Research paper

Struct Multidisc Optim 26, 200–208 (2004)
DOI 10.1007/s00158-003-0339-y

Structural design using Cellular Automata for eigenvalue problems

M.M. Abdalla and Z. Gürdal

Abstract Cellular Automata (CA) is an emerging paradigm for the analysis and design of complex systems. Recently, it has been successfully applied to structural design. In this work, an algorithm for designing structures for eigenvalue requirements is presented. The proposed algorithm, being fully local in nature, lends itself to CA-type implementation. To illustrate the effectiveness of the proposed approach, the design of Euler–Bernoulli columns for a prescribed buckling load is considered. Excellent agreement between the CA results and exact solutions is obtained. A more complex column design problem with local constraints is also considered, and the CA design is compared to the design obtained using a state-of-the-art structural optimization software.

Key words structural optimization, eigenvalues, cellular automata

1 Introduction

The use of the Cellular Automata (CA) paradigm is emerging as a powerful approach to the analysis and design of complex systems (Wolfram 1994). CA uses a *lattice* of regularly spaced cells to model physical phenomena. Each cell contains all the information needed to update its state. This information includes both field variables (e.g., displacements or stresses) as well as local design variables (e.g., local cross section area or thickness). The only external information to the cell comes directly from adjacent cells, which along with the cell itself form a *neighborhood*. By limiting computations to neighborhoods and using identical update rules for cell variables in the entire lattice, CA proves to be an inherently parallel algorithm. Moreover, since both field and design variables can be simultaneously updated, CA allows combined analysis and design. This greatly reduces the amount of computations required to reach an improved design.

Recently the CA concept has been successfully applied to structural design. Topology design using the finite element for global analysis and CA local rules for design is considered in Kita and Toyoda (2000). The design rules are obtained through the formulation of a local optimization problem. In truss design, considered in Gürdal and Tatting (2000), equilibrium equations at each truss joint (CA cell) are used as local update rules along with fully stressed design formulation for the update of cross section areas. CA is demonstrated to be superior to current finite-element-based optimization technology, especially when geometric nonlinearities are included. Continuum modeling and topology optimization have been attempted in Tatting and Gürdal (2000). The two-dimensional continuum is reduced to an equivalent truss representation to which the equilibrium update rules of Gürdal and Tatting (2000) are applicable and the von Mises stress is used for updating the thickness. However, the equivalence between the truss and the continuum structure is exact only for Poisson's ratio equal to one third. In this paper, an energy approach to the derivation of local equilibrium update rules is used, allowing for greater flexibility.

The main objective of this paper is to demonstrate the use of CA for the design of continuum structures for a specified eigenvalue. This important class of problems includes the design of structures for a given buckling load or natural frequency. Although the proposed method is general enough to solve more complex continuum problems, the paper specifically addresses the design of Euler–Bernoulli columns against buckling. Extension of the present work to multidimensional problem is a straightforward task.

Historically, two major approaches to the solution of the column buckling design problem can be identified.

Received: 31 August 2001
Revised manuscript received: 16 October 2002
Published online: 16 January 2004

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The first approach is based on the continuous optimality criteria method and employs the calculus of variations. In this approach, the continuous distribution of cross section area is the unknown to be determined, and the moment of inertia of the cross section is assumed to be proportional to the area raised to a fixed power. Both geometrically constrained (designs with minimum area constraint) and geometrically unconstrained problems are considered in the literature for various boundary conditions. When the optimality conditions (a set of integrodifferential equations) cannot be analytically solved, some numerical approximation method is employed such as the finite element method (Szyszkowski et al. 1989). Although this approach leads to analytic solutions and considerable insight into the buckling design problem (Gajewski and Zyczkowski 1989), it is not generally used in practice because of the limited freedom in the choice of the type of column cross section and the difficulty of incorporating local constraints.

The second traditional approach, which is generally used for practical problems, is based on mathematical optimization. The column is divided into a number of finite elements, and the cross section area and moment of inertia (or other geometric dimensions) of each element are used as design variables. The problem is formulated as a mathematical programming problem, and classical optimization methods (Haftka and Gürdal 1993) are used to find the optimal solution. When attacked in this manner, eigenvalue design problems require a repetitive determination of eigenvalues of a potentially large system of equations within an outer loop of a design optimization formulation. When a large number of structural properties are used as design variables, this formulation is computationally intensive. For that reason, approximation methodologies (Canfield 1993) are frequently employed to reduce the required number of eigenvalue evaluations. Simultaneous analysis and design (SAND), as used in Shin et al. (1988) for buckling design, attempts to simultaneously solve the finite element equations and the mathematical optimization problem. Although SAND obviates the need for nesting design and analysis, it tends to produce large nonlinear systems that are difficult to solve.

More recently, novel approaches to eigenvalue design problems have been introduced. A genetic algorithm (GA) is used for buckling design of columns in Ishida and Sugiyama (1995). The use of GA does not seem to introduce much computational savings since it is not well suited for problems with large numbers of continuous variables.

CA, by its combined local analysis and design approach, circumvents the inefficiency noted above by arriving at an improved design while simultaneously performing analysis. The algorithm presented here does not require the determination of eigenvalues or eigenvectors, thus potentially providing large savings in computational time. Massively parallel implementation, which is naturally consistent with the CA paradigm, is expected to improve computational efficiency in the future.

