has superior accuracy and accelerates the convergence of reliability-based design optimization. The probabilistic sufficiency factor also provides more information in regions of such low probability where the probability of failure or safety index cannot be estimated by Monte Carlo simulation with a given sample size, which is helpful in guiding the optimizer. Finally it was shown that the probabilistic sufficiency factor can be employed by the designer to estimate the required additional weight to achieve a target safety level, which might be difficult with probability of failure or safety index.

Acknowledgements This work was supported in part by grants NAG-1-2177 from NASA Langley Research Center, NSF grant DM5-9979711, and NASA Cooperative Agreement NCC3-994, the "Institute for Future Space Transport" University Research, Engineering and Technology Institute.

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Appendix: Contour plots of three design response surface approximations and test points along the curve of target reliability

To compare critical errors of the design response surface approximations, 11 test points were selected along a curve of target reliability (probability of failure = 0.00135) for

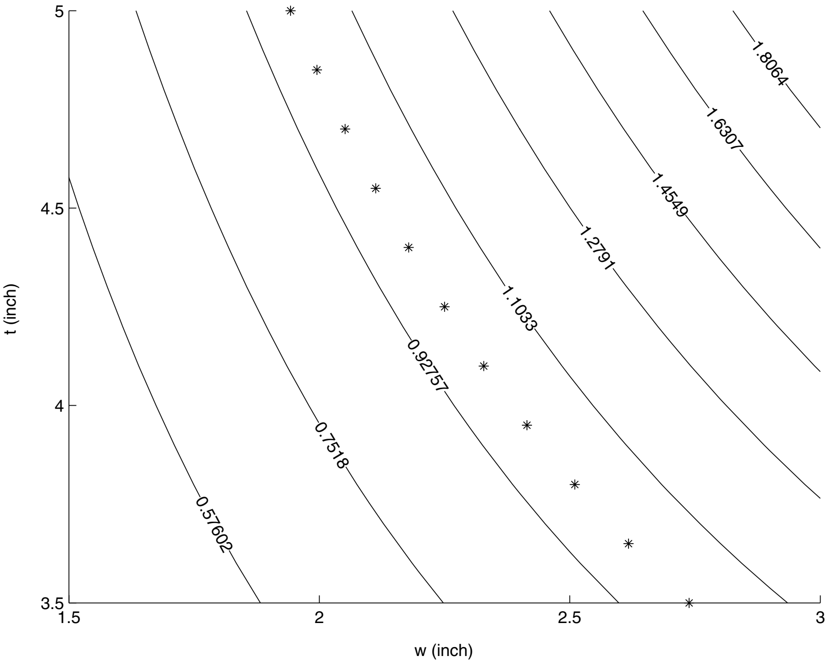

Fig. 4 Contour plot of the probabilistic safety factor design response surface approximation and test points along the curve of target reliability

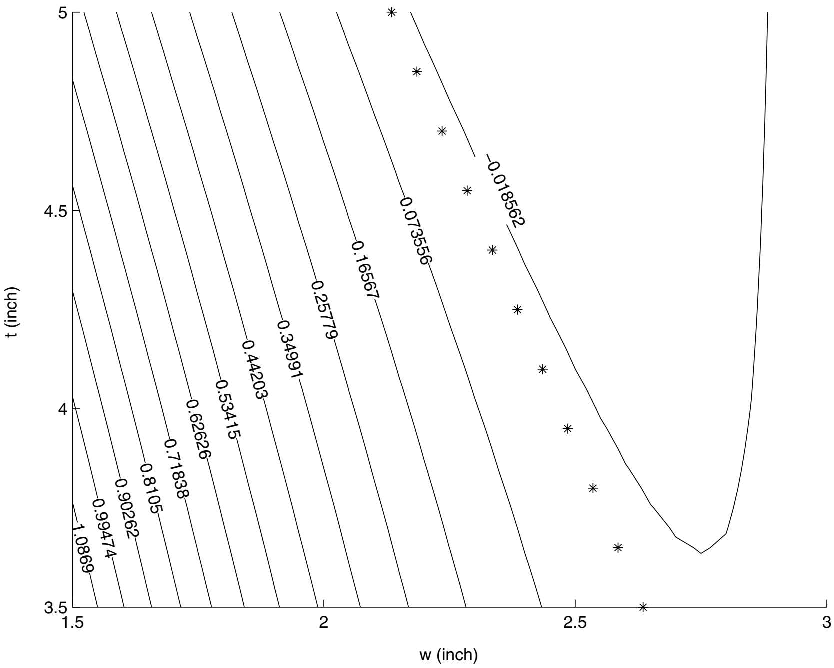

Fig. 5 Contour plot of probability of failure design response surface approximation and test points along the curve of target reliability. The negative values of probability of failure are due to the interpolation errors of the design response surface approximation

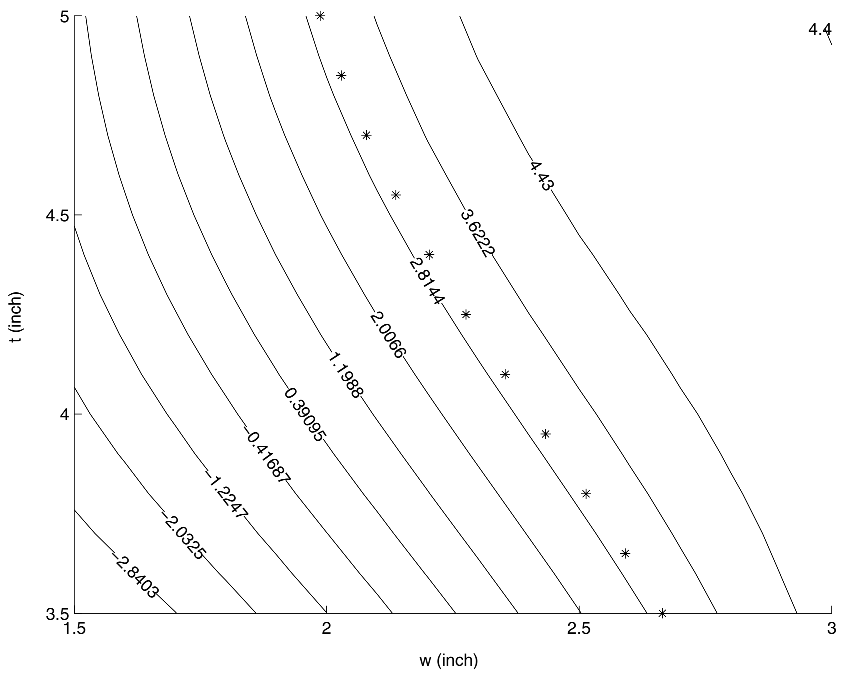

Fig. 6 Contour plot of the safety index design response surface approximation and test points along the curve of target reliability

each design response surface approximation. The contour plots and test points employed in error calculation for the probabilistic sufficiency factor, probability of failure and safety index response surface approximation are shown by the following figures. The average percentage errors at these test points are shown in Table 4.