2 Eigenvalue requirement design algorithm

The generic equations governing the structure are assumed to be of the form:

$$\mathcal{L}(\mathbf{d})\mathbf{u} = \lambda \mathcal{H}(\mathbf{d})\mathbf{u}, \tag{1}$$

where $\mathcal{L}$, $\mathcal{H}$ are operators and λ is a given eigenvalue. $\mathbf{u}$ represents the dependent (field) variables, while $\mathbf{d}$ represents the design variables; both are assumed to be defined over the domain of the problem Ω.

The operators $\mathcal{L}$ and $\mathcal{H}$ can be selected to describe structural problems in one, two, or three space dimensions. Proper selection of these operators depends on the structural theory being used. At this point they are left completely arbitrary.

To excite the system represented by Eq. 1, a fictitious source term $\mathbf{f}$ may be introduced to the right-hand side of Eq. 1 to give

$$\mathcal{L}(\mathbf{d})\mathbf{u} = \lambda \mathcal{H}(\mathbf{d})\mathbf{u} + \mathbf{f}. \tag{2}$$

Local to each point in the structure, a *stress measure* $\sigma(\mathbf{u}, \mathbf{d})$ is assumed to be defined in terms of the design and field variables. Also defined is a *strength measure*, $S(\mathbf{d})$. The weight of the structure (objective function to be minimized) is assumed to be represented in integral form as:

$$W = \int_{\Omega} \rho(\mathbf{d}) \, \mathrm{d}\Omega, \tag{3}$$

where $\rho(\mathbf{d})$ is a pointwise defined *density measure*. The iterative algorithm for the solution of both the field and design variables is:

Algorithm

1. Initialize $\mathbf{d}$, $\mathbf{f}$, and $\mathbf{u}$.
2. Solve the problem (at the $k+1$ iteration):

   $$\mathcal{L}\left(\mathbf{d}^k\right) \mathbf{u}^{k+1} = \lambda \mathcal{H}\left(\mathbf{d}^k\right) \mathbf{u}^k + \mathbf{f}^k, \tag{4}$$

   where $\left\{\mathbf{f}^k\right\}_{k=1}^{\infty}$ is such that

   $$\lim_{k\to\infty} \mathbf{f}^k = 0.$$

   This determines a new distribution of the field variables.
3. At each point solve the optimization problem:

   Minimize: $\rho\left(\mathbf{d}^{k+1}\right)$

   Subject to:

   $$\sigma\left(\mathbf{u}^{k+1}, \mathbf{d}^k\right) \le S\left(\mathbf{d}^{k+1}\right), \quad \mathbf{g}\left(\mathbf{d}^{k+1}\right) \le 0,$$

   where $\mathbf{g}(\mathbf{d})$ are pointwise defined side constraints.
4. Return to step 2 and repeat until convergence is achieved.

When local update rules are used in step 2, the algorithm becomes completely local in nature and thus completely consistent with the CA paradigm.

2.1
Example

To demonstrate the design algorithm in its simplest form, a single degree of freedom system is considered. Figure 1 depicts the system where a rigid pole of length L is hinged at the bottom and supported by two identical springs of stiffness K at its tip. The objective is to find the correct design of the springs to avoid instability under the specified compressive load P. The equilibrium equation for the system takes the form:

$$\underbrace{2\,K\,L\,\theta}_{\mathcal{L}\mathbf{u}} = \underbrace{P}_{\lambda}\,\underbrace{\theta}_{\mathcal{H}\mathbf{u}} + \underbrace{f}_{\mathbf{f}}, \qquad (5)$$

where the field variable is the pole rotation θ. It follows directly from Eq. 5 that the critical value of the spring stiffness is $K_{cr} = P/(2\,L)$.

To formulate the problem in the terms introduced earlier, the stress measure is taken as the force in the spring $\sigma = K\,L\,\theta$ and the strength measure is assumed to be proportional to the cross section area of the spring $S = S_o\,A$. The spring stiffness is also proportional to the spring area A used as the design variable (i.e., $K = C\,A$, where C is a constant). With these definitions, step 2 of the design algorithm takes the form

$$2\,K^k\,L\,\theta^{k+1} = P\,\theta^k + f^k. \qquad (6)$$

Solving for θ^{k+1} and simplifying we obtain:

$$\theta^{k+1} = \frac{K_{cr}}{K^k}\,\theta^k + \tilde{f}^k, \qquad (7)$$

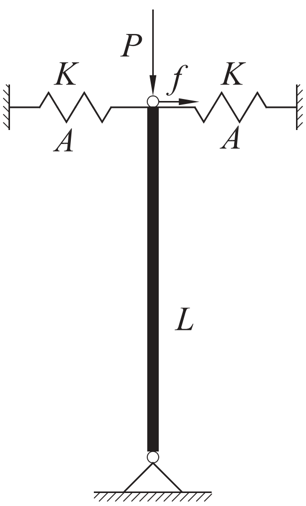

Fig. 1 A rigid pole supported by springs under compression

where $\tilde{f}^k = f^k/(2\,K^k\,L)$, and step 3 takes the form

Minimize: A^{k+1}.

Subject to: $K^k\,L\,\theta^{k+1} \le S_o A^{k+1}$.

The solution of this local optimization problem is simply:

$$A^{k+1} = K^k\,L\,\theta^k/S_o. \qquad (8)$$

From Eq. 7 we see that for $K^k < K_{cr}$ (*underdesign*) the response is *accentuated*, while for $K^k > K_{cr}$ (*overdesign*) the response is *attenuated*. This is the key point in the algorithm. Since the response of an underdesign is accentuated, the stress measure increases, and in the next design step the stiffness of the design is also increased, as seen from Eq. 8. The reverse happens when we have an overdesign. The net effect is that the algorithm converges to the correct stiffness to support the load.

Note that the introduction of a fictitious load f is not necessary for the algorithm to work. The only consequence of eliminating f altogether is that the solution procedure cannot be started from the undeflected position $\theta = 0$. Figure 2 shows the convergence of the algorithm for this simple case.

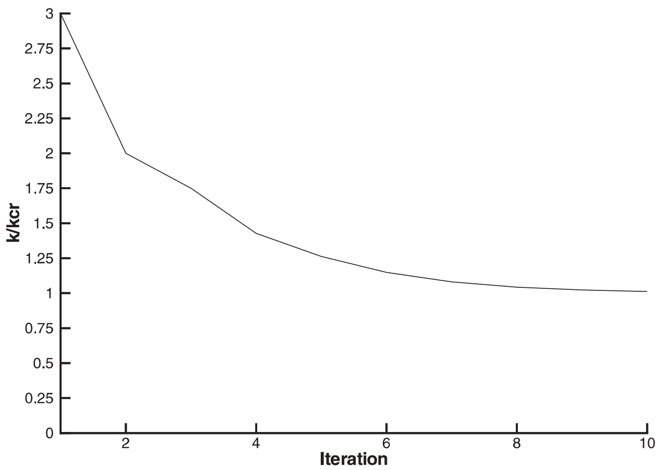

Fig. 2 Convergence of spring stiffness

3
Buckling design of columns

To illustrate the algorithm's ability to deal with practical problems, elastic column design for a specified buckling load is considered. Buckling design refers to finding the optimal material distribution of the column so that a given compressive load is supported without losing stability while minimizing the total volume of the column material. The governing equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI(\mathbf{d})\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = -P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + p(x), \quad 0 \le x \le L, \qquad (9)$$

where x is a coordinate along the column axis, E is Young's modulus of the column material, and I is the moment of inertia of the cross section that is assumed to be symmetric. Buckling in the plane of symmetry is exclusively considered. The dependent variable $w(x)$ is the lateral displacement, and $p(x)$ is a fictitious distributed load corresponding to $\mathbf{f}$ in Eq. 2. The stress measure is taken to be the bending moment defined by

$$M(x) = EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{10}$$

and the strength measure is taken as the maximum allowable stress S_{all} multiplied by the section modulus Z defined by

$$Z = \frac{I}{z}, \tag{11}$$

where z is the perpendicular distance between the extreme point of the cross section and the neutral axis. The weight of the column is given by

$$W = \int_0^L A(x)\,\mathrm{d}x; \tag{12}$$

thus the density measure is the cross section area $A(x)$.

In the absence of side constraints, the solution of the cell-level optimization problem is fully stressed; this gives the design update rule as

$$\frac{I}{z} = \frac{M}{S_{\text{all}}}. \tag{13}$$

3.1 Equilibrium update rules

Derivation of cell-level local update rules for field variables is a key step of any CA implementation. Each cell should be capable of finding its deformation state for any given deformation state of its neighbors. For column design, we obtain the update rules by discretizing the governing equation Eq. 9. The domain of the solution is divided into a number of cells. The beam cross section is assumed to be constant over each cell, and the field variables are associated with the midpoint of each cell as shown in Fig. 3. The field variables are bending displacement and rotation, $\mathbf{u} = (w, \theta)$. The neighborhood of each cell comprises the cell itself (C) and two neighbors, called left (L) and right (R) neighbors. The displacement field is considered to be of the form

$$w = w_i\,H_1(\xi) + h\,\theta_i\,H_2(\xi) + w_j\,H_3(\xi) + h\,\theta_j\,H_4(\xi), \tag{14}$$

where H_i are hermitian interpolation functions, $\xi = x/h$ is a nondimensional independent variable, and h is the lattice spacing. The displacement field is constructed in the form of Eq. 14 for the four different segments of the

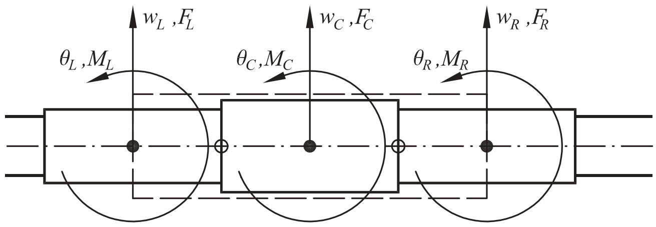

Fig. 3 Cell neighborhood. • cell variables, ○ auxiliary variables

control volume indicated by the dashed lines in Fig. 3 by introducing two auxiliary sets of cell variables associated with the middle left (ML) and the middle right (MR) points. Thus, the kinematic variables are

$$\mathbf{q} = \left(\mathbf{u}_C \,\middle|\, \mathbf{u}_{ML} \,\middle|\, \mathbf{u}_{MR}\right) \tag{15}$$

and the neighbor displacements are

$$\mathbf{p} = \left(\mathbf{u}_L \,\middle|\, \mathbf{u}_R\right). \tag{16}$$

The equilibrium equation (Eq. 9) is equivalent to the minimization of the total potential energy inside the control volume. The resulting equations are

$$\mathsf{K}\cdot\mathbf{q} = \mathsf{K}_{\mathrm{g}}\cdot\mathbf{q} + \mathsf{C}_{\mathrm{g}}\cdot\mathbf{p} - \mathsf{C}\cdot\mathbf{p} + \mathbf{f}_{ex}, \tag{17}$$

where the stiffness and geometric matrices $\mathbf{K}$ and $\mathbf{K}_{\mathrm{g}}$ are given by

$$\mathsf{K} = \frac{\partial^2 \Phi}{\partial\mathbf{q}\partial\mathbf{q}},\quad \mathsf{K}_{\mathrm{g}} = \frac{\partial^2 \Phi_{\mathrm{g}}}{\partial\mathbf{q}\partial\mathbf{q}}, \tag{18}$$

and the *clamp* matrices C and C_{g} are given by

$$\mathsf{C} = \frac{\partial^2 \Phi}{\partial\mathbf{p}\partial\mathbf{q}},\quad \mathsf{C}_{\mathrm{g}} = \frac{\partial^2 \Phi_{\mathrm{g}}}{\partial\mathbf{p}\partial\mathbf{q}}, \tag{19}$$

where the strain energies Φ and Φ_{g} are given by

$$\Phi = \int_{\Omega_c} EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x, \tag{20}$$

$$\Phi_{\mathrm{g}} = P\int_{\Omega_c}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x, \tag{21}$$

where Ω_c is the cell control volume. The external load vector $\mathbf{f}_{ex}$ represents the effect of $p(x)$. Since $p(x)$ is arbitrarily chosen, the load vector is assumed to consist of a concentrated force and couple at each cell, thus:

$$\mathbf{f}_{ex} = \left(F\ \ M \,\middle|\, 0\ \ 0 \,\middle|\, 0\ \ 0\right). \tag{22}$$

Since the external load at the auxiliary points is zero, equilibrium equations enable the elimination of the variables associated with these neighbors. This process is similar to static condensation. We start by partitioning $\mathbf{K}$ and $\mathbf{C}$ as

$$\mathsf{K} = \left[\begin{array}{c|c|c} \mathsf{K}_{11} & \mathsf{K}_{12} & \mathsf{K}_{13} \\ \hline \mathsf{K}_{12}^T & \mathsf{K}_{22} & 0 \\ \hline \mathsf{K}_{13}^T & 0 & \mathsf{K}_{33} \end{array}\right] \tag{23}$$

and

$$\mathsf{C} = \left[\begin{array}{c} \mathsf{C}_1 \\ \hline \mathsf{C}_2 \\ \hline \mathsf{C}_3 \end{array}\right] \tag{24}$$

with similar partitions for K_g and C_g. The auxiliary variables are thus eliminated as

$$\mathbf{u}_{ML} = -\mathsf{K}_{22}^{-1} \cdot \left(\mathsf{K}_{12}^T \cdot \mathbf{u}_C + \mathsf{C}_2 \cdot \mathbf{p}\right), \tag{25}$$

$$\mathbf{u}_{MR} = -\mathsf{K}_{33}^{-1} \cdot \left(\mathsf{K}_{13}^T \cdot \mathbf{u}_C + \mathsf{C}_3 \cdot \mathbf{p}\right). \tag{26}$$

This form of condensation neglects the geometric matrix contributions at the intermediate neighbors. This is deliberately done to make the geometric terms appear only as forcing terms. The consistent reduced equations of the system take the form

$$\tilde{\mathrm{K}} \cdot \mathbf{u}_C = \tilde{\mathrm{K}}_g \cdot \mathbf{u}_C + \tilde{\mathrm{C}}_g \cdot \mathbf{p} - \tilde{\mathrm{C}} \cdot \mathbf{p} + \tilde{\mathbf{f}}_{ex}, \tag{27}$$

where

$$\tilde{\mathsf{K}} = \mathsf{K}_{11} - \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{12}^T - \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{13}^T, \tag{28}$$

$$\begin{aligned}\tilde{\mathsf{K}}_g = {} & \mathsf{K}_{g_{11}} + \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{g_{22}} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{12}^T + \\ & \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{g_{33}} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{13}^T - \mathsf{K}_{g_{12}} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{12}^T - \\ & \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{g_{12}}^T - \mathsf{K}_{g_{13}} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{13}^T - \\ & \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{g_{13}}^T,\end{aligned} \tag{29}$$

$$\tilde{\mathsf{C}} = \mathsf{C}_1 - \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{C}_1 - \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{C}_3, \tag{30}$$

$$\begin{aligned}\tilde{\mathsf{C}}_g = {} & \mathsf{C}_{g_1} + \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{K}_{g_{22}} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{C}_2 + \\ & \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{K}_{g_{33}} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{C}_3 - \mathsf{K}_{g_{12}} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{C}_2 - \\ & \mathsf{K}_{12} \cdot \mathsf{K}_{22}^{-1} \cdot \mathsf{C}_{g_2} - \mathsf{K}_{g_{13}} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{C}_3 - \\ & \mathsf{K}_{13} \cdot \mathsf{K}_{33}^{-1} \cdot \mathsf{C}_{g_3},\end{aligned} \tag{31}$$

and

$$\tilde{\mathbf{f}}_{ex} = \left(F \;\; M\right). \tag{32}$$

Thus, after simplification the equilibrium relations for a cell, written exclusively in terms of its left and right neighbors, take the form

$$\frac{8EI_C}{h^3} \begin{bmatrix} S_{11} & -S_{12} \\ -S_{12} & S_{22} \end{bmatrix} \cdot \begin{Bmatrix} w_C \\ h\,\theta_C \end{Bmatrix} = \begin{Bmatrix} \tilde{F} \\ \tilde{M} \end{Bmatrix}, \tag{33}$$

where

$$\begin{aligned} S_{11} &= 12\,[c\,(1+c) + 2\,d\,(d-1) + d\,(28+15\,c)] \\ S_{12} &= 3\,(a-b)\,(3+c+11\,d) \\ S_{22} &= c\,(7+c) + d\,(196+21\,c) + 2\,d\,(d-1) \end{aligned} \tag{34}$$

$$a = EI_L/EI_C, \quad b = EI_R/EI_C, \tag{35}$$

$$c = a+b, \quad d = a\,b \tag{36}$$

$$\tilde{F} = F + F_g + F_e, \quad \tilde{M} = (M + M_g + M_e)/h \tag{37}$$

$$\begin{aligned} F_e = \frac{8EI_C}{h^3}[\,&6\,g_1(a)\,w_L + h\,g_2(a)\,\theta_L + 6\,g_1(b)\,w_R - \\ &h\,g_2(b)\,\theta_R\,] \end{aligned} \tag{38}$$

$$\begin{aligned} M_e = -\frac{8EI_C}{h^3}[\,&g_3(a)\,w_L - h\,g_1(a)\,\theta_L - g_3(b)\,w_R - \\ &h\,g_1(b)\,\theta_R\,] \end{aligned} \tag{39}$$

$$\begin{aligned} F_g = {} & \tilde{P}_1\,[\,f_2(a)\,w_C - h\,f_3(a)\,\theta_C - f_2(a)\,w_L - \\ & h\,f_4(a)\,\theta_L\,] + \tilde{P}_2\,[\,f_2(b)\,w_C + h\,f_3(b)\,\theta_C - \\ & f_2(b)\,w_R + h\,f_4(b)\,\theta_R\,], \end{aligned} \tag{40}$$

$$\begin{aligned} M_g = {} & -\tilde{P}_1\,[\,f_3(a)\,w_C - h\,f_5(a)\,\theta_C - f_3(a)\,w_L - \\ & h\,f_6(a)\,\theta_L\,] + \tilde{P}_2\,[\,f_3(b)\,w_C + h\,f_5(b)\,\theta_C - \\ & f_3(b)\,w_R - h\,f_6(b)\,\theta_R\,], \end{aligned} \tag{41}$$

$$\tilde{P}_1 = \frac{P}{30\,f_1^2(a)\,h}, \quad \tilde{P}_2 = \frac{P}{30\,f_1^2(b)\,h}, \tag{42}$$

and

$$\begin{aligned} f_1(r) &= 1 + 14\,r + r^2 \\ f_2(r) &= 72\,\left(1 + 12\,r + 102\,r^2 + 12\,r^3 + r^4\right) \\ f_3(r) &= 3\,\left(13 + 24\,r + 234\,r^2 - 16\,r^3 + r^4\right) \\ f_4(r) &= 3\,\left(1 - 16\,r + 234\,r^2 + 24\,r^3 + 13\,r^4\right) \\ f_5(r) &= 2\,\left(19 + 86\,r + 380\,r^2 + 26\,r^3 + r^4\right) \\ f_6(r) &= \left(1 - 100\,r - 58\,r^2 - 100\,r^3 + r^4\right) \\ g_1(r) &= 2\,r\,(1+r)/f_1(r) \\ g_2(r) &= 3\,r(1+3\,r)/f_1(r) \\ g_3(r) &= 3\,r(3+r)/f_1(r). \end{aligned} \tag{43}$$

When the applied axial load is not uniform (e.g., buckling of a column under its own weight) or design dependent (e.g., statically indeterminate frames), an additional axial degree of freedom can be added and updated using the axial equilibrium equation. In this paper, only columns with uniform applied compressive load are considered.

3.2 CA design algorithm

The equilibrium update rules of the previous section represent a discretization of the structural operators in Eq. 9. The next step in implementing the design algorithm is to calculate the stress measure, which will be used for the design update rule. Since boundary conditions are applied at the middle point of a cell, the bending moment is calculated at the endpoints of the cell to avoid numerical difficulties at free or hinged boundary conditions. The bending moment at the two ends of the cell is given by

$$M_L = \frac{EI_L}{f_1(a)h^2}\,(24\,(a-1)\,w_C - 4\,(a-5)\,h\,\theta_C -$$

$$24\,(a-1)\,w_L - 4\,(5\,a-1)\,\theta_L)\,, \qquad (44)$$

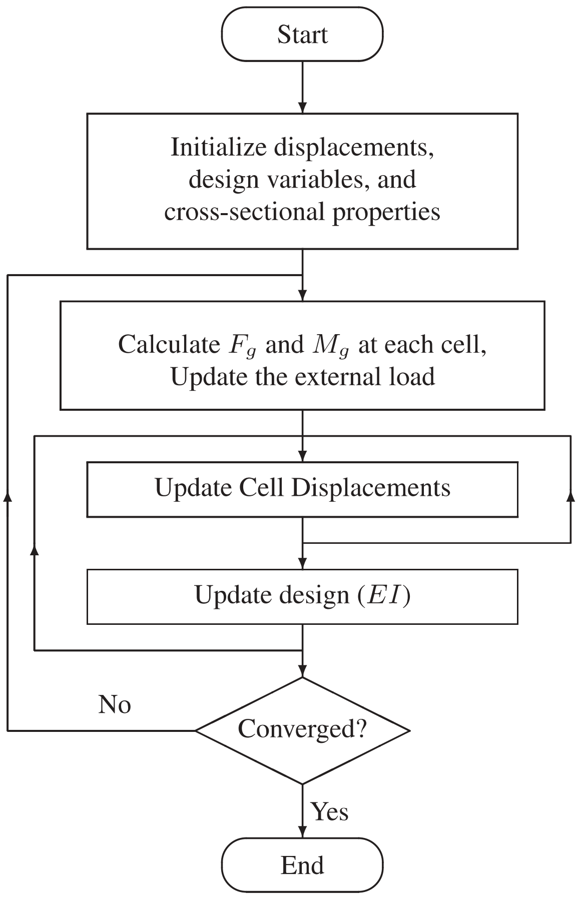

Fig. 4 Flow chart of the column design algorithm

$$M_R = \frac{EI_R}{f_1(b)h^2}\,(24\,(b-1)\,w_C + 4\,(b-5)\,h\,\theta_C -$$

$$24\,(b-1)\,w_R + 4\,(5\,b-1)\,h\,\theta_R). \qquad (45)$$

Thus, the stress measure is given by

$$M_{\max} = \max\{|M_L|, |M_R|\}. \qquad (46)$$

For geometrically similar cross sections, the design rule Eq. 13 simplifies to

$$A = k\left(\frac{M_{\max}}{S_{\text{all}}}\right)^{3/2}, \qquad (47)$$

where A is the area of the cross section and k is a constant that depends on the shape of the cross section (e.g., square, circular, ... etc.).

The flow chart of the design algorithm is shown in Fig. 4. Three nested loops can be identified. The innermost loop consists of cell-by-cell displacement (field variables) updates using Eq. 33. This loop is embedded in an intermediate loop in which the design is updated cell-by-cell using Eq. 13. Throughout these computations the loads (F, M, F_g and M_g) are kept fixed. The outermost loop comprises updating the local loads and checking for convergence. In our implementation, the inner loops are iterated a fixed number of times to reach a reasonable equilibrium distribution before updating the design. It was also found that underrelaxation of the geometric loads (F_g and M_g) is necessary. The amount of underrelaxation depends on the particular problem and boundary conditions.

4 Numerical examples

The following examples demonstrate the ability of the CA methodology to design continuum structures with constraints on eigenvalues. The examples cover a number of support conditions with and without geometric constraints.

4.1 Clamped-free column

First, we consider a clamped-free column. Following Tadjbakhsh and Keller (1962), we impose no minimum area constraint. The cross sections are assumed to be square, so that geometric similarity is satisfied. Figure 5 shows the normalized analytical optimal area distribution as compared to the CA design using 30 cells. The agreement is excellent except near the tip, where the cross section area vanishes and the analytic solution is singular. This is further illustrated in Fig. 6, which compares the mode shapes of the CA prediction (ten cells) and the analytically determined mode shape. The NASTRAN finite element simulation of the CA design is also shown;

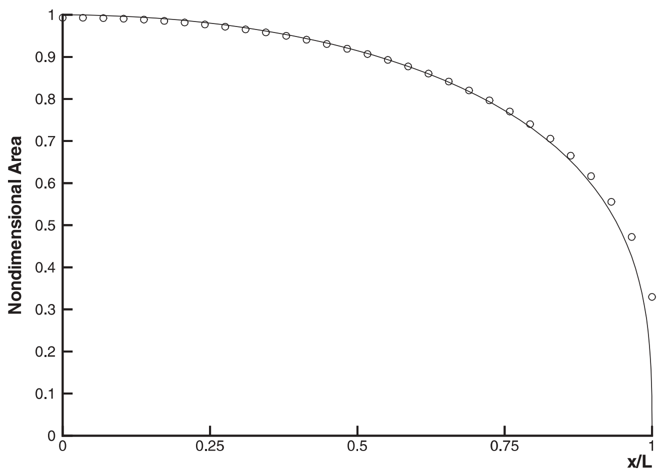

Fig. 5 Clamped-free column; area distribution of the optimal column. – Analytic solution, ∘ CA solution for 30 cells

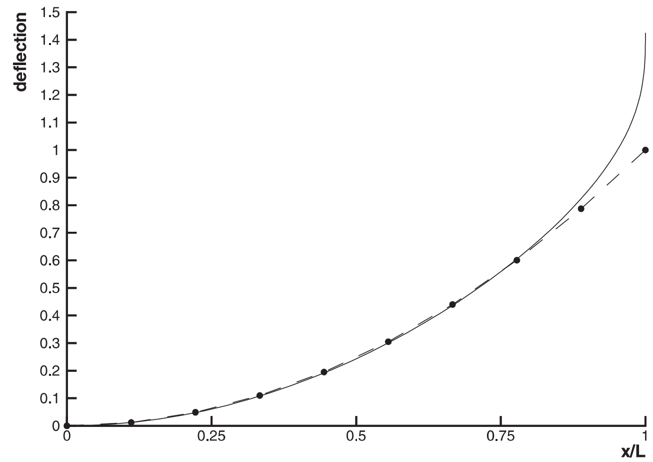

Fig. 6 Clamped-free column; mode shape for the optimal design. – Analytic solution, – – CA solution for 10 cells, • NASTRAN simulation of the CA design

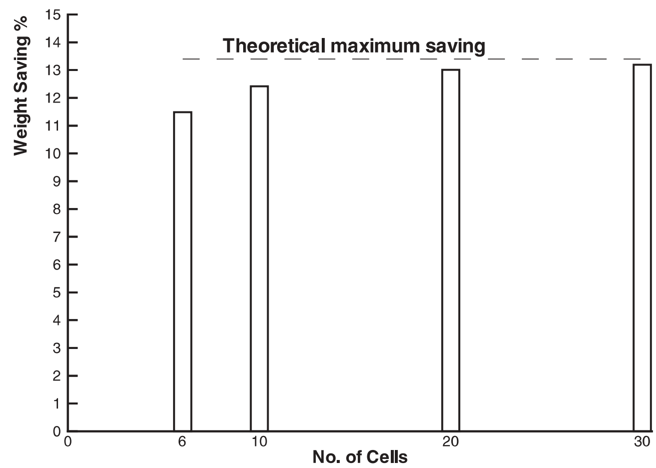

Fig. 7 Clamped-free column; design improvement vs. number of cells

it gives identical results to the CA predictions. Figure 7 depicts percentage weight saving (compared to uniform column design) using CA as the number of cells is varied. It is clear that, as the number of cells increases, the CA design approaches the theoretical maximum weight saving.

4.2
Simply supported column

The second example is a simply supported column with a minimum area constraint. The exact solution of Trahair and Booker (1970) is applicable to this case. A column of length $L = 1\,\text{m}$ made of aluminum (Young's modulus $E = 70\,\text{GPa}$ and maximum prebuckling stress $S_{\text{all}} = 270\,\text{MPa}$) is designed to support a $500\,\text{kN}$ compressive load. The cross sections are assumed to be square to maintain geometric similarity postulated in the analytic derivation. Due to symmetry, only half of the beam was discretized using ten cells. Figure 8 depicts the analytic and CA normalized area distributions. The CA design tends to add more material toward the pinned end and reduce the maximum cross section area below the analytic prediction. This is because of the coarse lattice used. The volume of the CA design is within 0.3% of the analytic optimal solution.

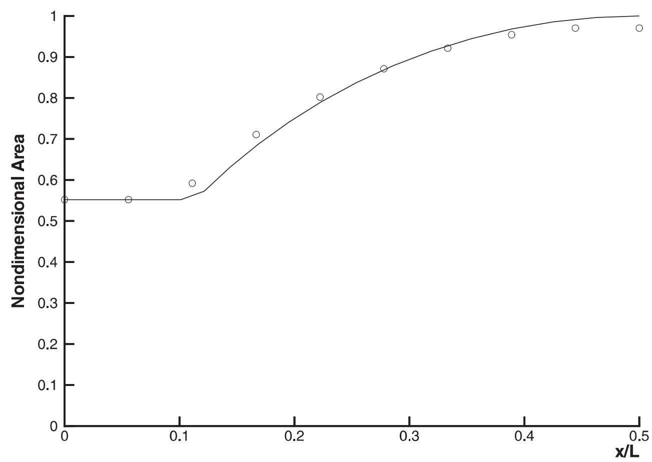

Fig. 8 Simply supported column; area distribution of the optimal column. – Analytic solution, ∘ CA solution for 10 cells

4.3
Clamped-clamped column

The third example is a clamped-clamped column. This problem was proclaimed solved in Tadjbakhsh and Keller (1962), but it was found later that this solution actually maximizes the second buckling mode and hence is not optimal. The actual optimum is bimodal as reported in Olhoff and Rasmussen (1977), meaning that the first two buckling modes have the same critical value. The column is discretized using 41 cells. The converged CA area distribution is plotted against the exact analytic solution of Olhoff and Rasmussen (1977) Fig. 9. The CA design is

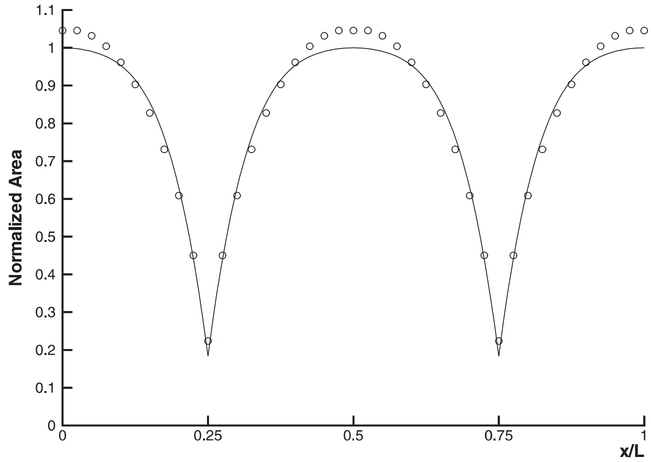

Fig. 9 Clamped-clamped column; area distribution of the optimal column. – Analytic solution, ○ CA solution for 41 cells

only 1% heavier than the analytic solution. A finite element analysis of the CA design using NASTRAN reveals that the critical load for the second buckling mode is only 3% higher than the first mode. The CA algorithm cannot handle bimodal optima without modification since it considers one mode only. For that reason, further refinement of the lattice causes the CA design to deviate from the bimodal optimum, following the symmetric mode (which becomes the second mode rather than the first). However, for practical purposes, the CA design is seen to approximate the bimodal optimum quite well.

4.4 Clamped-free column with manufacturing constraints

All the previous examples assume the cross sections are geometrically similar. This artificial restriction is removed in this final example. Consider a clamped-free column of rectangular cross section of height H and width W. The column length is 1 m and is made of aluminum (Young's modulus $E = 70$ GPa) and designed to support a 500 kN compressive load. The following manufacturing constraints are imposed:

$$H \geq H_{\min}, \quad W \geq W_{\min}, \quad \text{and} \ H \leq R\,W, \tag{48}$$

where $H_{\min} = 5$ cm, $W_{\min} = 5$ cm, and the maximum allowable aspect ratio $R = 10$.

The fully stressed condition (Eq. 13) evaluates to one of the following design points (see the sketch in Fig. 10 for the design domain arrangement for two different cases depending on the value of $M_{\max}$):

$$H = \left(\frac{6\,M_{\max}}{W_{\min}\,S_{\text{all}}}\right)^{1/2}, \quad W = W_{\min} \tag{49}$$

$$H = \left(\frac{6\,R\,M_{\max}}{S_{\text{all}}}\right)^{1/3}, \quad W = H/R \tag{50}$$

$$W = \left(\frac{6\,M_{\max}}{H_{\min}^2\,S_{\text{all}}}\right), \quad H = H_{\min}. \tag{51}$$

Another candidate solution is:

$$W = W_{\min}, \quad H = H_{\min}. \tag{52}$$

Of the feasible candidate solutions the one with minimum area is chosen.

Since to the authors' knowledge no analytic solution exists for this problem, the CA design is compared to the design obtained from traditional finite-element-based software GENESIS (Vandeplaats Research & Development 1998). The column is divided into 10 cells for the CA

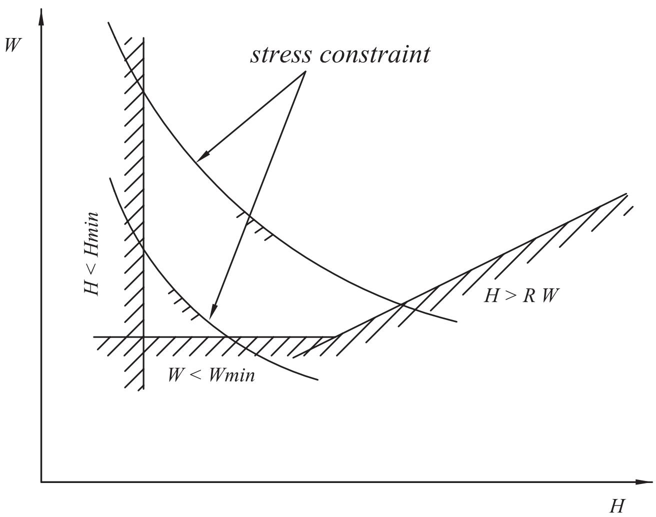

Fig. 10 Design domain with manufacturing constraints

Table 1 CA vs. GENESIS designs (dimensions in cm)

Cell	CA		GENESIS	
	H	W	H	W
1†	14.402	1.440	14.635	1.468
2†	14.402	1.440	14.687	1.473
3†	14.283	1.428	14.444	1.449
4†	14.040	1.404	14.078	1.412
5†	13.659	1.366	13.605	1.365
6†	13.116	1.312	13.004	1.304
7†	12.364	1.236	12.178	1.221
8†	11.313	1.131	10.959	1.099
9‡	9.631	1.000	9.385	1.000
10‡	5.731	1.000	5.168	1.000

† Aspect ratio constraint active.
‡ Minimum width constraint active

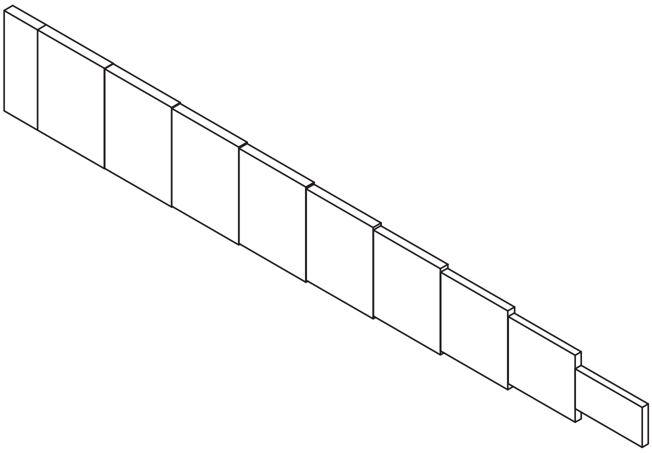

Fig. 11 Clamped-free column; 3D view of CA design

design and 21 elements for GENESIS linked to only 20 design variables to correspond directly to the CA model. Table 1 contains the results of both methods and indicates the active constraints. The total volume of the CA design, shown in Fig. 11, is 1640.7 cm^3 as compared to 1640 cm^3 for GENESIS design. The agreement is satisfactory between the two designs, and they predict the same active constraints.

5 Conclusion

An algorithm based on local rules for both analysis and design is proposed to solve structural design problems with eigenvalue requirements. The local nature of the algorithms lends it to Cellular Automata (CA)-type implementation. This gives the algorithm the benefit of being easily implemented on parallel architectures. To demonstrate the ability of the algorithm to handle practical problems, column design for buckling is considered. The numerical examples show that the CA design algorithm converges to the analytic optimum for columns made of geometrically similar cross sections, with and without geometric constraints. A design example with manufacturing constraints is also considered, and the CA design compares very favorably to the design determined through classical optimization coupled to finite element analysis. Although the algorithm is not yet implemented on parallel architecture, considerable savings can be gained by parallel implementation for large problems.

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